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Factors of independent and identically distributed processes with non-amenable group actions

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Abstract. Let X be a graph and let G be a subgroup of the automorphism group of X. We investigate the question of when the full 2-shift process on the vertices of X has a G-factor process on the same graph consisting of random variables which are independent and identically distributed (i.i.d.) uniform in $[0, 1]$. We show that such a factor always exists when X is a tree with bounded degrees, no leaves and at least three topological ends, and give a sufficient condition for when such a factor exists if X is a connected graph with infinitely many ends and G acts transitively on the vertices of X. We also show that the full 2^m-shift on any finitely-generated non-abelian group G has a G-factor which is i.i.d. uniform on $[0, 1]$ when m is sufficiently large.

1. *Introduction*

Consider a locally-finite graph $X = (V, E)$ and let G be a subgroup of the automorphism group of X, $\mathrm{Aut}(X)$. Let $\xi = \{\xi_x\}_{x \in V}$ be a process consisting of real-valued random variables on the vertices of X, which has a distribution that is invariant under the action of G on X. A G-factor map on ξ is a map F on the configurations of ξ such that F commutes with the action of G on ξ:

$$F(\gamma \cdot \xi) = \gamma \cdot F(\xi) \quad \text{for all } \gamma \in G.$$

We assume that the resulting factor process $F(\xi) = \zeta$ is also a G-process on the vertices of the same graph X. In very general terms, the question we consider here is the following.

Question 1.1. What types of processes can be achieved as G-factors of ξ?

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Let X be the one-dimensional integer lattice and G the natural $\mathbb{Z}$-action, so that the action of G on a process ξ is generated by the shift map σ_ξ. This is the realm of classical ergodic theory. If we restrict our attention to processes ξ for which σ_ξ is Bernoulli, then the answer to Question 1.1 is well understood. We use the term Bernoulli in the ergodic-theoretic sense, so that σ_ξ is Bernoulli if and only if it is conjugate to an independent and identically distributed (i.i.d.) process. In this case, a process ζ on X is a factor of ξ if and only if ζ defines a shift map σ_ζ which is also Bernoulli and $h(\sigma_\zeta) \leq h(\sigma_\xi)$, where h denotes the entropy of the dynamical system. This is Sinai's factor theorem [**13**].

Whenever G is an amenable group and ξ is an ergodic G-process, the entropy restriction above holds. In other words, if ζ is a factor of ξ, then the G-action with respect to ξ must have entropy which is greater than or equal to the entropy of the G-action on ζ. It is an area of active research to determine under what conditions such factor maps exist for symbolic $\mathbb{Z}^d$-actions (see, for example, [**11**]).

We are concerned here with the case that G is non-amenable and ξ is an i.i.d. G-process on X. Here we do not even have the entropy restriction since there is no known meaningful definition of entropy which differentiates between the i.i.d. processes we consider. Grigorchuk gives a definition for the entropy of a free semigroup action in [**5**], but we will see that under this definition the independent processes with free semigroup actions we consider will all have infinite entropy.

Let X be a graph and G a non-amenable subgroup of $\mathrm{Aut}(X)$. Let $\xi = \{\xi_x\}_{x \in X}$ be the full 2-shift process on (X, G), the i.i.d. G-process on the vertices of X where each ξ_x takes values in $\{0, 1\}$, each with probability $\frac{1}{2}$. (More generally, a full m-shift process on (X, G) is a G-process which is i.i.d. on the vertices of X and the random variables take values in $\{0, 1, \ldots, m-1\}$, each with equal probability.) We write $U[0, 1]$ for a random variable whose distribution is uniform in the interval $[0, 1]$. In this context, the specific question we address in this paper is the following.

Question 1.2. Under what conditions on X and G does the full 2-shift ξ have a G-factor ζ which is an i.i.d. G-process of X consisting of random variables which are $U[0, 1]$?

Our usual idea of entropy contains the idea that the entropy of an i.i.d. process should have entropy equal to the entropy of the distribution of the random variables used to define it. Thus, the question above is in a sense asking when a process with 'finite entropy' can have a factor with 'infinite entropy'. The idea that such factors exist runs counter to what we expect of entropy and points to the fact that outside of the realm of amenable group actions, the behavior of dynamical systems is not well understood. In fact, it is still an open problem whether the full 2-shift and the full 3-shift can be isomorphic for such systems.

In §2, we consider Question 1.2 for the case where X_{2n} is the standard Cayley graph of a free group on n generators and $G = \mathbb{F}_n$ is the free group action on X_{2n}. Of course, X_{2n} is just the regular tree of degree $2n$. We present an $\mathbb{F}_n$-factor of ξ due to Ornstein and Weiss [**9**] which is the full 4-shift on $(X_{2n}, \mathbb{F}_n)$. An adaptation of this factor is then given in Proposition 2.3 which produces an $\mathbb{F}_n$-factor of ξ which consists of i.i.d. $U[0, 1]$ random variables. This shows that there is a positive answer to Question 1.2 in this case.

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We next consider the full automorphism group of X_{2n}, which is larger than $\mathbb{F}_n$ since it allows a- and b-edges to be interchanged. Section 3 builds on the results in the previous section to give an $\mathrm{Aut}(X_{2n})$-factor which is the i.i.d. $U[0,1]$ process on X_{2n}, so that the answer to Question 1.2 is positive even with this larger group acting. We then generalize this result to the case where X is any tree with bounded degrees, no leaves and at least three ends. A leaf is a vertex of degree one and an end is essentially a way to go to infinity within the graph (see Definition 3.2). This result is Theorem 3.3.

The next natural step is to try to generalize the result in Theorem 3.3 to more general graphs. It turns out that there is a simple obstruction to doing so in some cases. In §4, we present an example of a graph X which is non-amenable and has nice properties (it is transitive and unimodular and has infinitely many ends) and yet ξ on this graph has no $\mathrm{Aut}(X)$-factor which is i.i.d. $U[0,1]$ on X. The reason this graph has no such $\mathrm{Aut}(X)$-factor is that there is a non-trivial automorphism of X which fixes all but finitely many vertices of X. No graph with such an automorphism will admit an $\mathrm{Aut}(X)$-factor of the desired type, as we will show.

With this counterexample in mind, we proceed to explore Question 1.2 for graphs which are not trees. The results in §§2 and 3 depend in an essential way on the tree structure of the graph. One natural way to try to extend these results to more general graphs is to construct a forest structure on X as a factor of ξ, so that the known techniques will work. A forest is a graph whose components are all trees. In order to apply our theorem about trees, the factor forest must contain a tree with at least three ends, almost surely.

We show in Theorem 5.3 in §5 that this can be done if X is a transitive or quasitransitive graph with infinitely many ends which has an asymmetry property which is designed to avoid the obstruction from Example 4.3.

It would follow from a conjecture of Benjamini and Schramm [**2**] that any one-ended transitive non-amenable graph X with the asymmetry property has an $\mathrm{Aut}(X)$-factor taking the full 2^m-shift on $(X, \mathrm{Aut}(X))$ to the i.i.d. $U[0,1]$ $\mathrm{Aut}(X)$-process on X for sufficiently large m. Enough is known about this conjecture to prove that if $\xi(m)$ is the full 2^m-shift on a finitely-generated non-abelian group G, then for m large enough there is a G factor of $\xi(m)$ which is i.i.d. $U[0,1]$ on G. These results are presented in §6.

We conclude with a list of open questions relating to Questions 1.1 and 1.2 in §7.

