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ASYMPTOTIC BEHAVIOUR NEAR EXTINCTION OF CONTINUOUS-STATE BRANCHING PROCESSES

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Abstract

In this paper we study the asymptotic behaviour near extinction of (sub-)critical continuous-state branching processes. In particular, we establish an analogue of Khintchine's law of the iterated logarithm near extinction time for a continuous-state branching process whose branching mechanism satisfies a given condition.

Keywords: Continuous-state branching process; Lamperti transform; Lévy process; conditioning to stay positive; rate of growth

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1. Introduction and main results

A continuous-state branching process (or CB-process for short) is a nonnegative-valued strong Markov process with probabilities $(\mathbb{Q}_x, x \geq 0)$ such that, for any $x, y \geq 0$, $\mathbb{Q}_{x+y}$ is equal in law to the convolution of $\mathbb{Q}_x$ and $\mathbb{Q}_y$. CB-processes were introduced by Jirina [6] and studied by many authors including Bingham [2], Grey [4], Grimvall [5], Lamperti [8], to name but a few. This class of processes may be thought of as the continuous (in time and space) analogues of classical Bienaymé–Galton–Watson branching processes. More precisely, a CB-process $Y = (Y_t, t \geq 0)$ is a Markov process taking values in $[0, \infty]$, where 0 and ∞ are two absorbing states, and satisfies the branching property; that is to say, its Laplace transform is given by

$$\mathbb{Q}_x[\mathrm{e}^{-\lambda Y_t}] = \mathrm{e}^{\{-xu_t(\lambda)\}} \quad \text{for } \lambda \geq 0, \tag{1.1}$$

for some function $u_t(\lambda)$. According to Silverstein [10], the function $u_t(\lambda)$ is determined by the integral equation

$$\int_{u_t(\lambda)}^{\lambda} \frac{1}{\psi(u)}\,\mathrm{d}u = t, \tag{1.2}$$

where ψ satisfies the celebrated Lévy–Khincthine formula

$$\psi(\lambda) = a\lambda + \beta\lambda^2 + \int_{(0,\infty)} (\mathrm{e}^{-\lambda x} - 1 + \lambda x \mathbf{1}_{\{x<1\}})\Pi(\mathrm{d}x), \tag{1.3}$$

where $a \in \mathbb{R}$, $\beta \geq 0$, and Π is a σ-finite measure such that $\int_{(0,\infty)}(1 \wedge x^2)\Pi(\mathrm{d}x)$ is finite. The function ψ is known as the branching mechanism of Y.

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Note that the first moment of Y_t can be obtained by differentiating (1.1) with respect to λ, i.e. $\mathbb{Q}_x[Y_t] = x\mathrm{e}^{-\psi'(0^+)t}$. Hence, in respective order, a CB-process is called supercritical, critical, or subcritical depending on $\psi'(0^+) < 0$, $\psi'(0^+) = 0$, or $\psi'(0^+) > 0$. Moreover, since

$$\mathbb{Q}_x\Big(\lim_{t\to\infty} Y_t = 0\Big) = \mathrm{e}^{-\eta x},$$

where η is the largest root of the branching mechanism ψ, the sign of $\psi'(0+)$ yields the criterion for almost sure extinction. More precisely, a CB-process Y with branching mechanism ψ has a finite extinction time almost surely (a.s.) if and only if

$$\int^{\infty} \frac{\mathrm{d}u}{\psi(u)} < \infty \quad \text{and} \quad \psi'(0+) \geq 0. \tag{1.4}$$

We denote by T_0 the extinction time of the CB-process Y, i.e. $T_0 := \inf\{t \geq 0 \colon Y_t = 0\}$.

In this paper we are interested in a detailed description of how CB-processes become extinct. Hence, in what follows we always assume that (1.4) is satisfied. One of the starting points of this paper are the results of Kyprianou and Pardo [7] for the CB-process in the self-similar case, i.e. when the branching mechanism is given by $\psi(\lambda) = c_+\lambda^\alpha$ for $1 < \alpha \leq 2$ and $c_+ > 0$. Observe that such a branching mechanism clearly satisfies (1.4). The authors in [7] described the upper and lower envelopes of the self-similar CB-process near extinction via integral tests. In particular, they obtained the following laws of the iterated logarithm (LIL) for the upper envelope of Y and its version reflected at its running infimum:

$$\limsup_{t\to 0} \frac{Y_{(T_0-t)^-}}{t^{1/(\alpha-1)}\log\log(1/t)} = c_+(\alpha-1)^{1/\alpha-1}, \qquad \mathbb{Q}_x\text{-a.s.}$$

and

$$\limsup_{t\to 0} \frac{(Y-\underline{Y})_{(T_0-t)^-}}{t^{1/(\alpha-1)}\log\log(1/t)} = c_+(\alpha-1)^{1/\alpha-1}, \qquad \mathbb{Q}_x\text{-a.s.},$$

where $\underline{Y}_t$ denotes the infimum of the CB-process $(Y, \mathbb{Q}_x)$ over $[0, t]$.

