THE PHYSICAL INTERPRETATION OF ENTITY-STRUCTURE

By D. J. NEWMAN

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1. *Introduction.* Several recent papers (Bastin and Kilmister (1), (2); Newman (3), (4)) have made the role of algebraic structures in physics the subject of detailed discussion. The general purpose of this work has not been to discover new physical laws or predict new results, but rather to give the theoretical physicist some further insight into the significance of familiar concepts. This is, of course, being done in the hope that it will lead to new refinements in our theoretical techniques.

In this paper we attack a specific problem: why does the kinematic reference frame used in physics have four dimensions, a quadratic metric and the signature $(-1, -1, -1, +1)$? These facts are a conceptual necessity to the theoretician, but are usually regarded as being incapable of further analysis. In a previous paper (4) the author gave a mathematical skeleton showing that such an analysis was possible. This paper is an attempt to clothe that skeleton with the flesh and blood of physics. If the resulting argument is acceptable, many aspects of theoretical physics will need reinterpretation. It will no longer be adequate to ensure that the fundamental theories of physics (such as Maxwell's electrodynamics) are expressible in a Lorentz covariant form; we shall also have to ask what these theories signify in terms of our new understanding of the space-time structure.

The motivation for the following considerations arose out of a modification of some of Eddington's ideas which has previously been described ((3), § 1). The starting point of the present paper can best be expressed in terms of a 'Postulate of Impotence'; it is impossible to tell whether the conceptual frameworks (i.e. the structures) which are employed in physics are derived initially from the way we look at the world, or from the intrinsic properties of the world itself. That is to say, we can know that part of the structure represents part of the observed physical world without knowing whether the world contains a 'reality' which corresponds to this structure.

A simple example of this impotence comes to mind; we cannot decide whether the three dimensionality of space is an intrinsic property of the world, or a property of our mode of thought, by a direct experimental test. This is most clearly shown by the fact that the language we need to describe such an experiment is the same as the language used to describe the result. The reason for this is that the three dimensionality of space is a part of the conceptual scheme of physics which has become absorbed in our method of description of the world. Later we shall see that the three dimensionality of space can be derived from a more primitive conceptual scheme, and is accordingly not as primitive as it is normally assumed to be in physics.

If the Postulate of Impotence is true, we should expect the structure of a set of

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results to be already contained in a complete description of the experimental situation that produces them. By a 'complete' description we mean that the experimental procedures can be put into a 1–1 correspondence with a closed symbolic structure. Such a description is only feasible when the experimental situation is of a particular fundamental type, namely that of a test particle in a specified environment. These considerations form the basis of the 'Structural Method' used in the present analysis, in which we examine certain procedures of the experimental physicist in order to determine the structures which are necessary to describe his results. As we are interested in explaining the properties of the space-time reference frame, it is clear that we shall have to concern ourselves with an examination of the procedures employed in the measurement of distances and times.

A large part of the procedure of the experimental physicist is aimed at idealizing and simplifying the physical conditions in such a way as to make them compatible with some theoretical model. The results of the experiments can only be meaningfully compared with calculations based on this model. But the model itself does not supply the experimental physicist with all the information he requires to make the necessary simplifications and adjustments. This information must also be derived from other models which are purely incidental to the experiment in hand. For example, the physicist who wishes to use a macroscopic 'test particle' must be prepared to identify this experimentally with the centre of gravity of a finite body.

In applying the Structural Method we shall be concerned with the analysis of certain experimental procedures which are common to all experiments of the fundamental type, mentioned above, and which are assumed to be necessities of technique by the experimenter rather than adjustments to a theoretical model. The practical application of this method is complicated by the fact that we shall have to isolate conceptually that portion of the total experimental situation which is amenable to mathematical analysis. We call this portion the *residual* experimental situation. Here we have a theoretical difficulty which is just the converse of the experimental difficulty encountered in conventional physics which was outlined in the preceding paragraph.

A simple residual experimental situation will correspond to a simple structure, and a complicated residual experimental situation will require a complex structure for its complete specification. Now it is apparent that a complicated residual situation will contain several simple residual situations as components of itself. Hence we shall expect the complex structures to contain several simpler structures. This is the basis of the idea due to Bastin and Kilmister that the structures form series ordered by inclusion. The discovery of the substructures of a theory enables us to make physical statements about that theory at their proper conceptual level in the ordered sequence. This determines the precise limits of applicability of such statements.

A warning is necessary at this point. It is clear from the above argument that certain simple structures (e.g. the quaternion structure of a spatial comparison) will recur again and again in all sorts of contexts. This fact, in conjunction with the usual human preoccupation with the mystical aspects of symbolism, has led many writers in the past to the mistaken belief that they had found a structure (usually quaternions) which gave some mysterious insight into the whole of physical 'reality'.

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The reader will find the symbolism employed in the following to be reminiscent of that used in quantum mechanics. Our interpretation will, however, be given in purely macroscopic terms. A further paper is being prepared which will attempt to show that the concepts and formalisms which are usually regarded as typically quantum mechanical are, in fact, a direct consequence of the macroscopic mould in which quantum phenomena are cast.

2. *The experimental situation.* The following analysis is self-contained. All discussion of the previous paper (4) is deferred until the last section.

We divide the arguments used into three categories according to their logical status in the theory. The first of these (**D**) includes descriptions of the external world which enable us to correlate the symbolic scheme with physical reality. The second category (**S**) contains statements which are linked directly to the symbolic scheme itself. The third category (**H**) contains various arguments which are designed to give the reader some insight into what is happening from an orthodox point of view. Statements made in category **H** will, of course, have no logical status in the development of the theory. For example, we shall have occasion to refer to space-time diagrams before the theory itself can give any meaning to such a concept.