2. *Free group factors*

The first case for which we will examine Question 1.2 is when X_{2n} is the standard Cayley graph of $\mathbb{F}_n$ = the free group on n generators ($n \geq 2$), and $G = \mathbb{F}_n$ acts on X_{2n} in the natural way. X_{2n} is then the regular tree of degree $2n$. Let $\xi = \{\xi(x)\}_{x \in X_n}$ be the full 2-shift process on $(X_{2n}, \mathbb{F}_n)$. In this case, it is not hard to show that ξ has an $\mathbb{F}_n$-factor which is i.i.d. $U[0,1]$ on X.

The important observation here is given by Ornstein and Weiss in [**9**]. They define a factor process ξ' of ξ which is the full 4-shift process on $(X_{2n}, \mathbb{F}_n)$. This factor is constructed as follows. Let a, b be the first two of the standard generators for $\mathbb{F}_n$. Then let

$$\xi'(x) = (\xi(x) \oplus \xi(ax), \xi(x) \oplus \xi(bx)),$$

where the addition is mod 2. These random variables are easily shown to be independent

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and all four possible outcomes are equally likely. By iterating, it is possible to get $\mathbb{F}_n$-factors of ξ which are full 2^m-shift processes on $(X_{2n}, \mathbb{F}_n)$ for arbitrarily large values of m.

In fact, this is enough to show that the answer to Question 1.2 is positive Cayley graphs of free groups with the free group action. The proof of the following proposition is due to Ádám Timár (personal communication).

PROPOSITION 2.1. (Timár) *When there is a G-factor F taking the full 2-shift ξ on a graph X to the full 4-shift on X, then there is a G-factor which is i.i.d. $U[0, 1]$ on X.*

Proof. Let $\xi^{(2)} = F(\xi)$, which by assumption is the full 4-shift, with symbols $\{00, 01, 10, 11\}$. Then we will define $\xi^{(n)}$ recursively. For each $x \in X$, $\xi^{(n)}(x)$ will be an n-bit binary sequence. Let $\bar{\xi}^{(n)}(x)$ be the first $n-1$ bits of this sequence and let $\tilde{\xi}^{(n)}(x)$ be the last bit. Then define

$$\xi^{(n)}(x) = (\bar{\xi}^{(n-1)}(x), F(\tilde{\xi}^{(n-1)})(x)).$$

What we are doing here is keeping the first $n-2$ bits of $\xi^{(n-1)}(x)$, and using the last bit of each to plug into F, getting 2 bits out, so in the limit as $n \to \infty$, we get a process ξ^{∞} where each $\xi^{\infty}(x)$ has infinitely many binary bits chosen uniformly and independently at random and so forms the binary expansion of a uniform random variable. □

Remark 2.2. One consequence of the existence of such a factor map from the full 2-shift to the full 4-shift is that it is easy to show that the free semigroup action given by ξ has infinite entropy if we take the definition of the entropy of a free semigroup action given by Grigorchuk in [**5**].

Let $(\overline{T} = T_1, \ldots, T_n, \Omega, \mu)$ be the dynamical system generated by non-commuting maps $T_1, \ldots T_n$, so that we get an action of the free semigroup on n generators. Let W denote a word in the alphabet $\{T_1, \ldots T_n\}$ with $|W|$ the length of the word. For a finite partition P of Ω, let P_W be the partition $\bigvee_{V<W} T_V^{-1} P$ where V runs through the initial segments of W and $T_V = T_{i_1} \circ \cdots \circ T_{i_k}$ when $V = T_{i_1}, \ldots T_{i_k}$. Then define the entropy of $\overline{T}$ with respect to the partition P by

$$h(\overline{T}, P) = \overline{\lim_{i \to \infty}} \frac{1}{i} \sum_{|W|=i} \frac{1}{2^i} H(P_W).$$

As usual, to get the entropy of the dynamical system, we take the sup over all partitions: $h(\overline{T}) = \sup_P h(\overline{T}, P)$.

Clearly, since ξ has factors which are full 2^m-shifts for all $m > 0$, there exist finite partitions with arbitrarily large entropy, so the semigroup action of $\mathbb{F}_n^+$ on ξ has infinite entropy in the sense of Grigorchuk. There is no known invariant that can distinguish between the action on $\mathbb{F}_n$ processes which are full m-shifts for distinct values of m. Grigorchuk gives no examples of free semigroup actions with finite positive entropy; such examples would be interesting.

The Ornstein–Weiss example shows that there is an $\mathbb{F}_n$-factor taking one independent bit at each vertex of X_{2n} to any finite number of independent bits at each vertex. By Proposition 2.1, this implies that there is an $\mathbb{F}_n$-factor taking one independent bit at

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each vertex to an infinite sequence of independent bits at each vertex. We give another proof here, which displays some of the techniques we will use to prove the existence of such factors for more general graphs and groups later in the paper. This construction is just a simple extension of the Ornstein–Weiss factor.

PROPOSITION 2.3. *There is an i.i.d. $U[0,1]$ $\mathbb{F}_n$-process $\zeta = \{\zeta(x)\}_{x\in X_{2n}}$ which is an $\mathbb{F}_n$-factor of ξ, the full 2-shift process on $(X_{2n}, \mathbb{F}_n)$.*

Proof. Let x be a vertex in X. We will construct the binary expansion of $\zeta(x)$ bit-by-bit. Let $\zeta_k(x)$ denote the kth bit of the binary expansion of $\zeta(x)$. As above, let a, b be the first two standard generators for $\mathbb{F}_n$. For $k \geq 1$ define

$$\zeta_k(x) = \xi(b^k a^k x) \oplus \xi(b^{-k} a^{-k} x)$$

where again the addition is taken mod 2. Since they depend on disjoint sets of independent random variables, it is clear that for fixed x, the sequence $\zeta_1(x), \zeta_2(x), \ldots$ is i.i.d. and takes values 0 and 1 with equal probability. Therefore, each $\zeta(x)$ is a uniform random variable. It only remains to show that they are independent as x varies.

Let B_m be the ball of radius m about the identity in X_{2n}. We wish to show that the random variables $\{\zeta(x)\}_{x\in B_m}$ are independent. Let x be any element of $\mathbb{F}_n$ at a distance of exactly m from the identity vertex in X_{2n}. Then since X_{2n} is a tree, either the vertex corresponding to $b^k a^k x$ or that corresponding to $b^{-k} a^{-k} x$ must be at a distance strictly greater than m from the identity vertex. We assume without loss of generality that the former is true. We claim that for all $y \in B_m$ with $y \neq x$, the value of $\zeta(y)$ does not depend on $\xi(b^k a^k x)$. Suppose that this is not the case and that $\zeta_l(y)$ depends on the value of $\xi(b^k a^k x)$ for some l. Then it must be that $b^k a^k x = b^l a^l y$. Since y is inside B_m and $b^k a^k x$ is outside B_m, the unique path in X_{2n} from y to $b^k a^k x = b^l a^l y$ must pass through x. This path must also switch from crossing a-edges to b-edges at the vertex $a^k x$ (otherwise it would end up in a different branch of the tree), so $a^k x = a^l y$. In order for this to happen, we must have $l > k$. However, then $b^k a^k x \neq b^l a^l y$ and so we have a contradiction.

Therefore, for each $k \geq 1$, $\zeta_k(x)$ is independent of $\zeta(y)$ for all $y \neq x$, $y \in B_m$. This is true for all x at distance m from the identity in X_{2n}, so it is sufficient to show that the random variables $\{\zeta(x)\}_{x\in B_{m-1}}$ are independent. An inductive argument then shows that the $\zeta(x)$ are independent and the proof is complete. □

This proof demonstrates the simple idea that will be at the heart of all of our factor constructions. For more general graphs and group actions, we will not have the a- and b-labels on the edges and eventually we will not even have a tree structure. Our approach will be to use data from ξ to put enough equivariant structure on the graph X to make the same independence argument work.

