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A GENERAL LOWER BOUND OF PARAMETER ESTIMATION FOR REFLECTED ORNSTEIN–UHLENBECK PROCESSES

QING-PEI ZANG,* ** *Zhejiang University and Huaiyin Normal University*
LI-XIN ZHANG,* *Zhejiang University*

Abstract

A reflected Ornstein–Uhlenbeck process is a process that returns continuously and immediately to the interior of the state space when it attains a certain boundary. It is an extended model of the traditional Ornstein–Uhlenbeck process being extensively used in finance as a one-factor short-term interest rate model. Under some mild conditions, this paper is devoted to the study of the analogue of the Cramer–Rao lower bound of a general class of parameter estimation of the unknown parameter in reflected Ornstein–Uhlenbeck processes.

Keywords: Reflected Ornstein–Uhlenbeck process; maximum likelihood estimation; Cramer–Rao lower bound

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1. Introduction

Given a filtered probability space $\Lambda := (\Omega, \mathcal{F}, \mathbb{P})$ with the filtration $(\mathcal{F}_t)_{t\geq 0}$ satisfying the usual conditions, the reflected Ornstein–Uhlenbeck (ROU) process $\{X_t, t \geq 0\}$ reflected at the boundary $b \in \mathbb{R}_+$ on Λ is defined as follows. Let $\{X_t, t \geq 0\}$ be the strong solution whose existence is guaranteed by an extension of the results of Lions and Sznitman [28] to the stochastic differential equation

$$\mathrm{d}X_t = -\alpha X_t\,\mathrm{d}t + \sigma\,\mathrm{d}W_t + \mathrm{d}L_t, \qquad X_t \geq b^L \quad \text{for all } t \geq 0, \qquad X_0 = x, \tag{1.1}$$

where $b^L \in \mathbb{R}_+$, $x \in [b^L, +\infty)$, $\sigma \in (0, +\infty)$, $\alpha \in \mathbb{R}_+$, and $\{W_t, t \geq 0\}$ is a one-dimensional standard Wiener process. Let $L = (L_t)_{t\geq 0}$ be the minimal nondecreasing and nonnegative process, which makes the process $X_t \geq b^L$ for all $t \geq 0$. The process L increases only when X hits the boundary b^L, so that

$$\int_{[0,\infty)} \mathbf{1}_{\{X_t > b^L\}}\,\mathrm{d}L_t = 0,$$

where $\mathbf{1}_{\{\cdot\}}$ denotes the indicator function. Sometimes L is called the regulator of the point b^L (see [18]) and by virtue of Ata *et al.* [3], the paths of the regulator are nondecreasing, right-continuous with left limits, and possess the support property

$$\int_0^t \mathbf{1}_{\{X_s = b^L\}}\,\mathrm{d}L_s = L_t.$$

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* Postal address: Department of Mathematics, Zhejiang University, Hangzhou, Zhejiang, 310027, P. R. China.
** Email address: zqphunhu@hytc.edu.cn

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It can be shown that (see, e.g. [18] and [35]) the process L has an explicit expression as

$$L_t = \max\left\{0, \sup_{s\in[0,\,t]}\left(-x + \alpha\int_0^s X_u\,\mathrm{d}u - \sigma W_s\right)\right\}. \tag{1.2}$$

It is often the case that the reflecting barrier is assumed to be 0 in applications to queueing systems, storage models, engineering, finance, etc. This is mainly due to the physical restriction of the state processes such as queue-length, inventory level, content process, stock prices, and interest rates, which take nonnegative values.

The traditional Ornstein–Uhlenbeck processes, which are Markovian, mean-reverting Gaussian processes, and also called the Vasicek model, well describe various real-life phenomena and so have found widespread use in a broad range of application domains as diverse as finance, life sciences, and operations research. However, unknown parameters or functions involved in all models need to be estimated from observations of the processes. The estimation of these processes is therefore a crucial step in all applications, in particular in applied finance. In the case of Ornstein–Uhlenbeck processes driven by Wiener processes, the statistical inference for these processes has been studied and a comprehensive survey of various methods was given in [8], [24], and [29], for example, for maximum likelihood estimation, minimum contrast estimation, maximum probability estimation, and nonparametric inference. By virtue of the Markovian and Gaussian properties of Wiener processes, both the maximum likelihood estimator (MLE) and the least squares estimator are easy to obtain and exhibit asymptotic unbiasedness, efficiency, and normality under the usual regularity conditions. We refer the reader to [7] and [9] and the references therein.