We shall denote changes of category by the presence of the appropriate symbol on the left of the text. Certain words and phrases in categories **D** and **S** will be in italics. This indicates that they are being defined in a technical sense by their context.

D. In experiments of the fundamental type mentioned in the previous section the physicist is concerned with observing and predicting the motions of test bodies, using the large-scale distribution of matter as a background. A region of the continuum defined by this large-scale distribution he divides into *parts* and he analyses the test bodies into *objects* (or test particles) which move from part to part. Such a characterization of the physical world can be further refined for theoretical purposes by making the abstractions:

(i) Parts $\rightarrow$ *States*, which are distinguishable regions in the continuum of our experience, and may therefore be numbered.

(ii) Objects $\rightarrow$ *Entities*, which can be followed (i.e. watched) without interference as they pass from state to state.

The states and entities may be material or not as our description requires, for although the experimenter only observes material things, it is not necessary that both the state and the entity should be materialized for an observation to be made. We shall use the term *coincidence* to describe the situation when an entity is noted at a given state.

H. The following discussion revolves round the fact that the material landmarks we use to describe the measurements of distance and time are capable of differentiating more than between positions in space-time. This extra degree of differentiation gives rise to the space-time structure we are seeking.

It should be noted that definition (ii) above excludes the possibility that the entities described are quantum particles. For a quantum entity would be subject to arbitrarily large interactions when its coincidences are being observed, and cannot therefore be

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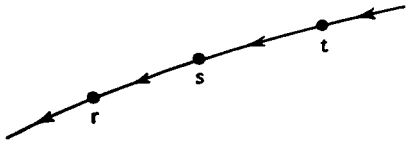
followed without interference. So we visualize entities as macroscopic test particles. A non-material entity is followed 'in the mind's eye', its path always being defined by the paths of other material entities.

S. Let us consider a finite set of N mutually disjoint states denoted by $\{r\}$, where r is an integer such that $1 \leqslant r \leqslant N$. The coincidence of an entity with state r will be written C_r.

A series of coincidences on the path of an entity will be called its *history* and written, for example, as

$$C_r \quad C_s \quad C_t,$$

where the entity is successively in coincidence with the states t, s, r in this order. We can also represent this situation diagrammatically:



D. Observations can be in the form of histories or counts of coincidences. In the latter case they can be represented as $n_i C_i$, where n_i is the number of coincidences observed at i.

The most important sort of situation we wish to describe is the transfer of an entity between two specified states in $\{r\}$. In order that this should be well defined it is necessary to have an explicit formalism to describe a history starting or finishing at a given state. It should be realized that the finite set of coincidences we write down to describe a history does not tell the whole story. We have assumed that an entity can be followed continuously through the coincidences which are actually noted. A 'start' or 'finish' would represent a definite boundary to the entity-following procedure of the experimental physicist.

We shall call the situation when an entity leaves state r an *exit* from r, and when an entity arrives at r an *entry* into r. Exits and entries will be collectively referred to as *events*. The symbolism corresponding to these ideas is developed in the next section.

3. *The concept of equivalence.*

D. If his experiments are to be explicable in general conceptual terms and so be amenable to repetition, the experimental physicist must be willing to accept certain experimental situations as being equivalent. In order that the criteria which distinguish two experimental situations should be considered irrelevant, the experimenter must

(a) record the same results irrespective of these criteria, and

(b) know that the abstracted situation can be isolated in a well-defined way from the effects of these criteria.

Condition (b) may hold for a variety of logical and physical reasons. It will be discussed in some detail for all the equivalences that we shall postulate.

S. We shall now use the symbol '=' to relate the *sequences* of symbols which represent equivalent histories, and later to relate the various combinations of such

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sequences. It follows from the above discussion that the usual axioms of equivalence hold:

(i) $a = a$ (all a),

(ii) if $a = b$ then $b = a$,

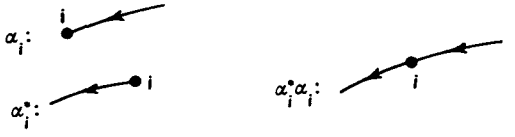
(iii) if $a = b$ and $b = c$ then $a = c$,

where the symbols a, b, c, represent experimental situations of any kind.

D. At the end of the last section we recognized the necessity for introducing the concepts of 'entry' and 'exit'. Let us represent entries with the symbols α_i and exits with α_i^* (relative to the state i). Now it is clear that an entry followed by an exit is just a coincidence, so we write

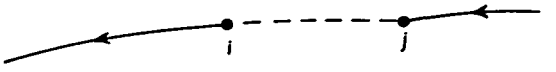
$$\alpha_i^* \alpha_i = C_i, \tag{3·1}$$

and condition (b) is satisfied on logical grounds. This can easily be seen if we use the diagrammatic representation



The *conjugate* of a history is that history traversed by the entity in the opposite order, so that exits and entries are interchanged and the order of events is reversed. We shall generalize the use of the superfix * to denote conjugates, e.g. $(\alpha_1 \alpha_2^*)^* = \alpha_2 \alpha_1^*$. Again we use the equivalence concept in a purely logical way.

S. Let us now redefine histories in terms of events. Any history consisting of alternate entries and exits is formally possible and will be called a *proper* history. Proper histories may be *continuous* or *discontinuous* according to whether we can follow the entity continuously from their beginning to their end, or not. Discontinuous histories contain factors like $\alpha_i^* \alpha_j$ $(i \neq j)$ and can be represented diagrammatically by



Sequences which are of the form $\alpha^* \ldots \alpha$ will be called *scalars*.