Actually, the proof above works for somewhat more general Cayley graphs.

COROLLARY 2.4. *Let X be the Cayley graph of a non-amenable group G. Assume further that G has a subgroup isomorphic to $\mathbb{F}_2$. Then the i.i.d. $U[0,1]$ G-process ζ on the vertices of X is a G-factor of the full 2-shift process ξ.*

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Proof. Let a and b be generators for a free subgroup of G. Then as above we can define

$$\zeta_k(x) = \xi(b^k a^k x) \oplus \xi(b^{-k} a^{-k} x)$$

for each $x \in X$ and $k \geq 1$. Then the proof given in Proposition 2.3 applies here as well. □

Remark 2.5. The $\mathbb{F}_2$-factor given by Ornstein and Weiss is easily seen to be uniformly two-to-one. The $\mathbb{F}_2$-factor given above in the proof of Proposition 2.3 is continuum-to-one. It would be interesting to know whether there exists a factor as in Proposition 2.3 which is uniformly bounded-to-one.

Remark 2.6. No known *invertible* $\mathbb{F}_n$-factor from the full 2-shift on $(X_{2n}, \mathbb{F}_n)$ to the full m-shift ($m \neq 2$) on the same graph is known. Such a factor would of course be of great interest since it would imply that such processes are conjugate. See Question 7.1.

3. *Factors of processes on trees*

As we have seen above, the free group on two generators, $\mathbb{F}_n$, acts on processes indexed by the regular tree of degree $2n$, X_{2n}. However, the full automorphism group of the regular tree of degree $2n$ is larger than $\mathbb{F}_n$. Clearly, the $\mathbb{F}_n$-factor given above is not a factor with respect to this full group of automorphisms of X_{2n}, since general automorphisms may exchange a- and b-edges of the graph.

Let X_n be the regular tree of degree n and let G_n be the full group of automorphisms of X_n. The next question we may ask is whether there is a G_n-factor of the full 2-shift process ξ on (X_n, G_n) which has random variables which are i.i.d. and uniform in $[0, 1]$. It turns out that this can also be done. The idea is to use some of the data from our process ξ to put enough equivariant structure on the edges of X_n to play the roles of the a and b labels on the edges that were used in the proof of Proposition 2.3. Proposition 3.1 will then be used to give a result for more general trees, which is given in Theorem 3.3.

PROPOSITION 3.1. *For $n \geq 3$, there is an i.i.d. $U[0, 1]$ G_n-process ζ on X_n which is a G_n-factor of the full 2-shift process ξ on (X_n, G_n).*

Proof. We first use some of the data in the process ξ to generate an ordering for the edges incident to each vertex. We reserve some bits from ξ to establish this ordering and use the rest to give the values for the factor process ζ. Fix a number $r \geq 4$. For this proof the choice of r is arbitrary, but we will want more flexibility later. Call a bit $\xi(x)$ a *directional bit* if

$$\xi(y) = \begin{cases} 0, & 0 < d_{X_n}(x, y) < r \\ 1, & d_{X_n}(x, y) = r \end{cases} \quad \text{or} \quad \xi(y) = \begin{cases} 1, & 0 < d_{X_n}(x, y) < r \\ 0, & d_{X_n}(x, y) = r \end{cases}$$

and if the distance to the next vertex with this property is at least $2r + 1$ (so that the regions defining directional bits may not overlap). Note that the definition of a directional bit is independent of the value of the bit itself. All other bits will be called *non-directional bits*.

We use the directional bits decide on an ordering of the edges at each vertex. There are many ways to do this; for concreteness we give one here. Consider a vertex $x \in X_n$. For each of the n edges e incident to x, we get an induced subtree of X_n consisting of those vertices y for which the unique path from x to y in X_n passes through e. If we consider

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such a subtree to be rooted at the vertex incident to e, then we can take the mod 2 sum of any directional bits on the same level of the subtree. Infinitely many levels will contain directional bits, and the resulting random variables (for those levels containing such bits) are independent. This sequence of bits can therefore be combined as the binary expansion of a random variable which is uniform on [0, 1] for each edge e, almost surely. The values of these n random variables will almost surely be distinct, and so give an ordering for the edges. There are only countably many vertices in X_n so these orderings are defined at each vertex, almost surely. Note that an edge e may have a different position in the ordering for each of its endpoints.

Once this ordering of the edges is established, we can define the factor map. To define the kth bit at x for the factor map given in the proof of Proposition 2.3 above, we considered two particular paths of length $2k$ from x: first go along k edges labeled a and then k edges labeled b, or otherwise first go along k edges labeled a^{-1} and then k edges labeled b^{-1}. In using G_n instead of $\mathbb{F}_n$, we have lost the a and b labelings for the edges. We use our orderings in their place.

For each vertex x let $p_k(x)$ be the vertex arrived at by taking the following path of length $6rk$ or $6rk+1$. Start at x and take its smallest edge e_1 to a vertex $z_1 = z_1(x,k)$. Then e_1 has a place in the ordering for z_1. Let e_2 be the next edge after e_1 in this ordering; if e_1 is the largest edge for z_1, let e_2 be the smallest edge for z_1. The other endpoint of e_2 is z_2. We call this method for choosing the next edge *moving straight though at* z_1. Repeat the process of moving straight through until $3rk$ steps have been taken and z_{3rk} has been reached. At z_{3rk}, we *make a turn*. This means that instead of taking the next largest edge out of z_{3rk}, we take the next smallest edge out of z_{3rk}. After the turn, we move straight though again until we have taken a total of $3rk$ steps after the turn.

We define $p_k(x)$ to be the vertex z_{6rk} unless $\xi(z_{6rk})$ is a directional bit, in which case we go straight through one more edge and $p_k(x)$ is the resulting vertex, z_{6rk+1}. Similarly, we get a vertex $q_k(x)$ by moving first along the largest edge for x, moving straight through for another $3rk-1$ steps, making a turn, and moving straight through for another $3rk-1$ steps to get to $q_k(x)$, unless this is a directional bit, in which case we move straight through one more edge to $q_k(x)$.

Again, $\zeta_k(x)$ will be the kth bit of the binary expansion of $\zeta(x)$. We define

$$\zeta_k(x) = \xi(p_k(x)) \oplus \xi(q_k(x)).$$

Clearly, this construction is equivariant under G_n. For fixed x, the $\zeta_k(x)$ depend on disjoint sets of independent random variables, so $\zeta(x)$ is a uniform random variable in [0, 1]. It only remains to show that they are independent. The proof is very similar to that given for Proposition 2.3.

Let B_m be the ball of radius m about some root vertex o in X_n. We inductively prove that for each m, the random variables $\zeta(x)$ for $x \in B_m$ are independent. This is clear for $m = 0$. We may then assume that it is true for $m-1$.

Now consider B_m and let x be a vertex which lies at a distance exactly m from o. Because X_n is a tree, at least one of $p_k(x)$ and $q_k(x)$ must lie at a distance strictly greater than m from o. We assume without loss of generality that it is $p_k(x)$. As in the proof of Proposition 2.3, if $y \neq x$, $y \in B_m$, then $\zeta(y)$ cannot depend on the value of $\xi(p_k(x))$.

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This is because X_n is a tree; any path from y to $p_k(x)$ must pass though x. Thus if $p_l(y) = p_k(x)$ (or $q_l(y) = p_k(x)$, but this case is similar), $l > k$. The path from y to $p_l(y)$ must enter $z_{3rk}(x, k)$ through the same edge as the path from x to $p_k(x)$. Both paths must then turn at $z_{3rk}(x, k)$ or else they will end up in different portions of the tree. However, the path from y to $p_l(y)$ must take at least $3r$ more steps after the turn than the path from x to $p_k(x)$ and hence $p_l(y) \neq p_k(x)$.