In order to state our main results, we first introduce some basic notation. Let us define the mapping

$$\phi(t) := \int_t^{\infty} \frac{\mathrm{d}u}{\psi(u)} \quad \text{for } t > 0,$$

and note that $\phi\colon (0,\infty) \to (0,\infty)$ is a bijection and thus its inverse exists, here denoted by φ. From (1.2), it is straightforward to obtain

$$u_t(\lambda) = \varphi(t + \phi(\lambda)), \qquad \lambda, t > 0.$$

Since $\phi(\infty) = 0$, we clearly have $u_t(\infty) = \varphi(t)$ for $t > 0$. Hence, we deduce that, for every $x, t > 0$,

$$\mathbb{Q}_x(T_0 \leq t) = \mathbb{Q}_x(Y_t = 0) = \lim_{\lambda\to\infty} \mathbb{Q}_x[\mathrm{e}^{-\lambda Y_t}] = \mathrm{e}^{-x\varphi(t)}.$$

Finally, following the ideas in [3], we introduce the following exponent:

$$\delta := \sup\{c \geq 0 \colon \text{ there exists } Q \in (0,\infty) \text{ such that } Q\psi(u)u^{-c} \leq \psi(v)v^{-c},\ 1 \leq u \leq v\}. \tag{1.5}$$

Observe that we necessarily have $1 \leq \delta \leq 2$. According to [3, Lemma 2.2], if ψ is regularly varying at ∞ with index $\alpha > 1$ then $\delta = \alpha$.

Our first result consists of a LIL at 0 for the upper envelope of the time-reversal process $(Y_{(T_0-t)^-}, 0 \leq t \leq T_0)$ under $\mathbb{Q}_x$.

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Theorem 1.1. *Assume that $\delta > 1$, then*

$$\limsup_{t\to 0} \frac{Y_{(T_0-t)^-}\varphi(t)}{\log\log\varphi(t)} = 1, \qquad \mathbb{Q}_x\text{-a.s.}$$

for every $x > 0$.

Recall that $\underline{Y}$ and $((Y-\underline{Y})_{(T_0-t)^-}, 0 \le t < T_0)$ denote the running infimum of Y and the time-reversal process of Y reflected at its running minimum, respectively. We also introduce the so-called scale function $W\colon [0,\infty) \mapsto [0,\infty)$ which is the unique absolutely continuous increasing function whose Laplace transform is $1/\psi$. Let us suppose that, for all $\beta < 1$, the scale function W satisfies the following hypothesis:

$$\limsup_{x\to 0} \frac{W(\beta x)}{W(x)} < 1. \tag{1.6}$$

The above hypothesis is satisfied, in particular, when ψ is regularly varying at ∞.

Theorem 1.2. *Suppose that $\delta > 1$, then under the hypothesis (1.6), we have*

$$\limsup_{t\to 0} \frac{(Y-\underline{Y})_{(T_0-t)^-}\varphi(t)}{\log\log\varphi(t)} = 1, \qquad \mathbb{Q}_x\text{-a.s.}$$

for every $x > 0$.

It is important to note that in the self-similar case, i.e. when $\psi(\lambda) = c_+\lambda^\alpha$ for $1 < \alpha \le 2$ and $c_+ > 0$, we have

$$\frac{\log\log\varphi(t)}{\varphi(t)} = \frac{\log\log c_+(\alpha-1)t)^{-1/(\alpha-1)}}{(c_+(\alpha-1)t)^{-1/(\alpha-1)}} \quad \text{for } t > 0.$$

Thus, we can replace the above function by

$$\frac{\log\log(1/t)}{\varphi(t)} = (c_+(\alpha-1)t)^{1/(\alpha-1)}\log\log\frac{1}{t} \quad \text{for } t > 0,$$

since they are asymptotically equivalent at 0. Similarly, we can take the previous function in the regularly varying case.

The rest of this paper is devoted to the proof of the main results. Our arguments are similar to those used in [1] for the asymptotic behaviour of subordinators.

