Math. Proc. Camb. Phil. Soc. (1988), **103**, 321
Printed in Great Britain

Fourier–Laplace transforms decaying exponentially

By H. KANETA
Department of Mathematics, Okayama University, Okayama 700, Japan

DEDICATED TO PROFESSOR TAKEYUKI HIDA ON HIS 60TH BIRTHDAY

(*Received* 19 *May* 1987)

1. Let $f(x)$ be a $\mathbf{C}$-valued function which is integrable with respect to a positive measure m on the Borel σ-field of $\mathbf{R} = (-\infty, \infty)$. We shall give necessary and sufficient conditions under which the following Fourier-Laplace transform $\tilde{f}(\lambda)$ of $f(x)$ decays exponentially as $\lambda \to \infty$:

$$\tilde{f}(\lambda) = \int_{[0,\infty)} f(x) \exp(-\lambda x e^{i\theta})\, m(dx),$$

where θ is a constant with $-\pi/2 \leqslant \theta \leqslant \pi/2$.

THEOREM. *The Fourier–Laplace transform decays exponentially, that is,* $\tilde{f}(\lambda) = O(e^{-\delta\lambda})$ *for some positive constant* δ *as* $\lambda \to \infty$, *if and only if* $f(x) = 0$ *almost everywhere on the interval* $[0, \delta/\cos\theta)$, *where* $\delta/0$ *stands for* ∞.

Remark 1. In case $\theta = 0$ and $f(x) \geqslant 0$, the celebrated Tauberian theorem says, roughly speaking, that $\tilde{f}(\lambda) = O(\lambda^{-\delta})$ for some $\delta > 0$ as $\lambda \to \infty$ if and only if $\int_{[0,x)} f(t)\, m(dt) = O(x^{\delta})$ as $x \to +0$ (see [1], p. 445).

Remark 2. In case $\theta = \pm\pi/2$, a much more general theorem holds. Namely, assume that $F(\lambda) = \int_{\mathbf{R}} f(x) \exp(-i\lambda x)\, m(dx)$ decays exponentially as $\lambda \to \infty$ and that $f(x)$ vanishes on an interval. Then $F(\lambda)$ vanishes identically on $\mathbf{R}$, hence $f(x)$ equals 0 almost everywhere on $\mathbf{R}$. This assertion is a special case of Levinson's theorem XXVII ([3], p. 84). Indeed, since $\int_{\mathbf{R}} F(\lambda) \exp(-\epsilon|\lambda| + i\lambda y)\, d\lambda$ is equal to $\int_{\mathbf{R}} 2\epsilon f(x)/(\epsilon^2 + (x-y)^2)\, m(x)$, the function $F(\lambda)$ above turns out to satisfy the condition of Levinson's theorem XXVII.

2. *Proof.* In view of Remark 2 we may confine ourselves to the case $|\theta| < \pi/2$. It suffices to show that the condition in the Theorem is necessary. Assume that $\tilde{f}(\lambda) = O(e^{-\delta\lambda})$ as $\lambda \to \infty$. Representing $\tilde{f}(\lambda) \exp(\lambda d e^{i\theta})$ (where $d = \delta/\cos\theta$) as

$$\int_{[0,d)} f(x) \exp(\lambda e^{i\theta}(d-x))\, m(dx) + \int_{[d,\infty)} f(x) \exp(\lambda e^{i\theta}(d-x))\, m(dx),$$

we denote the first integral by $G(\lambda)$, which is clearly entire in λ with $G(\lambda) = O(e^{a|\lambda|})$. By the assumption $G(\lambda)$ is bounded as $\lambda \to +\infty$, since so is the second integral above. On the other hand $G(\lambda)$ tends to zero as $\lambda \to -\infty$, since the integrand of $G(\lambda)$ converges to zero. In particular $G(\lambda)$ is bounded on the real line $\mathbf{R}$. We can easily see that $G(\lambda)$ is bounded on the line $\{te^{-i\theta} : t \in \mathbf{R}\}$, which differs from $\mathbf{R}$. We can now apply the Phragmen–Lindelöf theorem (see [2], p. 177) to $G(\lambda)$ to conclude that $G(\lambda)$ is

https://doi.org/10.1017/S0305004100064896 Published online by Cambridge University Press

bounded on the whole plane. Thus the entire function $G(\lambda)$ is a constant, which must equal zero (recall that $G(\lambda)$ tends to zero as $\lambda \to -\infty$). Differentiating $G(\lambda)$ at $\lambda = 0$, we obtain the following equality:

$$\int_{[0,d)} x^n f(x)\, m(dx) = 0 \quad \text{for} \quad n = 0, 1, 2, \ldots.$$

Thus $f(x)$ vanishes almost everywhere on $[0, d)$, as desired.

REFERENCES

[1] W. FELLER. *An Introduction to Probability Theory and its Applications*, Vol. II, 2nd ed. (Wiley, 1971).
[2] E. C. TITCHMARSH. *The Theory of Functions*, 2nd ed. (Oxford University Press, 1933).
[3] N. LEVINSON. *Gap and Density Theorems* (Amer. Math. Soc., 1940).

https://doi.org/10.1017/S0305004100064896 Published online by Cambridge University Press