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Effective λ-models versus recursively enumerable λ-theories

CHANTAL BERLINE[†], GIULIO MANZONETTO[‡] and ANTONINO SALIBRA[§]

[†]*CNRS, Laboratoire PPS, Université Paris 7, 2, place Jussieu (case 7014), 75251 Paris Cedex 05, France*
Email: chantal.berline@pps.jussieu.fr
[‡]*INRIA*
Email: giulio.manzonetto@inria.fr
[§]*Università Ca'Foscari di Venezia, Dipartimento di Informatica, Via Torino 155, 30172 Venezia, Italy*
Email: salibra@dsi.unive.it

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A longstanding open problem is whether there exists a non-syntactical model of the untyped λ-calculus whose theory is exactly the least λ-theory λ_β. In this paper we investigate the more general question of whether the equational/order theory of a model of the untyped λ-calculus can be recursively enumerable (r.e. for short). We introduce a notion of *effective model* of λ-calculus, which covers, in particular, all the models individually introduced in the literature. We prove that the order theory of an effective model is never r.e.; from this it follows that its equational theory cannot be λ_β or $\lambda_{\beta\eta}$. We then show that no effective model living in the stable or strongly stable semantics has an r.e. equational theory. For Scott's semantics, we investigate the class of graph models and prove that no order theory of a graph model can be r.e., and that there exists an effective graph model whose equational/order theory is the minimum among the theories of graph models. Finally, we show that the class of graph models enjoys a kind of downwards Löwenheim–Skolem theorem.

1. Introduction

1.1. *Lambda-theories and lambda-models*

λ-theories These are, by definition, the congruences on the set of λ-terms that contain β-conversion (λ_β); *extensional* λ-theories are those that contain $\beta\eta$-conversion ($\lambda_{\beta\eta}$). λ-theories can arise from syntactic or semantic considerations. Indeed, a λ-theory $\mathcal{T}$ may correspond to a possible operational (observational) semantics of λ-calculus, or it may be induced by a model $\mathscr{M}$ of λ-calculus through the kernel congruence relation of the interpretation function (then we will say that $\mathscr{M}$ represents $\mathcal{T}$, and we will write $Th(\mathscr{M}) = \mathcal{T}$). Although researchers have, until recently, mainly focused their attention on a limited number of them, the set of λ-theories ordered by inclusion constitutes a very rich, interesting and complex mathematical structure, whose cardinality is $2^{\aleph_0}$ (see Barendregt (1984), Berline (2000; 2006) and Lusin and Salibra (2004)).

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λ-models After the first model, found by Scott in 1969 in the category of complete lattices and Scott continuous functions, a large number of mathematical models for λ-calculus arising from syntax-free constructions have been introduced in various categories of domains and were classified into semantics according to the nature of their representable functions, see, for example, Barendregt (1984), Berline (2000) and Plotkin (1993). Scott's continuous semantics (Scott 1972) is given in the category whose objects are complete partial orders and morphisms are Scott continuous functions. The stable semantics (Berry 1978) and the strongly stable semantics (Bucciarelli and Ehrhard 1991b) are refinements of the continuous semantics, which were introduced to approximate the notion of 'sequential' Scott continuous function. Finally, 'weakly continuous' semantics have been introduced, either for modelling non-determinism or for foundational purposes. In each of these semantics all the models come equipped with a partial order, and some of them, called *webbed models*, are built from lower level structures called 'webs'. The simplest class of webbed models is the class of *graph models*, which was isolated in the seventies by Plotkin, Scott and Engeler within the continuous semantics. The class of graph models contains the simplest non-syntactical models of λ-calculus (beginning with Engeler's model $\mathscr{E}$) and is itself the easiest describable class, but, nevertheless, represents $2^{\aleph_0}$ (non-extensional) λ-theories. The results previously obtained for the class of graph models are surveyed in Berline (2006). Another example of a class of webbed models, and the most established one, is the class of *filter models*. It was isolated at the beginning of the eighties by Coppo, Dezani, Honsell and Longo in Scott's continuous semantics[†] (Coppo *et al.* 1984), after the introduction of the intersection type discipline by Coppo and Dezani (Coppo and Dezani-Ciancaglini 1980).

1.2. *The problems we are interested in*

The initial problem The question of the existence of a non-syntactical model of λ_β (or $\lambda_{\beta\eta}$) has been circulating since at least the beginning of the eighties[‡], but it was only first raised in print in Honsell and Ronchi Della Rocca (1992). This problem is still open, but it has generated a wealth of interesting research and results (surveyed in Berline (2000) and Berline (2006)) – we will only sketch below the work relevant to the present paper.

The first results In 1995, Di Gianantonio, Honsell and Plotkin succeeded in building an extensional model having theory $\lambda_{\beta\eta}$ and living in some weakly continuous semantics (Di Gianantonio *et al.* 1995). However, the construction of this model as an inverse limit starts from the term model of $\lambda_{\beta\eta}$, and hence involves the syntax of λ-calculus. Furthermore, the existence of a model living in Scott's semantics itself, or in one of its two refinements, remains completely open. Nevertheless, the authors also proved in Di Gianantonio *et al.* (1995) that the set of extensional theories representable by models living in Scott's semantics had a least element. At the same time, Selinger proved that if an

[†] Variations of some filter models adequate for the stable semantics were introduced later in Honsell and Ronchi Della Rocca (1990) and Pravato *et al.* (1997).

[‡] See Problem 22 in the list of TLCA open problems given in Honsell (2007).

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ordered model has theory λ_β or $\lambda_{\beta\eta}$, then the order must be discrete on the interpretations of λ-terms (Selinger 2003).

First extension: the minimality problem Given the second result of Di Gianantonio *et al.* (1995), it becomes natural to ask whether, given a (uniformly presented) class of models of λ-calculus, there is a minimum λ-theory represented in it; a question which was raised in Berline (2000). Bucciarelli and Salibra (2004; 2008) showed that the answer is also positive for the class of *graph models*, and that the least *graph theory* (theory of a graph model) was different from λ_β and, of course, $\lambda_{\beta\eta}$. The problem currently remains open for the other classes of models.

Each class of models represents and omits $2^{\aleph_0}$ *λ-theories* Ten years ago it was proved that in each of the known (uniformly presented) classes $\mathfrak{C}$ of models living in any of the above mentioned semantics, and beginning with the class of graph models, it is possible to build $2^{\aleph_0}$ (webbed) models inducing pairwise distinct λ-theories (Kerth 1998b; 2001). More recently, it was proved in Salibra (2003) that there are $2^{\aleph_0}$ theories that are omitted by all the $\mathfrak{C}$'s, among which $\aleph_0$ are finitely axiomatisable over λ_β.

From these results, and since there are only $\aleph_0$ recursively enumerable theories (r.e. for short), it follows that each $\mathfrak{C}$ represents $2^{\aleph_0}$ non-r.e. theories and omits $\aleph_0$ r.e. theories. Note also that there are only a very small number of theories of non-syntactical models that are known to admit an alternative description (for example, via syntactical considerations), and that all happen to coincide either with the theory $\mathcal{B}_\mathcal{T}$ of Böhm trees (Barendregt 1984) or some variations of it, and hence are non-r.e. This leads us to raise the following problem, which is a second natural generalisation of the initial problem.

Can a non-syntactical model have an r.e. theory? This problem was first raised in Berline (2006), where it was conjectured that no graph model can have an r.e. theory. But we expect that this could indeed be true for all λ-models living in the continuous semantics, or in its refinements (but of course not in its weakenings, because of Di Gianantonio *et al.* (1995)), and in the present paper we officially extend this conjecture.

Conjecture 1. No λ-model living in Scott's continuous semantics or in one of its refinements has an r.e. equational theory.

1.3. *Methodology*

1 *Look also at order theories.*
Since all the models we are interested in are partially ordered, and since, in this case, the equational theory $Th(\mathscr{M})$ is easily expressible from its order theory $Th_\sqsubseteq(\mathscr{M})$ (in particular, if $Th_\sqsubseteq(\mathscr{M})$ is r.e., then $Th(\mathscr{M})$ is r.e. also). We will also address the analogous problem for order theories.

2 *Look at models with built-in effectivity properties.*
There are several reasons for doing this. First, this approach allows us to apply the powerful tools that have been developed in the field of recursion theory to the study

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of the λ-models; second, because all the models that have been individually studied or given as examples in the literature are effective, in our sense. Starting from the known notion of an effective domain, we introduce an appropriate notion of an *effective model of λ-calculus* and study the main properties of these models[†]. Note that effective models happen to be rare in absolute terms, since each 'uniform' class $\mathfrak{C}$ of λ-models represents $2^{\aleph_0}$ theories, but contains only $\aleph_0$ non-isomorphic effective models! However, and this is a third *a posteriori* reason to work with them, it happens that they can be used to prove properties of non-effective models (Theorem 4 below is the first example we know of such a result).

3 A previous result obtained for *typed λ-calculus* also justifies the above methodology. Indeed, it was proved in (Berardi and Berline 2002) that there exists a (webbed) model of Girard's System F living in Scott's continuous semantics whose theory is the typed version of $\lambda_{\beta\eta}$, and whose construction does not involve the syntax of λ-calculus. Furthermore, this model can easily be checked to be 'effective' in the same spirit as in the present paper (see Berardi and Berline (2002, Appendix C) for a sketchy presentation of the model). Note that this model has no analogue in the stable semantics.

4 *Look at the class of graph models.*
Studying graph models illustrates the spirit of the tools we aim to develop, while keeping technicalities at the lowest possible level. Recall from a remark above that although a graph model may be effective, most of them are not.

5 *Prove a Löwenheim–Skolem theorem.*
Effective webbed models are, in particular, generated by countable webs. Hence, a key step for attacking the general conjecture is to prove that the order/equational theory of any webbed model can be represented by a model of the same kind but having a countable web. We will prove this here for graph models.

6 *Mention when the results extend to some other class of webbed models, and when they do not* (sometimes we do not know).
All the classes of webbed models do indeed appear to be (more or less) sophisticated variations of the class of graph models. We will not work out the details since this would lead us to far and would be tedious, and provides no special added interest. Our program is rather to search for generic tools, and our first success in this direction is a meta-Löwenheim–Skolem theorem whose proof will be given in a further paper.

1.4. *Main results and derived conjectures*

1.4.1. *For effective models*

The central technical device here is Visser's result (Visser 1980) stating that the complements of β-closed r.e. sets of λ-terms enjoy the finite intersection property (Theorem 5.4). We will be able to prove the following theorem.

[†] As far as we know, only Giannini and Longo have introduced a notion of an effective model (Giannini and Longo 1984). Moreover, their definition is *ad hoc* for two particular models (Scott's P_ω and Plotkin's T_ω) and their results depend on the fact that these models have a very special common theory, namely $\mathcal{B}_\mathcal{T}$.

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Theorem 1. Let $\mathscr{M}$ be an *effective* model of λ-calculus living in the continuous, stable or strongly stable semantics. Then:

(i) $Th_{\sqsubseteq}(\mathscr{M})$ is not r.e.
(ii) $Th(\mathscr{M}) \neq \lambda_\beta, \lambda_{\beta\eta}$.
(iii) If $\bot_{\mathscr{M}}$ is λ-definable, then $Th(\mathscr{M})$ is not r.e.

More generally:

(iv) If $\mathscr{M}$ has a λ-definable element that is 'decidable' and has only finitely many λ-definable elements below it, then $Th(\mathscr{M})$ is not r.e.

The items above correspond to Corollary 9.44, Corollary 9.46, Proposition 9.35 and Theorem 9.39, respectively, plus Claim 10.1.

As for the existence of a non-syntactical effective model with an r.e. equational theory, we are, furthermore, able to give a definite negative answer for the stable and strongly stable semantics.

Theorem 2 (Theorem 10.3). If $\mathscr{M}$ is an effective model living in the stable or in the strongly stable semantics, then $Th(\mathscr{M})$ is not r.e.

This theorem answers Conjecture 1 when restricted to effective models for these two semantics. The problem looks much more difficult for Scott's semantics, and we will just concentrate on the class of graph models.

1.4.2. *For graph models*

Theorem 3 (Theorem 11.12). There exists an *effective* graph model whose equational/order theory is the minimum graph theory.

Theorem 4 (Theorem 11.25). If $\mathscr{M}$ is a graph model, then $Th_{\sqsubseteq}(\mathscr{M})$ is not r.e.

We emphasise that Theorem 4, which happens to be a consequence of Theorem 3, plus the work on effective models concerns all the graph models and not only the effective ones. For the equational theories of graph models, we only give below, as Theorem 5, the flashiest example of the results we will prove in Section 11.3. However, the stronger versions are natural, and needed for covering all the traditional models (for example, the Engeler model is only covered by Theorem 5 below if it is generated from a finite set of atoms, though it is well known that its theory is $\mathcal{B}_{\mathcal{T}}$ independently of the number of its atoms).

Theorem 5 (Theorem 11.11). If $\mathscr{M}$ is a graph model that is 'freely generated from a finite partial web', then $Th(\mathscr{M})$ is not r.e.

Whether the minimum equational graph theory is r.e. remains an open question. Hence, the following instances of Conjecture 1 are still open; we state them in order from the weakest to the strongest.

Conjecture 2. The minimum equational graph theory is not r.e.

Conjecture 3. All the effective graph models have non-r.e. equational theories.

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Conjecture 4. All the effective models living in the continuous semantics have non-r.e. equational theories.

The following further theorem states that graph models with countable webs are sufficient for representing all graph theories. This can be viewed as a kind of Löwenheim–Skolem Theorem for graph models (see Section 8 for more comments).

Theorem 6 (Theorem 8.2). For any graph model $\mathscr{G}$ there is a graph model $\mathscr{G}'$ with a countable web and the same order theory (and hence the same equational theory).

This result provides a positive answer to Problem 12 in Berline (2006).

The more general problem concerning all known classes of webbed models appeared previously as Question 3 in Berline (2000, Section 6.3). We are now able to give a full positive answer to this question by relying on a more conceptual proof, but we will keep this development for a later work.

1.5. *Structure of the paper*

The remainder of this paper is divided into three parts plus some appendices.

Part 1 deals with preliminaries in Sections 2–5, which covers general definitions and results, the untyped λ-calculus, models of λ-calculus, and recursion in λ-calculus, respectively.

Part 2 covers graph models and partial pairs in Sections 6–8. These deal with the category of partial pairs, the minimum order and equational graph theories, and a Löwenheim–Skolem theorem for graph models, respectively.

Part 3 is concerned with effective λ-models in Sections 9–11. These cover effective λ-models in Scott-continuous semantics, effective stable and strongly stable λ-models, and effective graph models, respectively.

Appendix A considers the Visser topology on Λ, and Appendix B shows that gluings are sometimes weak products.

This paper is an expanded version of *Lambda theories of effective lambda-models* (Berline *et al.* 2007). As well as containing more proofs, explanations and examples, the current paper also contains some deeper results, such as Theorem 11.11 and its corollaries.

Part I: Preliminaries

2. Generalities

To keep this article as self-contained as possible, in this part of the paper we summarise some definitions and results that we will use later on. We will generally use the notation of Barendregt's classic work Barendregt (1984) for λ-calculus.

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2.1. *Sets, functions and groups of automorphisms*

We will use $\mathbb{N}$ to denote the set of natural numbers and p_k to denote the k-th prime number. If X is a set, $\mathscr{P}(X)$ (respectively, X^*) is the set of all subsets (respectively, finite subsets[†]) of X. We write $X \subseteq_{\mathrm{f}} Y$ to mean that X is a finite subset of Y.

For any function f, we write $\mathrm{dom}(f)$ for the domain of f, $\mathrm{rg}(f)$ for its range, $\mathrm{graph}(f)$ for its graph, and $f\restriction_X$ for its restriction to a subset $X \subseteq \mathrm{dom}(f)$. We define the *image* and the *inverse image* of X via f as $f^+(X) = \{f(x) : x \in X\}$ and $f^-(X) = \{x : f(x) \in X\}$, respectively. The *partial inverse* of an injective function f, denoted f^{-1}, is defined by $\mathrm{dom}(f^{-1}) = \mathrm{rg}(f)$ and $f^{-1}(x) = y$ if $f(y) = x$.

Let f, g be two partial functions. Then:

— f and g are *compatible* if $f(x) = g(x)$ for all $x \in \mathrm{dom}(f) \cap \mathrm{dom}(g)$.
— $f \cap g$ denotes the function whose graph is $\mathrm{graph}(f) \cap \mathrm{graph}(g)$.
— If f, g are compatible, we use $f \cup g$ to denote the function whose graph is $\mathrm{graph}(f) \cup \mathrm{graph}(g)$.
— $f(x) \simeq g(y)$ abbreviates:
  - $f(x)$ is undefined if and only if $g(y)$ is undefined; and
  - $f(x) = g(y)$ if they are both defined.

Given any mathematical structure $\mathcal{S}$ having a carrier set S, we use $Aut(\mathcal{S})$ to denote the group of all the automorphisms of $\mathcal{S}$. For all $s \in S$, the *orbit* $O(s)$ with respect to $Aut(\mathcal{S})$ is defined by $O(s) = \{\theta(s) : \theta \in Aut(\mathcal{S})\}$. A structure $\mathcal{S}$ is *finite modulo* $Aut(\mathcal{S})$ if the number of orbits of $\mathcal{S}$ with respect to $Aut(\mathcal{S})$ is finite.

2.2. *Recursion theory*

We write $\varphi_n : \mathbb{N} \to \mathbb{N}$ for the partial recursive function of index n and use $\mathcal{W}_n$ to denote the domain of φ_n. A set $E \subseteq \mathbb{N}$ is *recursively enumerable* (r.e. for short) if it is the domain of a partial recursive function. The complement E^c of an r.e. set E is said to be *co-r.e.* If both E and E^c are r.e., then E is said to be *decidable*. Note that the collection of all r.e. (co-r.e.) sets is closed under finite union and finite intersection.

We say that ν is an *encoding* of a countable set X if $\nu : X \to \mathbb{N}$ is bijective. A *numeration* γ is a pair (X, ν_X) such that $\nu_X : \mathbb{N} \to X$ is total and onto. Thus, the inverse of an encoding is a special case of numeration. A set $Y \subseteq X$ is *r.e.* (respectively, *co-r.e.*) with respect to ν_X if the set $\nu_X^-(Y)$ is r.e. (respectively, co-r.e.).

Given two numerations (X, ν_X) and (Y, ν_Y), we say that a partial recursive function φ *tracks* $f : X \to Y$ with respect to ν_X, ν_Y if the following diagram commutes:

$$\begin{array}{ccc} \mathbb{N} & \xrightarrow{\varphi} & \mathbb{N} \\ {\scriptstyle \nu_X}\downarrow & & \downarrow{\scriptstyle \nu_Y} \\ X & \xrightarrow[f]{} & Y \end{array}$$

[†] Following Krivine (1993), and many later papers, we use the notation X^* instead of the more usual notation $\mathscr{P}_{\mathrm{fin}}(X)$ because it is more concise and we will need to use it often.

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A function $f : X \to Y$ is said to be *computable* (with respect to ν_X, ν_Y) if there exists φ tracking f with respect to ν_X, ν_Y. From now on, we suppose that a computable encoding $\langle -, - \rangle : \mathbb{N}^2 \to \mathbb{N}$ for the pairs is fixed. Moreover, we fix an encoding $\#_* : \mathbb{N}^* \to \mathbb{N}$ that is effective in the sense that the relations $m \in \#_*^{-1}(n)$ and $m = card(\#_*^{-1}(n))$ are decidable in (m, n). Finally we set $\ll -, - \gg : \mathbb{N}^* \times \mathbb{N} \to \mathbb{N}$ defined as $\ll a, n \gg = \langle \#_*(a), n \rangle$. We recall here a basic property of recursion theory, which we will often use later in the paper.

Remark 2.1. The inverse image of an r.e. set via a computable map is r.e.

2.3. *Partial orderings*

Let $(\mathcal{D}, \sqsubseteq_{\mathcal{D}})$ be a partially ordered set (poset, for short). When there is no ambiguity, we write $\mathcal{D}$ instead of $(\mathcal{D}, \sqsubseteq_{\mathcal{D}})$. Two elements u and v of $\mathcal{D}$ are:

— *comparable* if either $u \sqsubseteq_{\mathcal{D}} v$ or $v \sqsubseteq_{\mathcal{D}} u$;
— *compatible* if they have an upper bound, that is, there exists z such that $u \sqsubseteq_{\mathcal{D}} z$ and $v \sqsubseteq_{\mathcal{D}} z$.

Let $A \subseteq \mathcal{D}$ be a set. A is *upward* (respectively, *downward*) *closed* if $v \in A$ and $v \sqsubseteq_{\mathcal{D}} u$ (respectively, $u \sqsubseteq_{\mathcal{D}} v$) imply $u \in A$. A is *directed* if for all $u, v \in A$ there exists $z \in A$ such that $u \sqsubseteq_{\mathcal{D}} z$ and $v \sqsubseteq_{\mathcal{D}} z$.