2. Proofs

In what follows, we assume that the branching mechanism ψ is of the form (1.3) and satisfies (1.4). Under these assumptions, the Lévy–Itô decomposition (see, for instance, [1, Theorem I.1]) implies that there exists a Lévy process with no negative jumps whose Laplace exponent is given by ψ and does not drift to $+\infty$; see, for instance, [1, Corollary VII.2]. More precisely, let X denote the trajectory of such a process and $\mathbb{P}$ its law issued from the origin, then

$$\mathbb{E}[\mathrm{e}^{-\lambda X_t}] = \mathrm{e}^{t\psi(\lambda)} \quad \text{for } \lambda \in \mathbb{R},$$

and $\lim_{t\to\infty} X_t = -\infty$ or oscillates. As X is necessarily a strong Markov process, for each $x \in \mathbb{R}$, we will need the probability $\mathbb{P}_x$ to denote the law of X when issued from x.

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Henceforth, we shall assume that $(X, \mathbb{P})$ is not a subordinator (recall that a subordinator is a Lévy process with increasing sample paths). In that case, it is known that the Laplace exponent ψ is strictly convex and tends to $+\infty$ as λ goes to ∞. In this case, we define, for $q \geq 0$,

$$\Phi(q) = \inf\{\lambda \geq 0 \colon \psi(\lambda) > q\}$$

the right-continuous inverse of ψ and then $\Phi(0)$ is the largest root of $\psi(\lambda) = 0$. Theorem VII.1 in [1] implies that under condition (1.4), we necessarily have $\Phi(0) = 0$.

Recall that CB-processes are connected to Lévy processes with no negative jumps by a simple time-change; see [8]. More precisely, consider a spectrally positive Lévy process $(X, \mathbb{P}_x)$ started at $x > 0$ and with Laplace exponent ψ. Now, we introduce the clock

$$A_t = \int_0^t \frac{\mathrm{d}s}{X_s}, \qquad t \in [0, \tau_0),$$

where $\tau_0 = \inf\{t \geq 0 \colon X_t \leq 0\}$, and its right-continuous inverse

$$\theta(t) = \inf\{s \geq 0 \colon A_s > t\}.$$

Then, the time-changed process $(X_{\theta(t)}, t \geq 0)$, under $\mathbb{P}_x$, is a CB-process with initial population of size x and branching mechanism given by ψ. The transformation described above will henceforth be referred to as the *CB-Lamperti representation*.

Now, define $\widehat{X} := -X$, the dual process of X and denote by $\widehat{\mathbb{P}}_x$ the law of $\widehat{X}$ when issued from x, so that $(X, \widehat{\mathbb{P}}_x) = (\widehat{X}, \mathbb{P}_{-x})$. For our purpose, we are also interested in the dual process conditioned to stay positive which can be defined as a Doob h-transform of $(X, \widehat{\mathbb{P}}_x)$ killed when it first exits $(0, \infty)$ with the harmonic function W, the so-called scale function. Recall that the scale function $W \colon [0, \infty) \to [0, \infty)$ is the unique absolutely increasing function with Laplace transform

$$\int_0^\infty \mathrm{e}^{-\lambda x} W(x)\, \mathrm{d}x = \frac{1}{\psi(\lambda)}, \qquad \lambda \geq 0;$$

see, for instance, [1, Theorem VII.8]. More precisely, since $\psi'(0+) \geq 0$, we can define a new probability measure

$$\widehat{\mathbb{P}}_x^{\uparrow}(\Lambda) = \frac{1}{W(x)} \widehat{\mathbb{E}}_x[W(X_t), \Lambda, t < \tau_0], \qquad \Lambda \in \mathcal{F}_t,$$

where $\mathcal{F}_t$ denotes the $\widehat{\mathbb{P}}$-completed sigma-field generated by $(X_s, s \leq t)$.

Under $\widehat{\mathbb{P}}_x^{\uparrow}$, X is a process taking values in $(0, \infty)$ satisfying the strong Markov property and its semigroup is given by

$$\widehat{\mathbb{P}}_x^{\uparrow}(X_t \in \mathrm{d}y) = \frac{W(y)}{W(x)} \widehat{\mathbb{P}}_x(X_t \in \mathrm{d}y, t < \tau_0), \qquad t \geq 0,\ x, y > 0.$$

It will be referred to as the dual Lévy process started at x and conditioned to stay positive. Moreover, the probability measures $\widehat{\mathbb{P}}_x^{\uparrow}$ converge weakly as $x \downarrow 0$ to a probability measure which we denote by $\widehat{\mathbb{P}}^{\uparrow}$; see [1, Proposition VII.14]. We refer the reader to [1, Section VII.3] for more details of the process conditioned to stay positive and for the proofs of the properties that we stated here.