In fact the two vertices must be at least a distance $3r$ from each other. This means that $p_l(y)$ and $p_k(x)$ cannot even lie in the same region of 0's and 1's which define a directional bit. Since there are two equally likely opposite patterns used to define directional bits which are independent of the values of the directional bits themselves, knowing that $p_k(x)$ lies in such a region gives no information about the value of the bit at $p_k(x)$.

Therefore, $\zeta(x)$ is independent of $\{\zeta(y)\}_{y \in B_m, y \neq x}$ and this is true for all x with $d_{X_n}(x, o) = m$. By the inductive assumption, we are done. The random variables $\zeta(x)$ are i.i.d. and uniform in $[0, 1]$. □

This proof depends on the tree structure of X_n, but not essentially on the regularity of the graph structure. We therefore get a more general theorem about trees with multiple ends, where an end is defined below.

Definition 3.2. If X is a tree, then a *ray* in X is a one-sidedly infinite path in X which does not cross any edge or vertex more than once. An *end* of X is an equivalence class of rays in X, where two rays are equivalent if they have infinitely many vertices in common.

More generally, if X is any infinite graph, then an *end* of X is an equivalence class of non-self-intersecting infinite paths in X, where two such paths are equivalent if for every finite set $A \subset X$, they are eventually contained in the same connected component of $X \setminus A$.

A leaf is a vertex of degree one in a tree. If a tree X has a vertex which is adjacent to two or more leaves x_1 and x_2, then there is an automorphism of X which just exchanges x_1 and x_2 and fixes all other vertices. If an $\mathrm{Aut}(X)$-process ξ has the property that $\xi(x_1) = \xi(x_2)$ with positive probability, then any $\mathrm{Aut}(X)$-factor of ξ will have the same property and so ξ cannot have an $\mathrm{Aut}(X)$-factor which is i.i.d. $U[0, 1]$. To avoid this kind of complication, we concentrate on trees with no leaves. We will see in §4 that for graphs which are not trees, we need to be even more careful to avoid this type of obstruction.

THEOREM 3.3. *Let X be a tree with bounded degree, no leaves and at least three ends. Then the full 2-shift process ξ on $(X, \mathrm{Aut}(X))$ has an $\mathrm{Aut}(X)$-factor ζ which is i.i.d. $U[0, 1]$ on X.*

Proof. A furcation of a tree X is a vertex $x \in X$ such that the induced graph on $X \setminus \{x\}$ has at least three infinite components. Let F be the set of all furcations in X. Since X has at least three ends, $F \neq \emptyset$. We will first define independent $U[0, 1]$ random variables $\zeta'(x)$ at each vertex x in F, then use ζ' to define ζ on all of X.

As in the proof of Proposition 3.1, select directional bits and use them to determine an ordering of the edges at each vertex of X. Here, we are using both the fact that X has bounded degree and that it has no leaves. Together, these guarantee that for each vertex x

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and each edge e incident to x, the subtree which is the component of $X \setminus \{e\}$ which does not contain x has infinitely many directional bits, which we use to define a $U[0, 1]$ random variable at (x, e). These random variables determine the order of the edges at x.

We define ζ' in two stages: first utilizing any rays in X which contain only a single furcation, and then applying Proposition 3.1 to the remaining furcations. The third stage is then to use ζ' to construct ζ.

Stage 1: use rays with only a single furcation. We now begin a (possibly countably transfinite) inductive construction. We start with a graph X', setting $X' = X$ in the first step. Let F' be the set of furcations of X'. If X' has no rays which start at some vertex $x \in F'$ and encounter no other vertices in F', then the construction is complete. Proceed to Stage 2. Otherwise, let $R \subset F'$ be the set of furcations of X' which have such a ray. For $x \in R$, the set of rays from x which encounter no other vertices in F' is non-empty and contains at most one ray which first traverses any particular edge out of x. Let r be the ray in this set which has the minimal first edge under the ordering at x determined by the directional bits. If $\zeta'(x)$ has not already been defined in an earlier stage of the construction, then for each $k \geq 1$ define $\zeta'_k(x)$ to be the $(k+1)$th non-directional bit in r.

Form a new graph $X'' \subset X'$ by removing from X' the rays starting from vertices in R and containing no other furcations from X' (but leaving the vertices in R in X''). Note that X'' is connected, all furcations of X'' must have been furcations of X', and if $x \in F'$ is not a furcation of X'', then $x \in R$ and $\zeta'(x)$ has already been defined. If x is a furcation of X' and $\zeta'(x)$ has not yet been defined, then x is a furcation of X''. If $x \in R$, it may still be a furcation of X''. For the next step of the construction, set $X' = X''$ and repeat Stage 1.

Since X has only countably many vertices, this inductive construction must halt after at most a countable number of steps. When it stops, either X' has no furcations or each ray in X' from a vertex x in F' encounters other furcations of X'. In the first case, ζ' is defined on all of F as desired. The random variables $\zeta'(x)$ defined by the end must all be independent since they depend on disjoint sets of bits from ξ.

Stage 2: apply Proposition 3.1. If X' as constructed at the conclusion of Stage 1 still contains furcations, we still need to define ζ' on F', the furcations of X'. These random variables will depend only on the bits at vertices in the final graph X' and so will be independent of all previously-defined values of ζ'.

Form a graph Y whose vertex set is F', with an edge between $x, y \in F'$ if and only if the unique path in X' from x to y does not pass through any other points in F'. Then Y is an infinite tree of bounded degree and each vertex has degree at least three. The factor from the proof of Proposition 3.1 defines ζ' on F' (where it is not already defined). Note that the proof of Proposition 3.1 does not require that the vertices have uniform degree, just that the degree of the vertices be bounded above and at least three everywhere. Clearly, all of the random variables $\{\zeta'_k(x) : x \in F', k \geq 1\}$ are independent by the same proof as before. We have now defined ζ' on all of F.

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Stage 3: use ζ' to construct ζ. All that remains is to use ζ', which is defined only on F, to define ζ on all of X. For each $y \in X$, let $c(y)$ be the vertex in F closest to y in graph distance. If there is a tie, $c(y)$ is chosen to be the one for which the path in X from y to $c(y)$ is minimal under the lexicographic ordering of finite paths from y defined by the ordering of the edges determined by the directional bits. Since this ordering well orders the paths starting at y, $c(y)$ is well defined. For $x \in F$, let $V_x = \{y : c(y) = x\}$. Within V_x, order the vertices $x = y_0 < y_1 < y_2 < \cdots$ according to graph distance from x and the order on the paths starting at x. Now distribute the bits in $\zeta'(x)$ to the vertices in V_x in a triangular array to define ζ:

$$\begin{array}{llll}
\zeta_1(y_0) = \zeta'_1(x) & & & \\
\zeta_2(y_0) = \zeta'_2(x) & \zeta_1(y_1) = \zeta'_3(x) & & \\
\zeta_3(y_0) = \zeta'_4(x) & \zeta_2(y_1) = \zeta'_5(x) & \zeta_1(y_2) = \zeta'_6(x) & \\
\zeta_4(y_0) = \zeta'_7(x) & \zeta_3(y_1) = \zeta'_8(x) & \zeta_2(y_2) = \zeta'_9(x) & \zeta_1(y_3) = \zeta'_{10}(x) \\
\vdots & & &
\end{array}$$

By the independence of the $\zeta'_k(x)$, ζ consists of i.i.d. random variables which are $U[0, 1]$. □

4. *A counterexample*

We have proven that for trees with bounded degree, no leaves and at least three ends, Question 1.2 has a positive answer. The proof used the tree structure in an essential way. How much more general is this phenomenon?