A poset $\mathcal{D}$ is a *complete partial order* (*cpo*, for short) if it has a least element (denoted $\bot_{\mathcal{D}}$) and every directed set $A \subseteq \mathcal{D}$ admits a least upper bound (denoted $\bigsqcup A$). A cpo is *bounded complete* if $\bigsqcup\{u, v\}$ exists for all compatible elements u, v. An element $d \in \mathcal{D}$ is said to be *compact* if for every directed $A \subseteq \mathcal{D}$ we have $d \sqsubseteq_{\mathcal{D}} \bigsqcup A$ implies $d \sqsubseteq_{\mathcal{D}} v$ for some $v \in A$. We write $\mathcal{K}(\mathcal{D})$ for the collection of compact elements of $\mathcal{D}$. A cpo $\mathcal{D}$ is *algebraic* if for every $u \in \mathcal{D}$ the set $\{d \in \mathcal{K}(\mathcal{D}) : d \sqsubseteq_{\mathcal{D}} u\}$ is directed and u is its least upper bound. An algebraic cpo $\mathcal{D}$ is said to be *ω-algebraic* when $\mathcal{K}(\mathcal{D})$ is countable. A bounded complete ω-algebraic cpo is called a *Scott domain*. A compact element $p \neq \bot_{\mathcal{D}}$ of a Scott domain $\mathcal{D}$ is *prime* if for all compatible $u, v \in \mathcal{D}$ we have $p \sqsubseteq_{\mathcal{D}} u \sqcup v$ implies $p \sqsubseteq_{\mathcal{D}} u$ or $p \sqsubseteq_{\mathcal{D}} v$. We use $\mathcal{P}(\mathcal{D})$ to denote the set of prime elements of $\mathcal{D}$. A Scott domain $\mathcal{D}$ is *prime algebraic* if for all $u \in \mathcal{D}$ we have $u = \bigsqcup\{p \in \mathcal{P}(\mathcal{D}) : p \sqsubseteq_{\mathcal{D}} u\}$.

The simplest examples of prime algebraic domains are the *flat domains* and the *powerset domains*. If D is a set and $\bot$ an element not belonging to D, the *flat domain* $D_\bot$ is, by definition, the poset $(\mathcal{D}, \sqsubseteq_{\mathcal{D}})$ such that $\mathcal{D} = D \cup \{\bot\}$ and for all $u, v \in \mathcal{D}$ we have $u \sqsubseteq_{\mathcal{D}} v$ if and only if $u = \bot$ or $u = v$. All elements of $\mathcal{D} - \{\bot\}$ are prime. For the full powerset domain $(\mathscr{P}(D), \subseteq)$, the compact elements are the finite subsets of D and the prime elements are the singleton sets. We have a Scott domain when D is countable.

3. The untyped λ-calculus

3.1. *λ-terms*

The set Λ of λ-terms over a countable set of variables is constructed as usual:

— every variable is a λ-term;
— if M and N are λ-terms, then so are (MN) and $\lambda x.M$ for each variable x.

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We use Λ^o to denote the set of closed λ-terms. We use the following notation for some specific λ-terms:

$$\mathbf{I} \equiv \lambda x.x, \quad \mathbf{1} \equiv \lambda xy.xy, \quad \mathbf{T} \equiv \lambda xy.x, \quad \mathbf{F} \equiv \lambda xy.y,$$
$$\mathbf{S} \equiv \lambda xyz.xz(yz), \quad \delta \equiv \lambda x.xx, \quad \mathbf{\Omega} \equiv \delta\delta, \quad \mathbf{\Omega}_3 \equiv (\lambda x.xxx)(\lambda x.xxx).$$

The symbol $\equiv$ denotes definitional equality up to α-equivalence. A more traditional notation for **T**, when not viewed as a boolean, is **K**. We will use λ_β to denote $\alpha\beta$-conversion and $\lambda_{\beta\eta}$ to denote $\alpha\beta\eta$-conversion.

Intuitively, contexts are λ-terms containing some occurrences of a hole, denoted []. A *context* is inductively defined as follows:

— [] is a context.
— Every variable x is a context.
— If C_1 and C_2 are contexts, then so are C_1C_2 and $\lambda x.C_1$ for each variable x.

If M is a λ-term, we will write $C[M]$ for the context C where all the occurrences of the hole [] have been simultaneously replaced (without α-conversion) by M.

A λ-term M is a *head normal form* (*hnf*) if $M \equiv \lambda x_1 \dots x_n.yM_1 \dots M_k$ for some $n, k \geqslant 0$. Let $M \equiv \lambda x_1 \dots x_n.yM_1 \cdots M_k$ and $N \equiv \lambda x_1 \dots x_{n'}.y'N_1 \cdots N_{k'}$ be two hnf's. Then M, N are *equivalent* if and only if $y \equiv y'$ and $k - n = k' - n'$.

A λ-term is *solvable* if it is β-convertible to an hnf, otherwise it is said to be *unsolvable*.

Notation 3.1. $\mathcal{U}$ denotes the set of all unsolvable λ-terms.

Definition 3.2. $M, N \in \Lambda^o$ are *separable* if there exists $S \in \Lambda^o$ such that $SM = \mathbf{T}$ and $SN = \mathbf{F}$; otherwise they are *inseparable*.

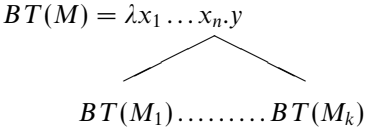
There are simple criteria implying separability or inseparability.

Proposition 3.3 (Böhm – Barendregt (1984, Lemma 10.4.1, Theorem 10.4.2)).

(i) Two hnf's are separable or equivalent (as hnf's).
(ii) Two normal λ-terms are separable or η-equivalent.

3.2. *Böhm trees*

The *Böhm tree* $BT(M)$ of a λ-term M is a finite or infinite labelled tree. If M is unsolvable, then $BT(M) = \bot$, that is, $BT(M)$ is a tree with a unique node labelled by $\bot$. If M is solvable and $\lambda x_1 \dots x_n.yM_1 \cdots M_k$ is the principal head normal form of M Barendregt (1984, Definition 8.3.20), we have:



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We call $\mathcal{BT}$ the set of all Böhm trees. Given $t, t' \in \mathcal{BT}$, we define $t \sqsubseteq_{\mathcal{B}} t'$ if and only if t results from t' by replacing some subtrees with $\bot$. It is easy to verify that $(\mathcal{BT}, \sqsubseteq_{\mathcal{B}})$ is an ω-algebraic cpo.

3.3. *λ-theories*

A *λ-theory* is a congruence that contains λ_β. If $\mathcal{T}$ is a λ-theory, we will write $M =_{\mathcal{T}} N$ for $(M, N) \in \mathcal{T}$ and $[M]_{\mathcal{T}}$ for the $\mathcal{T}$-equivalence class of M; for $V \subseteq \Lambda$, we use $V/\mathcal{T}$ to denote the quotient set of V modulo $\mathcal{T}$, that is, $V/\mathcal{T} = \{[M]_{\mathcal{T}} : M \in V\}$. A λ-theory $\mathcal{T}$ is:

— *consistent* if $\mathcal{T} \neq \Lambda \times \Lambda$;
— *extensional* if it contains the equation $\mathbf{I} = \mathbf{1}$;
— *recursively enumerable* if the set of Gödel numbers of all pairs of $\mathcal{T}$-equivalent λ-terms is r.e.

The λ-theory $\mathcal{H}$ generated by equating all the unsolvable λ-terms is consistent by Barendregt (1984, Theorem 16.1.3) and admits a unique maximal consistent extension $\mathcal{H}^*$ (Barendregt 1984, Theorem 16.2.6), which is an extensional λ-theory. A λ-theory $\mathcal{T}$ is *sensible* if $\mathcal{H} \subseteq \mathcal{T}$. Consistent sensible λ-theories are never r.e. (Barendregt 1984, Theorem 17.1.9). A λ-theory $\mathcal{T}$ is said to be *semi-sensible* if it contains no equations of the form $U = S$ where S is solvable and U unsolvable. Consistent sensible λ-theories are semi-sensible and $\mathcal{H}^*$ is also the unique maximal consistent semi-sensible λ-theory. The λ-theory $\mathcal{B}_{\mathcal{T}}$ that equates all λ-terms with the same Böhm tree is sensible, non-extensional and non-r.e. Moreover, $\mathcal{B}_{\mathcal{T}}$ is distinct from $\mathcal{H}$ and $\mathcal{H}^*$, so $\mathcal{H} \subsetneq \mathcal{B}_{\mathcal{T}} \subsetneq \mathcal{H}^*$.

4. Models of λ-calculus

4.1. *λ-models*

It is well known (Barendregt 1984, Chapter 5) that a model of untyped λ-calculus (abbreviated to *λ-model* here) is just a reflexive object of a Cartesian closed category (*ccc* for short) $\mathbf{C}$, that is to say a triple $\mathscr{M} = (\mathcal{D}, Ap, \lambda)$ such that $\mathcal{D}$ is an object of $\mathbf{C}$ and $Ap : \mathcal{D} \to [\mathcal{D} \to \mathcal{D}]$ and $\lambda : [\mathcal{D} \to \mathcal{D}] \to \mathcal{D}$ are two morphisms satisfying $Ap \circ \lambda = id_{[\mathcal{D} \to \mathcal{D}]}$. In the following we will only be interested in the case where $\mathbf{C}$ is a concrete Cartesian closed category whose objects are posets, possibly satisfying some constraints, and morphisms are (special) monotone functions between these sets. In fact we will mainly be interested in Scott semantics, but we will also draw conclusions for the stable and strongly stable semantics. In these three classes, all the λ-models have an underlying poset $\mathcal{D}$ that is a cpo; in this context, we will use $\sqsubseteq_{\mathscr{M}}$ to denote the partial order $\sqsubseteq_{\mathcal{D}}$. An *environment* with values in $\mathcal{D}$ is then a total function $\rho : Var \to \mathcal{D}$, where the carrier set of $\mathcal{D}$ is still denoted by $\mathcal{D}$.

We let $Env_{\mathcal{D}}$ be the set of environments with values in $\mathcal{D}$. $Env_{\mathcal{D}}$, ordered pointwise, is a cpo whose bottom element is the environment $\rho_\bot$ mapping everything to $\bot_{\mathcal{D}}$. Note also that $\rho \in Env_{\mathcal{D}}$ is compact if and only if $\mathrm{rg}(\rho) \subseteq \mathcal{K}(\mathcal{D})$ and $\rho(x) \neq \bot_{\mathcal{D}}$ for a finite number of $x \in Var$ only. For every $x \in Var$ and $d \in \mathcal{D}$, we use $\rho[x := d]$ to denote

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the environment ρ' that coincides with ρ except on x, where ρ' takes the value d. The interpretation $|M| : Env_{\mathcal{D}} \to \mathcal{D}$ of a λ-term M is defined by structural induction on M, as follows:

— $|x|_\rho = \rho(x)$
— $|MN|_\rho = Ap(|M|_\rho)(|N|_\rho)$
— $|\lambda x.M|_\rho = \lambda(d \in \mathcal{D} \mapsto |M|_{\rho[x:=d]})$.

This interpretation function generalises to terms with parameters in $\mathcal{D}$ (where an element of $\mathcal{D}$ is interpreted by itself) and to $\Lambda_\perp$ (recall that $\Lambda_\perp$ denotes the flat domain on $\Lambda \cup \{\perp\}$) by setting $|\perp|_\rho = \perp_{\mathcal{D}}$ for all $\rho \in Env_{\mathcal{D}}$. We use $\Lambda(\mathcal{D})$ (respectively, $\Lambda^o(\mathcal{D})$) to denote the set of all (respectively, all closed) terms with parameters in $\mathcal{D}$. If M is a closed λ-term, we write $|M|$ instead of $|M|_\rho$ since, clearly, $|M|_\rho$ only depends on the value of ρ on the free variables of M; in particular, $|M| = |M|_{\rho_\perp}$. In case of ambiguity, we will use $|M|^{\mathscr{M}}$ to denote the interpretation of the closed term M in the λ-model $\mathscr{M}$.

The *equational theory* $Th(\mathscr{M})$ and the *order theory* $Th_\sqsubseteq(\mathscr{M})$ of a λ-model $\mathscr{M}$ are defined by

$$Th(\mathscr{M}) = \{(M,N) : |M|_\rho = |N|_\rho \ \forall \rho \in Env_{\mathcal{D}}\}$$
$$Th_\sqsubseteq(\mathscr{M}) = \{(M,N) : |M|_\rho \sqsubseteq_{\mathscr{M}} |N|_\rho \ \forall \rho \in Env_{\mathcal{D}}\},$$

respectively. The model $\mathscr{M}$ is said to be *sensible* (respectively, *semi-sensible*) if $Th(\mathscr{M})$ is.

Any λ-model $\mathscr{M} = (\mathcal{D}, Ap, \lambda)$ can be viewed as the combinatory algebra $\mathscr{C} = (\mathcal{D}, \bullet, k, s)$ where $a \bullet b = Ap(a)(b)$ $(a, b \in \mathcal{D})$ and k and s are the interpretations of **K** and **S**, respectively, in $\mathscr{M}$. In the rest of this paper, we will write ab for $a \bullet b$, and when parentheses are omitted, we assume that association is made to the left, thus abc means $(ab)c$.

4.2. *Isomorphisms of λ-models*

Given two combinatory algebras $\mathscr{C} = (\mathcal{D}, \bullet, k, s)$ and $\mathscr{C}' = (\mathcal{D}', \bullet', k', s')$, a function $\Psi : \mathcal{D} \to \mathcal{D}'$ is a *morphism* from $\mathscr{C}$ to $\mathscr{C}'$ if $\Psi(u \bullet v) = \Psi(u) \bullet' \Psi(v)$ and $\Psi(k) = k'$, $\Psi(s) = s'$; it is an *isomorphism* if and only if Ψ is also a bijection.

It was proved in Manzonetto and Salibra (2006) that all homomorphisms between λ-models living in the continuous semantics or in its refinements are embeddings, that is to say, inclusions up to isomorphism.

Theorem 4.1. Given two λ-models $\mathscr{M} = (\mathcal{D}, Ap, \lambda), \mathscr{M}' = (\mathcal{D}', Ap', \lambda')$, the associated combinatory algebras $\mathscr{C}, \mathscr{C}'$ and a bijection $\Psi : \mathcal{D} \to \mathcal{D}'$ the following assertions are equivalent:

(i) Ψ is an isomorphism between $\mathscr{C}$ and $\mathscr{C}'$.
(ii) For all $M \in \Lambda^o$, we have $\Psi(|M|^{\mathscr{M}}) = |M|^{\mathscr{M}'}$.

So in this case we will also refer to this as an *isomorphism between the λ-models* $\mathscr{M}$ *and* $\mathscr{M}'$, or an *automorphism* when $\mathscr{M} = \mathscr{M}'$ (and hence $\mathscr{C} = \mathscr{C}'$).

The next remark is clear from the definition.

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Remark 4.2. If $\mathscr{M}$ and $\mathscr{M}'$ are isomorphic λ-models, then $Th(\mathscr{M}) = Th(\mathscr{M}')$.

Notation 4.3. $Aut(\mathscr{M})$ is the group of all automorphisms of $\mathscr{M}$.

4.3. *Scott continuous semantics*

The *Scott-continuous semantics* is the semantics of λ-calculus given in the category whose objects are cpo's and morphisms are Scott-continuous functions. If $\mathcal{D}$ is a cpo, we can define the *Scott topology* on $\mathcal{D}$. Given two cpo's $\mathcal{D}, \mathcal{D}'$ a function $f : \mathcal{D} \to \mathcal{D}'$ is Scott-continuous if and only if it is monotone and $f(\bigsqcup A) = \bigsqcup f^+(A)$ for all directed $A \subseteq \mathcal{D}$. We will use $[\mathcal{D} \to \mathcal{D}']$ to denote the set of all Scott continuous functions from $\mathcal{D}$ into $\mathcal{D}'$ considered as a cpo by pointwise ordering. If $d \in \mathcal{K}(\mathcal{D})$ and $e \in \mathcal{K}(\mathcal{D}')$, the step function $\varepsilon_{d,e}$ defined by

$$\varepsilon_{d,e}(x) = \begin{cases} e & \text{if } d \sqsubseteq_{\mathcal{D}} x \\ \bot_{\mathcal{D}'} & \text{otherwise} \end{cases}$$

is compact. If $\mathcal{D}, \mathcal{D}'$ are Scott domains, then $[\mathcal{D} \to \mathcal{D}']$ is a Scott domain and its compact elements are the functions of the form $\bigsqcup_{i \in I} \varepsilon_{d_i,e_i}$ for some I finite. Note that when $I \neq \emptyset$, such a least upper bound exists if and only if whenever $\{d_i : i \in I\}$ is bounded, then so is $\{e_i : i \in I\}$. For each function $f \in [\mathcal{D} \to \mathcal{D}']$, we define the *trace of* f as $tr(f) = \{(d,e) \in \mathcal{K}(\mathcal{D}) \times \mathcal{K}(\mathcal{D}') : e \sqsubseteq_{\mathcal{D}'} f(d)\}$. Note that $(d,e) \in tr(f)$ if and only if $\varepsilon_{d,e} \sqsubseteq_{[\mathcal{D} \to \mathcal{D}']} f$. If $\mathcal{D}'$ is prime algebraic, it is more interesting to work with $Tr(f) = \{(d,p) \in \mathcal{K}(\mathcal{D}) \times \mathcal{P}(\mathcal{D}') : p \sqsubseteq_{\mathcal{D}'} f(d)\}$. Hence, if $\mathcal{D} = \mathcal{D}' = \mathscr{P}(D)$, for some set D, we can use $Tr(f) = \{(a,\alpha) \in D^* \times D : \alpha \in f(a)\}$.

In the next section we will describe the simplest class of models living in Scott's continuous semantics, namely graph models.

4.4. *Definition of graph models*

The class of graph models belongs to the Scott continuous semantics, and is the simplest class of models of the untyped λ-calculus; nevertheless, it is very rich. All known classes of webbed λ-models can be presented as variations of this class (see Berline (2000)). The simplest graph model is Engeler's model $\mathscr{E}$ (Example 6.15(i)); it is, moreover, by far the simplest of all the non-syntactical λ-models. Historically, the first graph model to be isolated was Plotkin and Scott's $\mathscr{P}_\omega$, but it was soon followed by $\mathscr{E}$. The word *graph* refers to the fact that the continuous functions are encoded in the model via (a sufficient fragment of) their graphs, namely their traces, as recalled below. For more details, see Berline (2000; 2006).

Definition 4.4. A *total pair* $\mathcal{G}$ is a pair $(G, i_{\mathcal{G}})$ where G is an infinite set and $i_{\mathcal{G}} : G^* \times G \to G$ is an injective total function.

Definition 4.5. The *graph model* generated by the total pair $\mathcal{G}$ is the reflexive cpo

$$\mathscr{G} = ((\mathscr{P}(G), \subseteq), \lambda^{\mathcal{G}}, Ap^{\mathcal{G}}),$$

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where $\lambda^{\mathcal{G}} = i_{\mathcal{G}}^{+} \circ Tr$ and $Ap^{\mathcal{G}}$ is a left inverse of $\lambda^{\mathcal{G}}$. More explicitly:

(i) $\lambda^{\mathcal{G}}(f) = \{i_G(a,\alpha) : (a \in G^*)\ \alpha \in f(a)\}$.

(ii) $Ap^{\mathcal{G}}(X)(Y) = \{\alpha \in G : (\exists a \subseteq_{\mathrm{f}} Y)\ i_G(a,\alpha) \in X\}$.

The total pair $\mathcal{G} = (G, i_G)$ is called the '*web*' of the λ-model.

It is easy to check that in the case of a graph model $\mathscr{G}$, the interpretation $|M|^{\mathscr{G}} : Env_{\mathscr{P}(G)} \to \mathscr{P}(G)$ of $M \in \Lambda$ becomes:

— $|x|^{\mathscr{G}}_{\rho} = \rho(x)$

— $|MN|^{\mathscr{G}}_{\rho} = \{\alpha \in G : (\exists a \subseteq_{\mathrm{f}} |N|^{\mathscr{G}}_{\rho})\ i_G(a,\alpha) \in |M|^{\mathscr{G}}_{\rho}\}$

— $|\lambda x.M|^{\mathscr{G}}_{\rho} = \{i_G(a,\alpha) : (a \in G^*)\ \alpha \in |M|^{\mathscr{G}}_{\rho[x:=a]}\}$.

Example 4.6. Given a graph model $\mathscr{G}$:

$$|\mathbf{I}|^{\mathscr{G}} \equiv |\lambda x.x|^{\mathscr{G}} = \{i_G(a,\alpha) : a \in G^* \text{ and } \alpha \in a\},$$
$$|\mathbf{T}|^{\mathscr{G}} \equiv |\lambda xy.x|^{\mathscr{G}} = \{i_G(a, i_G(b,\alpha)) : a,b \in G^* \text{ and } \alpha \in a\},$$
$$|\mathbf{F}|^{\mathscr{G}} \equiv |\lambda xy.y|^{\mathscr{G}} = \{i_G(a, i_G(b,\alpha)) : a,b \in G^* \text{ and } \alpha \in b\}.$$

For $|\Omega|^{\mathscr{G}}$, we only use the following characterisation (the details of the proof are worked out, for example, in Berline and Salibra (2006, Lemma 4)).

Lemma 4.7. If $\mathscr{G}$ is a graph model, then $|\Omega|^{\mathscr{G}} \equiv |\delta\delta|^{\mathscr{G}} = \{\alpha : (\exists a \subseteq |\delta|^{\mathscr{G}})\ i_G(a,\alpha) \in a\}$.

In the following, we will use '*graph theory*' as an abbreviation for 'the λ-theory of a graph model'.

Proposition 4.8. For all graph models $\mathscr{G}$, we have $Th(\mathscr{G}) \neq \lambda_\beta, \lambda_{\beta\eta}$.