D. Let us suppose, for a given set of states $R = \{r\}$, that all continuous histories which begin and end with a specified pair of states $r = 1, 2$ are equivalent. Then the experimenter only records the *transition* of the entity from 1 to 2 without any reference to its intermediate coincidences (or path) in R. This has the representation

$$\alpha_r^* \alpha_r = C_r = U_R \quad (\text{all } r \text{ in } R), \tag{3·2a}$$

where U_R is the unit element defined by

$$\alpha_i U_R = \alpha_i, \quad U_R \alpha_i^* = \alpha_i^* \quad (\text{all } i \text{ in } R). \tag{3·2b}$$

These equations imply, respectively, that coincidences are not distinguished in R, and not recorded when they are associated historically with other events. In this case the equivalences are not purely logical, but refer to the physical characteristics of the

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situation. The coincidences in R may be really indistinguishable, or the distinction may be irrelevant. In the former case we shall say that R is an *infinitesimal region*, then (3·2 a, b) hold by definition. In the latter case we shall say that R is a *finite region* and then we require a physical justification for (3·2 a, b).

H. We have already brought to light the extra degree of differentiation that is needed to describe the intrinsic structure of an infinitesimal region of space-time. The orthodox description of the histories of test particles is entirely in terms of their coincidences with the infinitesimal regions of space-time (conventionally called 'events'). It has been shown in the above discussion that each coincidence may be regarded as the combination of an entry and an exit (i.e. a sequence of two events in the history). This refinement of our conceptual approach allows us to populate each infinitesimal region with several distinct states.

In order to determine the relationships between the various states populating an infinitesimal region we shall consider the (necessarily finite) apparatus used in making measurements. This approach can be justified only if the structure of an infinitesimal region is exactly reflected in the structure of the finite region in which our measurements are made. Being wise after the event, we know that this condition holds when (in the language of relativity theory) the space-time is flat. It follows that the measurement techniques that we shall describe algebraically can only be contemplated in a flat space-time. This is, however, a comment on the techniques that the experimental physicist uses, rather than on the limitations of the theory. The derived structure can be applied rigorously to the infinitesimal regions of any space-time.

D. The physical occurrence of a discontinuous history between two states is unlikely if the coincidences at these states are not equivalent, for this would imply that the region containing the states was multiply connected. Putting this possibility on one side we still have to consider discontinuous histories within the region R. The existence of such histories would indicate that the states were linked together by something besides the observed entity.

In the simple case, when the discontinuous history is (physically) impossible we shall say that the states concerned are *independent* and write

$$\alpha_i^* \alpha_j = O \quad (i \neq j). \tag{3·3}$$

If any part of a history is physically impossible, so is the whole. Therefore the symbol 'O' has the properties

$$Oa = O, \quad bO = O \quad \text{(all sequences } a, b). \tag{3·4}$$

4. *The theory of connects.*

H. The question now arises whether it is possible to specify unique relationships between certain pairs of states, so that it becomes meaningful to regard such pairs as single physical bodies. When this situation arises in practice the states are located at distinct points on a given 'rigid' body.

D. In the symbolic scheme as it now stands, there are two distinct entity transfers between any pair of states in R represented by $\alpha_1 \alpha_2^*$, $\alpha_2 \alpha_1^*$. There is no *a priori* relationship between these two expressions. Let us now suppose that the transfer of an entity from state 1 to state 2 in R ($\alpha_2 \alpha_1^*$) is equivalent to the transfer of an entity from state 2

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to state 1 via a possibly discontinuous history $(\alpha_1 \lambda \alpha_2^*)$ with an intermediate history, or *route*, denoted by λ. This condition is a formal possibility within our algebraic scheme and takes the form

$$\alpha_1 \lambda \alpha_2^* = \alpha_2 \alpha_1^*. \tag{4·1}$$

The equation conjugate to this is

$$\alpha_1 \alpha_2^* = \alpha_2 \lambda^* \alpha_1^*,$$

and relates the direct route from 2 to 1 to an indirect route between 1 and 2. If we are to avoid the states 1, 2 having an ordering significance relative to the indirect route it is also necessary to restrict

$$\lambda = \lambda^*. \tag{4·2}$$

S. Using equations (4·1) and (4·2) it is easy to show that

$$\lambda\lambda = U_R.$$

This has the solutions

$$\left.\begin{aligned} &\lambda = U_R \quad \text{(states 1 and 2 distinct)}, \\ &\lambda = (-), \quad \text{where} \quad (-)(-) = U_R, \quad (-) \neq U_R. \end{aligned}\right\} \tag{4·3}$$

H. It may make the situation clearer to the reader if we show explicitly the way in which equation (4·1) is linked up with our usual notions of rigidity. Consider two points A, B fixed on a rigid rule. We usually say that the displacement AB is opposite to BA so that if AB is associated with the number x, BA will be associated with $-x$. If, however, point B is moving along the rule the two displacements (each requiring a finite time) have the form $B_I A$ and AB_F, where B_I and B_F are the initial and final positions of B. Here the distinction between B_I and B_F requires the general criteria of stability supplied by the rigid rule. In this case there is no simple relation between the two displacements, for B_I and B_F are quite distinct.

Equation (4·1) postulates a simple relationship between the displacements $\alpha_1 \alpha_2^*$ and $\alpha_2 \alpha_1^*$ and thus implies that we are dealing with a rigid connexion. It is surprising (at least to the author) that the concept of rigidity should be representable in such a primitive algebraic scheme.

