Ergod. Th. & Dynam. Sys. (2009), **29**, 667–683
doi:10.1017/S0143385708080279
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Printed in the United Kingdom

Poisson suspensions and infinite ergodic theory

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(*Received* 7 *February* 2007 *and accepted in revised form* 25 *February* 2008)

Abstract. We investigate the ergodic theory of Poisson suspensions. In the process, we establish close connections between finite and infinite measure-preserving ergodic theory. Poisson suspensions thus provide a new approach to infinite-measure ergodic theory. Fields investigated here are mixing properties, spectral theory, joinings. We also compare Poisson suspensions to the apparently similar looking Gaussian dynamical systems.

1. *Introduction*

Poisson measures are classical constructions from probability theory. They are the discrete counterpart of Brownian motion and they together give way to an enormous proportion of probabilistic investigations. In measure-preserving ergodic theory, there is a functorial way to associate two dynamical systems: one associated to a probability measure (the Poisson measure) and one associated to a σ-finite measure (the base of the Poisson measure). The measure-preserving transformation T that sends a point to another on the base becomes a probability-preserving transformation T_* on points configurations by sending a configuration to another where each point is sent to its image according to T. A Poisson suspension is a Poisson measure with this kind of transformation.

After some generalities on Poisson suspensions, we recall the basics of infinite measure-preserving ergodic theory that are needed to study Poisson suspensions. Theorem 3.4 is the interpretation of mixing properties of the suspension in terms of asymptotic behaviours of the base. Part of these results was known for a long time but we believe it is worth giving the complete picture.

We then introduce Poissonian joinings, whose structure is completely clarified in Theorem 4.2 and based on strong connections with joinings at the bases level. These joinings (under different names) were introduced in parallel in [**6**] and in the PhD thesis of the author, with a different, totally probabilistic, but equivalent definition. In this paper we recall both definitions and extend the characterization of these joinings. A specific notion of disjointness (called ID-disjointness), weaker than the usual notion can then be introduced and provides a probabilistic analogue of the notion of strong disjointness

https://doi.org/10.1017/S0143385708080279 Published online by Cambridge University Press

introduced by Aaronson for infinite measure-preserving transformations. In particular (Theorem 4.8) we prove that two ergodic systems are strongly disjoint if and only if their Poisson suspensions are ID-disjoint. Since disjointness implies ID-disjointness, this result gives a useful 'finite-measure' criterion of strong disjointness. Surprisingly, ID-disjointness mixed with the theory of Gaussian dynamical systems allows us to derive a new result on spectral theory of infinite ergodic measure-preserving transformation: we prove that a whole family of continuous measures (among them, measures supported on Kronecker sets) cannot be realized as spectral measures.

In §5, as an illustration of what we have developed so far, we focus on Poisson suspensions with simple spectrum (in fact, the results in §5 are a bit more general) where many nice structural features appear very clearly, making the functorial relationship between the base and the suspension even closer. Throughout this work, we will see many results that are similar to the Gaussian case, proofs being often different. However, Section 5 shows some drastic differences between Poisson suspensions and Gaussian systems.

The tools developed in this paper will find further applications on papers in preparation, one specifically devoted to Poissonian joinings ([**15**] in collaboration with François Parreau), the other on the entropy of infinite-measure systems ([**9**], in collaboration with Elise Janvresse, Tom Meyerovitch and Thierry de la Rue).

2. *Poisson suspensions*

2.1. *Poisson measure.* Let $(X, \mathcal{A}, \mu)$ be a σ-finite Lebesgue space with a continuous infinite measure μ and consider $(X^*, \mathcal{A}^*)$ the space of measures on $(X, \mathcal{A})$ where $\mathcal{A}^*$ is the σ-algebra generated by the functions $N(A) : \nu \mapsto \nu(A)$, $A \in \mathcal{A}$, $\nu \in X^*$.