In many situations, though, the stochastic processes involved are not allowed to cross a certain boundary, or are even supposed to remain within two boundaries. The proposed reflected processes behave like standard Ornstein–Uhlenbeck processes in the interior of their domains. However, when they reach the boundary, the sample path returns to the interior in a manner such that the 'pushing' force is minimal. This kind of process, which can be applied to the field of queueing systems, financial engineering, and mathematical biology, has attracted the attention of scholars around the world.

Many attempts have been made to research ROU processes in both theory and application; see, for example, Ricciardi and Sacerdote [31], who applied ROU processes to the field of mathematical biology. Krugman [23] limited the currency exchange rate dynamics in a target zone by two reflecting barriers. Goldstein and Keirstead [16] explored the term structure of interest rates for the short-rate processes with reflecting boundaries. In Hanson *et al.* [17], the asset pricing models with truncated price distributions were investigated. Linetsky [27] studied the analytical representation of transition density for reflected diffusions in terms of their Sturm–Liouville spectral expansions. Bo *et al.* [11], [13] applied ROU processes to model the dynamics of asset prices in a regulated market, and the conditional default probability with incomplete (or partial) market information was calculated. Ward and Glynn [32]–[34] showed that ROU processes serve as a good approximation for a Markovian queue with reneging when the arrival rate is either close to or exceeds the processing rate and the reneging rate is small; and ROU processes also well approximate queues having renewal arrival and service processes in which customers have deadlines constraining the total sojourn time. Customers either renege from the queue when their deadline expires or balk if the conditional expected waiting time given the queue-length exceeds their deadline.

In practice, some important aspects of the performance of a queueing system (e.g. customers' waiting times, traffic intensities) may not be directly observable and therefore such performance

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measures and their related model parameters need be statistically inferred from the available observed data. Recently, based on continuous observations, Bo *et al.* [14] first presented the MLE for ROU processes (1.1) and derived the MLE of the unknown parameter α on the basis of the process $\{X_t\}$ up to a previously determined fixed time T. However, this estimator is biased and its mean-squared error depends on the parameter to be estimated. To overcome such limitations, Lee *et al.* [26] proposed the sequential estimation for ROU processes and pointed out that this kind of estimation is unbiased, exactly normally distributed, and optimal in the mean-square sense.

In this paper our interest lies in the analogue of the Cramer–Rao lower bound of a general class of estimator including biased and unbiased cases of the unknown parameter involved in ROU processes.

Noting that σ in our model is an unknown constant which is independent of the parameter α and the quadratic variation process $[X]_t$ equals to $\sigma^2 t$, $t \geq 0$, we assume that σ is known and set it equal to 1 in the situation of continuous observations.

The rest of this paper is organized as follows. In Section 2 the equivalent martingale measure of ROU processes is formulated. In Section 3 we propose a Cramer–Rao lower bound of a general class of parameter estimation for ROU processes, and some remarks are presented. In Section 4 the paper is concluded and some opportunities for future research are outlined.

2. Preliminaries

In this section we review some of the basic concepts concerning our context and formulate the equivalent martingale measure of ROU processes, which will be the basis of our further study.

Throughout this paper, we denote $\mathbb{P}_\alpha^T$ (also $\mathbb{P}_\beta^T, \ldots$) for the probability measure generated by the process $\{X_t^\alpha, 0 \leq t \leq T\}$ (also $\{X_t^\beta, 0 \leq t \leq T\}, \ldots$) on the space $(\mathcal{C}[0, T], \mathcal{B}_T)$, where $\mathcal{C}[0, T]$ denotes the space of continuous functions endowed with the supremum norm and $\mathcal{B}_T$ is the corresponding Borel σ-algebra. Let $\mathbb{E}_\alpha$ (also $\mathbb{E}_\beta, \ldots$) denote the expectation with respect to $\mathbb{P}_\alpha^T$ (also $\mathbb{P}_\beta^T, \ldots$) and $\mathbb{P}_{\mathrm{W}}^T$ be the probability measure induced by the standard Wiener process. Now we give the equivalent martingale measure of ROU processes.

Theorem 2.1. *Suppose that X^β and X^γ are two ROU processes satisfying the following stochastic differential equations, respectively:*

$$\mathrm{d}X_t^\beta = -\beta X_t^\beta \,\mathrm{d}t + \mathrm{d}W_t + \mathrm{d}L_t^\beta, \qquad X_t \geq b^L \quad \textit{for all } t \geq 0, \qquad X_0 = x,$$

and

$$\mathrm{d}X_t^\gamma = -\gamma X_t^\gamma \,\mathrm{d}t + \mathrm{d}W_t + \mathrm{d}L_t^\gamma, \qquad X_t \geq b^L \quad \textit{for all } t \geq 0, \qquad X_0 = x,$$

where $b^L \in \mathbb{R}_+$, $x \in [b^L, +\infty)$, and L^β and L^γ are the corresponding regulators at b^L. Then, for each $T > 0$, we have

$$\left.\frac{\mathrm{d}\mathbb{P}_\gamma^T}{\mathrm{d}\mathbb{P}_\beta^T}\right|_{\mathcal{F}_T^\beta} = \exp\left(-\int_0^T (\gamma - \beta) X_s^\beta \,\mathrm{d}W_s - \frac{1}{2}\int_0^T (\gamma - \beta)^2 (X_s^\beta)^2 \,\mathrm{d}s\right) := Y_T.$$

Here $\mathcal{F}^\beta$ denotes the natural filtration generated by X^β.