D. According to the results (4·3) given above there are two distinct solutions of the *connect* equation (4·1) with its subsidiary condition (4·2). The case $\lambda = (-)$ corresponds to the general concept of rigidity we were looking for, while the case $\lambda = U_R$ corresponds to the far stronger requirement that $\alpha_1 \alpha_2^* = \alpha_2 \alpha_1^*$. We shall call the physical situations that these solutions represent *indefinite* and *definite* connects, respectively. As both connects satisfy the rigidity requirement it appears that the extra condition necessary to ensure the definiteness of a connect must be linked to the type of entity involved.

H. An appeal to our intuitive ideas of space and time makes it easy to appreciate the significance of these two types of connect. Whereas spatial displacements can be in either direction and have a corresponding ambiguity of sign, temporal displacements have a characteristic direction and a definite sign. We therefore identify (provisionally) the definite connects as time displacements. Conventional relativity theory then indicates that the special entities we seek are light pulses *in vacuo*.

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D. The use of light pulses *in vacuo* as the physical realization of the entities in definite connects implies that no indefinite connect (with material entity) can be found which has its events superimposed on and simultaneous with those of a definite connect. It is an experimentally verifiable fact that events in the history of an inertial test particle cannot be superimposed upon events in the history of an undeflected light pulse. Such experiments do not require refined physical concepts for their interpretation and can be envisaged even in the primitive state of our conceptual scheme.

It should be noted that no direct experimental tests have been given which would enable the physicist to confirm the validity of equations (3·2) or (4·1) in the restricted physical contexts in which they have been stated to hold. This is because these equations represent the sort of conditions the experimenter chooses to work in, but which he has no prior means of testing. He can only check whether the results he obtains in these conditions are consistent. Our problem is to associate the various theoretical possibilities that arise with the corresponding physical necessities of the experimental physicist.

H. One of the most fundamental characteristics of physics is that it deals with quantitatively exact observations. That is to say, the results of observation are numerical. In the present section, we have seen how to represent some well-defined 'bits' of the physical world, but as yet we have no way of fitting them together. For this reason the numerical coefficients (which arise basically from counting) have not yet appeared.

Connects can be represented on a space-time diagram:

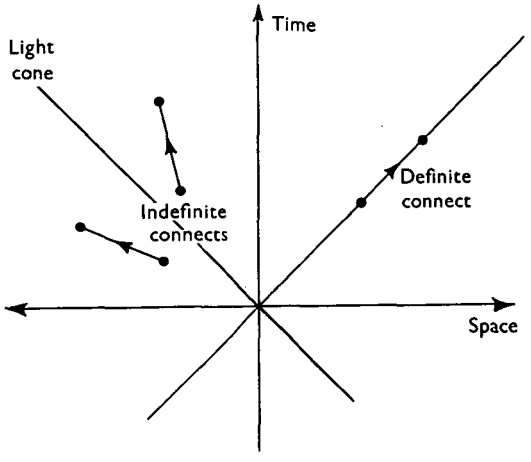


Fig. 1

The displacement along the light cone (which has a zero modulus) is definite in sign. All displacements with a finite modulus are indefinite in sign. Note that none of the displacements is capable of defining a precise direction in space-time.

S. Using equations (4·1–4·3) it is easy to derive the relations which hold between α_1 and α_2. When $\lambda = U_R$, $\alpha_1 = \alpha_2(-)$, where the properties of $(-)$ have already been given. When $\lambda = (-)$, $\alpha_1 = \alpha_2 i$, where i is defined by the relations

$$ii = (-), \quad i^* = (-)i = i(-).$$

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5. *The theory of comparison.*

D. In order to be able to represent the physical comparison of two connects (in the procedure of measurement) we require a new symbolic mode of combination. Consider two histories represented by 'a' and 'b' with their initial and final pairs of events exactly superimposed and occurring simultaneously. This total situation is called the *configuration* of the histories represented by a and b. It will be represented by '$a+b$' (*addition* of a and b).

H. The terms 'simultaneous' and 'superimposed' used in the above context refer to the immediate experience of the experimental physicist. We could say, alternatively, that the pairs of events occur within the same region R. Terms of this type should not be pictured in the context of a 'space-time' reference frame, which is a derivative concept in the present investigation.

D. Using the definition of addition given above it is easy to justify certain equivalences. The histories which make up a configuration have no simple intrinsic basis for ordering or grouping, so we write

$$\left.\begin{aligned} a+b &= b+a, \\ a+(b+c) &= (a+b)+c. \end{aligned}\right\} \tag{5·1}$$

Also
$$a+0 = a, \tag{5·2}$$

for the superposition of an impossible history cannot alter the observational context of 'a'. We can also define the following generalizations of conjugation

$$(a+b)^* = a^*+b^*. \tag{5·3}$$

Let us now consider the interpretation of those configurations which consist of two finite configurations in historic sequence, e.g.

$$(a_1+a_2+a_3+\ldots+a_m)(b_1+b_2+\ldots+b_n). \tag{5·4}$$

The initial events of all the a's are in a common R, and the final events of the b's must be in the same R if the expression is not to vanish. We cannot, therefore, distinguish the subsequent routes of the individual entities which have passed through the b configuration. So we are forced to interpret (5·4) as if each of the routes b were attached to all of the a's. This can be expressed by the distributive laws

$$\left.\begin{aligned} a(b+c) &= ab+ac, \\ (b+c)d &= bd+cd, \end{aligned}\right\} \tag{5·5}$$

providing that the expressions ab, cd, etc., represent proper histories. Then the expression (5·4) is a concise representation of a configuration of $n \times m$ histories which have certain sections in common allowing the given *factorization*.