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Let

$$\sigma_x = \sup\{t > 0 \colon X_t \le x\}$$

be the last passage time of X below $x \in \mathbb{R}$. The next proposition, whose proof may be found in [7], gives us a time-reversal property from extinction for the CB-process; see [7, Theorem 1, Proposition 4, and Remark 2]. Recall that, under $\widehat{\mathbb{P}}^{\uparrow}$, the canonical process X drifts towards ∞ and also that $X_t > 0$ for $t > 0$ a.s.

Proposition 2.1. *If (1.4) holds then, for each $x > 0$,*

$$\{(Y_{(T_0-t)^-} \colon 0 \le t < T_0), \mathbb{Q}_x\} \overset{\mathrm{D}}{=} \{(X_{\theta(t)}, 0 \le t < A_{\sigma_x}), \widehat{\mathbb{P}}^{\uparrow}\},$$

where '$\overset{\mathrm{D}}{=}$' denotes equality in distribution.

Recall that ψ is a branching mechanism satisfying (1.3) and (1.4); and whose exponent δ defined by (1.5) is strictly larger than 1. Therefore, since $\delta > 1$, there exist $c \in (1, \infty)$ and $C \in (0, \infty)$ such that

$$\psi(\lambda) \le C\psi(\lambda u)u^{-c} \quad \text{for any } u, \lambda \in [1, \infty). \tag{2.1}$$

Now, we introduce the so-called first and last passage times of the process $\{(X_{\theta(t)}, t \ge 0), \widehat{\mathbb{P}}^{\uparrow}\}$ by

$$S_y = \inf\{t \ge 0 \colon X_{\theta(t)} \ge y\} \quad \text{and} \quad U_y = \sup\{t \ge 0 \colon X_{\theta(t)} \le y\} \quad \text{for } y \ge 0.$$

Note that the processes $(S_y, y \ge 0)$ and $(U_y, y \ge 0)$ are increasing with independent increments since the process $\{(X_{\theta(t)}, t \ge 0), \widehat{\mathbb{P}}^{\uparrow}\}$ has no positive jumps.

Observe that A_{σ_x} and U_x are equal and that the latter, under $\widehat{\mathbb{P}}^{\uparrow}$, has the same law as T_0 under $\mathbb{Q}_x$. This clearly implies that the distribution of U_x, under $\widehat{\mathbb{P}}^{\uparrow}$, satisfies

$$\widehat{\mathbb{P}}^{\uparrow}(U_x \le t) = \mathrm{e}^{-x\varphi(t)} \quad \text{for every } x, t > 0. \tag{2.2}$$

Let $(J_t, t \ge 0)$ be the future infimum process of $(X_{\theta(t)}, t \ge 0)$ which is defined as

$$J_t = \inf_{s \ge t} X_{\theta(s)} \quad \text{for } t \ge 0.$$

Proposition 2.2. *Assume that $\delta > 1$, then*

$$\limsup_{t \to 0} \frac{J_t \varphi(t)}{\log\log \varphi(t)} = 1, \qquad \widehat{\mathbb{P}}^{\uparrow}\text{-a.s.}$$

For simplicity, we let

$$f(t) = \frac{\log\log \varphi(t)}{\varphi(t)} \quad \text{for } t > 0.$$

In order to prove this result, we need the following two technical lemmas. Recall that ϕ is the inverse function of φ.

Lemma 2.1. *For every integer $n \ge 1$ and $r > 1$, put $t_n = \phi(r^n)$ and $a_n = f(t_n)$.*

(i) *The sequence $(t_n \colon n \ge 1)$ decreases.*

(ii) *The series $\sum_n \widehat{\mathbb{P}}^{\uparrow}(J_{t_n} > r a_n)$ converges.*

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Proof. The first assertion follows readily from the fact that ϕ is decreasing. In order to prove Lemma 2.1(ii), we note that, for $n \geq 1$, $\varphi(t_n) = \varphi(\phi(r^n)) = r^n$. This entails

$$\log\log\varphi(t_n) = \log\log r^n. \tag{2.3}$$

Now, since the last passage times process is the right-continuous inverse of the future infimum of $(X_{\theta(t)}, t \geq 0)$, we have

$$\widehat{\mathbb{P}}^{\uparrow}(J_{t_n} > ra_n) \leq \widehat{\mathbb{P}}^{\uparrow}(U_{ra_n} \leq t_n).$$

Hence, (2.2) and (2.3) imply that

$$\sum_n \widehat{\mathbb{P}}^{\uparrow}(U_{ra_n} \leq t_n) = \sum_n \left(\frac{1}{n\log r}\right)^r,$$

which converges, and our statement follows. □

Lemma 2.2. *For every integer $n \geq 2$ and $r > 1$, put $s_n = \phi(\mathrm{e}^{n^r})$ and $b_n = f(s_n)$. Then the series $\sum_n \widehat{\mathbb{P}}^{\uparrow}(U_{b_n/r} \leq s_n)$ diverges.*

