Proc. Camb. Phil. Soc. (1967), **63**, 1141
PCPS 63–133
Printed in Great Britain

A class of infinitely divisible random variables

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(*Received* 16 *November* 1966)

The question has arisen in the theory of queues whether the product of two independent exponentially distributed random variables is infinitely divisible. In this note it is proved that the product of any non-negative random variable with an independent exponentially distributed variable is infinitely divisible. Such random variables are members of a wider class of infinitely divisible random variables, which will be exhibited first.

Given a non-negative random variable Z we shall refer to its Laplace transform $w(\theta) = E\{\exp(-\theta Z)\}$, for positive θ. The function w is analytic at θ,

$$w(\theta+h) = \sum_0^\infty (-)^n b_n(\theta) h^n \quad (|h| < \theta), \tag{1}$$

$$b_n(\theta) = E\{Z^n \exp(-\theta Z)\}/n! \quad (n \geqslant 0). \tag{2}$$

Since w has no positive real zeros, the function $r = -w'/w$ is also analytic at each positive θ,

$$r(\theta+h) = \sum_0^\infty (-)^n a_n(\theta) h^n. \tag{3}$$

The basic class of infinitely divisible variables is now established very simply in the following theorem. The condition of the theorem is equivalent to the assertion that *for each positive θ there exists $k(\theta)$ such that $\{k^n b_n/b_0\}$ is a renewal sequence.* (Renewal sequences are defined, for instance, in (6).)

Theorem 1. *If for each positive θ the quantities $f_n(\theta)$ $(n > 0)$, obtained successively from the relations*

$$\sum_{i=0}^{n-1} b_i f_{n-i} = b_n \quad (n > 0), \tag{4}$$

are all non-negative, then Z is infinitely divisible.

Proof. Clearly,

$$\sum_1^\infty (-)^n f_n h^n = 1 - [w(\theta+h)]^{-1} b_0 \quad (|h| < \theta),$$

so that $$r(\theta+h) = \left(\sum_0^\infty (-)^n (n+1) f_{n+1} h^n\right) \Big/ \left(1 - \sum_1^\infty (-)^n f_n h^n\right) \quad (|h| < \theta).$$

Therefore the coefficients a_n must satisfy the relations

$$a_0 = f_1,$$

$$a_n = \sum_{i=0}^{n-1} a_i f_{n-i} + (n+1) f_{n+1} \quad (n > 0),$$

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which show (by induction) that a_n is positive for all n, and so r is completely monotone. Thus $w = \exp(-u)$, where u has a completely monotone derivative. A standard theorem (for example, see Feller (1), p. 425) then proves that Z is infinitely divisible.

Attention will now be restricted to those non-negative variables whose sequences $\{b_n\}$ satisfy the condition

$$b_{n+1}b_{n-1} \geqslant b_n^2 \quad (n > 0, \quad \theta > 0). \tag{5}$$

That these random variables satisfy the conditions of Theorem 1 is a consequence of a theorem of Kaluza (2), as quoted by Lamperti (3). Alternatively, the next theorem gives an elementary direct proof that they are infinitely divisible.

THEOREM 2. *If the non-negative random variable Z satisfies*

$$E(Z^{n+1}e^{-\theta Z})\,E(Z^{n-1}e^{-\theta Z}) \geqslant (1+n^{-1})\{E(Z^n e^{-\theta Z})\}^2 \quad (n > 0, \quad \theta > 0), \tag{6}$$

then it is infinitely divisible.

Proof. The condition may be written $b_{n+1}/b_n \geqslant b_n/b_{n-1}$, which implies that

$$b_{n+1}/b_n \geqslant b_{i+1}/b_i \quad (0 \leqslant i < n).$$

Equating coefficients of powers of h in the identity

$$-w'(\theta+h) = w(\theta+h)\,r(\theta+h)$$

yields the relations

$$(n+1)b_{n+1} = \sum_{i=0}^{n} a_i b_{n-i} \quad (n \geqslant 0).$$

Suppose $a_i > 0$ $(0 \leqslant i \leqslant n)$, then

$$\begin{aligned}(n+2)b_{n+2} &> (b_{n+2}/b_{n+1})\,(n+1)b_{n+1}\\ &= (b_{n+2}/b_{n+1}) \sum_{i=0}^{n} a_i b_{n-i}\\ &\geqslant \sum_{i=0}^{n} a_i (b_{n-i+1}/b_{n-i})\,b_{n-i}\\ &= (n+2)b_{n+2} - a_{n+1}b_0,\end{aligned}$$

so that $a_{n+1} > 0$. Therefore, by induction, $a_n > 0$ $(n \geqslant 0)$. The result follows, as in Theorem 1.

If P is any probability distribution on $[0, \infty)$ then

$$w(\theta) = \int (1+\theta x)^{-1} P(dx) \quad (\theta > 0), \tag{7}$$

is clearly the Laplace transform of a non-negative variable, and we have (5), which is equivalent to (6), at once from the Cauchy–Schwartz inequality for integrals. This proves the next result.

COROLLARY. *If X is a non-negative random variable and Y is an independent exponentially distributed variable, then $Z = XY$ is infinitely divisible.*

Keilson and Runnenberg proved (4) that the stationary waiting time for the first-come, first-served $GI/G/1$ queue is infinitely divisible. Runnenberg then asked (4)

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whether the waiting times for other queue disciplines were infinitely divisible, in particular the limiting distribution in heavy traffic for the queue with service in random order. This is the distribution of the product of two independent unit exponential variables (Kingman (5), and Le Gall (7), for $M/G/1$; Le Gall (7), for $GI/M/1$) and so it *is* infinitely divisible.

My thanks are due to Prof. J. F. C. Kingman for his encouragement and suggestions. The work was carried out under a research studentship from the Science Research Council.

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