Entropy restrictions imply that we cannot get infinitely many independent bits at each vertex from finitely many bits at each vertex when G is an amenable group acting on its Cayley graph X. Similar statements can be formulated for amenable groups acting on more general amenable graphs. This indicates that we should be looking within the class of graphs with non-amenable group actions for the presence of such factors. In this section we show that there are non-amenable graphs X with very nice properties which have no $\mathrm{Aut}(X)$-factors from the full 2-shift to the i.i.d. $U[0, 1]$ process. We will show that such a factor is not possible when there is an automorphism of X which fixes all but finitely many vertices.

Several important and well-behaved classes of graphs are defined below.

Definition 4.1. A group G acts *transitively* on a graph X if for every $x, y \in X$, there is a $\gamma \in G$ such that $\gamma \cdot x = y$. A graph X is *transitive* if $\mathrm{Aut}(X)$ acts transitively on X. In other words, the orbit of a vertex x under $\mathrm{Aut}(X)$ is the whole graph X.

A group G acts *quasitransitively* on a graph X if there are only finitely many distinct orbits of G acting on the vertices of X and X is *quasitransitive* if $\mathrm{Aut}(X)$ acts quasitransitively on X.

Transitive graphs are highly symmetric, since each vertex looks the same as any other. A central class of transitive graphs are the Cayley graphs. Both transitivity and quasitransitivity arise naturally.

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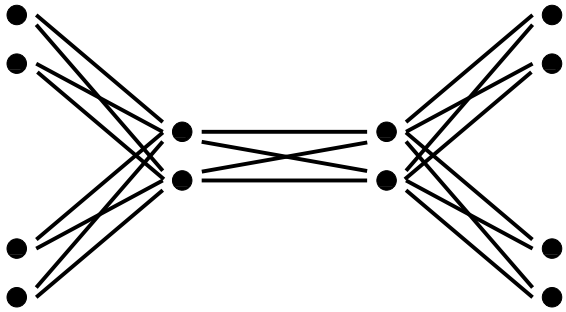


FIGURE 1. A portion of the graph S_3.

Definition 4.2. An automorphism group $\mathrm{Aut}(X)$ is *unimodular* if its left- and right-Haar measures are the same. A graph X is *unimodular* if $\mathrm{Aut}(X)$ is unimodular.

Unimodularity has played an important role in the recent study of percolation and random spanning forests on more general graphs than $\mathbb{Z}^d$. See [**1**] for a discussion of the roles of unimodularity, transitivity and quasitransitivity in percolation.

The purpose of this section is to give an example of a transitive unimodular graph with infinitely many ends for which the answer to Question 1.2 is negative; there is no $\mathrm{Aut}(X)$-factor which takes a single independent bit at each vertex of the graph to a sequence of infinitely many independent bits at each vertex of the same graph. There is an obstruction when vertices have too much symmetry, when there is an automorphism of X which fixes all but finitely many vertices.

Example 4.3. Start with T_3, the regular tree of degree three. Replace each vertex x in T_3 with two new vertices x_1, x_2. Now replace each edge (x, y) in T_3 with four new edges, (x_1, y_1), (x_1, y_2), (x_2, y_1) and (x_2, y_2). Call the resulting graph S_3, a portion of which is drawn in Figure 1. Clearly, this graph has infinitely many ends.

The automorphism group of T_3 acts on S_3 in the obvious way, but S_3 has extra symmetries. For instance, for any pair x_1, x_2 which correspond to a single vertex in T_3, there is an automorphism of S_3 which interchanges x_1 and x_2 and leaves the rest of the vertices fixed. Clearly, $\mathrm{Aut}(S_3)$ is transitive since any pair can be mapped to any other pair by a map from $\mathrm{Aut}(T_3)$, and then the pair interchanged if necessary.

S_3 is also unimodular. Unimodularity of a graph has a simple combinatorial characterization, as shown by Schlichting [**12**] and Trofimov [**15**]. Let

$$\mathrm{Stab}(x) := \{\gamma \in \mathrm{Aut}(X) : \gamma \cdot x = x\}$$

be the stabilizer of x in a graph X. Then X is unimodular if and only if for all x and y in the same $\mathrm{Aut}(X)$-orbit, $|\mathrm{Stab}(x) \cdot y| = |\mathrm{Stab}(y) \cdot x|$. This condition is easy to check for S_3.

Let ξ be the full 2-shift on S_3. Let x_1, x_2 be a pair of vertices coming from a single vertex $x \in T_3$. Condition on the case where $\xi(x_1) = \xi(x_2)$. Since there is an automorphism which takes x_1 to x_2 and fixes the rest of the graph, from an $\mathrm{Aut}(S_3)$ point of view, the configuration of ξ when seen from x_1 is *exactly the same* as the configuration of ξ when seen from x_2. Thus if F is a factor map from ξ to a process on S_3, $F(\xi)(x_1) = F(\xi)(x_2)$

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whenever $\xi(x_1) = \xi(x_2)$, which happens with probability $\frac{1}{2}$. Any process ζ which is i.i.d. uniform would have $\zeta(x_1) \neq \zeta(x_2)$ almost surely. Therefore, there exists no $\mathrm{Aut}(X)$-factor map from ξ to ζ. Note that this also shows that ξ cannot have a factor which is a full m-shift process for any $m \neq 2$.

This example shows that the existence of a factor map from a full 2-shift process to an i.i.d. $U[0, 1]$ process is not necessary for non-amenable transitive graphs, or even with the added assumption of unimodularity. A similar argument shows that there is no G-factor of this type for any graph X if there is an element of G which fixes a cofinite set of vertices in X.

In the next section we show that for a large class of transitive and quasitransitive graphs, a slight strengthening of the property of having no automorphism in G which fixes all but finitely many vertices is sufficient to prove that the desired G-factor does exist.

5. *Factors of processes on transitive graphs with infinitely many ends*

In §§2 and 3, we used the tree structure of the graphs studied in an essential way to generate an i.i.d. $U[0, 1]$ process as a factor of the full 2-shift process. To extend these results to more general types of graphs than trees, a natural approach is to try to generate a subgraph which is a forest as a factor of the full 2-shift process ξ.

Of course, we have to avoid the kind of obstruction in Example 4.3. For this proof to work, we require something which is possibly stronger than the lack of a non-trivial automorphism fixing a cofinite set of vertices. It is clear that any graph possessing such an automorphism must have symmetry pairs as defined below. It is not known whether the two conditions are equivalent.

Definition 5.1. Let X be a graph, $G \subseteq \mathrm{Aut}(X)$, and let x_1 and x_2 be vertices in X. Let

$$\mathrm{Stab}_G(x) := \{\gamma \in G : \gamma \cdot x = x\}.$$

We say that x_1 and x_2 form a *symmetry pair* of (X, G) if there exists ($\Longleftrightarrow$ all) $\gamma \in G$ with $\gamma \cdot x_1 = x_2$, the set

$$\{y \in X : \mathrm{Stab}_G(x_1) \cdot y = \mathrm{Stab}_G(x_2) \cdot \gamma y\}$$

is cofinite in X.

This definition is somewhat technical. Note that each set of the form $\mathrm{Stab}_G(x) \cdot y$ contains only points at a fixed radius r from x. Therefore, if x_1 and x_2 are a symmetry pair for some (X, G), they must eventually have the same spheres of radius r (where a sphere consists of those vertices at distance exactly r from the center). The converse is not true; there are graphs X which have points which eventually have the same spheres, but are not symmetry pairs for (X, G).