Indeed, it was long ago noticed that no graph model could be extensional, and recently noticed in Bucciarelli and Salibra (2004) that $|\Omega_3|^{\mathscr{G}} \subseteq |\mathbf{1}\Omega_3|^{\mathscr{G}}$ holds in all graph models $\mathscr{G}$ (because $|\Omega_3|^{\mathscr{G}} \subseteq \mathrm{rg}(i_G)$). Hence, Selinger's result (Selinger 1996, Corollary 4) stating that in any partially ordered model whose theory is λ_β or $\lambda_{\beta\eta}$ the interpretations of closed λ-terms are discretely ordered, implies that the theory of a graph model cannot be λ_β or $\lambda_{\beta\eta}$.

4.5. *The stable and strongly stable semantics*

The stable semantics and the strongly stable semantics are refinements of Scott's semantics, which were successively introduced by Berry (Berry 1978; 1979) and Ehrhard (Bucciarelli and Ehrhard 1991b), respectively, mainly for the purposes of proving some properties of typed λ-calculi with a flavour of sequentiality (Plotkin 1977; Berry 1978; 1979; Berry *et al.* 1985). For this paper it is enough to know the following:

— In the stable framework, the objects are special Scott domains called *DI-domains*[†]. The morphisms are the *stable functions* between DI-domains, that is, the Scott continuous functions satisfying $f(u \sqcap v) = f(u) \sqcap f(v)$, for all compatible elements u, v.

[†] See Amadio and Curien (1998) and Bucciarelli and Ehrhard (1991a) for a detailed description of DI-domains and DI-domains with coherences.

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— In the strongly stable case, the objects are *DI-domains with coherences*. A DI-domain with coherence is a pair $(\mathcal{D}, C)$ where $\mathcal{D}$ is a DI-domain and C is a non-empty set of finite subsets of $\mathcal{D}$ containing all singletons and satisfying some closure properties. The morphisms are the *strongly stable functions* between DI-domains with coherence, that is, stable functions preserving coherence.

The relevant order on the corresponding cpo's of functions, $[\mathcal{D} \to_s \mathcal{D}]$ and $[\mathcal{D} \to_{ss} \mathcal{D}]$, is, in both cases, Berry's order $\leqslant_s$, which is defined as follows.

Notation 4.9. $f \leqslant_s g$ if and only if $\forall x \forall y\ (x \sqsubseteq_{\mathcal{D}} y \Rightarrow f(x) = f(y) \sqcap g(x))$

The following basic properties of Berry's order are easy to check.

Remark 4.10.

(i) $f \leqslant_s g$ implies that f is pointwise smaller than g.
(ii) $f \leqslant_s g$ and g constant imply f constant.

When we are working with stable functions, the following alternative notion of trace makes sense and is more economical: $Tr_s(f)$ is defined in the same way as $Tr(f)$ in Section 4.3 (case where $\mathcal{D}$ is prime algebraic) but retains only the pairs (d, e) satisfying d is *minimal* such that $e \sqsubseteq_{\mathcal{D}} f(d)$; and similarly when one uses pairs (a, α). For example, if $\mathcal{D} = (\mathscr{P}(D), \subseteq)$, for some set D, then $Tr(id_{\mathcal{D}}) = \{(a, \alpha) : \alpha \in a \in D^*\}$ while $Tr_s(id_{\mathcal{D}}) = \{(\{\alpha\}, \alpha) : \alpha \in D\}$.

4.6. *Some classes of webbed models*

The informal notion of webbed models was introduced in Section 1. Here we will only introduce the classes of webbed models we will use in the rest of the paper:

1 K-models introduced by Krivine in Krivine (1993) (see also Berline (2000, Definition 126)), pcs-models (Berline 2000, Definition 153) and filter models (Coppo *et al.* 1987), all living in the continuous semantics;
2 Girard's reflexive coherences, called G-models in Berline (2000, Definition 150), living in the stable semantics;
3 Ehrhard's reflexive hypercoherences, called H-models in Berline (2000, Definition 160), living in the strongly stable semantics.

All the underlying domains of these models are prime algebraic, except for those of filter models, which may just be algebraic. The terminology of K-, G-, H- models will be used freely in this paper.

5. Recursion in λ-calculus

In this section we recall the main properties of recursion theory for λ-calculus that will be applied in the following sections.

Let $(-)_\omega : \Lambda \to \mathbb{N}$ be an arbitrary effective encoding of Λ. We use $(-)_\lambda$ to denote the inverse map of $(-)_\omega$, so we have $M_{\omega,\lambda} = M$.

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5.1. *Co-r.e. sets of λ-terms*

Definition 5.1. A set $V \subseteq \Lambda$ is *r.e.* (*co-r.e.*) if $\{(M)_\omega : M \in V\}$ is r.e. (co-r.e.). The set V is said to be *trivial* if either $V = \varnothing$ or $V = \Lambda$.

Notation 5.2. Let $\mathcal{T}$ be a λ-theory. An r.e. (co-r.e.) set of λ-terms closed under $=_\mathcal{T}$ will be called a $\mathcal{T}$-*r.e.* ($\mathcal{T}$-*co-r.e.*) set. If $\mathcal{T} = \lambda_\beta$, we simply refer to a β-r.e. (β-co-r.e.) set.

Definition 5.3. A family $(X_i)_{i\in I}$ of sets has the *FIP* (finite intersection property) if $X_{i_1} \cap \ldots \cap X_{i_n} \neq \varnothing$ for all $i_1, \ldots, i_n \in I$.

Our key tool for studying r.e. theories and effective models will be the following theorem (see Visser (1980, Theorem 2.5), or Barendregt (1984, Chapter 17)).

Theorem 5.4 (Visser). The family of all non-empty β-co-r.e. subsets of Λ has the FIP.

This theorem generalises the following classical result of Scott (see, for example, Barendregt (1984, Theorem 6.6.2)).

Theorem 5.5 (Scott). A set of λ-terms that is both β-r.e. and β-co-r.e. is trivial.

A topological reading of Theorem 5.4 is that the topology on Λ generated by the β-co-r.e. sets of λ-terms is hyperconnected (that is, the intersection of two non-empty open sets is non-empty).

Lemma 5.6.

(i) $\mathcal{U}$ is β-co-r.e. and thus non-r.e.,
(ii) $\mathcal{U}$ is $\mathcal{T}$-co-r.e. if and only if $\mathcal{T}$ is semi-sensible.

Proof.

(i) Since the set of λ-terms β-convertible to a hnf is r.e. (by Church–Rosser (Barendregt 1984, Theorem 3.2.8)) and β-closed.
(ii) Furthermore, it is easy to check that $\mathcal{U}$ is $\mathcal{T}$-closed exactly when $\mathcal{T}$ is semi-sensible. □

It follows from this lemma and Theorem 5.4 that every non-empty β-co-r.e. set of terms contains unsolvable λ-terms.

Lemma 5.7. If $O \neq \varnothing$ is β-co-r.e. and $\mathcal{T}$ is r.e., then $O/\mathcal{T}$ is infinite or $\mathcal{T}$ is inconsistent.

Proof. Let V be the $\mathcal{T}$-closure of O, and $O' = \Lambda - V$. If $\mathcal{T}$ is r.e. and $O/\mathcal{T}$ is finite, then V is r.e. and thus O' is β-co-r.e. Since $O' \cap O = \varnothing$, we have, by Theorem 5.4, that O' must be empty. Hence $V = \Lambda$, and $\Lambda/\mathcal{T}$ is finite. Hence $\mathcal{T}$ is inconsistent. □

5.2. *Separability revisited*

This section, which can be skipped at first reading, contains some other interesting examples of β-co-r.e. sets that should prove useful for later work, namely $M^{\mathcal{T}-ins}$ and $\Lambda_{\mathcal{T}-easy}$, as defined below, when $\mathcal{T}$ is r.e.

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Notation 5.8. Given a λ-theory $\mathcal{T}$, for all $M \in \Lambda^o$, we let

$$M^{\mathcal{T}-ins} = \{N \in \Lambda^o : \nexists S \in \Lambda^o (SM =_{\mathcal{T}} \mathbf{T} \wedge SN =_{\mathcal{T}} \mathbf{F})\}.$$

In this case M and N are said to be $\mathcal{T}$-*inseparable*. We will omit $\mathcal{T}$ when $\mathcal{T} = \lambda_\beta$.

Example 5.9.

(i) $\mathcal{U} \subseteq M^{ins}$ for all $M \in \Lambda^o$ (by the genericity lemma (Barendregt 1984, Proposition 14.3.24)).

Equivalently:

(ii) $U^{ins} = \Lambda^o$ for all unsolvable terms U.

More generally:

(iii) If $BT(M) \sqsubseteq_{\mathcal{B}} BT(N)$, then $M \in N^{ins}$ and $N \in M^{ins}$.

Remark 5.10. Let $\mathcal{T}$ be a λ-theory and $M \in \Lambda^o$, then:

(i) $M^{\mathcal{T}-ins} \subseteq M^{ins}$ is $\mathcal{T}$-closed.

More generally:

(ii) $M^{\mathcal{T}'-ins} \subseteq M^{\mathcal{T}-ins}$ if $\mathcal{T} \subseteq \mathcal{T}'$.

Proposition 5.11. Suppose $\mathcal{T}$ is a semi-sensible λ-theory and $M \in \Lambda^o$. Then $M^{\mathcal{T}-ins} = M^{ins}$.

Proof. Suppose that there are $S, N \in \Lambda^o$ such that $SM =_{\mathcal{T}} \mathbf{T}$ and $SN =_{\mathcal{T}} \mathbf{F}$. As $\mathcal{T}$ is semi-sensible, SM and SN are solvable, and, since $\mathcal{T}$ is necessarily consistent, their head-normal forms are non-equivalent. Hence SM and SN are separable, which implies that M and N are separable. □

Definition 5.12. Let $\mathcal{T}$ be a λ-theory. A λ-term $U \in \Lambda^o$ is said to be $\mathcal{T}$-*easy* if for all $M \in \Lambda^o$ we have $U = M$ is consistent with $\mathcal{T}$ (Barendregt 1984, Page 434). When $\mathcal{T} = \lambda_\beta$, we just say that U is *easy*.

Notation 5.13. We will use $\Lambda_{\mathcal{T}-easy}$ to denote the set of $\mathcal{T}$-easy terms.

It is clear that $\Lambda_{\mathcal{T}-easy} \subseteq \mathcal{U}$ and that $\Lambda_{\mathcal{T}-easy} \subseteq M^{\mathcal{T}-ins}$ for all $M \in \Lambda^o$.

Proposition 5.14. If $\mathcal{T}$ is a consistent r.e. λ-theory, then:

(i) $M^{\mathcal{T}-ins}$ and $\Lambda_{\mathcal{T}-easy}$ are $\mathcal{T}$-co-r.e. sets.
(ii) $M^{\mathcal{T}-ins}$ and $\Lambda_{\mathcal{T}-easy}$ are infinite modulo $\mathcal{T}$.

Proof. Recall that $\Lambda_{\mathcal{T}-easy} \subseteq M^{\mathcal{T}-ins}$ and $\Lambda_{\mathcal{T}-easy} \subseteq \mathcal{U}$ hold whether $\mathcal{T}$ is r.e. or not.

(i) This part follows easily from the definitions that both sets are co-r.e. if $\mathcal{T}$ is r.e.
(ii) This part follows from (i) and Lemma 5.7, using the fact that $\Lambda_{\mathcal{T}-easy} \neq \emptyset$ when $\mathcal{T}$ is r.e., which was proved by Visser (Visser 1980) (or see Barendregt (1984, Proposition 17.1.9)). □

Note that though it is obvious that $M^{\mathcal{T}-ins} \neq \emptyset$ holds for all $M \in \Lambda^o$ and $\mathcal{T}$, since $M \in M^{\mathcal{T}-ins}$, for r.e. theories $\mathcal{T}$, it is only known that $\Lambda_{\mathcal{T}-easy}$ is non-empty.

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Part II: Graph models and partial pairs

6. The category of partial pairs

The definition of graph models (and hence of total pairs) was recalled in Section 4.4. We now need to develop the wider framework of partial pairs.

In this section we will recall the definitions of partial pairs, interpretation with respect to a partial pair, free completion and gluings. We will also introduce the new notions of subpair relation, morphism of partial pairs and retract of partial pairs.

6.1. *Definition and ordering of partial pairs*

Definition 6.1. A *partial pair* $\mathcal{A}$ is a pair $(A, j_{\mathcal{A}})$ where A is a non-empty set and $j_{\mathcal{A}} : A^* \times A \to A$ is a partial (possibly total) injection.

In the rest of this paper the letters $\mathcal{A}, \mathcal{B}$ will always denote partial pairs. $\mathcal{A}$ is *finite* if A is finite, and it is total if $j_{\mathcal{A}}$ is total. The simplest example of a partial pair is $(A, \varnothing)$, where $\varnothing$ denotes the empty function.

Definition 6.2. $\mathcal{A}$ is a *subpair* of $\mathcal{B}$, written $\mathcal{A} \sqsubseteq \mathcal{B}$, if $A \subseteq B$ and $j_{\mathcal{A}}(a, \alpha) = j_{\mathcal{B}}(a, \alpha)$ for all $(a, \alpha) \in \mathrm{dom}(j_{\mathcal{A}})$. We will use $Sub(\mathcal{A})$ to denote the set of all the subpairs of $\mathcal{A}$.

It is clear that for all partial pairs $\mathcal{A}$, $(Sub(\mathcal{A}), \sqsubseteq)$ is a bounded complete algebraic cpo (provided we add the empty-pair) and that $\bigsqcup_{k\in K} \mathcal{A}_k = (\cup_{k\in K} A_k, \cup_{k\in K} j_{\mathcal{A}_k})$ if the $\mathcal{A}_k$'s are compatible. When A is countable, $(Sub(\mathcal{A}), \sqsubseteq)$ is even a DI-domain.

6.2. *Interpretation with respect to partial pairs*

Definition 6.3. An *$\mathcal{A}$-environment* is a function $\rho : Var \to \mathscr{P}(A)$.

We will use $Env_{\mathcal{A}}$, instead of $Env_{\mathscr{P}(A)}$, to denote the set of all $\mathcal{A}$-environments.

The definition of the *interpretation* $|M|^{\mathcal{A}}$ of a λ-term M with respect to a partial pair $\mathcal{A}$ generalises in the obvious way the one given for graph models in Section 4.4. For all $\rho \in Env_{\mathcal{A}}$ we let:

— $|x|^{\mathcal{A}}_{\rho} = \rho(x)$.
— $|PQ|^{\mathcal{A}}_{\rho} = \{\alpha : (\exists a \subseteq |Q|^{\mathcal{A}}_{\rho})\ (a, \alpha) \in \mathrm{dom}(j_{\mathcal{A}}) \wedge j_{\mathcal{A}}(a, \alpha) \in |P|^{\mathcal{A}}_{\rho}\}$.
— $|\lambda x.N|^{\mathcal{A}}_{\rho} = \{j_{\mathcal{A}}(a, \alpha) : (a, \alpha) \in \mathrm{dom}(j_{\mathcal{A}}) \wedge \alpha \in |N|^{\mathcal{A}}_{\rho[x:=a]}\}$.

Of course, if $\mathscr{G}$ is a graph model with web $\mathcal{G}$, then $|M|^{\mathscr{G}}_{\rho} = |M|^{\mathcal{G}}_{\rho}$ for all λ-terms M and environments ρ. Note that if $\mathcal{A}$ is not total, β-equivalent λ-terms do not necessarily have the same interpretation. The simplest example of such a situation is given by the partial pair $\mathcal{A} = (A, \varnothing)$ and the two β-equivalent λ-terms x and $\mathbf{I}x$. Indeed, it is easy to check that $|x|^{\mathcal{A}}_{\rho[x:=\{\alpha\}]} = \{\alpha\}$ and $|\mathbf{I}x|^{\mathcal{A}}_{\rho[x:=\{\alpha\}]} = \varnothing$.

Notation 6.4. If $\rho \in Env_{\mathcal{A}}, \sigma \in Env_{\mathcal{B}}$ and C is a set, then $\sigma = \rho \cap C$ means $\sigma(x) = \rho(x) \cap C$ for every variable x, and $\rho \subseteq \sigma$ means $\rho(x) \subseteq \sigma(x)$ for every variable x.

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We now provide two new lemmas expressing the continuity of the function defined from $Sub(\mathcal{A}) \times Env_{\mathcal{A}}$ to $\mathscr{P}(A)$ and mapping $(\mathcal{B}, \rho) \mapsto |M|^{\mathcal{B}}_{\rho \cap B}$.

Lemma 6.5. If $\mathcal{A} \sqsubseteq \mathcal{B}$, then $|M|^{\mathcal{A}}_{\rho} \subseteq |M|^{\mathcal{B}}_{\sigma}$ for all $\rho \in Env_{\mathcal{A}}$ and $\sigma \in Env_{\mathcal{B}}$ such that $\rho \subseteq \sigma$.

Proof. The proof is by a straightforward induction on the structure of M. □

Lemma 6.6. Let $M \in \Lambda$, $\mathcal{A}$ be a partial pair and $\rho \in Env_{\mathcal{A}}$. Suppose $\alpha \in |M|^{\mathcal{A}}_{\rho}$. Then there exists a finite pair $\mathcal{B} \sqsubseteq \mathcal{A}$ such that $\alpha \in |M|^{\mathcal{B}}_{\rho \cap B}$.

Proof. The proof is by induction on M.

— If $M \equiv x$, then $\alpha \in \rho(x)$, so we define $\mathcal{B} = (\{\alpha\}, \emptyset)$.
— If $M \equiv PQ$, then there is $a = \{\alpha_1, \ldots, \alpha_n\}$ for some $n \geqslant 0$ such that $(a, \alpha) \in \mathrm{dom}(j_{\mathcal{A}})$, $j_{\mathcal{A}}(a, \alpha) \in |P|^{\mathcal{A}}_{\rho}$ and $a \subseteq |Q|^{\mathcal{A}}_{\rho}$. By the induction hypothesis, there exist finite subpairs $\mathcal{B}_1, \ldots, \mathcal{B}_{n+1}$ of $\mathcal{A}$ such that $j_{\mathcal{A}}(a, \alpha) \in |P|^{\mathcal{B}_{n+1}}_{\rho \cap B_{n+1}}$ and $\alpha_k \in |Q|^{\mathcal{B}_k}_{\rho \cap B_k}$ for $k = 1, \ldots, n$. We define $\mathcal{B} \sqsubseteq \mathcal{A}$ as $\sqcup_{k=0 \ldots n+1} \mathcal{B}_k$ where $\mathcal{B}_0 = (a \cup \{\alpha\}, j_{\mathcal{A}} \restriction_{\{(a,\alpha)\}})$. The conclusion then follows from Lemma 6.5.
— If $M \equiv \lambda x.N$, then $\alpha = j_{\mathcal{A}}(b, \beta)$ for some b and β such that $(b, \beta) \in \mathrm{dom}(j_{\mathcal{A}})$ and $\beta \in |N|^{\mathcal{A}}_{\rho[x:=b]}$. By the induction hypothesis, there exists a finite pair $\mathcal{C} \sqsubseteq \mathcal{A}$ such that $\beta \in |N|^{\mathcal{C}}_{\rho[x:=b] \cap C}$. We define $\mathcal{B} \sqsubseteq \mathcal{A}$ as $\mathcal{C} \sqcup (b \cup \{\alpha, \beta\}, j_{\mathcal{A}} \restriction_{\{(b,\beta)\}})$. Then we have that $\mathcal{C} \sqsubseteq \mathcal{B}$ and $\rho[x := b] \cap C \subseteq \rho[x := b] \cap B$. From $\beta \in |N|^{\mathcal{C}}_{\rho[x:=b] \cap C}$ and from Lemma 6.5 it follows that $\beta \in |N|^{\mathcal{B}}_{\rho[x:=b] \cap B} = |N|^{\mathcal{B}}_{(\rho \cap B)[x:=b]}$. Then we conclude that $\alpha = j_{\mathcal{B}}(b, \beta) \in |\lambda x.N|^{\mathcal{B}}_{\rho \cap B}$. □

6.3. *Morphisms between partial pairs*

The following definition extends the definition of an isomorphism between total pairs, which was introduced in Longo (1983).

Definition 6.7. A total function $\theta : A \to B$ is a *morphism* from $\mathcal{A}$ to $\mathcal{B}$ if for all $(a, \alpha) \in A^* \times A$ we have

$$(a, \alpha) \in \mathrm{dom}(j_{\mathcal{A}}) \Longrightarrow [(\theta^+(a), \theta(\alpha)) \in \mathrm{dom}(j_{\mathcal{B}}) \text{ and } \theta(j_{\mathcal{A}}(a, \alpha)) = j_{\mathcal{B}}(\theta^+(a), \theta(\alpha))]$$

and it is an *endomorphism* if, in addition, $\mathcal{A} = \mathcal{B}$.

Remark 6.8.

(i) $\theta : A \to B$ is an *isomorphism* between $\mathcal{A}$ and $\mathcal{B}$ if and only if it is a bijection and both θ and θ^{-1} are morphisms; if, in addition, $\mathcal{A} = \mathcal{B}$, then θ is an *automorphism.*
(ii) $\mathcal{A} \sqsubseteq \mathcal{B}$ if and only if the inclusion mapping $\iota : A \to B$ is a morphism.

Notation 6.9.

(i) $Hom(\mathcal{A}, \mathcal{B})$ denotes the set of morphisms from $\mathcal{A}$ to $\mathcal{B}$.
(ii) $Iso(\mathcal{A}, \mathcal{B})$ denotes the set of isomorphisms between $\mathcal{A}$ and $\mathcal{B}$.
(iii) $Aut(\mathcal{A})$ denotes the group of automorphisms of $\mathcal{A}$.

We will also write $\theta : \mathcal{A} \to \mathcal{B}$ for $\theta \in Hom(\mathcal{A}, \mathcal{B})$.