Proof. In fact, note that from (1.1) and (1.2), we have

$$X_t = x - \alpha \int_0^t X_s \,\mathrm{d}s + W(t) + \sup_{s \in [0,\, t]} \left(-x + \alpha \int_0^s X_u \,\mathrm{d}u - W_s\right) \vee 0.$$

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If $\alpha > 0$, it follows that

$$\begin{aligned} X_t &\le x - \alpha \int_0^t X_s \,\mathrm{d}s + W(t) + \alpha \int_0^t X_s \,\mathrm{d}s + \sup_{s\in[0,\,t]} (-x - W_s) \vee 0 \\ &\le 2x + 2 \sup_{r\in[0,\,T]} |W(r)|. \end{aligned}$$

Similarly, for $\alpha < 0$, we have

$$\begin{aligned} X_t &\le x - \alpha \int_0^t X_s \,\mathrm{d}s + W(t) - \alpha \int_0^t X_s \,\mathrm{d}s + \sup_{s\in[0,\,t]} (-x - W_s) \vee 0 \\ &\le 2x - 2\alpha \int_0^t X_s \,\mathrm{d}s + 2 \sup_{r\in[0,\,T]} |W(r)|. \end{aligned}$$

Thus, for any $\alpha \in \mathbb{R}$, we obtain

$$|X_t| \le 2x + 2|\alpha| \int_0^t |X_s| \,\mathrm{d}s + 2 \sup_{r\in[0,\,T]} |W(r)|.$$

By Gronwall's inequality, it follows that

$$\sup_{t\in[0,\,T]} |X_t| \le 2x\mathrm{e}^{2|\alpha|T} + 2\mathrm{e}^{2|\alpha|T} \sup_{r\in[0,\,T]} |W(r)|. \tag{2.1}$$

Then, by Fernique's theorem, which claims the exponential integrability of the square of a seminorm for a Gaussian process (cf. Fernique [15]), there exists $\delta > 0$ such that

$$\mathbb{E}\exp\left(\delta \sup_{t\in[0,\,T]} |X_t|^2\right) < \infty.$$

Then $\mathbb{E}Y_t = 1$. Indeed, by Jensen's inequality, it follows that

$$\exp\left(\frac{1}{2}\int_0^T (\gamma-\beta)^2 (X_s^\beta)^2 \,\mathrm{d}s\right) \le \frac{1}{T}\int_0^T \exp\left(\frac{T(\gamma-\beta)^2(X_s^\beta)^2}{2}\right)\mathrm{d}s.$$

Hence, if $T \le 2\delta/(\gamma-\beta)^2$ then

$$\mathbb{E}\exp\left(\frac{1}{2}\int_0^T (\gamma-\beta)^2 (X_s^\beta)^2 \,\mathrm{d}s\right) \le \mathbb{E}\exp\left(\delta \sup_{t\in[0,\,T]} |X_t|^2\right) < \infty.$$

Therefore, $\mathbb{E}Y_T = 1$ and the fact that a supermartingale with constant expectation is a martingale (cf. [21], [22] or [30]).

Now, if $T \ge 2\delta/(\gamma-\beta)^2$, Y_T can be written as

$$Y_T = \prod_{i=0}^{n-1} Y_{t_i}^{t_{i+1}},$$

where $0 = t_0 < t_1 < \cdots < t_n = T$, $\max_{0\le i\le n-1}(t_{i+1} - t_i) \le 2\delta/(\gamma-\beta)^2$, and

$$Y_{t_i}^{t_{i+1}} = \exp\left(-\int_{t_i}^{t_{i+1}} (\gamma-\beta) X_s^\beta \,\mathrm{d}W_s - \frac{1}{2}\int_{t_i}^{t_{i+1}} (\gamma-\beta)^2 (X_s^\beta)^2 \,\mathrm{d}s\right).$$