For example, take the graph S_3 from Example 4.3 and let G be the group of automorphisms of S_3 generated by those coming from automorphisms of T_3 plus the automorphism which switches *all* pairs x_1 and x_2 coming from a single vertex in T_3. Then G acts transitively on S_3 and (S_3, G) has no symmetry pairs. However, since $(S_3, \mathrm{Aut}(S_3))$ has symmetry pairs, there are pairs of vertices with the same spheres eventually.

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Theorem 5.3 will show that the full 2-shift on (S_3, G) has a G-factor which consists of independent $U[0, 1]$ random variables.

The results in this section all require that the pair (X, G) have no symmetry pairs. Before we proceed, we prove that if X is a Cayley graph of G, then (X, G) has no symmetry pairs, giving a wide class of graphs where this condition holds.

PROPOSITION 5.2. *Let X be a Cayley graph of an infinite group G. Then there are no symmetry pairs of (X, G).*

Proof. Let $x_1, x_2 \in X$ and $\gamma \in G$ with $\gamma \cdot x_1 = x_2$. Suppose that for some $y \in X$,

$$\mathrm{Stab}_G(x_1) \cdot y = \mathrm{Stab}_G(x_2) \cdot \gamma y. \tag{1}$$

A group G acting on one of its Cayley graphs must have $\mathrm{Stab}_G(x) = \{\mathrm{Id}\}$ for all $x \in X$. Then (1) states that $y = \gamma y$, so $\gamma \in \mathrm{Stab}(y)$. This implies that $\gamma = \mathrm{Id}$ and $x_1 = x_2$. Therefore, two distinct vertices x_1 and x_2 cannot have any of the same stabilizer sets and so cannot form a symmetry pair. □

We now show that when X is a transitive graph with infinitely many ends and no symmetry pairs, the answer to Question 1.2 is positive. Note that any transitive graph with more than two ends has infinitely many ends.

THEOREM 5.3. *Let X be a connected graph with infinitely many ends. Suppose that $G \subseteq \mathrm{Aut}(X)$ acts transitively and there are no symmetry pairs of (X, G). If ξ is the full 2-shift on (X, G), then there is a G-factor of ξ which is i.i.d. $U[0, 1]$ on X.*

We start with a Lemma showing that we can get a forest percolation on X containing a tree with at least three ends as a G-factor of a subset of the bits in ξ. Benjamini *et al* proved [**1**] that if X is also unimodular, then any infinite component of such a percolation with at least three ends must have infinitely many ends. However, we do not make the assumption of unimodularity and Theorem 3.3 only requires at least three ends.

In the proof of Proposition 3.1, we reserved certain directional bits to give an equivariant structure to the tree X and then used this structure and the rest of the bits to define the factor map. Here, we want to reserve more bits to be used to define the forest structure. These bits will be called *seed bits* because they will be used to build a forest. The seed bits are defined similarly to the directional bits, as the bits at the center of certain configurations of bits on balls of a given radius.

LEMMA 5.4. *If X, G and ξ are as in Theorem 5.3, then there is an G-factor of ξ, depending only on the seed bits (defined in Step 1 of the proof below), which gives a percolation on the edges of a graph X for which almost every configuration is a forest and such that there is almost surely a component which is a tree with at least three ends.*

The construction will also give an G-equivariant random total ordering of the edges of X, $\prec$.

Proof. The proof will proceed in three steps. First, we define the seed bits of ξ, which will be used in constructing the forest percolation. Next, we use the seed bits to assign a random total ordering $\prec$ (depending on ξ) to the edges of X. The final step is to take the minimal spanning forest defined by $\prec$ on a modified version of X, which we can pull back

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to X to use as our percolation. Benjamini *et al* used the free minimal spanning forest for a very similar, but simpler, purpose in [**1**].

Step 1: define the seed bits. Let $B_r(o)$ be the ball of radius r centered at o in X. Take $r > 4$ large enough so that $X \setminus B_{r-2}(o)$ has at least three infinite components. Such an r is guaranteed to exist since X has infinitely many ends. By the transitivity of X, for any $x \in X$, $X \setminus B_r(x)$ has at least three infinite components. We call a bit $\xi(x)$ a *seed bit* if

$$\xi(y) = \begin{cases} 0, & 0 < d_X(x, y) < r - 1 \\ 1, & d_X(x, y) = r - 1, r \end{cases} \quad \text{or} \quad \xi(y) = \begin{cases} 1, & 0 < d_X(x, y) < r - 1 \\ 0, & d_X(x, y) = r - 1, r. \end{cases}$$

(Note that once the seed bits have been used to generate a forest contained in X, we will want to use Theorem 3.3 to define a factor of ξ which is i.i.d. $U[0, 1]$ on X. If we use r in the definition given in the proof of Proposition 3.1 of the directional bits used to construct this factor, they will be disjoint from the seed bits.)

Step 2: define the total ordering $\prec$. At each vertex x in X, we define a random variable $\theta(x)$. While these random variables will not in general be independent or uniform, we can ensure that they are almost surely distinct and so provide a total ordering of the vertices of X. We can then use this ordering to give a total ordering to the edges of X in an obvious way.

Fix a vertex o in x. Choose an ordering $o = y_0, y_1, y_2, \ldots$ for the vertices of X. Then at each vertex x, we can define $\prec_x$ to be a $\mathbb{Z}$-ordering on the sets $\mathrm{Stab}(x) \cdot y$ where

$$\mathrm{Stab}_G(x) \cdot y \prec_x \mathrm{Stab}_G(x) \cdot y' \quad \text{if and only if for some (all) } \gamma \in G \text{ with } \gamma(o) = x,$$
$$\min\{i : \gamma(y_i) \in \mathrm{Stab}_G(x) \cdot y\} < \min\{j : \gamma(y_j) \in \mathrm{Stab}_G(x) \cdot y'\}.$$

Note that this ordering is equivariant under G: for any $\gamma \in G$,

$$\mathrm{Stab}_G(x) \cdot y \prec_x \mathrm{Stab}_G(x) \cdot y' \quad \text{if and only if } \mathrm{Stab}_G(\gamma x) \cdot \gamma y \prec_{\gamma x} \mathrm{Stab}_G(\gamma x) \cdot \gamma y'.$$

Now, define $\theta_k(x)$ to be the mod-2 sum of all seed bits contained in the kth smallest (under $\prec_x$) set $\mathrm{Stab}_G(x) \cdot y$. If the set contains no seed bit, then $\theta_k(x) = 0$. We can take these bits $\theta_k(x)$ to be the binary expansion for a random variable $\theta(x)$. X contains no symmetry pairs, so for any pair of vertices x_1 and x_2, if $\gamma x_1 = x_2$, then $\mathrm{Stab}_G(x_1) \cdot y \neq \mathrm{Stab}_G(x_2) \cdot \gamma y$ infinitely often. Therefore, the symmetric difference between these two sets will contain a seed bit infinitely often and this guarantees that $\theta(x_1) \neq \theta(x_2)$ almost surely. Since there are only countably many pairs of vertices, θ gives a total ordering of the vertices of X almost surely. We use this ordering of the vertices to give a total ordering $\prec$ of the edges of X, in an obvious way.