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Lemma 6.10. Let $\phi \in Hom(\mathcal{A},\mathcal{B})$ and $\rho \in Env_{\mathcal{A}}$. Then:
(i) $\phi^+(|M|^{\mathcal{A}}_{\rho}) \subseteq |M|^{\mathcal{B}}_{\phi^+\circ\rho}$.
(ii) $\phi^+(|M|^{\mathcal{A}}_{\rho}) = |M|^{\mathcal{B}}_{\phi^+\circ\rho}$ if $\phi \in Iso(\mathcal{A},\mathcal{B})$.

Proof.
(i) By straightforward induction on M, we can prove that for all $\alpha \in \phi^+(|M|^{\mathcal{A}}_{\rho})$ we have $\phi(\alpha) \in |M|^{\mathcal{B}}_{\phi^+\circ\rho}$.
(ii) From Part (i), it is enough to prove that $|M|^{\mathcal{B}}_{\phi^+\circ\rho} \subseteq \phi^+(|M|^{\mathcal{A}}_{\rho})$. Let $\psi = \phi^{-1}$. Then $\phi^+\circ\psi^+ = id$, so $|M|^{\mathcal{B}}_{\phi^+\circ\rho} = \phi^+(\psi^+(|M|^{\mathcal{B}}_{\phi^+\circ\rho})) \subseteq \phi^+(|M|^{\mathcal{A}}_{\psi^+\circ\phi^+\circ\rho}) = \phi^+(|M|^{\mathcal{A}}_{\rho})$. □

Lemma 6.10(ii) implies that if $\phi \in Iso(\mathcal{A},\mathcal{B})$, with $\mathcal{A},\mathcal{B}$ total, then ϕ^+ is an isomorphism of λ-models (and of combinatory algebras). On the other hand, if ϕ is only a morphism of pairs, then ϕ^+ cannot be a morphism of combinatory algebras. Indeed, it is easy to check that $\phi^+(|\mathbf{K}|^{\mathcal{A}}) \subsetneq |\mathbf{K}|^{\mathcal{B}}$ if ϕ is not surjective and $\phi^+(|MN|^{\mathcal{A}}) \subsetneq \phi^+(|M|^{\mathcal{A}}) \bullet \phi^+(|N|^{\mathcal{A}})$ if ϕ is not injective.

6.4. *Free completions of partial pairs*

There are two known processes for building a graph model satisfying some additional requirements. Both consist of completing a partial pair $\mathcal{A}$ into a total pair. The *free completion*†, which is due to Longo (Longo 1983) and mimics the construction of $\mathscr{E}$, is a constructive way for building as freely as possible a total pair $\overline{\mathcal{A}}$ from a partial pair $\mathcal{A}$. The aim is to induce some properties of the graph model generated by $\overline{\mathcal{A}}$ from properties of $\mathcal{A}$. The other completion process, called *forcing completion* or simply 'forcing', originated in Baeten and Boerboom (1979). For all $M \in \Lambda^o$, Baeten and Boerboom built out of a partial pair $(G,\emptyset)$ a graph model $\mathscr{G}$ with web (G, i^M_G) such that $|\Omega|^{\mathscr{G}} = |M|^{\mathscr{G}}$, thus proving semantically that Ω is *easy* (that is, that Ω can be consistently equated with any λ-term $M \in \Lambda^o$). This technique is, in general, non-constructive, but it can be effective in some degenerate but interesting cases (this contradicts a remark in (Berline 2006, Section 5.3.5)). Forcing was, in particular, generalised in Berline and Salibra (2006) to a class of terms M (called *generalised terms*), which goes far beyond $M \in \Lambda^o$ and even $M \in \Lambda^o(\mathcal{D})$. In the present paper forcing will only have an auxiliary role allowing us to produce examples; hence 'completion' will mean 'free completion' unless otherwise stated.

Definition 6.11 (Longo). Let $\mathcal{A} = (A, j_{\mathcal{A}})$ be a partial pair. The *free completion*† of $\mathcal{A}$ is the total pair $\overline{\mathcal{A}} = (\overline{A}, i_{\overline{A}})$, where $\overline{A} = \cup_{n\in\mathbb{N}} A_n$, with $A_0 = A$, $A_{n+1} = A \cup ((A^*_n \times A_n) - \mathrm{dom}(j_{\mathcal{A}}))$ and $i_{\overline{A}}$ is defined by

$$i_{\overline{A}}(a,\alpha) = \begin{cases} j_{\mathcal{A}}(a,\alpha) & \text{if } (a,\alpha) \in \mathrm{dom}(j_{\mathcal{A}}) \\ (a,\alpha) & \text{otherwise.} \end{cases}$$

† Free completion is termed *canonical completion* in Bucciarelli and Salibra (2008) and *Engeler completion* in Bucciarelli and Salibra (2003; 2004).
† In fact, this completion construction requires that $((A^* \times A) - \mathrm{dom}(j_{\mathcal{A}})) \cap \mathrm{rg}(j_{\mathcal{A}}) = \emptyset$, otherwise $i_{\overline{A}}$ would not be injective, so we will always suppose that no element of A is a pair. This is not restrictive because partial pairs can be considered up to isomorphism.

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An element of A has rank 0, whilst an element $\alpha \in \overline{A} - A$ has rank n if $\alpha \in A_n - A_{n-1}$.

Notation 6.12. $\mathscr{G}_{\mathcal{A}}$ denotes the graph model whose web is $\overline{\mathcal{A}}$, and it will be said to be *freely generated* by $\mathcal{A}$.

Theorem 6.13 (Bucciarelli and Salibra 2008, Theorem 29). If $\mathcal{A}$ is a partial pair that is not total, then $\mathscr{G}_{\mathcal{A}}$ is semi-sensible.

Remark 6.14. Let $\mathcal{A}, \mathcal{B}$ be two partial pairs. If $\mathcal{A} \sqsubseteq \mathcal{B} \sqsubseteq \overline{\mathcal{A}}$, then $\overline{\mathcal{A}} = \overline{\mathcal{B}}$ and thus $\mathscr{G}_{\mathcal{A}} = \mathscr{G}_{\mathcal{B}}$.

Example 6.15. By definition:

(i) The Engeler model $\mathscr{E}$ is freely generated by $\mathcal{A} = (A, \emptyset)$, where A is a non-empty set. Thus, in fact, we have a family of graph models $\mathscr{E}_A$.
(ii) The graph-Scott models are freely generated by $\mathcal{A} = (A, j_{\mathcal{A}})$, where $j_{\mathcal{A}}(\emptyset, \alpha) = \alpha$ for all $\alpha \in A$.
(iii) The graph-Park models are freely generated by $\mathcal{A} = (A, j_{\mathcal{A}})$, where $j_{\mathcal{A}}(\{\alpha\}, \alpha) = \alpha$ for all $\alpha \in A$.
(iv) The mixed-Scott-Park graph models are freely generated by $\mathcal{A} = (A, j_{\mathcal{A}})$ where $j_{\mathcal{A}}(\emptyset, \alpha) = \alpha$ for all $\alpha \in Q$, $j_{\mathcal{A}}(\{\beta\}, \beta) = \beta$ for all $\beta \in R$ and Q, R form a non-trivial partition of A.

Remark 6.16 (Longo 1983). The model $\mathscr{P}_\omega$ is isomorphic to the graph-Scott model associated with any singleton set A.

Theorem 6.17 (Kerth 1994; 1998b). There exist $2^{\aleph_0}$ graph models of the form $\mathscr{G}_{\mathcal{A}}$ with distinct theories, among which $\aleph_0$ are freely generated by finite pairs. The same is still true for sensible graph models.

Proof. Among the continuum of distinct graph models given in Kerth (1994; 1998b), countably many are freely generated by finite partial pairs. The result for sensible graph theories follows from Kerth (1998a) plus David (2001). □

Lemma 6.18.

(i) For all $\theta \in Hom(\mathcal{A}, \mathcal{B})$ there is a unique $\bar{\theta} \in Hom(\overline{\mathcal{A}}, \overline{\mathcal{B}})$ such that $\bar{\theta}{\restriction_A} = \theta$.
(ii) If $\theta \in Iso(\mathcal{A}, \mathcal{B})$, then $\bar{\theta} \in Iso(\overline{\mathcal{A}}, \overline{\mathcal{B}})$.

Proof. The definition of $\bar{\theta}$ and verification of the first part are by straightforward induction on the rank of the elements of $\overline{A}$. It is also easy to check that if θ is an isomorphism then $\bar{\theta}^{-1}$ is the inverse of $\bar{\theta}$. □

A morphism $\theta : \mathcal{A} \to \mathcal{B}$ does not, in general, induce a morphism of λ-models. But this is not the case when θ is an isomorphism. In other words, the next corollary holds.

Corollary 6.19. Let $\theta \in Iso(\mathcal{A}, \mathcal{B})$. Then:

(i) $\bar{\theta}^+ \in Iso(\mathscr{G}_{\mathcal{A}}, \mathscr{G}_{\mathcal{B}})$.
(ii) $Th_{\sqsubseteq}(\mathscr{G}_{\mathcal{A}}) = Th_{\sqsubseteq}(\mathscr{G}_{\mathcal{B}})$.
(iii) $Th(\mathscr{G}_{\mathcal{A}}) = Th(\mathscr{G}_{\mathcal{B}})$.

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Proof.

(i) This part follows from Lemmas 6.10 and 6.18.
(ii) This part follows from (i) and Remark 4.2.
(iii) This part follows from (ii). □

We now come to the key (and new) result of this subsection.

Proposition 6.20. Let $\mathscr{G}$ be a graph model with web $\mathcal{G}$, and suppose $\alpha \in |M|^{\mathscr{G}} - |N|^{\mathscr{G}}$ for some $M, N \in \Lambda^o$. Then there exists a finite $\mathcal{A} \sqsubseteq \mathcal{G}$ such that $\alpha \in \mathcal{A}$ and for all pairs $\mathcal{C} \sqsupseteq \mathcal{A}$, if there is a morphism $\theta : \mathcal{C} \to \mathcal{G}$ such that $\theta(\alpha) = \alpha$, then $\alpha \in |M|^{\mathcal{C}} - |N|^{\mathcal{C}}$.

Proof. By Lemma 6.6 there is a finite $\mathcal{A} \sqsubseteq \mathcal{G}$ such that $\alpha \in |M|^{\mathcal{A}}$. By Lemma 6.5 we have $\alpha \in |M|^{\mathcal{C}}$. Now, if $\alpha \in |N|^{\mathcal{C}}$, then, by Lemma 6.10, $\alpha = \theta(\alpha) \in |N|^{\mathscr{G}}$, which gives a contradiction. □

Corollary 6.21. Let $\mathscr{G}$ be a graph model, and suppose $\alpha \in |M|^{\mathscr{G}} - |N|^{\mathscr{G}}$ for some $M, N \in \Lambda^o$. Then there exists a finite $\mathcal{A} \sqsubseteq \mathcal{G}$ such that $\alpha \in \mathcal{A}$, and for all pairs $\mathcal{B}$ satisfying $\mathcal{A} \sqsubseteq \mathcal{B} \sqsubseteq \mathcal{G}$, we have:

(i) $\alpha \in |M|^{\mathcal{B}} - |N|^{\mathcal{B}}$; and
(ii) $\alpha \in |M|^{\mathscr{G}_{\mathcal{B}}} - |N|^{\mathscr{G}_{\mathcal{B}}}$.

Proof. We apply Proposition 6.20, taking for θ the inclusion mapping $\iota : \mathcal{B} \to \mathcal{G}$ for (i), and $\bar{\iota}$ given by Lemma 6.18 for (ii). □

6.5. *Retracts*

Definition 6.22. Given two partial pairs $\mathcal{A}$ and $\mathcal{B}$, we say that $\mathcal{A}$ *is a retract of* $\mathcal{B}$, and write $\mathcal{A} \lhd \mathcal{B}$, if there are morphims $e \in Hom(\mathcal{A}, \mathcal{B})$ and $\pi \in Hom(\mathcal{B}, \mathcal{A})$ such that $\pi \circ e = id_A$. In this case we will also write $e, \pi : \mathcal{A} \lhd \mathcal{B}$.

Notation 6.23. Given two graph models $\mathscr{G}, \mathscr{G}'$, we write $\mathscr{G} \lhd \mathscr{G}'$ if $\mathcal{G} \lhd \mathcal{G}'$.

From Lemma 6.18(i) and the fact that $id_{\overline{\mathcal{A}}}$ is the only endomorphism of $\overline{\mathcal{A}}$ whose restriction to A is the identity id_A, we get the following lemma.

Lemma 6.24. Let $\mathcal{A}, \mathcal{B}$ be two partial pairs. Then $\mathcal{A} \lhd \mathcal{B}$ implies $\overline{\mathcal{A}} \lhd \overline{\mathcal{B}}$.

Proposition 6.25. If $\mathcal{G} \lhd \mathcal{G}'$, then $Th_{\sqsubseteq}(\mathscr{G}') \subseteq Th_{\sqsubseteq}(\mathscr{G})$ and $Th(\mathscr{G}') \subseteq Th(\mathscr{G})$.

Proof. Let $\pi, e : \mathcal{G} \lhd \mathcal{G}'$. It is enough to prove that for all $M, N \in \Lambda^o$, if $\alpha \in |M|^{\mathscr{G}} - |N|^{\mathscr{G}}$, then $e(\alpha) \in |M|^{\mathscr{G}'} - |N|^{\mathscr{G}'}$. Now, applying Lemma 6.10 twice, we get $e(\alpha) \in |M|^{\mathscr{G}'}$ and $e(\alpha) \in |N|^{\mathscr{G}'}$ would imply $\alpha = \pi(e(\alpha)) \in |N|^{\mathscr{G}}$. □

7. The minimum order and equational graph theories

Bucciarelli and Salibra (2004; 2008) define a notion of 'weak product' for graph models. In this paper we prefer to call this construction *gluing* since, in general, it does not satisfy the categorical definition of a weak product (in Appendix B we present some special cases in which gluing acts as a weak product).

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Definition 7.1. The *gluing* $\diamond_{k\in K}\mathscr{G}_k$ of a family $(\mathscr{G}_k)_{k\in K}$ of graph models with pairwise disjoint webs is the graph model freely generated by the partial pair $\sqcup_{k\in K}\mathcal{G}_k$; we use $\diamond_{k\in K}\mathcal{G}_k$ instead of $\overline{\sqcup_{k\in K}\mathcal{G}_k}$ to denote its web. More generally, for any family $(\mathscr{G}_k)_{k\in K}$ of graph models, we use $\diamond_{k\in K}\mathscr{G}_k$ to denote any gluing of isomorphic copies of the $\mathscr{G}_k$'s with pairwise disjoint webs.

Note that gluing is commutative and associative up to isomorphism (of graph models).

Lemma 7.2. Let $(\mathscr{G}_k)_{k\in K}$, be a family of graph models such that $\mathscr{G}_k = \mathscr{G}_{\mathcal{A}_k}$ for some family $(\mathcal{A}_k)_{k\in K}$ of pairwise disjoint partial pairs. Then $\diamond_{k\in K}\mathscr{G}_k = \mathscr{G}_{\mathcal{A}}$, where $\mathcal{A} = \overline{\sqcup_{k\in K}\mathcal{A}_k}$.

Proof. The conclusion follows from Remark 6.14 since, clearly, $\sqcup_{k\in K}\mathcal{A}_k \sqsubseteq \sqcup_{k\in K}\overline{\mathcal{A}_k} \sqsubseteq \overline{\sqcup_{k\in K}\mathcal{A}_k}$, and $\sqcup_{k\in K}\overline{\mathcal{A}_k} = \sqcup_{k\in K}\mathcal{G}_k$. □

Proposition 7.3 (Bucciarelli and Salibra 2003, Proposition 2). Let $\mathscr{G} = \diamond_{k\in K}\mathscr{G}_k$, where $(\mathscr{G}_k)_{k\in K}$ is a family of graph models. Then:

(i) $|M|^{\mathscr{G}_k} = |M|^{\mathscr{G}} \cap G_k$ for any $M \in \Lambda^o$.

Hence:

(ii) $Th_{\sqsubseteq}(\mathscr{G}) \subseteq Th_{\sqsubseteq}(\mathscr{G}_k)$,

(iii) $Th(\mathscr{G}) \subseteq Th(\mathscr{G}_k)$.

The proof of the above proposition is complex in the general case, but we will see in Appendix B that it becomes trivial when gluing acts as a weak product. Using this result, we now derive the existence of a minimum order graph theory (the following proof is analogous to the one given in Bucciarelli and Salibra (2003) for equational graph theories).

Theorem 7.4. There exists a graph model whose order theory is the minimum among all order graph theories (hence, the analogue holds for its equational theory).

Proof. Let $(\mathcal{A}_k)_{k\in\mathbb{N}}$ be a family of pairwise disjoint finite partial pairs such that all other finite pairs are isomorphic to at least one $\mathcal{A}_k$. Take $\mathscr{G} = \diamond_{k\in K}\mathscr{G}_k$, where $\mathscr{G}_k = \mathscr{G}_{\mathcal{A}_k}$; by Lemma 7.2, $\mathscr{G} = \mathscr{G}_{\mathcal{A}}$ where $\mathcal{A} = \sqcup_{k\in\mathbb{N}}\mathcal{A}_k$.

We now prove that the order theory, and hence also the equational theory, of $\mathscr{G}$ is the minimum one. Let e be an inequation that fails in some graph model. By Corollary 6.21(ii), e fails in some $\mathscr{G}_{\mathcal{B}}$ where $\mathcal{B}$ is some finite pair, hence it fails in some $\mathscr{G}_k$. Then, by Proposition 7.3(ii), we get that e fails in $\mathscr{G}$. □

Recall that, by Proposition 4.8, the minimum equational graph theory cannot be λ_β or $\lambda_{\beta\eta}$.

8. A Löwenheim–Skolem theorem for graph models

In this section we prove a kind of downwards Löwenheim–Skolem theorem for graph models: every equational/order graph theory is the theory of a graph model having a countable web. This result gives a positive answer to Question 3 in Berline (2000, Section 6.3) for the class of graph models. Note that applying the classical Löwenheim–Skolem theorem to a graph model $\mathscr{G}$, viewed as a combinatory algebra $\mathscr{C}$, would only

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give a countable elementary substructure $\mathscr{C}'$ of $\mathscr{C}$. Such a $\mathscr{C}'$ does not correspond to any graph model since there exists no countable graph model.

Let us first note that the class of total subpairs of a total pair $\mathcal{G}$ is closed under (finite or infinite) intersections and increasing unions.

Definition 8.1. If $\mathcal{A} \sqsubseteq \mathcal{G}$ is a partial pair, then the *total subpair of* $\mathcal{G}$ *generated by* $\mathcal{A}$ is defined as the intersection of all the total pairs $\mathcal{G}'$ such that $\mathcal{A} \sqsubseteq \mathcal{G}' \sqsubseteq \mathcal{G}$.

Theorem 8.2 (Löwenheim–Skolem Theorem for graph models). For all graph models $\mathscr{G}$ there exists a graph model $\mathscr{G}'$ with a countable web $\mathcal{G}' \sqsubseteq \mathcal{G}$ such that $Th_{\sqsubseteq}(\mathscr{G}') = Th_{\sqsubseteq}(\mathscr{G})$, and hence such that $Th(\mathscr{G}') = Th(\mathscr{G})$.

Proof. We will define an increasing sequence of countable subpairs $\mathcal{A}_n$ of $\mathcal{G}$, and take for $\mathcal{G}'$ the total subpair of $\mathcal{G}$ generated by $\mathcal{A} = \bigsqcup_{n\in\mathbb{N}} \mathcal{A}_n$.

We start by defining $\mathcal{A}_0$. Let I be the countable set of inequalities between closed λ-terms that fail in $\mathscr{G}$. Let $e \in I$. By Corollary 6.21(i), there exists a finite partial pair $\mathcal{A}_e \sqsubseteq \mathcal{G}$ such that e fails in every partial pair $\mathcal{B}$ satisfying $\mathcal{A}_e \sqsubseteq \mathcal{B} \sqsubseteq \mathcal{G}$. Then we define $\mathcal{A}_0 = \bigsqcup_{e\in I} \mathcal{A}_e \sqsubseteq \mathcal{G}$. Assume now that $\mathcal{A}_n$ has been defined, and we define $\mathcal{A}_{n+1}$ as follows. Let $\mathscr{G}'_n$ be the graph model whose web $\mathcal{G}'_n$ is the total subpair of $\mathcal{G}$ generated by $\mathcal{A}_n$. For each inequality $e = M \sqsubseteq N$ that holds in $\mathscr{G}$ and fails in $\mathscr{G}'_n$, we consider the set $L_e = \{\alpha \in G'_n : \alpha \in |M|^{\mathscr{G}'_n} - |N|^{\mathscr{G}'_n}\}$. Let $\alpha \in L_e$. Since $\mathcal{G}'_n \sqsubseteq \mathcal{G}$ and $\alpha \in |M|^{\mathscr{G}'_n}$, by Lemma 6.5, we have that $\alpha \in |M|^{\mathscr{G}}$. By $|M|^{\mathscr{G}} \subseteq |N|^{\mathscr{G}}$, we also have $\alpha \in |N|^{\mathscr{G}}$. By Lemma 6.6, there exists a partial pair $\mathcal{C}_{\alpha,e} \sqsubseteq \mathcal{G}$ such that $\alpha \in |N|^{\mathcal{C}_{\alpha,e}}$. We define $\mathcal{A}_{n+1}$ as the union of the partial pair $\mathcal{A}_n$ and the partial pairs $\mathcal{C}_{\alpha,e}$ for every $\alpha \in L_e$.