Proof. First we note that, for $n \geq 1$, $\varphi(s_n) = \varphi(\phi(\mathrm{e}^{n^r})) = \mathrm{e}^{n^r}$. This entails $\log\log\varphi(s_n) = \log n^r$. Hence, (2.2) implies that

$$\sum_n \widehat{\mathbb{P}}^{\uparrow}(U_{b_n/r} \leq s_n) = \sum_n \frac{1}{n},$$

which diverges, and our claim follows. □

We are now able to establish the LIL of Proposition 2.2.

Proof of Proposition 2.2. We fix $r > 1$ and consider the sequences $(t_n : n \geq 1)$, $(a_n : n \geq 1)$, $(s_n : n \geq 1)$, and $(b_n : n \geq 1)$ as in Lemmas 2.1 and 2.2. We first prove the upper bound using Lemma 2.1. We fix $t \in [t_{n+1}, t_n]$, so, provided that n is large enough,

$$f(t) \geq \frac{\log\log\varphi(t_n)}{\varphi(t_{n+1})},$$

since φ decreases. Note that the denominator is equal to r^{n+1} and the numerator is equal to $\log\log r^n$. We thus have

$$\limsup_{t\to 0} \frac{f(t_n)}{f(t)} \leq r. \tag{2.4}$$

On the other hand, an application of the Borel–Cantelli lemma to Lemma 2.1 shows that

$$\limsup_{n\to\infty} \frac{J_{t_n}}{f(t_n)} \leq r, \qquad \widehat{\mathbb{P}}^{\uparrow}\text{-a.s.},$$

and we deduce from the monotonicity of J that

$$\limsup_{t\to 0} \frac{J_t}{f(t)} \leq \left(\limsup_{n\to\infty} \frac{J_{t_n}}{f(t_n)}\right)\left(\limsup_{t\to 0} \frac{f(t_n)}{f(t)}\right) \leq r^2, \qquad \widehat{\mathbb{P}}^{\uparrow}\text{-a.s.}$$

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To prove the lower bound, we use Lemma 2.2 and observe that the sequence $(b_n, n \geq 2)$ decreases. First, from Lemma 2.2, we have

$$\sum_n \widehat{\mathbb{P}}^{\uparrow}(U_{b_n/r} - U_{b_{n+1}/r} \leq s_n) \geq \sum_n \widehat{\mathbb{P}}^{\uparrow}(U_{b_n/r} \leq s_n) = \infty,$$

so by the Borel–Cantelli lemma for independent events, we obtain

$$\liminf_{n\to\infty} \frac{U_{b_n/r} - U_{b_{n+1}/r}}{s_n} \leq 1, \qquad \widehat{\mathbb{P}}^{\uparrow}\text{-a.s.}$$

If we admit for a while that

$$\limsup_{n\to\infty} \frac{U_{b_{n+1}/r}}{s_n} = 0, \qquad \widehat{\mathbb{P}}^{\uparrow}\text{-a.s.}, \tag{2.5}$$

we can conclude that

$$\liminf_{n\to\infty} \frac{U_{b_n/r}}{s_n} \leq 1, \qquad \widehat{\mathbb{P}}^{\uparrow}\text{-a.s.}$$

This implies that the set $\{1/s\colon U_{f(s)/r} \leq s\}$ is unbounded $\widehat{\mathbb{P}}^{\uparrow}$-a.s. Plainly, the same then holds for $\{1/s\colon J_s \geq f(s)/r\}$, and as a consequence

$$\limsup_{t\to 0} \frac{J_t}{f(t)} \geq \frac{1}{r}, \qquad \widehat{\mathbb{P}}^{\uparrow}\text{-a.s.} \tag{2.6}$$

Next we establish the behaviour in (2.5). First since $\delta > 1$, we observe from (2.1) that there exist $c \in (1, \infty)$ and $C \in (0, \infty)$ such that

$$\varepsilon\phi(t) = \varepsilon \int_t^{\infty} \frac{\mathrm{d}u}{\psi(u)} \geq \left(\frac{C}{\varepsilon}\right)^{1/(c-1)} \int_t^{\infty} \frac{\mathrm{d}u}{\psi((C/\varepsilon)^{1/(c-1)}u)} = \phi\left(\left(\frac{C}{\varepsilon}\right)^{1/(c-1)} t\right) \tag{2.7}$$

for any $t \geq 1$ and $0 < \varepsilon < \min(C, 1)$. On the other hand, we see that for large enough n and $0 < \varepsilon < \min(C, 1)$,