Step 3: define the percolation on X. We need to modify the graph X a bit to get around the fact that our random variables used to determine $\prec$ are not i.i.d. and uniform. Let

$$W = W(\xi) = \{x \in X : \xi(x) \text{ is a seed bit}\}.$$

To form the new graph $X' = X'(\xi)$, identify all of the vertices in $B_{r-2}(x)$ for each $x \in W$, leaving all other vertices of X untouched. By the way in which the seed bits were defined, these balls of radius $r - 2$ are disjoint and so only finitely many vertices are identified

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to a single vertex in X'. Include an edge e from X in X' if: (1) neither endpoint of e is in $\bigcup_{x\in W} B_{r-2}(x)$; or (2) e is the smallest edge connecting $B_{r-2}(x)$ to a component of $X \setminus B_{r-2}(x)$ for some $x \in W$, so that there is exactly one edge in X' from x to each such component. The vertices and edges in X' will be referred to by their labels from X; in particular, the vertex in X' formed by identifying everything in $B_{r-2}(x)$ will still be called x. We also denote by $\prec$ the total ordering of the edges of X' inherited from $\prec$ on X.

We now construct the minimal spanning forest $T'(\prec)$ on X'. To form $T'(\prec)$, remove an edge e from X' if and only if e is the maximal edge contained in some simple cycle of X'. A simple cycle is a cycle in the graph which crosses each edge and each vertex only once. It is easy and standard that each component of $T'(\prec)$ is an infinite tree. We will show that for each $x \in W$, the component $K(x)$ containing x in $T'(\prec)$ must have at least three ends.

Let $A_1, A_2, \ldots, A_m$ be the components of $X' \setminus \{x\}$. If e_i is the unique edge connecting x to A_i, then e_i is in $T'(\prec)$ since there are no simple cycles containing e_i. Consider any finite set A in A_i which contains the endpoint of e_i which is not x. Let e be the smallest edge from A to $A_i \setminus A$ in X'. Then e must also be in $T'(\prec)$ because any simple cycle in X' containing e must contain another, larger edge connecting A and $A_i \setminus A$. Therefore, A cannot be disconnected from its complement in A_i and there must be an infinite non-self-intersecting path in $A_i \cup \{x\}$ starting from x. This proves that $K(x)$ has at least three ends.

Now we pull back $T'(\prec)$ to get a percolation $T(\prec)$ on X. The only place where the pull-back is not well defined is on $B_{r-2}(x)$ in X when $x \in W$. On each such finite set $B_{r-2}(x)$, we take the minimal spanning tree using $\prec$ again. The resulting percolation is a spanning forest of X and contains a component tree with at least three ends. □

Theorem 5.3 follows easily from Lemma 5.4 and Theorem 3.3.

Proof of Theorem 5.3. Apply Lemma 5.4 to get a forest percolation and an ordering $\prec$ of the edges of X. Let $K = K(\xi)$ be any component of the forest percolation with at least three ends. Successively remove all of the leaves of K until we get a subgraph $K' = K'(\xi) \subseteq X$ with no leaves. Since K has at least three ends, K' is still an infinite tree with at least three ends. Conditioning on the seed bits and hence on the forest percolation, we can apply our tree theorem, Theorem 3.3, to each K' and get independent uniform random variables $\zeta'(x)$ at each vertex x in K'. The only modification that needs to be made to the proof of Theorem 3.3 is that both directional and seed bits need to be avoided when computing $\zeta'(x)$.

To define random variables at all vertices of X, let $F = F(\xi)$ be the set of all vertices in X which are in some set K' as constructed above. We will distribute the bits from the random variables ζ' as we did in the proof of Theorem 3.3. For each $y \in X$, let $c(y)$ be the closest vertex in F to y. In case of a tie, we can take the smallest such vertex under the lexicographic order on the paths starting from y given by $\prec$. Let $V_x = \{y : c(y) = x\}$. The graph distance together with $\prec$ allow us to list the vertices in V_x: $x = y_0, y_1, y_2, \ldots$ in a way that is G-equivariant. We can now distribute the bits in $\zeta'(x)$ in a triangular array to all of the vertices in V_x, as was done in the proof of Theorem 3.3.

The resulting random variables $\zeta(x)$, defined on all of X, are independent and $U[0, 1]$ for each configuration of seed bits. Therefore, the unconditioned process ζ must also consist of independent $U[0, 1]$ random variables and the proof is complete. □

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Remark 5.5. This proof can be easily extended to connected graphs X with no symmetry pairs of (X, G) and infinitely many ends, on which G acts quasitransitively.

The author knows of no example of a graph X with a closed group G acting transitively such that there are symmetry pairs of (X, G), but no non-trivial automorphism $\gamma \in G$ fixes all but finitely many vertices of X, although this may be possible. Recall that the existence of such a γ was the obstruction discussed in §4 to the existence of a G-factor from the full 2-shift to the independent $U[0, 1]$ process on (X, G).

6. *Factors of processes on transitive graphs with one end*

We now have a reasonably good idea of how to answer Question 1.2 in the case of graphs with infinitely many ends and a transitive group acting, with the only gap the possibility of a graph with symmetry pairs but no non-trivial automorphism which fixes a cofinite set of vertices. We turn our attention to transitive graphs with finitely many ends. A locally-finite transitive non-amenable graph with finitely many ends must have exactly one end, so we are left with the following question.

Question 6.1. Let X be a one-ended non-amenable transitive or quasitransitive graph with no symmetry pair. Does there exist an m such that there is an $\mathrm{Aut}(X)$-factor which takes the full 2^m-shift process ξ on $(X, \mathrm{Aut}(X))$ to the i.i.d. $U[0, 1]$ process on $(X, \mathrm{Aut}(X))$?

If there is a factor percolation of a full 2^m-shift process ξ on X which has infinitely many ends, the proof of Theorem 5.3 can be adapted fairly easily to show that the i.i.d. $U[0, 1]$ process on X is an $\mathrm{Aut}(X)$-factor of the full 2^{m+1}-shift on X, since the extra bit can be used to define the factor using the many-ended percolation. Theorem 6.3 below is an example of this. A theorem from percolation theory is relevant to the existence of such a factor with infinitely many ends. First, some terminology. By Bernoulli(p) percolation, we mean Bernoulli site percolation: each vertex is included in the percolation independently with probability p.

The critical value $p_c(x)$ for Bernoulli percolation on a graph is the supremum of all p such that there are no infinite clusters in Bernoulli(p) percolation on X. Another phase transition may happen when the percolation goes from having infinitely many infinite components to having a unique infinite component. The infimum of all p such that Bernoulli(p) percolation has a unique infinite cluster is called p_u.

For amenable graphs such as $\mathbb{Z}^d$, $p_c = p_u$ (this was proved by Burton and Keane [**4**] for the d-dimensional lattice and Benjamini and Schramm note in [**2**] that their proof works for general quasitransitive amenable graphs). However, this is not always the case for non-amenable quasitransitive graphs. We are interested in the case where $p_c < p_u$ because of the following theorem.

THEOREM 6.2. (Häggström *et al* [**6**]) *If X is a quasitransitive connected graph and $p_c < p < p_u$, then each of the infinitely many infinite components of* Bernoulli(p) *percolation on X has infinitely many ends almost surely.*

This theorem shows that if X is a connected transitive graph with $p_c < p_u$, then for m sufficiently large the full 2^m-shift process on (X, G) has a G-factor percolation with a

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component which has infinitely many ends. This allows us to give a positive answer to Question 6.1 when $p_c < p_u$ in the absence of symmetry pairs.

THEOREM 6.3. *Let X be a graph with $p_c < p_u$ and let G be a subgroup of the automorphism group of X which acts transitively and suppose that there are no symmetry pairs of (X, G). Let $\xi(m)$ be the full 2^m-shift on (X, G). Then for m sufficiently large, there is a G-factor of $\xi(m)$ which gives the i.i.d. $U[0, 1]$ process on X.*

Proof. Take m large enough so that there is a positive integer k with $p_c < k/2^{m-1} < p_u$. By Theorem 6.2, Bernoulli$(k/2^{m-1})$ percolation on X has infinitely many infinite components, each with infinitely many ends. This percolation can be achieved as a G-factor of $\xi(m)$, depending only on the first $m-1$ bits of the process. The remaining mth bit will be used in the rest of the proof to give the factor, as in the proof of Theorem 5.3.