As mentioned above, we take for $\mathcal{G}'$ the total subpair of $\mathcal{G}$ generated by $\mathcal{A} = \bigsqcup_{n\in\mathbb{N}} \mathcal{A}_n$. By construction, we have for every inequality e that fails in $\mathscr{G}$ that $\mathcal{A}_e \sqsubseteq \mathcal{G}'_n \sqsubseteq \mathcal{G}' \sqsubseteq \mathcal{G}$. Now, $Th_{\sqsubseteq}(\mathscr{G}') \subseteq Th_{\sqsubseteq}(\mathscr{G})$ follows from Corollary 6.21(i) and the choice of $\mathcal{A}_e$.

Now, in order to show a contradiction, we suppose that there exists an inequality $M \sqsubseteq N$ that fails in $\mathscr{G}'$ but not in $\mathscr{G}$. Then there is an $\alpha \in |M|^{\mathscr{G}'} - |N|^{\mathscr{G}'}$. By Corollary 6.21(i), there is a finite partial pair $\mathcal{B} \sqsubseteq \mathcal{G}'$ satisfying the following condition: for every partial pair $\mathcal{C}$ such that $\mathcal{B} \sqsubseteq \mathcal{C} \sqsubseteq \mathcal{G}'$, we have $\alpha \in |M|^{\mathcal{C}} - |N|^{\mathcal{C}}$. Since $\mathcal{B}$ is finite, we have that $\mathcal{B} \sqsubseteq \mathcal{G}'_n$ for some n. This implies that $\alpha \in |M|^{\mathscr{G}'_n} - |N|^{\mathscr{G}'_n}$. By the construction of $\mathscr{G}'_{n+1}$, we have that $\alpha \in |N|^{\mathscr{G}'_{n+1}}$, which implies $\alpha \in |N|^{\mathscr{G}'}$, which gives a contradiction. □

As we mentioned at the end of Sections 1.3/1.4, an alternative and more conceptual proof of Theorem 8.2 could be given that can be much more easily and transparently adapted to the other classes of webbed models.

Part III: Effective λ-models

9. Effective λ-models in Scott-continuous semantics

In this section we recall the definition of *effective domains*, which are also called 'effectively given domains' in the literature. Then we introduce the new notion of *effective λ-models*

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and *weakly effective λ-models* and prove some properties of these models using methods of recursion theory. In particular, we prove that:

(i) The equational theory of an effective λ-model cannot be λ_β or $\lambda_{\beta\eta}$ (Corollary 9.46).
(ii) The order theory of an effective λ-model cannot be r.e. (Corollary 9.44).

9.1. *Effective Scott domains*

All the material developed in this subsection can be found in Stoltenberg-Hansen *et al.* (1994, Chapter 10); its adaptation to DI-domains and DI-domains with coherences can be found in Gruchalski (1996).

9.1.1. *The category* **ED** *of effective Scott domains and continuous functions*

Definition 9.1. A triple $\mathcal{D} = (\mathcal{D}, \sqsubseteq_\mathcal{D}, d)$ is said to be an *effective domain* if $(\mathcal{D}, \sqsubseteq_\mathcal{D})$ is a Scott domain and $d : \mathbb{N} \to \mathcal{K}(\mathcal{D})$ is a numeration of $\mathcal{K}(\mathcal{D})$ such that:

(i) The relation 'd_m and d_n have an upper bound' is decidable in (m, n).
(ii) The relation '$d_n = d_m \sqcup d_k$' is decidable in (m, n, k).

We can equivalently replace (ii) by:

(ii′) The join operator restricted to pairs of compact elements is total recursive and '$d_n = d_m$' (or, equivalently, '$d_m \sqsubseteq_\mathcal{D} d_n$') is decidable in (m, n).

The equivalence holds because '$d_m \sqsubseteq_\mathcal{D} d_n$' is equivalent to '$d_n = d_m \sqcup d_n$', and because '$d_m = d_n$' is equivalent to '$d_m \sqsubseteq_\mathcal{D} d_n$ and $d_n \sqsubseteq_\mathcal{D} d_m$'.

As usual, when there is no ambiguity, we will use $\mathcal{D}$ to denote the effective domain $(\mathcal{D}, \sqsubseteq_\mathcal{D}, d)$.

Notation 9.2. For all $v \in \mathcal{D}$, we set $\widehat{v} = \{n : d_n \sqsubseteq_\mathcal{D} v\}$.

Definition 9.3. An element v of an effective domain $\mathcal{D}$ is said to be *r.e.* (respectively, *decidable*) if the set $\widehat{v}$ is r.e. (respectively, decidable).

In the literature, *r.e. elements* (of domains) are called 'computable elements', while our decidable elements have apparently not been addressed. We choose the alternative terminology of r.e. elements for the following reasons:

(1) It is more coherent with the usual terminology for elements of $\mathscr{P}(\mathbb{N})$ (see Example 9.14).
(2) It emphasises the difference between r.e. elements and decidable elements of $\mathcal{D}$.

Notation 9.4. We use $\mathcal{D}^{r.e.}$ (respectively, $\mathcal{D}^{dec}$) to denote the set of r.e. (respectively, decidable) elements of the effective domain $\mathcal{D}$.

Note that $\mathcal{K}(\mathcal{D}) \subseteq \mathcal{D}^{r.e.}$, and that, in general, $\mathcal{D}^{r.e.}$ is not a cpo.

Example 9.5. Given an effective numeration of a countable set D, the flat domain $D_\perp$ is effective and all its elements are decidable since they are compact. In particular, $\Lambda_\perp$ is

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effective for the following numeration:

$$\nu_{\Lambda_\perp}(n) = \begin{cases} \perp & \text{if } n = 0 \\ (n-1)_\lambda & \text{if } n > 0, \end{cases}$$

where $(-)_\lambda$ was defined at the beginning of Section 5.

Definition 9.6. ED is the category with effective domains as objects and all continuous functions as morphisms.

ED is a full subcategory of the category of Scott domains. It is Cartesian closed since if $\mathcal{D}, \mathcal{D}'$ are effective domains, then $\mathcal{D} \times \mathcal{D}'$ and $[\mathcal{D} \to \mathcal{D}']$ with appropriate numerations are effective domains also. It is easy to check these facts, but the proofs can be found in Escardó (1996).

Remark 9.7. It is clear that the composition of r.e. functions is an r.e. function, moreover it is straightforward to check that the maps *curry* and *eval*, with the usual behaviour, and the composition operator $C(f, g) = g \circ f$ are r.e. at all types. Hence, by Theorem 9.13 below, their restrictions to r.e. elements are computable.

9.1.2. *Characterisations of r.e. continuous functions* The next proposition gives two other characterisations of r.e. functions.

Proposition 9.8. Let $f : \mathcal{D} \to \mathcal{D}'$ be a continuous function where $\mathcal{D}, \mathcal{D}'$ are effective domains for d, d'. The following conditions are equivalent:

(a) $f \in [\mathcal{D} \to \mathcal{D}']^{r.e.}$.
(b) The relation $d'_m \sqsubseteq_{\mathcal{D}'} f(d_n)$ is r.e. in (m, n).
(c) $\{(m, n) : (d_m, d'_n) \in Tr(f)\}$ is r.e.

and the same holds when 'decidable' replaces 'r.e.'.

See Stoltenberg-Hansen *et al.* (1994, Chapter 10, Proposition 3.7) for a proof of this proposition in the r.e. case; of course (c) is just a reformulation of (b).

9.1.3. *Adequate numerations of* $\mathcal{D}^{r.e.}$

Definition 9.9. A natural number $n \in \mathbb{N}$ *represents* $v \in \mathcal{D}^{r.e.}$ if $\mathcal{W}_n = \widehat{v}$.

The surjection $\zeta' : \mathbb{N} \to \mathcal{D}^{r.e.}$ defined by $\zeta'(n) = v$ if and only if n represents v is not (in general) a numeration since it can be partial. This partiality would create technical difficulties. However, using standard techniques of recursion theory, it is not difficult to get a *total* numeration $\zeta^{\mathcal{D}}$ of $\mathcal{D}^{r.e.}$ in a uniform way (Stoltenberg-Hansen *et al.* 1994, Chapter 10, Theorem 4.4). In the rest of this paper we will need some further constraints on $\zeta^{\mathcal{D}}$, and their satisfiability is guaranteed by the following proposition.

Proposition 9.10. There exists a total numeration $\zeta^{\mathcal{D}} : \mathbb{N} \to \mathcal{D}^{r.e.}$ such that:

(i) $d_n \sqsubseteq_{\mathcal{D}} \zeta^{\mathcal{D}}_m$ is r.e. in (m, n).
(ii) The inclusion mapping $\iota : \mathcal{K}(\mathcal{D}) \to \mathcal{D}^{r.e.}$ is computable with respect to $d, \zeta^{\mathcal{D}}$.

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Definition 9.11. A numeration $\zeta^{\mathcal{D}}$ of $\mathcal{D}^{r.e.}$ is said to be *adequate* if it fulfills conditions (i) and (ii) of Proposition 9.10.

Lemma 9.12. For all adequate $\zeta^{\mathcal{D}}, \zeta'^{\mathcal{D}}$, there is a total recursive function $\varphi : \mathbb{N} \to \mathbb{N}$ such that $\zeta^{\mathcal{D}} = \zeta'^{\mathcal{D}} \circ \varphi$.

From now on we will always suppose that $\zeta^{\mathcal{D}}$ is an adequate numeration of $\mathcal{D}^{r.e.}$.

Given two effective domains $(\mathcal{D}, \sqsubseteq_{\mathcal{D}}, d)$ and $(\mathcal{D}', \sqsubseteq_{\mathcal{D}'}, d')$, it is essentially straightforward to obtain, in a canonical way, a numeration $\nu_{(d,d')}$ of the compact elements of $[\mathcal{D} \to \mathcal{D}']$ (see Stoltenberg-Hansen *et al.* (1994, Chapter 10, Theorem 3.6)), which is then used to give $[\mathcal{D} \to \mathcal{D}']$ a structure of effective domain.

Theorem 9.13. Let $(\mathcal{D}, \sqsubseteq_{\mathcal{D}}, d)$ and $(\mathcal{D}', \sqsubseteq_{\mathcal{D}'}, d')$ be effective domains. Then a continuous function $f : \mathcal{D} \to \mathcal{D}'$ is r.e. (with respect to $\nu_{(d,d')}$) if and only if its restriction $f\!\upharpoonright\!: \mathcal{D}^{r.e.} \to \mathcal{D}'^{r.e.}$ is computable with respect to $\zeta^{\mathcal{D}}, \zeta^{\mathcal{D}'}$.

Proof. The proof is relatively easy (the details are worked out in Stoltenberg-Hansen *et al.* (1994, Chapter 10, Proposition 4.14)). Note that the right-hand side of the equivalence only depends on d, d' because of Lemma 9.12. □

In particular, the previous theorem states that r.e. functions preserve the r.e. elements.

Example 9.14. The key example of an effective domain is $(\mathscr{P}(\mathbb{N}), \subseteq, d)$ where d is some standard bijective numeration $d : \mathbb{N} \to \mathbb{N}^*$ of the finite subsets of $\mathbb{N}$. Here the r.e. (respectively, decidable) elements are the r.e. (respectively, decidable) sets, and the usual map $n \mapsto \mathcal{W}_n$ is an adequate numeration of the r.e. elements. It follows from Proposition 9.8 that a Scott continuous function $f : \mathscr{P}(\mathbb{N}) \to \mathscr{P}(\mathbb{N})$ is r.e. if and only if its trace $Tr(f) = \{(a, n) : a \in \mathbb{N}^* \text{ and } n \in f(a)\}$ is an r.e. set.

9.2. *Completely co-r.e. subsets of* $\mathcal{D}^{r.e.}$

Our aim is to infer properties of weakly effective λ-models using methods of recursion theory. To this end, given an effective domain $\mathcal{D} = (\mathcal{D}, \sqsubseteq_{\mathcal{D}}, d)$ and an adequate numeration $\zeta^{\mathcal{D}} : \mathbb{N} \to \mathcal{D}^{r.e.}$, we study the properties of the completely co-r.e. subsets of $\mathcal{D}^{r.e.}$. The work done here could also be easily adapted to DI-domains and DI-domains with coherences.

Definition 9.15. $A \subseteq \mathcal{D}^{r.e.}$ is said to be *completely r.e.* if A is r.e. with respect to $\zeta^{\mathcal{D}}$; it is said to be *trivial* if $A = \varnothing$ or $A = \mathcal{D}^{r.e.}$. We define *completely co-r.e. sets* and *completely decidable sets* in a similar way.

This terminology (that is, the use of 'completely r.e.' where one would expect 'r.e.') is coherent with the terminology classically used in recursion theory (see, for example, Odifreddi (1989)). We will see in Corollary 9.20(iii) below that there exist no non-trivial completely decidable sets.

Notation 9.16. We write $\mathscr{P}^{core}(\mathcal{D}^{r.e.})$ for the set of completely co-r.e. subsets of $\mathcal{D}^{r.e.}$.

Remark 9.17. $\{\bot_{\mathcal{D}}\} \in \mathscr{P}^{core}(\mathcal{D}^{r.e.})$ (therefore $\{\bot_{\mathcal{D}}\}$ is not completely decidable).

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Theorem 9.18. The family of all non-empty completely co-r.e. subsets of $\mathcal{D}^{r.e.}$ has the FIP.

Proof. The conclusion follows from Corollary 9.20(ii) below, which is itself a consequence of the following extension of the Rice–Myhill–Shepherdson theorem (for details, see Odifreddi (1989, Theorem 10.5.2)). □

Theorem 9.19. Let $\mathcal{D}$ be an effective domain and let $A \subseteq \mathcal{D}^{r.e.}$. Then A is completely r.e. if and only if there is an r.e. set $E \subseteq \mathbb{N}$ such that $A = \{v \in \mathcal{D}^{r.e.} : \exists n \in E\ (\zeta_n^{\mathcal{D}} \in \mathcal{K}(\mathcal{D})$ and $\zeta_n^{\mathcal{D}} \sqsubseteq_{\mathcal{D}} v)\}$.

Proof. See Stoltenberg-Hansen *et al.* (1994, Theorem 5.2). □

Corollary 9.20. With respect to the partial order $\sqsubseteq_{\mathcal{D}}$:

(i) Completely r.e. sets are upward closed (in $\mathcal{D}^{r.e.}$).
(ii) Completely co-r.e. sets are downward closed (in $\mathcal{D}^{r.e.}$).
(iii) Completely decidable sets are trivial.

9.3. *Weakly effective continuous λ-models*

In this section we will consider λ-models living in the Scott semantics; but analogous notions can be defined for the stable and the strongly stable semantics. The following definition of weakly effective λ-models is completely natural in this context. However, in order to obtain stronger results, we will need a slightly more powerful notion. This is why we only speak of 'weak effectivity' here.

Definition 9.21. A λ-model $\mathscr{M} = (\mathcal{D}, Ap, \lambda)$ is a *weakly effective λ-model* if:

(i) $\mathscr{M}$ is a reflexive object in the category **ED**.
(ii) Ap and λ are r.e.

Remark 9.22. Let $\mathbf{ED}^{\mathbf{r.e.}}$ be the subcategory of **ED** with the same objects as **ED** (and the same exponential objects) but r.e. continuous functions as morphisms. It is easy to check using Remark 9.7 that $\mathbf{ED}^{\mathbf{r.e.}}$ inherits the structure of ccc from **ED**. The weakly effective models of Definition 9.21 above are exactly the reflexive objects of $\mathbf{ED}^{\mathbf{r.e.}}$. We prefer to use the category **ED**:

1 because we think that it is more coherent with the definition of the exponential objects to take all continuous functions as morphisms; and
2 to put in major evidence the only effectiveness conditions required.

We recall a consequence of Theorem 9.13 that we will use often later in the paper.

Remark 9.23. Let $\mathscr{M}$ be weakly effective, then:

(i) If $u, v \in \mathcal{D}^{r.e.}$, then $Ap(u)(v) \in \mathcal{D}^{r.e.}$.
(ii) If $f \in [\mathcal{D} \to \mathcal{D}]^{r.e.}$, then $\lambda(f) \in \mathcal{D}^{r.e.}$.

Convention 9.24. In the rest of this section we will assume that we are speaking of a fixed λ-model $\mathscr{M} = (\mathcal{D}, Ap, \lambda)$, where $\mathcal{D} = (\mathcal{D}, \sqsubseteq_{\mathcal{D}}, d)$ is an effective domain and that $\mathcal{T} = Th(\mathscr{M})$.

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Furthermore, we fix a bijective map v_{Var} from $\mathbb{N}$ to the set Var of variables of λ-calculus. This gives $Env_{\mathcal{D}}$ the structure of an effective domain.

Proposition 9.25. If $\mathscr{M}$ is weakly effective, then $(\mathcal{D}^{r.e.}, \bullet, |\mathbf{K}|, |\mathbf{S}|)$ is a combinatory subalgebra of $(\mathcal{D}, \bullet, |\mathbf{K}|, |\mathbf{S}|)$.

Proof. It follows from Remark 9.23 that $|\mathbf{K}| \in \mathcal{D}^{r.e.}$ and that $\mathcal{D}^{r.e.}$ is closed under $\bullet$. The fact that $|\mathbf{S}| \in \mathcal{D}^{r.e.}$ is a consequence of the next result. □

Theorem 9.26. If $\mathscr{M}$ is weakly effective, then $|M| \in \mathcal{D}^{r.e.}$ for all $M \in \Lambda^o_\perp$.

Proof. The statement is a consequence of the following proposition. □

Proposition 9.27. If $\mathscr{M}$ is weakly effective, then, for all $M \in \Lambda_\perp$, the function $|M| : Env_{\mathcal{D}} \to \mathcal{D}$ is r.e.

Proof.

— If $M \equiv \perp$, then $|\perp|$ is the constant function mapping ρ to $\perp$, which is obviously r.e. Otherwise, the proof is by structural induction over M.
— If $M \equiv x$, then $|M|$ is the map $\rho \mapsto \rho(x)$, that is, the evaluation of the environment ρ on the variable x. It is easy to check that this function is r.e.
— If $M \equiv NP$, then $|M| = eval \circ (Ap \circ |N|, |P|)$. By the induction hypothesis and Remark 9.7, $|M|$ is a composition of r.e. functions, hence it is r.e.
— If $M \equiv \lambda x.N$, then $|M| = \lambda \circ C \circ (curry(f_x), k)$, where f_x is the function $(\rho, d) \mapsto \rho[x := d]$ and k is the constant function mapping ρ to $|N|$. Note that f_x is r.e. because its restriction f'_x to $Env^{r.e.}_{\mathcal{D}} \times \mathcal{D}^{r.e.}$ is computable. Indeed $f'_x(\rho, d)$ differs from the r.e. environment ρ only on x where it takes as value the r.e. element d. This case then follows again from induction hypothesis and Remark 9.7. □

Theorem 9.28. If $\mathscr{M}$ is weakly effective, then the function $\theta : \Lambda_\perp \times Env_{\mathcal{D}} \to \mathcal{D}$ defined by $\theta(M, \rho) = |M|_\rho$ is r.e.

Proof. Since the function $|M|$ is r.e. for every M, we have that $|M|_\rho \in \mathcal{D}^{r.e.}$ for all λ-terms M and all r.e. environments ρ. Moreover, whenever $M \in \Lambda$ and $\rho \in Env^{r.e.}_{\mathcal{D}}$, the proof of Proposition 9.27 gives an effective algorithm to compute the code of $|M|_\rho$ in a uniform way starting from the codes of M and ρ. □

Corollary 9.29. If $\mathscr{M}$ is weakly effective and $\rho \in Env^{r.e.}_{\mathcal{D}}$, then the function $|-|_\rho : \Lambda_\perp \to \mathcal{D}$ is r.e. and its restriction to Λ is computable with respect to $(-)_\lambda, \zeta^{\mathcal{D}}$.

Corollary 9.30. If $\mathscr{M}$ is weakly effective and $V \subseteq \mathcal{D}^{r.e.}$ is completely co-r.e., then $\{M \in \Lambda^o : |M| \in V\}$ is β-co-r.e.

Proof. Let $\rho \in Env^{r.e.}_{\mathcal{D}}$. By Corollary 9.29 there exists a recursive map $\varphi_\rho : \mathbb{N} \to \mathbb{N}$ tracking the interpretation function $M \mapsto |M|_\rho$ of λ-terms from Λ into $\mathcal{D}^{r.e.}$. By Remark 2.1 and since the set $E = \{n : \zeta^{\mathcal{D}}_n \in V\}$ is co-r.e., it follows that $\varphi^-_\rho(E) = \{M_\omega : |M|_\rho \in V\}$ is also co-r.e. We then get the conclusion because Λ^o is a decidable subset of Λ. □

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Notation 9.31. Given an ordered λ-model $\mathscr{M}$ and $\mathcal{T} = Th(\mathscr{M})$ we set:

(i) $O_M = \{N \in \Lambda^o : |N| \sqsubseteq_{\mathscr{M}} |M|\}$, for all $M \in \Lambda^o(\mathcal{D})$. Note that O_M is a union of $\mathcal{T}$-classes and that $M \in O_M$.
(ii) $O_\perp = \{N \in \Lambda^o : |N| = \perp_{\mathcal{D}}\}$. Note that either $O_\perp = \varnothing$ or $O_\perp$ consists of a single $\mathcal{T}$-class,
(iii) $O_\perp^\omega = \cup_{n\in\mathbb{N}} O_\perp^n$ where $O_\perp^n$ is O_L for $L \equiv \lambda x_1 \dots x_n.\perp_{\mathcal{D}}$.