There exists an isomorphism φ between $(X, \mathcal{A}, \mu)$ and $(\mathbb{R}, \mathcal{B}, \lambda)$ where λ is the Lebesgue measure. Let $\widetilde{X^*} \subset X^*$ be the measurable set defined by the relations $\nu \in \widetilde{X^*} \iff \forall n \in \mathbb{Z},\ \nu(\varphi^{-1}[n, n+1]) \in \mathbb{N}$. On $(\widetilde{X^*}, \widetilde{\mathcal{A}^*})$, we can easily form the probability measure μ^* such that for every finite family of disjoint finite μ-measure sets $A_1, \ldots, A_k$, the $N(A_i)$ are independent and distributed as Poisson random variables of parameter $\mu(A_i)$. We can extend μ^* to $(X^*, \mathcal{A}^*)$ by setting $\mu^*(X^* \setminus \widetilde{X^*}) = 0$. If we consider that $\mathcal{A}^*$ is completed with respect to μ^*, $(X^*, \mathcal{A}^*, \mu^*)$ is a Lebesgue space and called the Poisson measure over the base $(X, \mathcal{A}, \mu)$. This construction is independent of the choice of the isomorphism φ.

2.2. *Poissonian map, Poissonian factor, Fock space structure.* If Ψ is a measure-preserving map from $(X_1, \mathcal{A}_1, \mu_1)$ to $(X_2, \mathcal{A}_2, \mu_2)$, call the map $\Psi_* : \nu \mapsto \nu \circ \Psi^{-1}$ a *Poissonian map*. Ψ_* is a measure preserving map from $(X_1^*, \mathcal{A}_1^*, \mu_1^*)$ to $(X_2^*, \mathcal{A}_2^*, \mu_2^*)$.

Let T be an automorphism of $(X, \mathcal{A}, \mu)$; the Poissonian map T_* is an automorphism of $(X^*, \mathcal{A}^*, \mu^*)$ called *Poissonian automorphism*. The dynamical system $(X^*, \mathcal{A}^*, \mu^*, T_*)$ is the *Poisson suspension* over the base $(X, \mathcal{A}, \mu, T)$.

Consider $\mathcal{F}$ a factor of $\mathcal{A}$ (where μ restricted to $\mathcal{A}$ is not necessarily σ-finite), then $\mathcal{F}^* := \sigma\{N(F),\ F \in \mathcal{F}\}$ is a factor of $\mathcal{A}^*$ called the Poissonian factor associated to $\mathcal{F}$. Two factors of $\mathcal{A}$ which do not differ on positive finite measure sets have the same Poissonian factor.

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It is well known that $L^2(\mu^*)$ is the Fock space over $L^2(\mu)$, i.e.

$$L^2(\mu^*) \simeq \mathbb{C} \oplus L^2(\mu) \oplus L^2(\mu)^{\otimes 2} \oplus \cdots \oplus L^2(\mu)^{\otimes n} \oplus \cdots ,$$

the isomorphism being given through multiple stochastic integrals. Considered as a subspace of $L^2(\mu^*)$, $L^2(\mu)^{\otimes n}$ is called the nth *chaos*.

An operator Ψ with norm less than 1 on $L^2(\mu)$ gives way to an operator $\widetilde{\Psi}$ on $L^2(\mu^*)$ which acts as $\Psi^{\otimes n}$ on $L^2(\mu)^{\otimes n}$; we call $\widetilde{\Psi}$ the *exponential* of Ψ. In particular, U_{T_*} is the exponential of U_T. Therefore, if σ is the maximal spectral type of T, $\sum_{k\geq 1}(1/k!)\sigma^{*k}$ is the reduced maximal spectral type of T_*.

If $\mathcal{F}$ is a σ-finite factor and $E_{\mathcal{F}}$ the corresponding conditional expectation, it is easy to see that $E_{\mathcal{F}^*}$, the conditional expectation corresponding to $\mathcal{F}^*$, is the exponential of $E_{\mathcal{F}}$.

3. *Infinite ergodic theory and mixing properties of suspensions*

3.1. *Categories of dynamical systems.* Consider a system $(X, \mathcal{A}, \mu, T)$ where $(X, \mathcal{A}, \mu)$ is a Lebesgue space and T a measure-preserving automorphism.

Most of the following terminology is taken from [**11**].