$$\widehat{\mathbb{P}}^{\uparrow}(U_{b_{n+1}/r} > \varepsilon s_n) = 1 - \exp\left\{-\frac{b_{n+1}}{r}\varphi(\varepsilon s_n)\right\} \leq \varepsilon' \frac{b_{n+1}}{r}\varphi(\varepsilon s_n)$$

for some $\varepsilon' > 1$. Hence, from (2.7), we have

$$\begin{aligned}\widehat{\mathbb{P}}^{\uparrow}(U_{b_{n+1}/r} > \varepsilon s_n) &\leq \varepsilon'\left(\frac{C}{\varepsilon}\right)^{1/(c-1)} \mathrm{e}^{\{n^r - (n+1)^r\}} \log(n+1) \\ &\leq \varepsilon'\left(\frac{C}{\varepsilon}\right)^{1/(c-1)} \mathrm{e}^{-rn^{r-1}} \log(n+1),\end{aligned}$$

where the last identity follows since $(n+1)^r - n^r \geq rn^{r-1}$. In conclusion, it follows that the series $\sum_n \widehat{\mathbb{P}}^{\uparrow}(U_{b_{n+1}/r} > s_n)$ converges, and according to the Borel–Cantelli lemma,

$$\limsup_{n\to\infty} \frac{U_{b_{n+1}/r}}{s_n} \leq \varepsilon, \qquad \widehat{\mathbb{P}}^{\uparrow}\text{-a.s.},$$

which establishes (2.5) since $0 < \varepsilon < \min(C, 1)$ can be chosen arbitrarily small. The proof of (2.6) is now complete. The two preceding bounds show that

$$\frac{1}{r} \leq \limsup_{t\to 0} \frac{J_t}{f(t)} \leq r^2, \qquad \widehat{\mathbb{P}}^{\uparrow}\text{-a.s.}$$

Hence, the result follows by letting r go to 1. □

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Proof of Theorem 1.1. In order to establish our result, we prove that the following LIL holds:

$$\limsup_{t\to 0} \frac{X_{\theta(t)}}{f(t)} = 1, \qquad \widehat{\mathbb{P}}^{\uparrow}\text{-a.s.} \tag{2.8}$$

and then our claim follows by applying Proposition 2.1.

The lower bound of (2.8) is easy to deduce from Proposition 2.2. More precisely,

$$1 = \limsup_{t\to 0} \frac{J_t}{f(t)} \le \limsup_{t\to 0} \frac{X_{\theta(t)}}{f(t)}, \qquad \widehat{\mathbb{P}}^{\uparrow}\text{-a.s.}$$

Now, we prove the upper bound. We denote by $\overline{X}_{\theta(\cdot)}$ for the supremum process of $\{(X_{\theta(s)}, s \ge 0), \widehat{\mathbb{P}}^{\uparrow}\}$, which is defined by $\overline{X}_{\theta(t)} = \sup_{0\le s\le t} X_{\theta(s)}$ for $t \ge 0$.

Lemma 2.3. *Let $c' > 1, 0 < \epsilon < 1 - 1/c'$, and $r > 1$. For $t_n = \phi(r^n), n \ge 1$, then there exists a positive real number K which does not depend on c', ϵ, and r, such that*

$$\widehat{\mathbb{P}}^{\uparrow}(J_{t_n} > (1-\epsilon)c' f(t_n)) \ge \epsilon K^2 \widehat{\mathbb{P}}^{\uparrow}(\overline{X}_{\theta(t_n)} > c' f(t_n)).$$

Proof. From the strong Markov property, we have

$$\begin{aligned}
\widehat{\mathbb{P}}^{\uparrow}(J_{t_n} > (1-\epsilon)c' f(t_n)) &\ge \widehat{\mathbb{P}}^{\uparrow}(\overline{X}_{\theta(t_n)} > c' f(t_n),\, J_{t_n} > (1-\epsilon)c' f(t_n)) \\
&= \int_0^{t_n} \widehat{\mathbb{P}}^{\uparrow}(S_{c' f(t_n)} \in \mathrm{d}t) \widehat{\mathbb{P}}^{\uparrow}_{c' f(t_n)}(J_{t_n - t} > (1-\epsilon)c' f(t_n)) \\
&\ge \widehat{\mathbb{P}}^{\uparrow}(\overline{X}_{\theta(t_n)} > c' f(t_n)) \widehat{\mathbb{P}}^{\uparrow}_{c' f(t_n)}\Big(\inf_{0\le s} X_{\theta(s)} > (1-\epsilon)c' f(t_n)\Big).
\end{aligned}$$

Now from the Lamperti transform and [1, Lemma VII.12], we have

$$\widehat{\mathbb{P}}^{\uparrow}_{c' f(t_n)}\Big(\inf_{0\le s} X_{\theta(s)} > (1-\epsilon)c' f(t_n)\Big) = \widehat{\mathbb{P}}^{\uparrow}_{c' f(t_n)}(\tau_{[0,(1-\epsilon)c' f(t_n))} = \infty) = \frac{W(c'\epsilon f(t_n))}{W(c' f(t_n))},$$

where $\tau_{[0,z)} = \inf\{t \ge 0 \colon X_t \in [0, z)\}$.