So $O_\perp$ is just a simplified notation for $O_\perp^0$. The close notations for $O_\perp$ and $\mathcal{D}_\perp$ will not be confusing.

Lemma 9.32. For all ordered models $\mathscr{M}$, and $M, N \in \Lambda^o$ we have:

(i) $O_\perp \cup [M]_{\mathcal{T}} \subseteq O_M \subseteq M^{\mathcal{T}-ins}$, where $\mathcal{T} = Th(\mathscr{M})$.
(ii) If M, N are non-equivalent hnf's, then $O_M \cap O_N \subseteq \mathcal{U}$.

Proof.

(i) The first inclusion is obvious. The second follows from the observation that the interpretations of two separable λ-terms are incomparable in any non-trivial partially ordered λ-model.
(ii) This part is immediate once we have noted that no solvable λ-term belongs to $M^{ins} \cap N^{ins}$, since an hnf in the intersection should be simultaneously equivalent to M and N. □

Note that under the hypothesis of Lemma 9.32 it can be true that $O_M \cap O_N = \varnothing$ for all M, N that are not $\beta\eta$-equivalent, as shown by the model of Di Gianantonio *et al.* (1995).

It is interesting to note the following related result, which holds only for graph models and which, as in the preceeding case, does not need any hypothesis of effectivity.

Lemma 9.33. For all graph models $\mathscr{G}$, if $N \in O_{\mathbf{I}}$, then either $N =_{\lambda_\beta} \mathbf{I}$ or N is unsolvable.

Proof. Suppose that $N \in O_{\mathbf{I}}$ and N is solvable. Without loss of generality, N is an hnf equivalent to $\mathbf{I}$, and hence of the form $N \equiv \lambda x.\lambda\vec{z}.x\vec{N}$, with $\vec{z}$ and $\vec{N}$ of the same length $k \geqslant 0$. If $k \geqslant 1$, it is easy to check that $\gamma = i_{\mathcal{G}}(\{i_{\mathcal{G}}(\varnothing^k, \alpha)\}, i_{\mathcal{G}}(\{\alpha\}, i_{\mathcal{G}}(\varnothing^{k-1}, \alpha))) \in |N|^{\mathscr{G}} - |\mathbf{I}|^{\mathscr{G}}$, where $i_{\mathcal{G}}(\varnothing^n, \alpha)$ is a shorthand for $i_{\mathcal{G}}(\varnothing, i_{\mathcal{G}}(\varnothing \dots, i_{\mathcal{G}}(\varnothing, \alpha)\dots))$. □

Note that unlike the other results of this section, this lemma is false for the other classes of models that have been introduced in the literature, whether living in the continuous, stable or strongly stable semantics. Indeed, all these classes contain extensional models (we will give more details on them later). Looking at the proof, we can observe that the reasons for its failing differ according to the semantics: $\gamma \in |\mathbf{I}|$ in the case of K-models, and $\gamma \notin |N|$ in the stable and strongly stable case (because the injective function i of the web is defined via Tr_s).

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Proposition 9.34. If $\mathscr{M}$ is weakly effective and $\mathcal{T} = Th(\mathscr{M})$, then:

(i) $O_\perp \subseteq \mathcal{U}$.
(ii) $O_\perp$ is $\mathcal{T}$-co-r.e.

Proof.

(i) This part is true by Lemma 9.32(ii) since $O_\perp \subseteq O_M \cap O_N$ for all M, N.
(ii) This part follows from Remark 9.17 and Corollary 9.30. □

Any sensible model satisfies $\mathcal{U} = [\Omega]_\mathcal{T} \subseteq O_\Omega$. Thus, in all sensible models that interpret Ω by $\perp_\mathcal{D}$, we have $O_\perp = O_\Omega = \mathcal{U}$ (for example, this is the case for all sensible graph models). On the other hand, it is easy to build models satisfying $O_\Omega = \Lambda^o$: for example, finding a graph model $\mathscr{G}$ with carrier set G such that $|\Omega|^\mathscr{G} = G$ is a straightforward exercise, which also appears as the simplest application of the generalised forcing developed in Berline and Salibra (2006). Finally, (the usual) forcing also allows us to build, for all $M \in \Lambda^o$, a graph model satisfying $\Omega = M$ and hence $O_\Omega = O_M$, and this is still true for $M \in \Lambda^o(\mathcal{D})$, and beyond, using generalised forcing.

Proposition 9.35. If $\mathscr{M}$ is weakly effective and $\mathcal{T} = Th(\mathscr{M})$ is r.e., then $O_\perp^\omega = \varnothing = O_\perp$.

Proof. Since $O_\perp$ consists of zero or one $\mathcal{T}$-classes, it follows from Lemma 5.7 and Proposition 9.34 that $O_\perp = \varnothing$. Now it follows by an easy induction on n that $O_\perp^n = \varnothing$ for all n since, if $N \in O_\perp^{k+1}$, then $N\mathbf{I} \in O_\perp^k$. □

Notation 9.36. Given a λ-model $\mathscr{M}$ and $\mathcal{T} = Th(\mathscr{M})$, for all $E \subseteq \mathbb{N}$, we set $\Lambda_E^o = \{N \in \Lambda^o : \widehat{|N|} \subseteq E\}$, where $\widehat{|N|} = \{n : d_n \sqsubseteq_\mathscr{M} |N|\}$. Note that Λ_E^o is a union of $\mathcal{T}$-classes, which depends on $\mathscr{M}$ (and not only on $\mathcal{T}$). Furthermore, for all $M \in \Lambda^o$, $\Lambda^o_{\widehat{|M|}} = O_M$.

Theorem 9.37. Let $\mathscr{M}$ be weakly effective, $\mathcal{T} = Th(\mathscr{M})$ and $E \subseteq \mathbb{N}$.

(i) If E is co-r.e., then Λ_E^o is $\mathcal{T}$-co-r.e.
(ii) If E is decidable, then either $\Lambda_E^o = \varnothing$ or $\Lambda_{E^c}^o = \varnothing$.

Proof.

(i) First note that $E' = \{n : (\exists m \notin E)\ d_m \sqsubseteq_\mathcal{D} \zeta_n^\mathcal{D}\}$ is r.e. Hence $\{\zeta_n : n \notin E'\}$ is completely co-r.e. Corollary 9.30 then gives the conclusion.
(ii) This part follows from the FIP since $\Lambda_E^o \cap \Lambda_{E^c}^o = \varnothing$. □

Theorem 9.38. Let $\mathscr{M}$ be weakly effective, $\mathcal{T} = Th(\mathscr{M})$ and $M_1, \ldots, M_n \in \Lambda^o$. If $|M_i| \in \mathcal{D}^{dec}$ for all $1 \leqslant i \leqslant n$, then $O_{M_1} \cap \cdots \cap O_{M_n}$ is a $\mathcal{T}$-co-r.e. set, which contains a non-empty β-co-r.e. set $\mathcal{V}$ of unsolvable terms.

Proof. Since, for all $1 \leqslant i \leqslant n$, the set $\widehat{|M_i|}$ is decidable and $O_{M_i} = \Lambda^o_{\widehat{|M_i|}}$, we get that every O_{M_i} is a non-empty $\mathcal{T}$-co-r.e. set by Theorem 9.37(i). Hence $V = O_{M_1} \cap \cdots \cap O_{M_n}$ is also a $\mathcal{T}$-co-r.e. set containing $\mathcal{V} = V \cap \mathcal{U}$, which is β-co-r.e. since $\mathcal{U}$ is β-co-r.e., and is non-empty by the FIP. □

Theorem 9.39. Let $\mathscr{M}$ be weakly effective and $\mathcal{T} = Th(\mathscr{M})$. If there exists $M \in \Lambda^o$ such that $|M| \in \mathcal{D}^{dec}$ and $O_M - [M]_\mathcal{T}$ is finite modulo $\mathcal{T}$, then $\mathcal{T}$ is not r.e.

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Proof. By Lemma 5.7, if $\mathcal{T}$ is r.e., then $O_M/\mathcal{T}$ is infinite. □

9.4. *Effective continuous λ-models*

As proved in Proposition 9.27, weakly effective λ-models interpret λ-terms by r.e. elements. The notion of an effective λ-model introduced below has the further key advantage that normal terms are interpreted by decidable elements and this leads to interesting consequences. As we will see in Sections 10 and 11, all the models living in the continuous semantics, or in one of its refinements, and introduced *individually* in the literature are effective. Furthermore, in the case of webbed models, easy sufficient conditions can be given at the level of the web in order to guarantee the effectiveness of the λ-model.

Definition 9.40. A weakly effective λ-model $\mathscr{M} = (\mathcal{D}, Ap, \lambda)$ is said to be *effective* if it satisfies the following two conditions:

(i) If $d \in \mathcal{K}(\mathcal{D})$ and $e_1, \ldots, e_k \in \mathcal{D}^{dec}$, then $de_1 \cdots e_k \in \mathcal{D}^{dec}$.
(ii) If $f \in [\mathcal{D} \to \mathcal{D}]^{r.e.}$ and $f(d) \in \mathcal{D}^{dec}$ for all $d \in \mathcal{K}(\mathcal{D})$, then $\lambda(f) \in \mathcal{D}^{dec}$.

Remark 9.41. The key condition is the second one. Indeed, many λ-models, and, in particular, all graph models and all extensional algebraic λ-models, automatically satisfy a property that is stronger than (i), namely (i′) below.

Proposition 9.42. Let $\mathscr{M} = (\mathcal{D}, Ap, \lambda)$ be either a graph model, or a λ-model such that $\mathcal{D}$ is a Scott domain and $\mathscr{M}$ satisfies $|\mathbf{I}| \sqsubseteq_{\mathcal{D}} |\varepsilon|$ (equivalently, $id \sqsubseteq_{[\mathcal{D}\to\mathcal{D}]} \lambda \circ Ap$). Then we have:

(i′) If $d \in \mathcal{K}(\mathcal{D})$ and $e_1, \ldots, e_k \in \mathcal{D}$, then $de_1 \cdots e_k \in \mathcal{K}(\mathcal{D})$.

Proof. If $\mathscr{M}$ is a graph model, then the compact elements of $\mathcal{D}$ are exactly the finite subsets of its web. From this and the definition of application in the case of graph models, it follows that de is compact for every compact d and arbitrary e. If $\mathscr{M}$ lives in Scott semantics, then $f(e)$ is compact for every compact continuous function f and $e \in \mathcal{D}$. Indeed, such an f is a finite sup of step functions $\varepsilon_{h,k}$ where $h, k \in \mathcal{K}(\mathcal{D})$. Thus, it only remains to prove that $d \in \mathcal{K}(\mathcal{D})$ entails that $Ap(d)$ is also compact. Let $\{f_i : i \in I\}$ be a non-empty directed set of continuous functions on $\mathcal{D}$, and suppose $Ap(d) \sqsubseteq \bigsqcup_{i\in I} f_i$. Then $\lambda(Ap(d)) \sqsubseteq \lambda(\bigsqcup_{i\in I} f_i) = \bigsqcup_{i\in I} \lambda(f_i)$ by continuity of λ. Since $id \sqsubseteq \lambda \circ Ap$, we get $d \sqsubseteq \bigsqcup_{i\in I} \lambda(f_i)$ and, by the compactness of d, there is a $j \in I$ such that $d \sqsubseteq \lambda(f_j)$. It then follows that $Ap(d) \sqsubseteq Ap(\lambda(f_j)) = f_j$, and hence $Ap(d)$ is compact. □

Theorem 9.43. If $\mathscr{M}$ is effective, then for all normal λ-terms $M \in \Lambda^o$, we have $|M| \in \mathcal{D}^{dec}$.

Proof. Since the interpretation of a closed λ-term is independent of the context, it is enough to show that $|M|_\rho \in \mathcal{D}^{dec}$ for all normal $M \in \Lambda$ and for all $\rho \in \mathcal{K}(Env_{\mathcal{D}})$. This proof is done by induction over the complexity of M.

— If $M \equiv x$, then $|M|_\rho = \rho(x)$ is a compact element, and thus decidable.
— Suppose $M \equiv yN_1 \cdots N_k$ with N_i normal for all $1 \leqslant i \leqslant k$. By definition, $|M|_\rho$ is equal to $|y|_\rho \cdot |N_1|_\rho \cdots |N_k|_\rho$. Hence this case follows from Definition 9.40(i), the fact that $\rho(y)$ is compact, and the induction hypothesis.

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— If $M \equiv \lambda x.N$, then $|M|_\rho = \lambda(d \mapsto |N|_{\rho[x:=d]})$. Note that since $\rho \in \mathcal{K}(Env_\mathcal{D})$, we also have that $\rho[x := d]$ is compact for all $d \in \mathcal{K}(\mathcal{D})$. Hence the result follows from the induction hypothesis and Definition 9.40(ii). □

Corollary 9.44. If $\mathscr{M}$ is effective, then $Th_\sqsubseteq(\mathscr{M})$ is not r.e.

Proof. Let $M \in \Lambda^o$ be normal. If $Th_\sqsubseteq(\mathscr{M})$ were r.e., then we could enumerate the set O_M. However, by Theorem 9.43 and Theorem 9.38, this set is co-r.e. and it is non-empty because clearly $M \in O_M$. Hence O_M would be a non-empty decidable set of λ-terms closed under β-conversion, that is, $O_M = \Lambda^o$. Since the model is non-trivial and M is arbitrary, this leads us to a contradiction. □

Corollary 9.45. If $\mathscr{M}$ is effective and $Th(\mathscr{M})$ is r.e., then $\sqsubseteq_\mathscr{M}$ induces a non-trivial partial order on the interpretations of closed λ-terms.

Corollary 9.46. If $\mathscr{M}$ is effective, then $Th(\mathscr{M}) \neq \lambda_\beta, \lambda_{\beta\eta}$.

Proof. By Selinger's result stating that in any partially ordered model whose theory is λ_β or $\lambda_{\beta\eta}$, the interpretations of closed λ-terms are discretely ordered (Selinger 1996, Corollary 4). □

Recall that in the case of graph models we know a much stronger result since we already know from Proposition 4.8 that for *all* graph models $\mathscr{G}$ we have $Th(\mathscr{G}) \neq \lambda_\beta, \lambda_{\beta\eta}$.

10. Effective stable and strongly stable λ-models

The definitions of effective and weakly effective continuous λ-models can be slightly modified to capture the notions of effective and weakly effective stable (and strongly stable) λ-models. It is enough to change the underlying category **ED** as follows: for the stable semantics we take the category **EDID** having effective DI-domains as objects and stable functions as morphisms; for the strongly stable semantics we take the category $\mathbf{EDID}^{\mathbf{coh}}$ having effective DI-domains with coherences as objects and strongly stable functions as morphisms.

Claim 10.1. The analogues of all the results proved for weakly effective and effective continuous λ-models (Subsections 9.3 and 9.4) hold in the stable and strongly stable semantics. The only exception is Lemma 9.33, as noted there.

This is easy to check from the proofs that we gave in detail for the continuous case.

The purpose of this section is to show that no effective model living in the stable or strongly stable semantics can have an r.e. theory, even though these two semantics, like the continuous one, happen to contain many effective models (see the end of the section).

Lemma 10.2. If $\mathscr{M}$ belongs to the stable or strongly stable semantics, then

$$O_\mathbf{F} \cap O_\mathbf{T} \subseteq \{N : \mathbf{1}N \in O_\perp^2\}.$$

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Proof. Suppose $N \in O_{\mathbf{F}} \cap O_{\mathbf{T}}$. Let $f, g, h \in [\mathcal{D} \to_s \mathcal{D}]$ (or $[\mathcal{D} \to_{ss} \mathcal{D}]$) be $f = Ap(|\mathbf{T}|)$, $g = Ap(|\mathbf{F}|)$ and $h = Ap(|N|)$. By monotonicity of Ap, we have $h \leqslant_s f, g$. Now g is the constant function taking value $|\mathbf{I}|$, and $f(\bot_{\mathcal{D}}) = |\lambda y.\bot_{\mathcal{D}}|$. The first assertion forces h to be a constant function (Remark 4.10) and the fact that h is pointwise smaller than f forces $\lambda(h) = |\lambda x.\lambda y.\bot_{\mathcal{D}}|$. Therefore, $\lambda(h) \in O_{\bot}^2$. It is now enough to note that in all λ-models $\mathscr{M} = (\mathcal{D}, Ap, \lambda)$ we have $\lambda(Ap(u)) = |\mathbf{1}u|$ for all $u \in \mathcal{D}$ and, in particular, $\lambda(h) = |\mathbf{1}N|$. Hence $\mathbf{1}N \in O_{\bot}^2$. □

Theorem 10.3. If $\mathscr{M}$ is effective and belongs to the stable or to the strongly stable semantics, then $Th(\mathscr{M})$ is not r.e.

Proof. Since $\mathscr{M}$ is effective and $\mathbf{F}, \mathbf{T}$ are closed and normal, we have $O_{\mathbf{F}} \cap O_{\mathbf{T}} \neq \varnothing$ by Theorem 9.38. Now, if $Th(\mathscr{M})$ is r.e., then, by Proposition 9.35, every $O_{\bot}^n$ is empty. In particular, this implies, by Lemma 10.2, that $O_{\mathbf{F}} \cap O_{\mathbf{T}} = \varnothing$, which leads us to a contradiction. □

It is easy to check that Lemma 10.2 is false for the continuous semantics. We can even give a counter-example in the class of graph models. Indeed, we know from Berline and Salibra (2006) that there exists a graph model $\mathscr{G}$ (built by forcing) in which $\mathbf{\Omega}$ acts like intersection (and $|\mathbf{\Omega}|^{\mathscr{G}} = |\mathbf{1\Omega}|^{\mathscr{G}}$). Equivalently, in $\mathscr{G}$ we have $|\mathbf{\Omega}|^{\mathscr{G}} = |\mathbf{T}|^{\mathscr{G}} \cap |\mathbf{F}|^{\mathscr{G}}$, and thus $\mathbf{\Omega} \in (O_{\mathbf{T}} \cap O_{\mathbf{F}}) - O_{\bot}^2$ and $\mathbf{1\Omega} \notin O_{\bot}^2$.

Let us now justify our statement that there are many effective stable and strongly stable models. In the next section we will introduce the notion of an effective pair and we will prove that all the graph models freely generated by effective partial pairs are effective models, showing, in particular, that there are many effective graph models.

The stable semantics contains a class that is analogous to the class of graph models, namely Girard's class of *reflexive coherent spaces*, called *G-models* in Berline (2000). G-models are generated by webs that are not 'pairs', but a slightly sophisticated version of them (see Berline (2000, Definition 149)).

The material developed in Sections 11.2 and 11.3 below for graph models could be adapted for *G*-models, though completing partial webs is more delicate in this case (the free completion process was described in Kerth (1995; 2001)).

This material could also be developed for *H*-models (that is, reflexive hypercoherences, which belong to the strongly stable semantics); the free completion process has only been worked out in print for particular *H*-models (Gouy 1995; Bastonero 1996), but it does work in greater generality[†], even though working in the strongly stable semantics certainly adds technical difficulties.

In fact, the material developed in Sections 11.2 and 11.3 could be developed for each class of webbed models, starting each time from the appropriate notion of webs, then defining the notion of an effective web, and so on. Unfortunately, it does not seem possible at the moment to give a uniform treatment of all known classes of webbed models simultaneously.

[†] R. Kerth and O. Bastonero, *private communication*, 1997.

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We conclude by recalling that all the λ-models introduced individually in the literature are (or could be) generated by effective partial webs.

11. Effective graph models

A side effect of this section is to show that effective models are omni-present in the continuous semantics. In Section 11.1 we will introduce a notion of weakly effective (respectively, effective) partial pairs, and in Section 11.2 we will prove that they generate weakly effective (respectively, effective) λ-models. An analogue of the work done in these two sections could clearly be developed for each of the other classes of webbed models, for example, using the terminology of Berline (2000), K-models, pcs-models, filter models (for the continuous semantics), G-models and H-models (for the stable and strongly stable semantics, respectively). Note that all the λ-models that have been introduced individually in the literature, beginning with $\mathscr{P}_\omega$, $\mathscr{E}$ (graph models) and Scott's $\mathscr{D}^\infty$ (K-model) are (or could be) presented as generated by webs that happen to be effective in our sense.

11.1. *Weakly effective and effective pairs*

Definition 11.1. A partial pair $\mathcal{A}$ is *weakly effective* if it is isomorphic to some pair (E,ℓ) where E is a decidable subset of $\mathbb{N}$ and ℓ is partial recursive with decidable domain. It is *effective* if, in addition, $\mathrm{rg}(\ell)$ is decidable.

In the case of *total* pairs, being 'weakly effective' (respectively, 'effective') admits the following characterisation. The interest of this characterisation is that it allows us to assume, without loss of generality, that the underlying set of the web of a weakly effective total pair is the set $\mathbb{N}$ of natural numbers.

Lemma 11.2. A total pair $\mathcal{G}$ is weakly effective if and only if it is isomorphic to a total pair $(\mathbb{N},\ell)$ where ℓ is *total* recursive. It is effective if, in addition, we can choose ℓ with a decidable range.

Proof. The proof is straightforward. □

Throughout this section we will assume that all effective and weakly effective pairs have as underlying set a subset of $\mathbb{N}$; when these pairs are also total, we will assume that their underlying set is exactly $\mathbb{N}$.

Example 11.3. An example is $\mathscr{P}_\omega$ in its original definition (see, for example, Barendregt (1984)), since ℓ is defined by $\ell(a,n) = \ll a,n \gg$ (with the notation of Section 2.2); here ℓ is also surjective.