Definition 3.1. T is said to be:

- *dissipative*, if there exists a set W such that $T^n W \cap T^m W = \emptyset$ (W is called a *wandering set*) if $n \neq m$ and $X = \bigcup_{n\in\mathbb{Z}} T^n W$ mod μ;
- *conservative*, if there is no wandering set;
- *remotely infinite*, if there exists a σ-finite sub-σ-field $\mathcal{G} \subset \mathcal{A}$ such that, mod μ,

$$T^{-1}\mathcal{G} \subseteq \mathcal{G},$$
$$\bigvee_{n\in\mathbb{Z}} T^{-n}\mathcal{G} = \mathcal{A},$$

and

$$\mathcal{A}_f \bigcap_{n\in\mathbb{Z}} T^{-n}\mathcal{G} = \emptyset,$$

where $\mathcal{A}_f$ are the elements of $\mathcal{A}$ of finite positive μ-measure;
- of *zero type*, if, for all sets A of finite measure,

$$\lim_{n\to\infty}\mu(A \cap T^n A) = 0;$$

- of *type* $\mathbf{II}_\infty$, if it is conservative and if there is no T-invariant probability measure $\tilde{\mu}$ such that $\tilde{\mu} \ll \mu$;
- *rigid*, if there exists a sequence $n_k \uparrow \infty$ such that $\lim_{k\to\infty}\mu(A \triangle T^{n_k} A) = 0$.

The notion of rigidity allows us to introduce a new notion for type $\mathbf{II}_\infty$ systems, parallel to mild mixing for probability-measure-preserving systems.

Definition 3.2. Consider a dynamical system $(X, \mathcal{A}, \mu, T)$.

T is said to be *rigidity free* if for every sequence $n_k \uparrow \infty$ and set A of finite measure, $\underline{\lim}_{k\to\infty}\mu(A \triangle T^{n_k} A) > 0$.

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3.2. *Mixing properties of Poisson suspensions.* We will make the connections between the above definitions and the mixing properties of the corresponding Poisson suspensions. First we need this simple continuity lemma.

LEMMA 3.3. *Let $\{\mathcal{A}_n\}_{n\in\mathbb{N}}$ be a collection of σ-finite sub-σ-fields of $\mathcal{A}$ such that $\mathcal{A}_n \subset \mathcal{A}_m$ if $n \geq m$. We have*

$$\left(\bigcap_{n\in\mathbb{N}} \mathcal{A}_n\right)^* = \bigcap_{n\in\mathbb{N}} \mathcal{A}_n^*$$

and, with $\{\mathcal{A}_n\}_{n\in\mathbb{N}}$ a collection of σ-finite sub-σ-fields of $\mathcal{A}$ such that $\mathcal{A}_n \subset \mathcal{A}_m$ if $n \leq m$,

$$\left(\bigvee_{n\in\mathbb{N}} \mathcal{A}_n\right)^* = \bigvee_{n\in\mathbb{N}} \mathcal{A}_n^*.$$

Proof. First note that $E_{\mathcal{A}_n}$ tends to $E_{\bigcap_{n\in\mathbb{N}} \mathcal{A}_n}$ weakly on $L^2(\mu)$ and $E_{\mathcal{A}_n^*}$ tends to $E_{\bigcap_{n\in\mathbb{N}} \mathcal{A}_n^*}$ weakly on $L^2(\mu^*)$. Since $E_{\mathcal{A}_n^*}$ acts as $E_{\mathcal{A}_n}^{\otimes n}$ on $L^2(\mu)^{\otimes n}$ which tends to $E_{\bigcap_{n\in\mathbb{N}} \mathcal{A}_n}^{\otimes n}$, we deduce that $E_{\bigcap_{n\in\mathbb{N}} \mathcal{A}_n^*}$ coincide with $E_{\bigcap_{n\in\mathbb{N}} \mathcal{A}_n}^{\otimes n}$ on $L^2(\mu)^{\otimes n}$. That is $E_{\bigcap_{n\in\mathbb{N}} \mathcal{A}_n^*}$ $= E_{(\bigcap_{n\in\mathbb{N}} \mathcal{A}_n)^*}$, which proves the first point. The proof of the second assertion is similar. □