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Then $\mathbb{E}(Y_{t_i}^{t_{i+1}}) = 1$, and it follows that $\mathbb{E}[(Y_{t_i}^{t_{i+1}}) \mid \mathcal{F}_{t_i}^{\beta}] = 1$. Hence,

$$\mathbb{E}Y_T = \mathbb{E}[\mathbb{E}(Y_T \mid \mathcal{F}_{t_{n-1}}^{\beta})] = \mathbb{E}Y_{t_{n-1}} = \cdots = \mathbb{E}Y_{t_1} = 1.$$

Thus far we have proved that Y_t is a martingale. For a given $T > 0$, we define the equivalent martingale measure $\tilde{\mathbb{P}}$ as $\mathrm{d}\tilde{\mathbb{P}} = Y_T \,\mathrm{d}\mathbb{P}$. Then, by the usual Girsanov theorem, we know that the process

$$\tilde{W}_t = W_t + (\gamma - \beta) \int_0^t X_s^{\beta} \,\mathrm{d}s, \qquad t \in [0, T]$$

is a standard Brownian motion under $\tilde{\mathbb{P}}$ and

$$\begin{aligned} \mathrm{d}X_t^{\beta} &= -\beta X_t^{\beta} \,\mathrm{d}t + \mathrm{d}W_t + \mathrm{d}L_t^{\beta} \\ &= -\beta X_t^{\beta} \,\mathrm{d}t + \beta X_t^{\beta} \,\mathrm{d}t - \gamma X_t^{\beta} \,\mathrm{d}t + \mathrm{d}\tilde{W}_t + \mathrm{d}L_t^{\beta} \\ &= -\gamma X_t^{\beta} \,\mathrm{d}t + \mathrm{d}\tilde{W}_t + \mathrm{d}L_t^{\beta} \quad \text{for } t \in [0, T]. \end{aligned}$$

Thus, X^{β} under $\tilde{\mathbb{P}}$ has the same distribution as X^{γ} under $\mathbb{P}$. Therefore, for each measurable subset A of the space $(\mathcal{C}[0, T], \mathcal{B}_T)$, we have

$$\tilde{\mathbb{P}}(X^{\beta} \in A) = \mathbb{P}(X^{\gamma} \in A),$$

and then by our preliminaries and the definition of conditional expectation, we have

$$\begin{aligned} \mathbb{P}_{\gamma}^{T}(A) &= \mathbb{P}(X^{\gamma} \in A) \\ &= \tilde{\mathbb{P}}(X^{\beta} \in A) \\ &= \int_{[X^{\beta} \in A]} Z(T) \,\mathrm{d}\mathbb{P} \\ &= \int_{[X^{\beta} \in A]} \mathbb{E}[Z(T) \mid \mathcal{F}_t^{\beta}] \,\mathrm{d}\mathbb{P} \\ &= \int_A \mathbb{E}[Z(T) \mid \mathcal{F}_t^{\beta}] \,\mathrm{d}\mathbb{P}_{\beta}^{T}. \end{aligned}$$

Thus, the proof is complete. □

3. Lower bound of the estimator for the ROU model

In this section we state our main results concerning an analogue of the Cramer–Rao lower bound for any estimator satisfying some regular conditions.

Theorem 3.1. *Suppose that $\hat{\alpha}_T$ is any estimator of α in (1.1) such that $\mathbb{E}_{\alpha}(\hat{\alpha}_T - \alpha)^2 < \infty$ over a compact set Θ. Let $b(\alpha) = \mathbb{E}_{\alpha}(\hat{\alpha}_T) - \alpha$ be differentiable with respect to α. Then*

$$\mathbb{E}_{\alpha}(\hat{\alpha}_T - \alpha)^2 \geq \frac{(1 + b'(\alpha))^2}{\mathbf{1}_{\{\alpha,\, \alpha\}}} + b^2(\alpha). \tag{3.1}$$

Here $\mathbf{1}_{\{\alpha_1,\, \alpha_2\}} = \mathbb{E}_{\alpha_1}(\int_0^T (X_t^{\alpha_2})^2 \,\mathrm{d}t) > 0$ and it is continuous in $\Theta \times \Theta$.

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Remark 3.1. Based on continuous observations, Bo *et al.* [14] derived the maximum likelihood estimator $\hat{\alpha}_T^{\mathrm{MLE}}$ for an unknown drift parameter $\alpha \in (0,\ \infty)$ of the ROU process, and established strong consistency and asymptotic normality of this estimation. However, the classical Cramer–Rao lower bound for this MLE may not be attained through our result. In fact, by virtue of Theorem 2.1, we can define the log-likelihood function under the measure $\mathbb{P}_\alpha^T$ as

$$l_T(\alpha) = \log \frac{\mathrm{d}\mathbb{P}_\alpha^T}{\mathrm{d}\mathbb{P}_{\mathrm{W}}^T} = -\alpha \int_0^T X_t \,\mathrm{d}X_t - \frac{\alpha^2}{2} \int_0^T X_t^2 \,\mathrm{d}t + \alpha \int_0^T X_t \,\mathrm{d}L_t.$$