On the other hand, an application of [1, Proposition III.1] shows that there exists a positive real number K such that

$$K \frac{1}{x\psi(1/x)} \le W(x) \le K^{-1} \frac{1}{x\psi(1/x)} \quad \text{for all } x > 0,$$

then it is clear that

$$\frac{W(c'\epsilon f(t_n))}{W(c' f(t_n))} \ge K^2 \epsilon^{-1} \frac{\psi(1/c' f(t_n))}{\psi(\epsilon^{-1}/c' f(t_n))}. \tag{2.9}$$

Now, we will need the next result whose proof can be found in [9, Lemma 3].

Lemma 2.4. *For every $\delta > 1$, we have*

$$\delta^2 \psi(x) \ge \psi(\delta x) \quad \textit{for all } x \ge 0.$$

Finally, an application of the previous lemma allows us to conclude that from (2.9)

$$\widehat{\mathbb{P}}^{\uparrow}_{c' f(t_n)}\Big(\inf_{0\le s} X_{\theta(s)} > (1-\epsilon)c' f(t_n)\Big) \ge \epsilon K^2,$$

which clearly implies our result. □

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Now, we prove the upper bound for the LIL of $((X_{\theta(t)}, t \geq 0), \widehat{\mathbb{P}}^{\uparrow})$. Let $c' > 1$ and fix $0 < \epsilon < 1 - 1/c'$. Recall from (2.2) that

$$\widehat{\mathbb{P}}^{\uparrow}(J_{t_n} > (1-\epsilon)c' f(t_n)) \leq \widehat{\mathbb{P}}^{\uparrow}(U_{(1-\epsilon)c' f(t_n)} \leq t_n) = (n \log r)^{-(1-\epsilon)c'}.$$

Hence, from Lemma 2.3, we deduce that

$$\sum_{n \geq 1} \widehat{\mathbb{P}}^{\uparrow}(\overline{X}_{\theta(t_n)} > c' f(t_n)) \leq K^{-2}\epsilon^{-1} \sum_{n} (n \log r)^{-(1-\epsilon)c'} < \infty,$$

since $(1-\epsilon)c' > 1$. Therefore, an application of the Borel–Cantelli lemma shows that

$$\limsup_{n \to \infty} \frac{\overline{X}_{\theta(t_n)}}{f(t_n)} \leq c', \qquad \widehat{\mathbb{P}}^{\uparrow}\text{-a.s.}$$

Using (2.4), we deduce that

$$\limsup_{t \to 0} \frac{\overline{X}_{\theta(t)}}{f(t)} \leq \left(\limsup_{n \to \infty} \frac{\overline{X}_{\theta(t_n)}}{f(t_n)}\right)\left(\limsup_{t \to 0} \frac{f(t_n)}{f(t)}\right) \leq r c', \qquad \widehat{\mathbb{P}}^{\uparrow}\text{-a.s.}$$

Hence, from the first part of the proof and letting r go to 1 above, we obtain

$$\limsup_{t \to 0} \frac{X_{\theta(t)}}{f(t)} \in [1, c'], \qquad \widehat{\mathbb{P}}^{\uparrow}\text{-a.s.}$$

By the Blumenthal 0-1 law, it must be a constant number k, $\widehat{\mathbb{P}}^{\uparrow}$-a.s. On the other hand, since c' was chosen arbitrarily, we observe that, for all $c' > 1$, we have

$$\widehat{\mathbb{P}}^{\uparrow}\left(\limsup_{t \to \infty} \frac{X_{\theta(t)}}{f(t)} \in [1, c']\right) = 1.$$

Therefore, our claim follows by letting $c' \to 1$. □

Proof of Theorem 1.2. Here we follow similar arguments as those used in the last part of the previous result. Recall that thanks to Proposition 2.1, the process $X_{\theta(\cdot)} - J$ is equal in distribution to $Y - \underline{Y}$ backward in time from T_0, so we only need to compute

$$\limsup_{t \to 0} \frac{X_{\theta(t)} - J_t}{f(t)} = 1, \qquad \widehat{\mathbb{P}}^{\uparrow}\text{-a.s.}$$