Histories will be said to be of the *same type* when they contain the same sequence of exits and entries (without regard to the specific states involved). As we shall only be concerned with the comparison of histories of the same type (particularly the connects), configurations will only be defined in this case. Then either all the terms generated in (5·4) will be proper histories, or none of them. This avoids the difficulty entailed by the use of (5·5) to form expressions which contain both conceptually possible and

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conceptually impossible terms. For example, the configuration represented by $\alpha_1 + \alpha_2^*$ could not be joined to a prior or posterior event of any kind without giving rise to a conceptually impossible situation.

S. In order to complete the identification between the algebra of complex numbers and the scalars associated with a single indefinite connect within R it is necessary to show that

$$U_R + (-) = 0.$$

This result follows if we write $q = U_R + (-)$, notice that $(-)q = q(-) = q$ and assume that q^{-1} exists such that $q^{-1}q = U_R$ for all scalars $q \neq 0$.

D. This assumption, which is obviously necessary from a mathematical point of view, is also justifiable on physical grounds. It corresponds (given the process of conjugation) to the assumption that all the physical configurations described are reversible in the historic sense. That is to say, for any given configuration (scalar or otherwise) there always exists an *inverse* configuration which restores the entities to their former states.

We now return to the development of the representation of physically significant situations which was begun in the previous section. Suppose that the connect represented by $\alpha_1\alpha_3^* = \alpha_3(-)\alpha_1^*$ exists in R; also the independent state 2 ($\alpha_1^*\alpha_2 = 0$). This enables us to define a state 4 connected to 2 in exactly the same way as 3 is connected to 1 so that

$$\alpha_2\alpha_4^* = \alpha_4(-)\alpha_2^*,$$

where $\alpha_4 = \alpha_2 i$, i being given by $\alpha_1^*\alpha_3$. Here we have the symbolic representation of a situation that is physically obvious, namely that given a connect in R we can determine any number of independent connects.

H. In conventional terms we recognize the familiar situation where the rigidity of one rod is used to determine the rigidity of other rods and displacements. For example, the experimenter habitually uses rigid rods to determine the distance moved by free particles which, in themselves, can give no criterion for rigidity.

D. Given these two indefinite connects we can define their *comparison* in two distinct ways (apart from a trivial choice of overall sign)

$$E = \alpha_1\alpha_3^* + \alpha_4\alpha_2^*, \tag{5·6a}$$

or

$$\alpha_1\alpha_3^* + \alpha_2\alpha_4^*. \tag{5·6b}$$

The choice between (a) and (b) determines the relative sign of the two indefinite connects and is easily settled if we require that the self-comparison of a connect is impossible. This is a physical (rather than a logical) impossibility, because we are denying the existence of a physical configuration which corresponds to a direct self-comparison. So, just as in the former example of physical impossibility, we require that $E \to 0$ when the states 1 and 2 are identified. This determines (5·6a) as the correct definition of E.

S. We shall find it convenient to use the notation

$$\phi - \alpha_1\alpha_3^* \quad \text{for} \quad \phi + \alpha_1(-)\alpha_3^*$$

and

$$-E \quad \text{for} \quad \alpha_1(-)\alpha_3^* + \alpha_4(-)\alpha_2^* = \alpha_3\alpha_1^* - \alpha_4\alpha_2^*.$$

This simplifies the algebra without implying any further assumptions.

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D. An exactly similar argument to that given for indefinite connects applies to the comparison of definite connects. In this case we can write the comparison as

$$E' = \alpha_1\alpha_3^* + \alpha_2(-)\alpha_4^* = \alpha_1\alpha_3^* + \alpha_2\alpha_2^*, \tag{5·7}$$

where $\alpha_1\alpha_3^* = \alpha_3\alpha_1^*$ and $\alpha_2\alpha_4^* = \alpha_4\alpha_2^*$. The most noticeable difference between this expression and (5·6a) is that here one of the terms ($\alpha_2\alpha_2^*$) does not represent a connect at all, but is merely a self-transition in state 2.

We have already identified definite connects as the connexion of rigidly related states by a light pulse *in vacuo*. If we rewrite (5·7) in the form

$$E' = \alpha_1(-)\alpha_1^* + \alpha_2\alpha_2^*$$

a possible physical realization becomes apparent. States 1 and 2 are superimposed. An entity leaves state 2 and returns to it by a continuous route (1); simultaneously a light pulse leaves state 1, is reflected ($-$), and then returns to it.

H. If we portray these comparisons on a space-time diagram it becomes clear how they are able to determine directions:

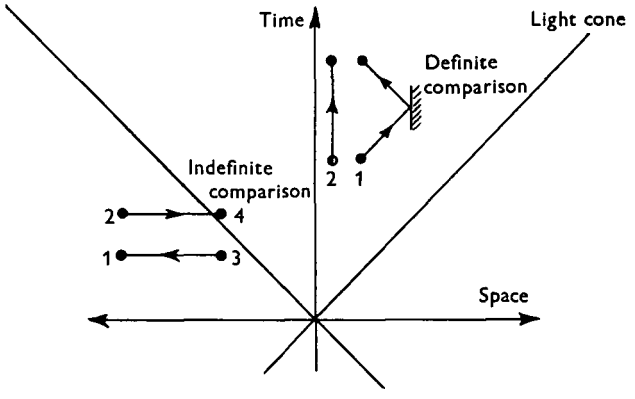


Fig. 2

The light pulse determines a precise directed displacement because it returns to the same state. The comparison of rigid rods determines a precise direction because, owing to the opposed directions of the displacements, all events in the connects must be simultaneous.

6. *Orientation of comparisons.*

S. The comparison of indefinite connects

$$E = \alpha_1\alpha_3^* + \alpha_4\alpha_2^*,$$

where $\alpha_1^*\alpha_3 = \alpha_2^*\alpha_4 = i$, keeps the same form (apart from the trivial change $E \to -E$) under several permutations of states: (13)(24), (14)(23), (12)(34). These permutations form an Abelian group of order 4 if adjoined to the unit element.