Proposition 11.4. If $\mathcal{G}$ is a weakly effective (respectively, effective) total pair, then $\mathscr{G}$ is a weakly effective (respectively, effective) λ-model.

Proof. By Lemma 11.2, it is enough to prove the statement for weakly effective pairs of the form $(\mathbb{N},\ell)$. Then it is easy to check, using Definition 4.5, that $Ap^{\mathcal{G}}, \lambda^{\mathcal{G}}$ are r.e.

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Furthermore, condition (i) of Definition 9.40 (effective λ-models) is satisfied for all graph models. Finally, it is straightforward to check that condition (ii) holds when $\mathrm{rg}(\ell)$ is decidable. □

Next we show that the free completion process preserves the effectivity of the partial pairs.

11.2. *Free completions of (weakly) effective pairs*

Theorem 11.5. If $\mathcal{A}$ is weakly effective (respectively, effective), then $\overline{\mathcal{A}}$ is weakly effective (respectively, effective).

Proof. Suppose $\mathcal{A} = (A, j_{\mathcal{A}})$ is a weakly effective partial pair. Without loss of generality, we can suppose $A = \{2^k : k < card(A)\}$. For all $n \in \mathbb{N}$, we will use j_n to denote the restriction $i_{\bar{A}}{\restriction}_{A_n^* \times A_n}$ where A_n was introduced in Definition 6.11.

We now build $\theta : \overline{A} \to \mathbb{N}$ as an increasing union of functions $\theta_n : A_n \to \mathbb{N}$ defined by induction on n. At each step we set $E_n = \mathrm{rg}(\theta_n)$ and define $\ell_n : E_n^* \times E_n \to E_n$ such that θ_n is an isomorphism between (A_n, j_n) and (E_n, ℓ_n). We will take $E = \cup_{n\in\mathbb{N}} E_n$ and $\ell = \cup_{n\in\mathbb{N}} \ell_n$.

— Case $n = 0$. We take for θ_0 the identity on A, then $E_0 = A$ and we take $\ell_0 = j_{\mathcal{A}}$. By hypothesis, E_0 is decidable and $j_{\mathcal{A}}$ has a decidable domain, and if $\mathcal{A}$ is also effective, a decidable range also.

— Case $n + 1$. We define

$$\theta_{n+1}(x) = \begin{cases} \theta_n(x) & \text{if } x \in A_n \\ p_{n+1}^{\ll a,\alpha \gg} & \text{if } x = (a, \alpha) \in (A_{n+1} - A_n) \end{cases}$$

where p_{n+1} denotes the $(n+1)$-th prime number and $\ll -, - \gg$ is defined in Section 2.2. Since A and $\mathrm{dom}(j_{\mathcal{A}})$ are decidable by hypothesis and A_n is decidable by the induction hypothesis, we also have $A_{n+1} = A \cup ((A_n^* \times A_n) - \mathrm{dom}(j_{\mathcal{A}}))$ is decidable. θ_{n+1} is injective by construction and the induction hypothesis.

Moreover, θ_{n+1} is computable and $E_{n+1} = \mathrm{rg}(\theta_{n+1})$ is decidable since A_n and $A_{n+1} - A_n$ are decidable and θ_n and $\ll -, - \gg$ are computable with decidable range.

We define $\ell_{n+1} : E_{n+1}^* \times E_{n+1} \to E_{n+1}$ by

$$\ell_{n+1}(a, \alpha) = \begin{cases} \ell_n(a, \alpha) & \text{if } q(a, \alpha) \in E_n \\ q(a, \alpha) & \text{if } q(a, \alpha) \in E_{n+1} - E_n \end{cases}$$

where $q = \theta_{n+1} \circ (\theta_{n+1}^-, \theta_{n+1}^{-1})$. The map ℓ_{n+1} is partial recursive since ℓ_n is partial recursive by the induction hypothesis, E_n and $E_{n+1} - E_n$ are decidable and $\theta_{n+1}, \theta^+, \theta^{-1}$ are computable.

It is clear that for all $(a, \alpha) \in A_{n+1}^* \times A_{n+1}$ we have

$$\theta_{n+1}(j_{n+1}(a, \alpha)) \simeq \ell_{n+1}(\theta_{n+1}^+(a), \theta_{n+1}(\alpha)),$$

so θ_{n+1} is an isomophism between (A_{n+1}, j_{n+1}) and (E_{n+1}, ℓ_{n+1}). Note that if ℓ_n has a decidable range, then ℓ_{n+1} has a decidable range also.

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So $\theta = \cup_{n\in\mathbb{N}}\theta_n$ is an isomorphism between $(\overline{A}, i_{\bar{A}})$ and (E,ℓ) where $E = \cup_{n\in\mathbb{N}}\mathrm{rg}(\theta_n)$ and $\ell = \cup_{n\in\mathbb{N}}\ell_n$. It is now routine to check that θ is computable, $E = \mathrm{rg}(\theta)$ is decidable, $\ell : E^* \times E \to E$ is partial recursive, $\mathrm{dom}(\ell)$ is decidable and, in the case of effectivity, that $\mathrm{rg}(\ell)$ is decidable. □

Corollary 11.6. If $\mathcal{A}$ is weakly effective (respectively, effective), then $\mathscr{G}_{\mathcal{A}}$ is a weakly effective (respectively, effective) graph model.

Proof. The conclusion follows from Proposition 11.4 and Theorem 11.5. □

Corollary 11.7. If $\mathcal{A}$ is finite, then $\mathscr{G}_{\mathcal{A}}$ is effective.

Lemma 11.8. If $\mathcal{A}$ is weakly effective, then O_A is $\mathcal{T}$-co-r.e., where $\mathcal{T} = Th(\mathscr{G}_{\mathcal{A}})$.

Proof. $O_A = \Lambda^o_E$ for $E = \{n : d_n \subseteq A\}$. If A is decidable, then E is decidable, hence O_A is co-r.e. by Theorem 9.37. Moreover, it is obviously $\mathcal{T}$-closed. □

All the results of this Section would hold for G- and H- models (even though the corresponding partial pairs and free completion process are somewhat more complex than for graph models).

11.3. *Can there be any r.e. graph theories?*

We will now prove several instances of our conjecture(s). We will prove, in particular, that Conjecture 1 holds for all free completions of finite partial pairs. Recall that $O_M = \{N \in \Lambda^o : |N| \sqsubseteq_{\mathscr{M}} |M|\}$ for all $M \in \Lambda^o(\mathcal{D})$ and that the domain associated with a graph model $\mathscr{G}$ has the form $\mathcal{D} = (\mathscr{P}(G), \subseteq)$.

Lemma 11.9. If $\mathcal{A}$ is a partial pair, then $|\Omega|^{\mathscr{G}_{\mathcal{A}}} \subseteq A$, hence $O_A \neq \emptyset$ for the model $\mathscr{G}_{\mathcal{A}}$.

Proof. It is well known, and provable in a few lines, that $\alpha \in |\Omega|^{\mathscr{G}_{\mathcal{A}}}$ implies that $i_{\bar{A}}(a,\alpha) \in a$ for some $a \in \overline{A}^*$ (the details are, for example, worked out in Berline and Salibra (2006)). Immediate considerations on the rank show that this is possible only if $(a,\alpha) \in \mathrm{dom}(j_{\mathcal{A}})$, which forces $\alpha \in A$. □

Corollary 11.10. If $\mathcal{A}$ is a non-total pair and $|U|^{\mathscr{G}_{\mathcal{A}}} \subseteq |\Omega|^{\mathscr{G}_{\mathcal{A}}}$, with $U \in \Lambda^o$, then U is unsolvable.

Proof. In models $\mathscr{G}_{\mathcal{A}}$ such that $\mathcal{A}$ is not total, a solvable λ-term has an interpretation that contains elements of any rank, while $|\Omega|^{\mathscr{G}_{\mathcal{A}}}$ contains only elements of rank 0. □

Theorem 11.11. Let $\mathcal{A}$ be a weakly effective partial pair. If there exists $E \subseteq A$ such that E is co-r.e., $O_E \neq \emptyset$ and $E/Aut(\mathcal{A})$ is finite (possibly empty), then $\mathcal{T} = Th(\mathscr{G}_{\mathcal{A}})$ is not r.e.

Proof. We first show that if $card(E/Aut(\mathcal{A})) = k$ for some $k \in \mathbb{N}$, then we have $card(O_E/\mathcal{T}) \leqslant 2^k$.

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Assume $M \in O_E$ and $\alpha \in |M|^{\mathscr{G}_\mathcal{A}} \subseteq E$. Then $O(\alpha)$ is included in $|M|^{\mathscr{G}_\mathcal{A}}$ where $O(\alpha)$ is the orbit of α in A modulo $Aut(\mathcal{A})$. Indeed, if $\theta \in Aut(\mathcal{A})$, then $\theta(\alpha) = \bar{\theta}(\alpha) \in \bar{\theta}^+(|M|^{\mathscr{G}_\mathcal{A}}) = |M|^{\mathscr{G}_\mathcal{A}}$ since $\bar{\theta}^+ \in Aut(\mathscr{G}_\mathcal{A})$ (Lemma 6.18(ii), Theorem 4.1(ii)). By hypothesis, the number of orbits is k, so the number of all possible interpretations $|M|^{\mathscr{G}_\mathcal{A}} \subseteq E$ cannot overcome 2^k, and thus O_E is a finite union of $\mathcal{T}$-classes.

Since O_E is co-r.e. by Theorem 9.37 and $O_E \neq \Lambda^o$, it cannot be decidable, so $\mathcal{T}$ cannot be r.e. □

From Theorem 11.11 and Lemma 11.9 we get the following corollaries, whose use will be illustrated by the examples following them.

Corollary 11.12. If $\mathcal{A}$ is finite, then $Th(\mathscr{G}_\mathcal{A})$ is not r.e.

Corollary 11.13. If $\mathcal{A}$ is weakly effective and $A/Aut(\mathcal{A})$ is finite, then $Th(\mathscr{G}_\mathcal{A})$ is not r.e.

Corollary 11.14. If $\mathcal{A}$ is weakly effective and there is a co-r.e. set E such that $|\Omega|^{\mathscr{G}_\mathcal{A}} \subseteq E \subseteq A$ and $E/Aut(\mathcal{A})$ is finite, then $Th(\mathscr{G}_\mathcal{A})$ is not r.e.

Corollary 11.15. If $\mathcal{A}$ is weakly effective, $|\Omega|^{\mathscr{G}_\mathcal{A}}$ is decidable and $|\Omega|^{\mathscr{G}_\mathcal{A}}/Aut(\mathcal{A})$ is finite, then $Th(\mathscr{G}_\mathcal{A})$ is not r.e.

Corollary 11.16. If $\mathcal{A}$ is effective, $|\Omega|^{\mathscr{G}_\mathcal{A}}$ is decidable and $|\Omega|^{\mathscr{G}_\mathcal{A}} \cap |N_1|^{\mathscr{G}_\mathcal{A}} \cap \ldots \cap |N_k|^{\mathscr{G}_\mathcal{A}}/Aut(\mathcal{A})$ is finite (possibly empty) for some normal terms $N_1, \ldots, N_k \in \Lambda^o$, with $k \in \mathbb{N}$, then $Th(\mathscr{G}_\mathcal{A})$ is not r.e.

We now give some applications of these corollaries.

Example 11.17. Corollary 11.13 applies to all the usual graph (or webbed) models, indeed:

(i) Engeler's model $\mathscr{E}$ is freely generated by $\mathcal{A} = (A, \emptyset)$, thus all the elements of A play exactly the same role and any permutation of A is an automorphism of $\mathcal{A}$, so the pair has only one orbit whatever the cardinality of A. Of course, if $\mathcal{A}$ is finite, Corollary 11.12 also applies.

(ii) The same considerations apply for the graph-Scott models (including $\mathscr{P}_\omega$) and the graph-Park models introduced in Example 6.15. Similarly, the graph model freely generated by $(\{\alpha, \beta\}, j)$ where $j(\{\alpha\}, \beta) = \beta$ and $j(\{\beta\}, \alpha) = \alpha$ has only one orbit.

(iii) Now consider the mixed-Scott-Park graph models defined in Example 6.15(iv). Only the permutations of A that leave Q and R invariant will be automorphisms of $(A, j_\mathcal{A})$, and we will have two orbits.

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Example 11.18. Corollary 11.15 (and hence Corollary 11.14) applies to the following effective pair $\mathcal{A}$:

$$A = \{\alpha_1, \ldots, \alpha_n, \ldots, \beta_1, \ldots, \beta_n, \ldots\}$$

and $j_{\mathcal{A}}$ defined by:

$$\begin{aligned} j_{\mathcal{A}}(\{\beta_n\}, \beta_n) &= \beta_n, \text{ for every } n \in \mathbb{N}, \\ j_{\mathcal{A}}(\{\alpha_1\}, \alpha_2) &= \alpha_2, \\ j_{\mathcal{A}}(\{\alpha_1, \alpha_2\}, \alpha_3) &= \alpha_3, \\ &\vdots \\ j_{\mathcal{A}}(\{\alpha_1, \ldots, \alpha_{n+1}\}, \alpha_{n+2}) &= \alpha_{n+2}. \end{aligned}$$

Here we have that $|\Omega|^{\mathcal{G}_{\mathcal{A}}} = \{\beta_n : n \in \mathbb{N}\}$ is decidable and that $|\Omega|^{\mathcal{G}_{\mathcal{A}}}/Aut(\mathcal{A})$ has cardinality 1 since every permutation of the β_n extends into an automorphism of $\mathcal{A}$. Note that the orbits of A are $|\Omega|$ and all the singletons $\{\alpha_n\}$. In particular, $A/Aut(\mathcal{A})$ is infinite.

Example 11.19. Corollary 11.15 applies to the following pair (despite appearances, it is an effective pair). Consider the set $A = \{\beta_1, \ldots, \beta_n, \ldots\}$ and the function $j_{\mathcal{A}}$ defined by $j_{\mathcal{A}}(\{\beta_n\}, \beta_n) = \beta_n$ if and only if n belongs to a non-co-r.e. set $E \subseteq \mathbb{N}$.

Then $|\Omega|^{\mathcal{G}_{\mathcal{A}}} = \{\beta_n : n \in E\}$ consists of only one orbit but is not co-r.e. However, starting, for example, from any bijection between E and the set of even numbers, it is easy to find an isomorphism of pairs such that $j_{\mathcal{A}}$ is partial recursive with decidable range, and hence $|\Omega|^{\mathcal{G}_{\mathcal{A}}}$ becomes decidable.

Example 11.20. Corollary 11.16 (but no other corollary) applies to the following effective pair $\mathcal{A}$.

$$\begin{aligned} A = \{&\alpha_1, \ldots, \alpha_n, \ldots, \beta_1, \ldots, \beta_n, \ldots, \\ &\alpha'_1, \ldots, \alpha'_n, \ldots, \beta'_1, \ldots, \beta'_n, \ldots\} \end{aligned}$$

and $j_{\mathcal{A}}$ is defined by:

$$\begin{aligned} j_{\mathcal{A}}(\{\alpha'_n\}, \beta_n) &= \alpha'_n \\ j_{\mathcal{A}}(\{\alpha_1, \ldots, \alpha_n\}, \beta_n) &= \beta'_n \end{aligned}$$

for all $n \in \mathbb{N}$. In this case $|\Omega|^{\mathcal{G}_{\mathcal{A}}} = \{\beta_n : n \in \mathbb{N}\}$ is decidable and $|\Omega|^{\mathcal{G}_{\mathcal{A}}} \cap |\mathbf{I}|^{\mathcal{G}_{\mathcal{A}}} = \varnothing$ (note that $|\Omega|^{\mathcal{G}_{\mathcal{A}}}/Aut(\mathcal{A})$ is infinite).

Example 11.21. (This is an example of an effective graph model beyond the scope of the corollaries of Theorem 11.11). Take for $\mathcal{A}$ the total pair $\mathcal{G} = (\mathbb{N}, \ell)$ where ℓ is defined as follows:

$$\ell(a, m) = \begin{cases} 2k & \text{if } a = \{2k\} \text{ for } k \in \mathbb{N} \\ 3^{\#_*(a)}5^m & \text{otherwise} \end{cases}$$

where $\#_* : \mathbb{N}^* \to \mathbb{N}$ is the effective encoding introduced in Section 2.2. It is easy to check that $\mathcal{G}$ is effective and that $|\Omega|^{\mathcal{G}} = \mathbb{N}$. Then $O_\Omega = \Lambda^o$, and thus $|\Omega|^{\mathcal{G}}/Aut(\mathcal{G})$ is infinite.

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Another example of an effective graph model to which the corollaries of Theorem 11.11 are not applicable will be provided by Theorem 11.12. We do not know whether these two counterexamples could enter in the scope of Theorem 11.11 or not.

11.4. *What about the other classes of webbed models?*

To give a first idea of the strength of Theorem 11.11, note that all the webbed models that have been introduced individually in the literature are (or can be) generated by a weakly effective partial web W *such that* $W/Aut(W)$ *is finite*. Of course, the notion of (effective partial) webs, and of automorphism of these webs, should be defined case by case for each class of models.

It should be noted that the results and proofs of Section 11.3 hold not only for graph models but also for G- and H- models. The situation is much less clear for Scott continuous semantics as soon as we go beyond graph models: the problems already occur at the level of K-models (not to mention filter models!).

The difficulty for Lemma 11.9 is that the web of a K-model is a tuple $(D, \preceq, i)$ where $\preceq$ is a preorder on D and $i : D^* \times D \to D$ is an injection compatible with $\preceq$ in a certain sense. The elements of the associated reflexive domain are the downward closed subsets of D, so we should already change the hypothesis for $|\Omega| \subseteq A\downarrow$, where $A\downarrow$ is the downwards closure of A. But the real problem is that the control we have on $|\Omega|$ in K-models is much looser than in graph models. The only thing we know (from Ying Jiang's thesis (Jiang 1993)) is that if $\alpha \in |\Omega|$, there are two sequences $\alpha_n \in D$ and $a_n \in D^*$ such that $\alpha = \alpha_0 \preceq \alpha_1 \preceq \ldots \alpha_n \preceq \ldots$, $|\delta| \supseteq a_0\downarrow \supseteq a_1\downarrow \supseteq \ldots \supseteq a_n\downarrow \supseteq \ldots$ and $\beta_n = i(a_{n+1}, \alpha_{n+1}) \in a_n$ for all n. This forces β_n to be an increasing sequence included in $\cap_{n\in\mathbb{N}}(a_n\downarrow)$. Moreover, if the model is extensional, we have that $\alpha_n = \alpha$ for all $n \in \mathbb{N}$. This does not seem to be enough to get an analogue of Lemma 11.9.

Finally, in any statement of Theorem 11.11 for K-models we should replace $E \subseteq A$ by $E \subseteq A\downarrow$ to have any chance of getting $O_E \neq \emptyset$ (since the interpretation of terms is downward closed).

11.5. *An effective graph model having the minimum graph theory*

In this section we show another main theorem of the paper, namely that the minimum order graph theory is the theory of an *effective* graph model. As we will see in the next section, this result implies that:

(i) No order graph theories can be r.e.

(ii) For any closed normal term M, there exists a non-empty β-co-r.e. set $\mathscr{V}$ of unsolvable terms whose interpretations are below that of M in all graph models.

Theorem 11.12. There exists an effective graph model whose order/equational theory is the minimum order/equational graph theory.

Proof. It is not difficult to define an effective numeration $\mathcal{N}$ of all the finite partial pairs whose carrier set is a subset of $\mathbb{N}$. We now make the carrier sets N_k, for $k \in \mathbb{N}$,

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pairwise disjoint. Let p_k be the k-th prime number. Then we define another finite partial pair $\mathcal{A}_k$ as follows:

$$\begin{aligned} A_k &= \{p_k^{x+1} : x \in N_k\} \\ j_{\mathcal{A}_k}(\{p_k^{\alpha_1+1}, \ldots, p_k^{\alpha_n+1}\}, p_k^{\alpha+1}) &= p_k^{j_{\mathcal{N}_k}(\{\alpha_1,\ldots,\alpha_n\},\alpha)+1} \\ &\quad \text{for all } (\{\alpha_1, \ldots, \alpha_n\}, \alpha) \in \mathrm{dom}(j_{\mathcal{N}_k}) \end{aligned}$$

In this way we get an effective bijective numeration of all the finite partial pairs $\mathcal{A}_k$.

Let us take $\mathcal{A} = \sqcup_{k\in\mathbb{N}}\mathcal{A}_k$. It is an easy matter to prove that A is a decidable subset of $\mathbb{N}$ and that $j_{\mathcal{A}}$ is a computable map with decidable domain and range. It follows from Theorem 11.5 that $\mathscr{G}_{\mathcal{A}}$ is an effective graph model.

Finally, with the same reasoning as in the proof of Theorem 7.4, we can conclude that $Th_{\sqsubseteq}(\mathscr{G}_{\mathcal{A}})$ (respectively, $Th(\mathscr{G}_{\mathcal{A}})$) is the minimum order graph theory (respectively, equational graph theory). □

Let $\mathcal{T}^{min}$ and $\mathcal{T}^{min}_{\sqsubseteq}$ be the minimum equational graph theory and the minimum order graph theory, respectively.

Proposition 11.23. $\mathcal{T}^{min}$ and $\mathcal{T}^{min}_{\sqsubseteq}$ are in fact the theories of countably many non-isomorphic effective graph models.