Let ω be a configuration of this Bernoulli$(k/2^{m-1})$ percolation. The graph given by ω is not transitive in general, so we cannot apply Theorem 5.3 directly. However, most of the proof will go through to prove that, conditioned on ω, there is a factor taking the 2-shift (defined by the mth bit of $\xi(m)$) to the i.i.d. $U[0, 1]$ process. We sketch the steps below, highlighting the points where the proofs differ.

Use the mth bit of $\xi(m)$ to define seed bits as in Step 1 of the proof of Lemma 5.4, where r will be chosen below. Define the function θ on the vertices of X as in Step 2 of the proof of the same lemma. Note that the construction of θ did not require the presence of at least three ends, so we use the full graph X here rather than the percolation. Since (X, G) has no symmetry pairs, the values of θ at the vertices of X are almost surely distinct. θ then gives an ordering $\prec$ of the edges of X, which is inherited by the edges of the configurations ω.

In Step 3 of Lemma 5.4, we used the minimal spanning forest defined by $\prec$ on a modified graph to get a forest percolation containing a tree with at least three ends. In this part of the proof, we used the fact that the graph was transitive to choose an r such that $X \setminus B_{r-2}(x)$ had at least three infinite components for all $x \in X$.

Such an r may not exist for an infinite component of the configuration ω of our percolation. However, we can choose r sufficiently large that there is a positive probability p that for fixed x, if $K(x)$ is the component of x in ω, then $K(x) \setminus B_{r-2}(x)$ has at least three infinite components. The seed bits are chosen independent of the percolation graphs ω, so there is a positive probability that a vertex x with a seed bit at x will have the property that $K(x) \setminus B_{r-2}(x)$ has at least three infinite components and we will call these good seeds. Construct a forest percolation $T(\prec) \subset \omega$ using the method mapped out in Step 3 of the proof of Lemma 5.4. Any good seed will be contained in a tree with at least three ends in $T(\prec)$.

The rest of the proof proceeds as in the proof of Theorem 5.3. We trim the trees in $T(\prec)$ with three or more ends so that they have no leaves and apply Theorem 3.3 (again using just the mth bit of $\xi(m)$), avoiding the seed bits as well as the directional bits. The infinitely many independent bits collected on the nodes which are contained in the trimmed trees can be distributed to the rest of the graph X in a triangular array, as was done in Stage 3 of Theorem 3.3. This completes the construction of the desired G-factor. □

Benjamini and Schramm conjectured in [**2**] that all non-amenable transitive graphs X have $p_c < p_u$ and this conjecture is still open. A proof of the full conjecture would give

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a positive answer to Question 6.1 for all (X, G) with no symmetry pairs. Some partial progress has been made on this conjecture, which is summarized in [**8**]. In particular, it has been shown for the following special cases.

- X is a planar graph. This is due to Lalley [**7**] in certain cases and to Benjamini and Schramm [**3**] in general.
- X has a vertex-isoperimetric constant which is at least $d/\sqrt{2}$, where d is the degree of X. This is due to Schonmann [**14**].

Pak and Smirnova-Nagnibeda [**10**] have also shown that if G is a finitely-generated non-amenable group, then it has a Cayley graph X with $p_c < p_u$. Together with Theorem 6.3 and Proposition 5.2, this gives the following result.

THEOREM 6.4. *Let G be any finitely-generated non-amenable group. Let $\xi(m) = \{\xi_g(m)\}_{g\in G}$ be the full 2^m-shift G-process on the elements of G. Then for m sufficiently large, there is a G-factor of $\xi(m)$ which gives the i.i.d. $U[0, 1]$ process on G.*

7. *Open questions*

There are many more open questions regarding when one process can be obtained as a G-factor of another process. We list a few of these here.

The example of Ornstein and Weiss given in §2 shows that for any m,n, the full m-shift on the Cayley graph of $\mathbb{F}_n$ is an $\mathbb{F}_n$-factor of the full n-shift on the same graph. This raises the question of whether these processes are isomorphic.

Question 7.1. (Ornstein and Weiss [**9**]) Let X be a transitive non-amenable graph and let G be a non-amenable subgroup of $\mathrm{Aut}(X)$. If $m \neq n$, can there be an *invertible* G-factor map from the full m-shift process on X to the full n-shift process on X? Probably the simplest version of this question is, are there distinct full m-shifts on the standard Cayley graph of $\mathbb{F}_n$ which are $\mathbb{F}_n$-conjugate?

Another approach to the question above is to try to find a conjugacy invariant that distinguishes between m-shifts and n-shifts when $m \neq n$ on, for instance, $(X_{2n}, \mathbb{F}_n)$. Such an invariant would be very interesting.

The factors given in this paper are not invertible and so do not give an answer to Question 7.1. In fact, they are often continuum-to-one (this is the case for the factor given in Proposition 2.3 for instance). This gives rise to the following question.

Question 7.2. Assume that X is such that the full 2-shift process on X has a factor which is i.i.d. $U[0, 1]$. Does there exist such a factor which is almost surely finite-to-one/bounded-to-one/uniformly finite-to-one/invertible?

In §5, we constructed a forest as a factor of the full 2-shift on certain graphs X. This raises questions about what kinds of graph-theoretic structures can be achieved as factors of i.i.d. processes on the vertices of a graph X. Below are examples of this type of question.

Question 7.3. (A. Kechris, personal communication to R. Lyons) Let ζ be an i.i.d. $U[0, 1]$ process on X, the standard Cayley graph of $\mathbb{Z}_3 * \mathbb{Z}_3$. Does there exist a $(\mathbb{Z}_3 * \mathbb{Z}_3)$-factor of ζ of X which gives an independent set of 'density' $\frac{1}{3}$ in X? An independent subset of a

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graph is a set of vertices span no edge of the graph, so such a factor graph would choose exactly one vertex from each triangle in X. An answer of 'no' to this question would have interesting consequences.

Since T_3, the regular tree of degree three, is the line graph of the Cayley graph of $\mathbb{Z}_3 * \mathbb{Z}_3$, $\mathbb{Z}_3 * \mathbb{Z}_3$ acts on T_3. As noted by Bálint Virág, the question above is equivalent to the existence of a $(\mathbb{Z}_3 * \mathbb{Z}_3)$-factor bond percolation of the i.i.d. $U[0, 1]$ process on T_3 which gives a perfect matching of T_3 almost surely. Recall that a subgraph of a graph is a perfect matching if all of the vertices are spanned by exactly one edge of the subgraph.

More generally, one can ask which graphs X and groups G have a G-factor taking the i.i.d. $U[0, 1]$ process on (X, G) to a perfect matching on X?

Question 7.4. (R. Lyons, personal communication) Let T_4 be the regular tree of degree 4 and let ζ be i.i.d. $U[0, 1]$ on T_4. Does there exist an $\mathrm{Aut}(T_4)$-factor of ξ which gives almost every configuration the structure of the Cayley graph of $\mathbb{F}_2$? In other words, is there a factor which gives each edge of T_4 a labelling of a or b and a direction in such a way that at each vertex there is one of each of the four possible outcomes, a, a^{-1}, b, b^{-1}? This question is really asking about how different the actions of $\mathbb{F}_2$ and $\mathrm{Aut}(T_4)$ are on T_4.

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