H. Groups of this (Klein quadratic) type have already been introduced into the context of structural physics by Bastin and Kilmister (1) from a general philosophical standpoint. The weakness in their argument was that no precise physical significance could be attached to the group, which now appears in a specific context.

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S. The symmetries of E expressed by this group can be shown explicitly, and independently of the connect equations, if we write

$$E = \tfrac{1}{2}(\alpha_1\alpha_3^* - \alpha_3\alpha_1^* + \alpha_4\alpha_2^* - \alpha_2\alpha_4^*) = E_1 \quad \text{(say)}. \tag{6·1}$$

The permutations (234), (243) generate two similar expressions, respectively

$$\left.\begin{aligned} E_2 &= \tfrac{1}{2}(\alpha_1\alpha_2^* - \alpha_2\alpha_1^* + \alpha_3\alpha_4^* - \alpha_4\alpha_3^*), \\ E_3 &= \tfrac{1}{2}(\alpha_1\alpha_4^* - \alpha_4\alpha_1^* + \alpha_2\alpha_3^* - \alpha_3\alpha_2^*). \end{aligned}\right\} \tag{6·2}$$

These also keep the same form, apart from $E \to -E$, under the Abelian group of order 4 defined above.

D. We now have a set of three symbols E_1, E_2, E_3 which is closed under all the even permutations of the states 1–4. If only one odd permutation is adjoined to this group all the others can be generated. But the odd permutation (13) changes E into a physically forbidden form. Hence all odd permutations of the states have no physical significance.

S. Using the algebraic relations established in the previous section for the indefinite connects $\alpha_1\alpha_3^*$, $\alpha_2\alpha_4^*$ we have

$$\left.\begin{aligned} \alpha_2\alpha_3^* &= \alpha_4(-)\alpha_1^*, \\ \alpha_1\alpha_2^* &= \alpha_3\alpha_4^*, \end{aligned}\right\} \tag{6·3}$$

and the conjugate equations. Substituting these into the above expressions for E_2 and E_3 we get

$$\left.\begin{aligned} E_2 &= \alpha_1\alpha_2^* - \alpha_2\alpha_1^*, \\ E_3 &= \alpha_1\alpha_4^* - \alpha_4\alpha_1^*. \end{aligned}\right\} \tag{6·4}$$

It is now easy to show that

$$\left.\begin{aligned} E_1^2 = E_2^2 = E_3^2 &= -I, \\ E_2E_3 = -E_3E_2 &= E_1, \end{aligned}\right\} \tag{6·5}$$

where I is the identical transition

$$I = \alpha_1\alpha_1^* + \alpha_2\alpha_2^*, \quad IE_1 = E_1, \quad \text{etc.}$$

Now it is clear that if we were just given the expressions (6·1) and (6·2) we could choose any of the three states 2, 3, 4 as that making an indefinite connect with 1, and this choice would not affect the algebraic results set out in (6·5). For this reason we shall refer to E_2 and E_3 as the *alternative orientations* of the comparison.

H. Equation (6·5) reproduces the defining relationships of real quaternions which, as is well known, can be used to represent the properties of the three unit directions in three-dimensional Euclidean space. We have therefore shown that the procedure of comparison of indefinite connects, when full account is taken of the possible state permutations, generates explicitly the structure of three-dimensional Euclidean geometry.

D. It is illuminating to carry out the process analogous to that given above for the comparison of definite connects $E' = \alpha_1\alpha_3^* - \alpha_2\alpha_4^*$. This keeps the same form under the state permutations (13), (24), (13)(24), (12)(34) and (13)(24). Hence the complete permutation group of E' contains eight elements, both odd and even.

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As before, we can define three symmetrized expressions related by the permutation group I, (234), (243):

$$E' = E'_A = \tfrac{1}{2}(\alpha_1\alpha_3^* + \alpha_3\alpha_1^* - \alpha_2\alpha_4^* - \alpha_4\alpha_2^*),$$
$$E'_B = \tfrac{1}{2}(\alpha_1\alpha_4^* + \alpha_4\alpha_1^* - \alpha_3\alpha_2^* - \alpha_2\alpha_3^*),$$
$$E'_C = \tfrac{1}{2}(\alpha_1\alpha_2^* + \alpha_2\alpha_1^* - \alpha_4\alpha_3^* - \alpha_3\alpha_4^*).$$

The imposition of a connect relation on any one of these immediately ensures that the remaining two vanish. Here we discover one of the most important properties of a temporal comparison—it has no alternative orientations.

We can, however, define a symmetrized transition between the states 1 and 2:

$$E'_2 = \alpha_1\alpha_2^* + \alpha_2\alpha_1^*, \quad E'_2E'_2 = I.$$

If we write $E' = E'_3$ it then follows that

$$E'_2E'_3 = \alpha_2\alpha_1^* - \alpha_1\alpha_2^* = E'_1 \quad \text{(say)},$$

which is the corresponding antisymmetric transition.