Proof. The conclusion follows from the fact that in the proof of Theorem 11.12, there exist countably many choices for the effective numeration $\mathcal{N}$ that give rise to non-isomorphic graph models having minimal theory. For example, for every recursive sequence $(n_k)_{k\in\mathbb{N}}$, take a recursive numeration that repeats n_k-times the pair $\mathcal{N}_k$. □

Proposition 11.24. $\mathcal{T}^{min}$ is an intersection of a countable set of non-r.e. equational graph theories.

Proof. By the proof of Theorem 7.4, $\mathcal{T}^{min}_{\sqsubseteq} = \cap Th_{\sqsubseteq}(\mathscr{G}_{\mathcal{A}_k})$ where $\mathcal{A}_k$ ranges over all finite pairs. By Corollary 11.12, these theories are not r.e. □

11.6. *Applications to the class of all graph models*

The following two results are consequences of Theorem 11.12.

Theorem 11.25. For all graph models $\mathscr{G}$, we have that $Th_{\sqsubseteq}(\mathscr{G})$ is not r.e.

Proof. Let $\mathscr{G}_{min}$ be some fixed effective graph model having minimum order theory and M be any closed normal λ-term. Since $\mathscr{G}_{min}$ is effective, Theorem 9.43 implies that $|M|^{\mathscr{G}_{min}}$ is decidable, so $O_M^{\mathscr{G}_{min}} = \{N \in \Lambda^o : |N|^{\mathscr{G}_{min}} \subseteq |M|^{\mathscr{G}_{min}}\}$ is β-co-r.e. by Theorem 9.38.

Suppose now that $\mathscr{G}$ is a graph model such that $Th_{\sqsubseteq}(\mathscr{G})$ is r.e. Then $O_M^{\mathscr{G}} = \{N \in \Lambda^o : |N|^{\mathscr{G}} \subseteq |M|^{\mathscr{G}}\}$ is a β-r.e. set that contains the co-r.e. set $O_M^{\mathscr{G}_{min}}$. Thus, by the FIP, we get $O_M^{\mathscr{G}} = \Lambda^o$.

From the arbitrariness of M, it follows that $\mathbf{F} \in O_{\mathbf{T}}$, and *vice versa*, so we get $|\mathbf{F}|^{\mathscr{G}} = |\mathbf{T}|^{\mathscr{G}}$, which gives a contradiction. □

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Proposition 11.26. For all normal $M_1, \ldots, M_n \in \Lambda^o$ there exists a non-empty β-co-r.e. set $\mathcal{V}$ of closed unsolvable terms such that:

$$\text{For all graph models } \mathscr{G}\text{: } \forall U \in \mathcal{V}\ (|U|^{\mathscr{G}} \subseteq |M_1|^{\mathscr{G}} \cap \ldots \cap |M_n|^{\mathscr{G}}).$$

Proof. Let $\mathscr{G}_{min}$ be some fixed effective graph model having a minimum order theory. Since $\mathscr{G}_{min}$ is effective and $M_1, \ldots, M_n$ are closed normal λ-terms, Theorem 9.43 implies that every $|M_i|^{\mathscr{G}_{min}}$ is decidable. Thus, it follows from Theorem 9.38 that there exists a set $\mathcal{V}$ of unsolvable terms such that for all $U \in \mathcal{V}$ we have $|U|^{\mathscr{G}_{min}} \subseteq |M_i|^{\mathscr{G}_{min}}$ for all $1 \leqslant i \leqslant n$. The conclusion then follows since $Th_{\sqsubseteq}(\mathscr{G}_{min})$ is the minimum order graph theory. □

Remark 11.27. The authors do not know any concrete example of an unsolvable term U satisfying the above inclusion; not even of an unsolvable U such that, for all graph models $\mathscr{G}$, we have $|U|^{\mathscr{G}} \subseteq |\mathbf{I}|^{\mathscr{G}}$.

Nevertheless, Proposition 11.26 has the following interesting corollary, which intuitively expresses the fact that there are lots of easy terms for which we will never be able to give a semantic proof of their easiness (which by no means implies that we could prove their easiness syntactically), at least using graph models. Following the terminology of Berline and Salibra (2006), we say that $U \in \Lambda^o$ is *graph easy* if for all $M \in \Lambda^o$ there is a graph model $\mathscr{G}$ satisfying $|U|^{\mathscr{G}} = |M|^{\mathscr{G}}$. The best-known example of a graph easy term is, of course, Ω from Baeten and Boerboom (1979), and, consequently, all the ΩN for $N \in \Lambda^o$.

Corollary 11.28. There exists a non-empty β-co-r.e. set $\mathcal{V}'$ of easy terms that are not graph-easy.

Proof. Let $\mathcal{V}$ be the β-co-r.e. set such that for all $U \in \mathcal{V}$ and all graph models $\mathscr{G}$, we have $|U|^{\mathscr{G}} \subseteq |\mathbf{I}|^{\mathscr{G}}$ (using Proposition 11.26). It is then sufficient to take $\mathcal{V}' = \mathcal{V} \cap \Lambda_{easy}$. □

Appendix A. The Visser topology on Λ

It may be possible to give a topological flavour to the framework of effective λ-models using the Visser topology. However, to obtain the results in this paper, no real topological manipulation would be needed. In particular, the topology itself is never used: just its base.

The general definition of the Visser topology on an enumerated set (X, ν_X) was introduced in Visser (1980), along with its application to Λ. We recall this definition below.

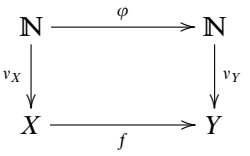
A.1. *General definition of the Visser topology*

If $\gamma = (X, \nu_X)$ is a numeration, we write $m \sim_\gamma n$ for $\nu_X(m) = \nu_X(n)$. A numeration γ is said to be *precomplete* if for every partial recursive φ there is a total recursive ψ such that $\forall n \in \mathrm{dom}(\varphi)$ we have $\varphi(n) \sim_\gamma \psi(n)$.

If $\gamma_1 = (X, \nu_X)$ and $\gamma_2 = (Y, \nu_Y)$ are numerations, then f is a *morphism* from γ_1 to γ_2 if f is a function from X to Y and if there is a total recursive φ such that the following

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diagram commutes:



The numeration $(-)_\lambda$ introduced in Section 5.1 of Λ is precomplete.

Definition A.1. Let $\gamma = (v_X, X)$ be a numeration. The Visser topology on X is the topology generated by the base

$$\mathcal{O}_X = \{O \subseteq X : v_X^-[O] \text{ is co-r.e.}\}.$$

The fact that $\mathcal{O}_X$ is a topological base follows from the closure of co-r.e. sets under finite intersection.

Theorem A.2. If the numeration $\gamma = (v_X, X)$ is precomplete, the Visser topology on X is hyperconnected, that is, every two non-empty open sets have non-empty intersection.

Proof. See Visser (1980, Theorem 2.5). □

The hyperconnectedness is the most important and useful property of the Visser topology.

A.2. *The Visser topology on* Λ

We now provide the instance of the definition in the context of λ-calculus.

Definition A.3. The Visser topology on Λ is the topology whose basic open sets are the subsets O such that:

— O is closed under β-conversion.
— $\{(M)_\omega : M \in O\}$ is a co-r.e. set.

Proposition A.4. The Visser topology on Λ is hyperconnected.

Proof. The numeration $((-)_\lambda, \Lambda)$ is precomplete by Visser (1980, Theorem 1.6.1.1), so the result follows from Theorem A.2. □

A.3. *Visser-continuity*

The second main property of the Visser topology on Λ is the following.

Proposition A.5. Every λ-definable map on Λ is Visser continuous.

Proof. The inverse image of every r.e. set via a computable function is an r.e. set. □

Proposition A.5 is also true for n-ary functions for the same reason if we consider the Visser topology $\mathcal{W}_n$ on Λ^n. But it is important to bear in mind that in general it is false for the product topology $\mathcal{V}^n$ on Λ^n. Note that all non-empty open sets of $\mathcal{V}^2$ meet the diagonal $\Delta = \{(M, N) : M =_{\lambda\beta} N\}$ (by the hyperconnectedness of $\mathcal{V}$), hence Δ^c, which

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is obviously basic open in $\mathcal{W}$, is not even open in $\mathcal{V}^2$, which proves that $\mathcal{V}^2 \neq \mathcal{W}_2$. The following result is a variation of this remark.

Proposition A.6. The application function on Λ is:

(i) Visser continuous in each coordinate;
(ii) continuous with respect to $\mathcal{W}_2$; but
(iii) not continuous with respect to $\mathcal{V}^2$.

Proof. (i) and (ii) follow from the fact that application is computable.

To prove (iii), let ψ be defined on Λ^2 by $\psi(M,N) = \Omega MN$, and let $\mathcal{T}$ be the r.e. λ-theory generated by $\Omega xx = \Omega$. Since $\mathcal{T} \subseteq \mathcal{H}$, it is obvious that $\mathcal{T}$ is consistent. Let $V = \{(M,N) : \Omega MN \neq_{\mathcal{T}} \Omega\}$. Since $\mathcal{T}$ is r.e., $[\Omega]_{\mathcal{T}}$ is r.e. and hence its complement O is Visser open (and non-empty). If the application were $\mathcal{V}^2$-continuous, then ψ would be $\mathcal{V}^2$-continuous, and $V = \psi^{-}(O)$ would be $\mathcal{V}^2$-open. But, V is non-empty because $(M,N) \in V$ if and only if $M \neq_{\mathcal{T}} N$ (Salibra 2003), so V would meet the diagonal, which contradicts the definition of $\mathcal{T}$. □

Appendix B. Sometimes gluings are weak products

Although there exist obvious morphisms $\iota_n : \mathcal{G}_n \to \diamond_{k\in K}\mathcal{G}_k$, there are, in general, no 'projection-like' morphisms. However, there are good cases where projections $\pi_n : \diamond_{k\in K}\mathcal{G}_k \to \mathcal{G}_n$ can be found, and, moreover, $\mathcal{G}_n \lhd \diamond_{k\in K}\mathcal{G}_k$ for all $n \in K$. In these cases it is easy to show that $Th(\diamond_{k\in K}\mathcal{G}_k) \subseteq \cap_{k\in K} Th(\mathcal{G}_k)$, an inclusion which was proved in full generality in Bucciarelli and Salibra (2003) with a much more intricate proof.

The first example of such a situation is 'self-gluing'. We use $\mathcal{G}^k$ to denote the gluing of $\mathcal{G}$ with itself k-times, that is, $\mathcal{G}\diamond\cdots\diamond\mathcal{G}$. It is easy to check the next result.

Proposition B.1. For all graph models $\mathcal{G}$ we have $\mathcal{G} \lhd \mathcal{G}^k$.

Definition B.2. Given a partial pair $\mathcal{A}$, an element $\alpha \in A$ is *isolated* if $j_{\mathcal{A}}(\{\alpha\},\alpha) = \alpha$ and *critical* if $(\emptyset,\alpha) \in \mathrm{dom}(j_{\mathcal{A}})$.

Definition B.3. A partial pair $\mathcal{A}$ is *good* if it has at least one isolated element and no critical elements. A graph model $\mathcal{G}$ is *good* if there exists a good pair $\mathcal{A}$ such that $\mathcal{G} = \mathcal{G}_{\mathcal{A}}$.

Remark B.4. If $\mathcal{A}$ is not total, then $\overline{\mathcal{A}}$ contains critical elements, so it is not good, but $\mathcal{G}_{\mathcal{A}}$ can be.

For example, the graph-Park models are good, whilst the graph-Scott models (including $\mathcal{P}_\omega$) are not: both of these classes were introduced in Example 6.15.

Remark B.5. The class of good graph models is closed under $\diamond$ (by Lemma 7.2).

Proposition B.6. If $(\mathcal{A}_k)_{k\in K}$ is a family of pairwise disjoint good partial pairs, then $\mathcal{G}_{\mathcal{A}_n} \lhd \diamond_{k\in K}\mathcal{G}_{\mathcal{A}_k}$ for all $n \in K$.

Proof. By Lemma 7.2, we recall that $\diamond_{k\in K}\mathcal{G}_{\mathcal{A}_k} = \mathcal{G}_{\mathcal{A}}$, where $\mathcal{A} = \overline{\sqcup_{k\in K}\mathcal{A}_k}$. For all $k \in K$, let $\alpha_k \in A_k$ be an isolated element. We can now define, for any n, a map $\pi_n : \sqcup_{k\in K}\mathcal{A}_k \to \mathcal{A}_n$

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as follows: $\pi_n(x) = x$ if $x \in A_n$ and $\pi_n(x) = \alpha_n$ otherwise. Since the pairs have no critical elements, it is easy to prove that $\pi_n, \iota_n : A_n \lhd \sqcup_{k\in K} A_k$ where $\iota_n : A_n \to \sqcup_{k\in K} A_k$ is the inclusion mapping, so we get the conclusion using Lemma 6.24. □

Corollary B.7. If $(\mathcal{G}_k)_{k\in K}$ is a family of good graph models, then $Th(\diamond_{k\in K}\mathcal{G}_k) \subseteq \cap_{k\in K} Th(\mathcal{G}_k)$.

Corollary B.8. There is a minimal theory of a good graph model and it is semi-sensible.

Proof. Let $(\mathcal{A}_k)_{k\in K}$ be a family of pairwise disjoint good finite pairs such that all other finite good pairs are isomorphic to at least one $\mathcal{A}_k$. Take $\mathcal{G} = \diamond_{k\in K}\mathcal{G}_{\mathcal{A}_k}$. It is easy to check that $Th(\mathcal{G})$ is minimal, and it is semi-sensible by Theorem 6.13. □

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References

Amadio, R. and Curien, P.-L. (1998) *Domains and lambda-calculi*, Cambridge Tracts in Theoretical Computer Science **46**, Cambridge University Press.

Baeten, J. and Boerboom, B. (1979) Omega can be anything it should not be. In: Proc. Koninklijke Netherlandse Akademie van Wetenschappen. *Indag. Matematicae* Serie A **41** 111–120.

Barendregt, H. (1984) *The Lambda calculus: Its syntax and semantics*, North-Holland.

Bastonero, O. (1996) *Modèles fortement stables du λ-calcul et résultats d'incomplétude*, Ph.D. thesis, Université de Paris 7.

Berardi, S. and Berline, C. (2002) $\beta\eta$-complete models for System F. *Mathematical Structures in Computer Science* **12** 823–874.

Berline, C. (2000) From computation to foundations via functions and application: The λ-calculus and its webbed models. *Theoretical Computer Science* **249** 81–161.

Berline, C. (2006) Graph models of λ-calculus at work, and variations. *Mathematical Structures in Computer Science* **16** 1–37.

Berline, C., Manzonetto, G. and Salibra, A. (2007) Lambda theories of effective lambda models. In: 16th EACSL Annual Conference on Computer Science and Logic (CSL'07). *Springer-Verlag Lecture Notes in Computer Science* **4646** 268–282.

Berline, C. and Salibra, A. (2006) Easiness in graph models. *Theoretical Computer Science* **354** 4–23.

Berry, G. (1978) Stable models of typed lambda-calculi. In: Proceedings of the Fifth Colloquium on Automata, Languages and Programming. *Springer-Verlag Lecture Notes in Computer Science* **62**.

Berry, G. (1979) *Modèles complètement adéquats et stable des λ-calculs typés*, Ph.D. thesis, Université de Paris 7.

Berry, G., Curien, P.-L. and Lévy, J.-J. (1985) Full abstraction for sequential languages: the state of the art. *Algebraic Methods in Semantics* 89–132.

Bucciarelli, A. and Ehrhard, T. (1991a) Extensional embedding of a strongly stable model of pcf. In: *ICALP '91: Proceedings of the 18th International Colloquium on Automata, Languages and Programming, London, U.K.*, Springer-Verlag 35–46.

https://doi.org/10.1017/S0960129509990053 Published online by Cambridge University Press

Bucciarelli, A. and Ehrhard, T. (1991b) Sequentiality and strong stability. In: *Sixth Annual IEEE Symposium on Logic in Computer Science*, IEEE Computer Society Press 138–145.

Bucciarelli, A. and Salibra, A. (2003) The minimal graph model of lambda calculus. In: MFCS'03. *Springer-Verlag Lecture Notes in Computer Science* **2747** 300–307.

Bucciarelli, A. and Salibra, A. (2004) The sensible graph theories of lambda calculus. In: *19th Annual IEEE Symposium on Logic in Computer Science (LICS'04)*, IEEE Computer Society Publications 276–285.

Bucciarelli, A. and Salibra, A. (2008) Graph lambda theories. *Mathematical Structures in Computer Science* **18** (5) 975–1004.

Coppo, M. and Dezani-Ciancaglini, M. (1980) An extension of the basic functionality theory for the λ-calculus. *Notre-Dame Journal of Formal Logic* **21** (4) 685–693.

Coppo, M., Dezani-Ciancaglini, M., Honsell, F. and Longo, G. (1984) Extended Type Structures and Filter Lambda Models. In: Lolli, G., Longo, G. and Marcja, A. (eds.) *Logic Colloquium 82, Amsterdam, the Netherlands*, North-Holland 241–262.

Coppo, M., Dezani-Ciancaglini, M. and Zacchi, M. (1987) Type theories, normal forms and D_∞ λ-models. *Information and Computation* **72** 85–116.

David, R. (2001) Computing with Böhm trees. *Fundam. Inform.* **45** (1–2) 53–77.

Di Gianantonio, P., Honsell, F. and Plotkin, G. (1995) Uncountable limits and the lambda calculus. *Nordic Journal of Computing* **2** (2) 126–145.

Escardó, M. (1996) Pcf extended with real numbers. *Theoretical Computer Science* **162** (1) 79–115.

Giannini, P. and Longo, G. (1984) Effectively given domains and lamba-calculus models. *Information and Control* **62** 36–63.

Gouy, X. (1995) *Etude des théories équationnelles et des propriétés algébriques des modèles stables du λ-calcul*, Ph.D. thesis, Université de Paris 7.

Gruchalski, A. (1996) Computability on di-domains. *Inf. Comput.* **124** (1) 7–19.

Honsell, F. (2007) TLCA list of open problems: Problem # 22. (Available at `http://tlca.di.unito.it/opltlca/problem22.pdf`.)

Honsell, F. and Ronchi Della Rocca, S. (1990) Reasoning about interpretations in qualitative lambda models. In: Broy, M. and Jones, C. B. (eds.) *Proceedings of the IFIP Working Group 2.2/2.3*, North-Holland 505–521.

Honsell, F. and Ronchi Della Rocca, S. (1992) An approximation theorem for topological lambda models and the topological incompleteness of lambda calculus. *Journal of Computer and System Sciences* **45** 49–75.

Jiang, Y. (1993) *Consistence et inconsistence de théories de λ-calcul étendus*, Ph.D. thesis, Université de Paris 7.

Kerth, R. (1994) $2^{\aleph_0}$ modèles de graphes non-équationnellement équivalents. *Notes de comptes-rendus de l'Académie des Sciences* **318** 587–590.

Kerth, R. (1995) *Isomorphism et équivalence équationnelle entre modèles du λ-calcul*, Ph.D. thesis, Université de Paris 7.

Kerth, R. (1998a) The interpretation of unsolvable terms in models of pure λ-calculus. *J. Symbolic Logic* **63** 1529–1548.

Kerth, R. (1998b) Isomorphism and equational equivalence of continuous lambda models. *Studia Logica* **61** 403–415.

Kerth, R. (2001) On the construction of stable models of λ-calculus. *Theoretical Computer Science* **269** 23–46.

Krivine, J.-L. (1993) *Lambda-calculus, types and models*. Ellis Horwood. (Translated from the French original: *Lambda-calcul, Types et Modeles* (1990) Masson, Paris.)

https://doi.org/10.1017/S0960129509990053 Published online by Cambridge University Press

Longo, G. (1983) Set-theoretical models of λ-calculus: theories, expansions, isomorphisms. *Ann. Pure Appl. Logic* **24** (2) 153–188.

Lusin, S. and Salibra, A. (2004) The lattice of lambda theories. *Journal of Logic and Computation* **14** 373–394.

Manzonetto, G. and Salibra, A. (2006) Boolean algebras for lambda calculus. In: *Proc. 21th IEEE Symposium on Logic in Computer Science (LICS 2006)* 139–148.

Odifreddi, P. (1989) *Classical Recursion Theory*, Elsevier.

Plotkin, G. (1977) LCF considered as a programming language. *Theoretical Computer Science* **5** 223–255.

Plotkin, G. (1993) Set-theoretical and other elementary models of the lambda-calculus. *Theoretical Computer Science* **121** (1–2) 351–409.

Pravato, A., Bastonero, O. and Ronchi Della Rocca, S. (1997) Structures for lazy semantics.

Salibra, A. (2003) Topological incompleteness and order incompleteness of the lambda calculus. In: LICS'01 Special Issue. *ACM Transactions on Computational Logic* **4** 379–401.

Scott, D. (1972) Continuous lattices. In: *Toposes, algebraic geometry and logic*, Springer-Verlag.

Selinger, P. (1996) Order-incompleteness and finite lambda models – extended abstract. In: *LICS'96: Proceedings of the 11th Annual IEEE Symposium on Logic in Computer Science*, IEEE Computer Society 432.

Selinger, P. (2003) Order-incompleteness and finite lambda reduction models. *Theoretical Computer Science* **309** 43–63.

Stoltenberg-Hansen, V., Lindström, I. and Griffor, E. (1994) *Mathematical theory of domains*, Cambridge University Press.

Visser, A. (1980) Numerations, λ-calculus, and arithmetic. In: Hindley, J. R. and Seldin, J. P. (eds.) *Essays on Combinatory Logic, Lambda-Calculus, and Formalism*, Academic Press 259–284.

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