Assume that hypothesis (1.6) is satisfied. From Theorem 1.1, it is clear that

$$\limsup_{t \to 0} \frac{X_{\theta(t)} - J_t}{f(t)} \leq \limsup_{t \to 0} \frac{X_{\theta(t)}}{f(t)} = 1, \qquad \widehat{\mathbb{P}}^{\uparrow}\text{-a.s.}$$

Fix $\varepsilon \in (0, \frac{1}{2})$ and define

$$R_n = \inf\left\{\frac{1}{n} \leq s : \frac{X_{\theta(s)}}{f(s)} \geq (1-\epsilon)\right\}.$$

First note that, for sufficiently large n, $1/n < R_n < \infty$ $\widehat{\mathbb{P}}^{\uparrow}$-a.s. Moreover, from Theorem 1.1 it follows that R_n converge to 0 as n goes to ∞, $\widehat{\mathbb{P}}^{\uparrow}$-a.s.

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From [1, Lemma VII.12], the strong Markov property, and since $\{(X_{\theta(t)}, t \geq 0), \widehat{\mathbb{P}}^{\uparrow}\}$ has no positive jumps, we have

$$\begin{aligned}\widehat{\mathbb{P}}^{\uparrow}\left(\frac{X_{\theta(R_n)} - J_{R_n}}{f(R_n)} \geq (1-2\epsilon)\right) &= \widehat{\mathbb{P}}^{\uparrow}\left(J_{R_n} \leq \frac{\epsilon}{1-\epsilon} X_{\theta(R_n)}\right) \\ &= \widehat{\mathbb{E}}^{\uparrow}\left(\widehat{\mathbb{P}}^{\uparrow}\left(J_{R_n} \leq \frac{\epsilon}{1-\epsilon} X_{\theta(R_n)} \;\middle|\; X_{\theta(R_n)}\right)\right) \\ &= 1 - \widehat{\mathbb{E}}^{\uparrow}\left(\frac{W(\ell(\epsilon) X_{\theta(R_n)})}{W(X_{\theta(R_n)})}\right),\end{aligned}$$

where $\ell(\epsilon) = (1-2\epsilon)/(1-\epsilon)$. Since the hypothesis (1.6) is satisfied, an application of the Fatou–Lebesgue theorem shows that

$$\limsup_{n\to+\infty} \widehat{\mathbb{E}}^{\uparrow}\left(\frac{W(\ell(\epsilon) X_{\theta(R_n)})}{W(X_{\theta(R_n)})}\right) \leq \widehat{\mathbb{E}}^{\uparrow}\left(\limsup_{n\to\infty} \frac{W(\ell(\epsilon) X_{\theta(R_n)})}{W(X_{\theta(R_n)})}\right) < 1,$$

which implies that

$$\lim_{n\to\infty} \widehat{\mathbb{P}}^{\uparrow}\left(\frac{X_{\theta(R_n)} - J_{R_n}}{f(R_n)} \geq (1-2\epsilon)\right) > 0.$$

Next, we note that

$$\widehat{\mathbb{P}}^{\uparrow}\left(\frac{X_{\theta(R_p)} - J_{R_p}}{f(R_p)} \geq (1-2\epsilon) \text{ for some } p \geq n\right) \geq \widehat{\mathbb{P}}^{\uparrow}\left(\frac{X_{\theta(R_n)} - J_{R_n}}{f(R_n)} \geq (1-2\epsilon)\right).$$

Since R_n converges to 0, $\widehat{\mathbb{P}}^{\uparrow}$-a.s. as n goes to ∞, it is enough to take limits on both sides. Therefore, for all $\epsilon \in (0, \frac{1}{2})$,

$$\begin{aligned}&\widehat{\mathbb{P}}^{\uparrow}\left(\frac{X_{\theta(t)} - J_t}{f(t)} \geq (1-2\epsilon) \text{ infinitely often as } t \to 0\right) \\ &\quad \geq \lim_{n\to+\infty} \widehat{\mathbb{P}}^{\uparrow}\left(\frac{X_{\theta(R_n)} - J_{R_n}}{f(R_n)} \geq (1-2\epsilon)\right) \\ &\quad > 0.\end{aligned}$$

Then,

$$\limsup_{t\to 0} \frac{X_{\theta(t)} - J_t}{f(t)} \geq 1 - 2\epsilon, \qquad \widehat{\mathbb{P}}^{\uparrow}\text{-a.s.}$$

and since ϵ can be chosen arbitrarily small, we obtain the result. □

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