Thus, if $f(x)$ is not real-rooted, then the quadratic form Q_f is not positive semi-definite. This completes the proof.

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105.07 A golden ratio enigma

The ancient golden ratio

$$\phi = \frac{1 + \sqrt{5}}{2}$$

arises in several ways, upon comparing number ratios [1], or a solution to a quadratic, or the limiting ratio in Fibonacci recursion, or the dominant eigenvalue of a certain matrix [2]. But a seemingly harmless generalisation given by

$$\phi_n = \frac{1 + \sqrt{1 + \frac{4}{n}}}{2},$$

so that $\phi_1 = \phi$, leads to an enigma.

Euclid considered comparing numbers (segments) $a > b > 0$ as 'greater-to-lesser' in two ways:

$$\frac{a + b}{a} = \frac{a}{b} \equiv \phi$$

which leads to the quadratic

$$\phi^2 - \phi - 1 = 0$$

whose positive solution is indeed the Golden Ratio. The generalisation to ϕ_n comes about by altering Euclid's construction as

$$\frac{na + b}{na} = \frac{a}{b} \equiv \phi_n$$

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so that the quadratic

$$\phi_n^2 - \phi_n - \frac{1}{n} = 0$$

produces ϕ_n as a solution. But here's the problem.

Clearly

$$\lim_{n \to \infty} \phi_n = 1.$$

Curiously then, 'stretching' a in this generalisation – while retaining the Golden Ratio Rule – eventually 'shrinks' a to b, though strict equality is clearly impossible; so this Rule breaks down in the limit.

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105.08 An infinitesimal definition of limit

The concept of limit is crucial in calculus. The pioneers of calculus assumed that they had a system that included the real numbers and some extremely tiny 'numbers', which we call infinitesimals. However, they could not provide a convincing justification for their assertions. In order to make calculus more rigorous, mathematicians have used the epsilon–delta definition of limit. However, many students find this difficult.

In the 1990s, Robinson [1] provided a justification of the pioneers' approach (and much more). The aim of this Note is to provide a simpler and easier justification. We will define a set S, which has a copy of the real numbers and also has elements that we call 'infinitesimals'. Then we provide a criterion for the limit of $f(x)$ as x tends to a, which is equivalent to the epsilon-delta definition.

Definition 1: We define S to be the set of all infinite sequences of real numbers. If α is a sequence equal to $(a_1, a_2, a_3, \ldots)$ then we call a_i the ith term of the sequence α. If β is a sequence where the ith term is b_i, then we define $\alpha + \beta$ to be the sequence whose ith term is $a_i + b_i$, and we define $\alpha\beta$ to be the sequence whose ith term is $a_i b_i$.

We note that S satisfies rules such as the distributive law, i.e. $\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma$. To see this, let α, β, γ have ith terms a_i, b_i, c_i respectively. Then the ith term of $\alpha(\beta + \gamma)$ is a_i times the ith term of $\beta + \gamma$, i.e. $a_i(b_i + c_i) = a_i b_i + a_i c_i$. This is the ith term of $\alpha\beta + \alpha\gamma$. Hence $\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma$. In a similar way, we can show that $\alpha\beta = \beta\alpha$, $\alpha + \beta = \beta + \alpha$, $\alpha(\beta\gamma) = (\alpha\beta)\gamma$.

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