The estimator $\hat{\alpha}_T^{\mathrm{MLE}}$ is naturally defined as $\hat{\alpha}_T := \arg\sup_{\alpha\in\Theta} l_T(\alpha)$. Then we can derive the MLE of the parameter α by solving the likelihood equation $l_T'(\alpha) = 0$. On the other hand, it follows that

$$l_T'(\alpha) = -\int_0^T X_t \,\mathrm{d}W_t$$

and

$$l_T''(\alpha) = -\int_0^T X_t^2 \,\mathrm{d}t. \tag{3.2}$$

Thus, in view of Taylor's expansion, it follows that

$$l_T'(\alpha) = l_T'(\hat{\alpha}_T^{\mathrm{MLE}}) + (\alpha - \hat{\alpha}_T^{\mathrm{MLE}}) l_T''(\xi_T),$$

where $|\xi_T| \le 1$. Then

$$b(\alpha) = \mathbb{E}_\alpha(\hat{\alpha}_T^{\mathrm{MLE}} - \alpha) = \mathbb{E}_\alpha\left(\frac{l_T'(\alpha)}{\int_0^T X_t^2 \,\mathrm{d}t}\right),$$

which, together with (3.2), implies that

$$b'(\alpha) = \mathbb{E}_\alpha\left(\frac{l_T''(\alpha)}{\int_0^T X_t^2 \,\mathrm{d}t}\right) = -1.$$

Therefore, (3.1) turns out to be trivial and it is obvious to observe that the equality does not follow. Hence, the maximum likelihood estimator $\hat{\alpha}_T^{\mathrm{MLE}}$ may not attain this class of lower bound.

Remark 3.2. In the case of an unbiased estimator, the lower bound is the inverse of the Fisher information. Lee *et al.* [26] established the sequential maximum likelihood estimator (SMLE) $\hat{\alpha}_{\tau(H)}$ of the drift parameter in a one-dimensional ROU process and researched some important properties of $\hat{\alpha}_{\tau(H)}$. The SMLE is given by

$$\hat{\alpha}_{\tau(H)} = \frac{1}{H}\left[bL_{\tau(H)} - \int_0^{\tau(H)} X_s \,\mathrm{d}X_s\right],$$

where

$$\tau(H) = \inf\left\{t \ge 0 \;\middle|\; \int_0^t X_s^2 \,\mathrm{d}s \ge H\right\}, \qquad 0 < H < \infty.$$

In view of the results of Lee *et al.* [26], $\hat{\alpha}_{\tau(H)}$ is an unbiased estimation and

$$\mathbb{E}_\alpha\left(\int_0^{\tau(H)} X_t^2 \,\mathrm{d}t\right) = H, \qquad \mathbb{E}_\alpha(\hat{\alpha}_{\tau(H)} - \alpha)^2 = \frac{1}{H}.$$

Thus, the SMLE can attain the lower bound and, hence, is efficient in the mean-squared error sense, which was revealed by Lee *et al.* [26, Theorem 2.3].

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Proof of Theorem 3.1. Let α be the true parameter and

$$Z_T := \left(\frac{\mathrm{d}\mathbb{P}^T_{\alpha_2}}{\mathrm{d}\mathbb{P}^T_{\alpha_1}}\right)^{1/2},$$

where $\alpha_1, \alpha_2 \in \Theta$, and $\alpha_1 \neq \alpha_2$. Then, according to Theorem 2.1, we have

$$Z_T = \exp\left(-\frac{1}{2}\int_0^T (\alpha_2-\alpha_1)X_s^{\alpha_1}\,\mathrm{d}W_s - \frac{1}{4}\int_0^T (\alpha_2-\alpha_1)^2(X_s^{\alpha_1})^2\,\mathrm{d}s\right).$$

Denote $Z_T = \exp(Y(T))$, where

$$Y(T) = -\frac{1}{2}\int_0^T (\alpha_2-\alpha_1)X_s^{\alpha_1}\,\mathrm{d}W_s - \frac{1}{4}\int_0^T (\alpha_2-\alpha_1)^2(X_s^{\alpha_1})^2\,\mathrm{d}s.$$