S. The comparisons, alternative comparison orientations and symmetrized transitions just defined have a simple matrix representation. In the case of the comparison of indefinite connects this is given by

$$\alpha_1 = \begin{pmatrix}1\\0\end{pmatrix}, \quad \alpha_2 = \begin{pmatrix}0\\1\end{pmatrix}, \quad \alpha_3 = \begin{pmatrix}i\\0\end{pmatrix}, \quad \alpha_4 = \begin{pmatrix}0\\i\end{pmatrix},$$

the α_i^* being Hermitian conjugates of the α_i. Then

$$E_1 = \begin{pmatrix}-i & 0\\0 & i\end{pmatrix}, \quad E_2 = \begin{pmatrix}0 & -1\\1 & 0\end{pmatrix}, \quad E_3 = \begin{pmatrix}0 & i\\i & 0\end{pmatrix}.$$

Similarly, for the definite connects,

$$\alpha_1 = \begin{pmatrix}1\\0\end{pmatrix}, \quad \alpha_2 = \begin{pmatrix}0\\1\end{pmatrix}, \quad \alpha_3 = \begin{pmatrix}-1\\0\end{pmatrix}, \quad \alpha_4 = \begin{pmatrix}0\\-1\end{pmatrix},$$
$$E'_3 = \begin{pmatrix}-1 & 0\\0 & 1\end{pmatrix}, \quad E'_2 = \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}, \quad E'_1 = \begin{pmatrix}0 & 1\\-1 & 0\end{pmatrix}.$$

These matrix forms enable us to picture very easily the significance of a configuration. They will be used extensively in the next section.

Using as a basis the unit scalar U_R, or the identical transition I, it is easy to construct the full arithmetic of complex numbers. We shall use the same symbols in both cases, their position in the sequence indicating the unit element involved. For example, we shall write $\alpha_i 2\alpha^* = 2\alpha_i\alpha_j^*$.

Let us now define the transformation

$$\alpha_1 \to \alpha_A = \alpha_1 a + \alpha_2 b, \tag{6·6}$$

where a and b are any complex numbers which satisfy $a^*a + b^*b = U_R$. If 1 and 2 are states of an indefinite comparison, then A is a state related to these by (6·6).

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Given state A we can easily determine the form of the three other states required to complete a comparison:

$$\alpha_B = \alpha_1(-b^*)+\alpha_2 a^*,$$
$$\alpha_C = \alpha_1 ai+\alpha_2 bi,$$
$$\alpha_D = \alpha_1(-ib^*)+\alpha_2(ia^*).$$

Then the following relations are satisfied

$$\alpha_C = \alpha_A i, \quad \alpha_D = \alpha_B i, \quad \alpha_A^*\alpha_B = 0.$$

The comparison has the form

$$E_X = \alpha_A\alpha_C^*-\alpha_B\alpha_D^* = (\alpha_1\alpha_3^*-\alpha_2\alpha_4^*)(a^*a-b^*b)-\alpha_1\alpha_2^*(2aib^*)-\alpha_2\alpha^*(2a^*ib)$$
$$= E_1(a^*a-b^*b)+E_2 s+E_3 r,$$

where $r+is = 2ab^*$ (r, s real).

D. This transformation of states allows us to specify the continuous process which changes one orientation into another. The comparisons E_X (which may carry an arbitrary modulus, x) traverse the whole orientation space defined by the set of standard independent orientations E_1, E_2 and E_3. We shall call this set a *reference frame* (if we may borrow the orthodox term). Using this reference frame the most general indefinite displacement within R takes the form

$$X_R = xE_X = E_1x_1+E_2x_2+E_3x_3,$$

where x_1, x_2, x_3 are real parameters.

7. *The complete description of an arbitrary displacement.*

D. We have already discussed two modes of comparison applicable to the displacements of a material particle, according to whether it is at rest relative to the states in R, or in motion. The argument given below will show how the (definite) comparison appropriate to rest material particles can be generalized to give a complete description of all displacements in a continuum with four degrees of freedom.

Let us suppose that the definite displacement of a rest particle is described by the comparison

$$E_3' = \alpha_1\alpha_3^*+\alpha_2\alpha_2^* = \begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}.$$

As we have seen, this comparison is associated with the antisymmetric transition

$$E_1' = \alpha_1\alpha_2^*-\alpha_2\alpha_1^* = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}.$$

Such antisymmetric expressions have already occurred in the forms of the indefinite comparison with explicit symmetries given in (6·1) and (6·2). In that case the expressions were either connects or alternative orientations of connects. Now it is clear that E_1' cannot be reduced to a comparison, because the states 1, 2 are necessarily independent. We must therefore suppose that E_1' represents an alternative orientation for an indefinite connect.

H. The potentiality for spatial relationships between states 1 and 2 is apparent if we examine Fig. 2, which shows these states with an arbitrary spatial separation. Our problem is, of course, to find a way of linking the temporal and spatial comparisons already found to form an integrated space-time.

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D. In the previous section we have seen that a given orientation can be expressed as a linear form in terms of a set of standard orientations E_1, E_2, E_3. Applying this idea to E_3' and E_1' we have the expression

$$T = E_3' t + E_1' x, \tag{7·1}$$

where t is the modulus of the definite comparison and x is the modulus of the anti-symmetric part E_1'. We already know that no alternative orientations exist for the direction of a definite comparison. This is the reason that we omit E_2' from the above expression. It satisfies $E_2' E_2' = 1$ and therefore behaves like an alternative orientation of the comparison.