By Itô's formula, we have

$$\begin{aligned}
\mathrm{d}Z(t) &= \exp(Y(T))\,\mathrm{d}Y(t) + \tfrac{1}{2}\exp(Y(T))\tfrac{1}{4}(\alpha_2-\alpha_1)^2(X_t^{\alpha_1})^2\,\mathrm{d}t\\
&= Z(t)\,\mathrm{d}Y(t) + \tfrac{1}{8}Z(t)(\alpha_2-\alpha_1)^2(X_t^{\alpha_1})^2\,\mathrm{d}t\\
&= Z(t)\big[-\tfrac{1}{2}(\alpha_2-\alpha_1)X_t^{\alpha_1}\,\mathrm{d}W_t - \tfrac{1}{4}(\alpha_2-\alpha_1)^2(X_t^{\alpha_1})^2\,\mathrm{d}t\\
&\qquad + \tfrac{1}{8}(\alpha_2-\alpha_1)^2(X_t^{\alpha_1})^2\,\mathrm{d}t\big]\\
&= -\frac{1}{8}(\alpha_2-\alpha_1)^2 Z_t(X_t^{\alpha_1})^2\,\mathrm{d}t - \frac{(\alpha_2-\alpha_1)}{2}Z_t X_t^{\alpha_1}\,\mathrm{d}W_t.
\end{aligned}$$

Hence, Z_T can be represented as

$$Z_T = 1 - \frac{1}{8}(\alpha_2-\alpha_1)^2\int_0^T Z_t(X_t^{\alpha_1})^2\,\mathrm{d}t - \frac{(\alpha_2-\alpha_1)}{2}\int_0^T Z_t X_t^{\alpha_1}\,\mathrm{d}W_t.$$

On the other hand, Z_t^2 is a martingale under the measure $\mathbb{P}^T_{\alpha_1}$. Furthermore, from the $(\mathcal{F}_t)_{t\geq 0}$-measurability of $X_t^{\alpha_1}$ for each $t \in [0,\ T]$ and Fubini's theorem, we have

$$\begin{aligned}
\mathbb{E}_{\alpha_1}\int_0^T Z_t^2(X_t^{\alpha_1})^2\,\mathrm{d}t &= \mathbb{E}_{\alpha_1}\int_0^T \mathbb{E}_{\alpha_1}(Z_T^2 \mid \mathcal{F}_t)(X_t^{\alpha_1})^2\,\mathrm{d}t\\
&= \mathbb{E}_{\alpha_1} Z_T^2\int_0^T (X_t^{\alpha_1})^2\,\mathrm{d}t\\
&= \int_{\mathcal{C}[0,T]} Z_T^2\left(\int_0^T (X_t^{\alpha_1})^2\,\mathrm{d}t\right)\mathrm{d}\mathbb{P}^T_{\alpha_1}\\
&= \int_{\mathcal{C}[0,T]}\left(\int_0^T (X_t^{\alpha_1})^2\,\mathrm{d}t\right)\mathrm{d}\mathbb{P}^T_{\alpha_2}\\
&= \mathbb{E}_{\alpha_2}\left(\int_0^T (X_t^{\alpha_1})^2\,\mathrm{d}t\right).
\end{aligned}$$

Meanwhile, by virtue of the maximal inequality and the existence of the second moment of $\sup_{r\in[0,\ T]}|W(r)|$, it is easy to see that, from (2.1),

$$\mathbb{E}_{\alpha_2}\left(\int_0^T (X_t^{\alpha_1})^2\,\mathrm{d}t\right) < \infty,$$

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i.e.

$$\mathbb{E}_{\alpha_1}\int_0^T Z_t^2 (X_t^{\alpha_1})^2\,\mathrm{d}t < \infty.$$

Hence,

$$\mathbb{E}_{\alpha_1}\int_0^T Z_t X_t^{\alpha_1}\,\mathrm{d}W(t) = 0.$$

Therefore, by the elementary inequality $a \le (a^2+1)/2$, we have

$$\begin{aligned}
\mathbb{E}_{\alpha_1} Z_T &= 1 - \frac{1}{8}(\alpha_2-\alpha_1)^2\mathbb{E}_{\alpha_1}\int_0^T Z_t (X_s^{\alpha_1})^2\,\mathrm{d}t \\
&\ge 1 - \frac{1}{16}(\alpha_2-\alpha_1)^2\mathbb{E}_{\alpha_1}\int_0^T (Z_t^2+1)(X_s^{\alpha_1})^2\,\mathrm{d}s \\
&= 1 - \frac{1}{16}(\alpha_2-\alpha_1)^2\mathbb{E}_{\alpha_1}\int_0^T (X_s^{\alpha_1})^2\,\mathrm{d}s - \frac{1}{16}(\alpha_2-\alpha_1)^2\mathbb{E}_{\alpha_1}\int_0^T Z_t^2 (X_s^{\alpha_1})^2\,\mathrm{d}s \\
&= 1 - \frac{1}{16}(\alpha_2-\alpha_1)^2\mathbb{E}_{\alpha_1}\int_0^T (X_s^{\alpha_1})^2\,\mathrm{d}s - \frac{1}{16}(\alpha_2-\alpha_1)^2\mathbb{E}_{\alpha_2}\int_0^T (X_s^{\alpha_1})^2\,\mathrm{d}s.
\end{aligned}$$

While, due to $\mathbb{E}_{\alpha_1}(Z_T^2 - 1) = 0$, we have

$$\begin{aligned}
(\mathbb{E}_{\alpha_2}(\hat\alpha_T) - \mathbb{E}_{\alpha_1}(\hat\alpha_T))^2 &= (\mathbb{E}_{\alpha_1}(Z_T^2\hat\alpha_T) - \mathbb{E}_{\alpha_1}(\hat\alpha_T))^2 \\
&= (\mathbb{E}_{\alpha_1}(Z_T^2 - 1)\hat\alpha_T)^2 \\
&= \big(\mathbb{E}_{\alpha_1}(Z_T^2-1)\big\{\hat\alpha_T - \tfrac{1}{2}(\mathbb{E}_{\alpha_1}(\hat\alpha_T) + \mathbb{E}_{\alpha_2}(\hat\alpha_T))\big\}\big)^2 \\
&\le \mathbb{E}_{\alpha_1}\big((1+Z_T)\big\{\hat\alpha_T - \tfrac{1}{2}(\mathbb{E}_{\alpha_1}(\hat\alpha_T) + \mathbb{E}_{\alpha_2}(\hat\alpha_T))\big\}\big)^2 \\
&\quad \times \mathbb{E}_{\alpha_1}(1-Z_T)^2 \quad \text{(by Hölder's inequality)} \\
&\le \Big\{2\mathbb{E}_{\alpha_1}\big\{\hat\alpha_T - \tfrac{1}{2}(\mathbb{E}_{\alpha_1}(\hat\alpha_T) + \mathbb{E}_{\alpha_2}(\hat\alpha_T))\big\}^2 \\
&\quad + 2\mathbb{E}_{\alpha_1} Z_T^2\big\{\hat\alpha_T - \tfrac{1}{2}(\mathbb{E}_{\alpha_1}(\hat\alpha_T) + \mathbb{E}_{\alpha_2}(\hat\alpha_T))\big\}^2\Big\}2(1-\mathbb{E}_{\alpha_1} Z_T) \\
&= \Big\{2\mathbb{E}_{\alpha_1}\big\{\hat\alpha_T - \tfrac{1}{2}(\mathbb{E}_{\alpha_1}(\hat\alpha_T) + \mathbb{E}_{\alpha_2}(\hat\alpha_T))bigr\}^2 \\
&\quad + 2\mathbb{E}_{\alpha_2}\big\{\hat\alpha_T - \tfrac{1}{2}(\mathbb{E}_{\alpha_1}(\hat\alpha_T) + \mathbb{E}_{\alpha_2}(\hat\alpha_T))\big\}^2\Big\}2(1-\mathbb{E}_{\alpha_1} Z_T).
\end{aligned}$$

Note that

$$\begin{aligned}
\mathbb{E}_{\alpha_1}\big\{\hat\alpha_T - \tfrac{1}{2}(\mathbb{E}_{\alpha_1}(\hat\alpha_T) + \mathbb{E}_{\alpha_2}(\hat\alpha_T))\big\}^2 &= \mathbb{E}_{\alpha_1}\big\{\hat\alpha_T - \mathbb{E}_{\alpha_1}(\hat\alpha_T) - \tfrac{1}{2}(\mathbb{E}_{\alpha_2}(\hat\alpha_T) - \mathbb{E}_{\alpha_1}(\hat\alpha_T))\big\}^2 \\
&= \mathbb{E}_{\alpha_1}\{\hat\alpha_T - \alpha_1\}^2 - b^2(\alpha_1) + \tfrac{1}{4}(\mathbb{E}_{\alpha_1}(\hat\alpha_T) - \mathbb{E}_{\alpha_2}(\hat\alpha_T))^2.
\end{aligned}$$

Similarly,

$$\mathbb{E}_{\alpha_2}\big\{\hat\alpha_T - \tfrac{1}{2}(\mathbb{E}_{\alpha_1}(\hat\alpha_T) + \mathbb{E}_{\alpha_2}(\hat\alpha_T))\big\}^2 = \mathbb{E}_{\alpha_2}\{\hat\alpha_T - \alpha_1\}^2 - b^2(\alpha_2) + \tfrac{1}{4}(\mathbb{E}_{\alpha_1}(\hat\alpha_T) - \mathbb{E}_{\alpha_2}(\hat\alpha_T))^2.$$