Equation (7·1) can be generalized to contain indefinite comparison if we introduce the *duplicated* comparison defined, in matrix form, by

$$E_3' \to E_3' \oplus E_3' = \left(\begin{array}{cc:cc} -1 & 0 & 0 & \\ 0 & 1 & & \\ \hdashline & & -1 & 0 \\ 0 & & 0 & 1 \end{array}\right) = \tilde{E}_3' \quad \text{(say)},$$

where $\oplus$ is to be interpreted as the outer addition of matrices. This just corresponds to the superposition of two definite comparisons defined in terms of independent states. Similarly

$$E_1' \to E_1' \oplus E_1' = \left(\begin{array}{cc:cc} 0 & 1 & 0 & \\ -1 & 0 & & \\ \hdashline & & 0 & 1 \\ 0 & & -1 & 0 \end{array}\right) = \tilde{E}_1'.$$

Given two expressions like T defined in terms of independent states we can formulate an indefinite comparison between the E_1' parts by giving them opposite signs relative to the E_3' parts. This is just an alternative procedure to that used originally in the formulation of an indefinite comparison where the signs of the connects were intrinsic. Accordingly we write

$$\begin{aligned} \tilde{T} &= (E_3' t + E_1' x) \oplus (E_3' t - E_1' x) \\ &= \left(\begin{array}{cc:cc} -1 & 0 & 0 & \\ 0 & 1 & & \\ \hdashline & & -1 & 0 \\ 0 & & 0 & 1 \end{array}\right) t + \left(\begin{array}{cc:cc} 0 & 1 & 0 & \\ -1 & 0 & & \\ \hdashline & & 0 & -1 \\ 0 & & 1 & 0 \end{array}\right) x. \end{aligned}$$

This expression can be arranged in a more convenient form by state interchanges to become

$$\begin{aligned} \tilde{T} &= \left(\begin{array}{cc:cc} -1 & 0 & 0 & \\ 0 & -1 & & \\ \hdashline & & 1 & 0 \\ 0 & & 0 & 1 \end{array}\right) t + \left(\begin{array}{cc:cc} 0 & & 0 & 1 \\ & 0 & -1 & 0 \\ \hdashline 0 & 1 & & \\ -1 & 0 & & 0 \end{array}\right) x \\ &= (E_3' \times I) t + (E_2' \times E_2) x, \end{aligned}$$

in the direct product notation.

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The derivation of the full structure of a displacement is now simple. All we have to do is to allow for all possible orientations of the indefinite comparison instead of restricting this to the E_2 direction. Hence the final expression, giving the complete structure required, takes the form

$$S = (E_3' \times I)\, t + (E_2' \times E_1)\, x_1 + (E_2' \times E_2)\, x_2 + (E_2' \times E_3)\, x_3, \tag{7·2}$$

whence

$$S^2 = t^2 - x_1^2 - x_2^2 - x_3^2. \tag{7·3}$$

The state interchanges referred to above also rearrange the form of $\tilde{E}_1'$. This is given in the revised matrix notation as

$$\tilde{E}_1' = \left(\begin{array}{cc:cc} & 0 & 1 & 0 \\ & & 0 & 1 \\ \hdashline -1 & 0 & & \\ 0 & -1 & 0 & \end{array}\right) = E_1' \times I.$$

It should be noted that this expression is quite distinct from the matrix coefficients of x_1, x_2, x_3 in (7·2).

H. The essential character of space-time was already apparent in (7·1), for $T^2 = t^2 - x^2$. The subsequent developments were introduced to show how the characteristic three dimensionality of space once again appeared from the structure of the indefinite comparison.

Equation (7·2) shows explicitly the following properties of the displacements in R: They are completely described by a reference frame with

(a) four dimensions,

(b) a quadratic metric

and (c) the signature $(-1, -1, -1, +1)$.

8. *Concluding remarks.* This section is concerned with fitting the contents of an earlier paper (Newman (4)) into the theoretical framework that has just been developed. When the previous paper was written, very little was understood about the detailed physical significance of the steps involved, although the general pattern of the argument was clear. For this reason that paper was given a very formal construction which displayed each step in the argument explicitly without requiring an adequate physical justification.

In broad outline both papers follow the same algebraic pattern of development. The stages of this development and its dependence on experimental procedures (of the fundamental kind referred to in § 1) may be summarized as follows:

(a) The experimenter dissects displacements into definite sections—which we describe by breaking coincidences into entries and exits.

(b) The experimenter characterizes the sections entirely by their end points. He therefore confines his experiments to regions of the world which have this property.

(c) The experimenter assumes that every section has a definite relationship with its opposite. This we describe by the algebra of connects which characterizes the distinction between space and time.

(d) The experimenter compares physically separate sections. We describe this by the algebra of comparisons which determines the dimensionality of the physical continuum.

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The final algebraic scheme, developed in the former paper and set out in it as the table of matrices, (8·9), remains unchanged. Certain algebraic derivations omitted in the above work will be found in the earlier paper. The algebra of the theory of connects was given in § 5 and the algebra of the theory of comparisons was given in § 7.

The special terminology which was formerly employed has now been simplified as much as is consistent with clarity. Some changes have also been made in cases where the previous terminology implied associations which require careful justification. For example, we have replaced the term 'interval' by the less suggestive expression 'definite connect'.

There were several errors of intepretation in the previous paper which tended to obscure the physical significance of the algebraic manipulations. It seems important to the author that such errors can be detected, for this suggests that the theoretical construction has an objective significance.

The main errors are corrected below:

(*a*) The definitions (PD 3·2, 3·3) made it appear that the entities and states were subsets of objects and parts, respectively. We have seen, however, that this process is one of abstraction rather than selection.

(*b*) All discussion of the region R was omitted, and this made the physical interpretation of $\alpha_i^* \alpha_i = 1$ and the addition concept quite inadequate. In particular, the concepts of 'compound entity' and 'compound state' (PD 6·1, 6·2), introduced to explain the distributive law, were superfluous.

(*c*) The concepts of 'superposition' and 'duality' (§ 8) which were introduced to explain the construction of a space-time comparison from 4×4 matrices has now been replaced by the more coherent argument given in the previous section.

Some of the physical definitions (PD) given in the previous paper were inadequate and have now been replaced by more explicit formulations. For example, the definitions of equivalence (PD 4·1) and independence (PD 4·12) have been reformulated.

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Queen Mary College
London

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