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Therefore, it follows that

$$\begin{aligned}
&(\mathbb{E}_{\alpha_2}(\hat{\alpha}_T) - \mathbb{E}_{\alpha_1}(\hat{\alpha}_T))^2 \\
&\quad \le \big\{2\mathbb{E}_{\alpha_1}\{\hat{\alpha}_T - \alpha_1\}^2 - 2b^2(\alpha_1) + \tfrac{1}{2}(\mathbb{E}_{\alpha_1}(\hat{\alpha}_T) - \mathbb{E}_{\alpha_2}(\hat{\alpha}_T))^2 \\
&\qquad + 2\mathbb{E}_{\alpha_2}\{\hat{\alpha}_T - \alpha_1\}^2 - 2b^2(\alpha_2) + \tfrac{1}{2}(\mathbb{E}_{\alpha_1}(\hat{\alpha}_T) - \mathbb{E}_{\alpha_2}(\hat{\alpha}_T))^2\big\} 2(1 - \mathbb{E}_{\alpha_1} Z_T) \\
&\quad \le \{2\mathbb{E}_{\alpha_1}\{\hat{\alpha}_T - \alpha_1\}^2 - 2b^2(\alpha_1) + 2\mathbb{E}_{\alpha_2}\{\hat{\alpha}_T - \alpha_1\}^2 - 2b^2(\alpha_2) \\
&\qquad + (\mathbb{E}_{\alpha_1}(\hat{\alpha}_T) - \mathbb{E}_{\alpha_2}(\hat{\alpha}_T))^2\} \\
&\qquad \times \left(\frac{1}{8}(\alpha_2 - \alpha_1)^2 \mathbb{E}_{\alpha_1} \int_0^T (X_s^{\alpha_1})^2 \, \mathrm{d}s + \frac{1}{8}(\alpha_2 - \alpha_1)^2 \mathbb{E}_{\alpha_2} \int_0^T (X_s^{\alpha_1})^2 \, \mathrm{d}s\right).
\end{aligned}$$

Let $\alpha_2 = \alpha$. Then

$$\begin{aligned}
&(\mathbb{E}_{\alpha}(\hat{\alpha}_T) - \mathbb{E}_{\alpha_1}(\hat{\alpha}_T))^2 \\
&\quad \le \{2\mathbb{E}_{\alpha_1}\{\hat{\alpha}_T - \alpha_1\}^2 - 2b^2(\alpha_1) + 2\mathbb{E}_{\alpha}\{\hat{\alpha}_T - \alpha_1\}^2 - 2b^2(\alpha) + (\mathbb{E}_{\alpha_1}(\hat{\alpha}_T) - \mathbb{E}_{\alpha}(\hat{\alpha}_T))^2\} \\
&\qquad \times \left(\frac{1}{8}(\alpha - \alpha_1)^2 \mathbb{E}_{\alpha_1} \int_0^T (X_s^{\alpha_1})^2 \, \mathrm{d}s + \frac{1}{8}(\alpha - \alpha_1)^2 \mathbb{E}_{\alpha} \int_0^T (X_s^{\alpha_1})^2 \, \mathrm{d}s\right).
\end{aligned}$$

On the other hand,

$$(\mathbb{E}_{\alpha}(\hat{\alpha}_T) - \mathbb{E}_{\alpha_1}(\hat{\alpha}_T))^2 = (b(\alpha_1) - b(\alpha) + \alpha_1 - \alpha)^2.$$

Hence, divide the inequality by $(\alpha_1 - \alpha)^2$ on both sides and let $\alpha_1 \to \alpha$. Then it follows that

$$\mathbb{E}_{\alpha}(\hat{\alpha}_T - \alpha)^2 \ge \frac{(1 + b'(\alpha))^2}{\mathbf{1}_{\{\alpha,\, \alpha\}}} + b^2(\alpha),$$

which completes the proof. □

4. Conclusion

Based on the continuous observations of $\{X_t,\ t \ge 1\}$, the main findings in this paper concern the lower bound of estimation of the unknown parameter in ROU processes. Our main result holds for both unbiased and biased estimation. However, as shown in Remark 3.2, the MLE derived by Bo *et al.* [14] cannot attain our lower bound. Investigations on more statistical properties related to the estimation can be regarded as a future research topic, for example, a similar topic for our model based on discrete observations, as well as its consistency and asymptotic distribution (see, e.g. Hu *et al.* [20]).

On the other hand, some future work may investigate some other estimators for the other reflected diffusions. For example, Lee *et al.* [26] proposed an SMLE of the unknown drift of the ROU process without jumps; the reflected jump-diffusion or Levy processes have been extensively investigated in the literature; see [1], [2], [4]–[6], [10]–[12], and [36]); other authors are concerned with the problem of statistical parameter estimation for reflected fractional Brownian motion; see [19] and [25]).

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