THEOREM 3.4. *The following facts hold.*

(1) *T is of type $\mathbf{II}_\infty$ on its conservative part if and only if T_* is ergodic (and then weakly mixing).*
(2) *T is rigid (for the sequence n_k) if and only if T_* is rigid (for the sequence n_k).*
(3) *T is rigidity free if and only if T_* is mildly mixing.*
(4) *T is of zero type if and only if T_* is mixing (and then mixing of any order).*
(5) *If T is remotely infinite, then T_* is* K.
(6) *If T is dissipative, then T_* is Bernoulli.*

Proof. Points (1) and (4) are apparently due to Marchat in his PhD dissertation as pointed out in [**8**]. We have given our proof of these facts in [**18**]. Point (6) is folklore.

If T is rigid for the sequence n_k, any vector of the first chaos of T_* is a rigid vector for the sequence n_k. But since the smallest σ-algebra containing the first chaos is the whole σ-algebra $\mathcal{A}^*$, T_* is rigid for the sequence n_k. The converse is obvious.

If T is rigidity free, it means that its maximal spectral type σ does not put mass on weak Dirichlet sets. It is easily checked that this property is also shared by its successive convolution powers, and then by the measure $\sum_{k\geq 1}(1/k!)\sigma^{*k}$. This implies that T_* is mildly mixing. Once again, the converse is obvious.

If T is remotely infinite, by taking $\mathcal{G}$ as in the definition, $\mathcal{G}^*$ satisfies:

$$T_*^{-1}\mathcal{G}^* \subseteq \mathcal{G}^*,$$
$$\bigvee_{n\in\mathbb{Z}} T_*^{-n}\mathcal{G}^* = \mathcal{A}^*,$$

and

$$\bigcap_{n\in\mathbb{Z}} T_*^{-n}\mathcal{G}^* = \{\emptyset, X^*\}$$

thanks to Lemma 3.3. Thus T_* is K. □

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4. *Poissonian joinings*

We present here a natural family of joinings between Poisson suspensions. These joinings were introduced in [**6**] and in parallel, with a different point of view in [**17**] under the name *ID joinings*.

4.1. *Infinite divisibility of a Poisson suspension.* Loosely speaking, a probability measure is infinitely divisible (abbreviated ID) if it is, for all positive integer n, the nth convolution power of certain probability measure (see [**19**] for infinite divisibility on $\mathbb{R}^d$).

Consider two bases $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$. The sum of two elements in $(X^*, \mathcal{A}^*)$ and $(Y^*, \mathcal{B}^*)$ is defined as the usual sum of two measures. The convolution operation $*$ is thus well defined for distributions on $(X^*, \mathcal{A}^*)$ and $(Y^*, \mathcal{B}^*)$, and so is infinite divisibility. The well-known formula $\mu^* = (((1/n)\mu)^*)^{*n}$ for all $k \geq 1$ makes the intrinsic ID nature of a Poisson measure precise.

Infinite divisibility is also well defined on $(X^* \times Y^*, \mathcal{A}^* \otimes \mathcal{B}^*)$, the convolution operation coming from the sum coordinate-wise.

Definition 4.1. A *Poissonian joining* of $(X^*, \mathcal{A}^*, \mu^*, T_*)$ and $(Y^*, \mathcal{B}^*, \nu^*, S_*)$ is any joining $(X^* \times Y^*, \mathcal{A}^* \otimes \mathcal{B}^*, \rho, T_* \times S_*)$ such that ρ is ID.

4.2. *KLM measure.* To derive results on the structure of Poissonian joinings of Poisson suspensions, we need to consider the notion of KLM measure (see [**14**] and [**5**]) of an ID point process which plays exactly the same role as does the Lévy measure for ID processes (see [**13**]). KLM measures are measures on $(X^*, \mathcal{A}^*)$ and their existence is ensured if $(X, \mathcal{A})$ is a complete separable metric space. As it can be checked from the definition in [**14**], KLM measures can be extended to the product space $(X^* \times X^*, \mathcal{A}^* \otimes \mathcal{A}^*)$ corresponding to bivariate ID point processes.

Here $(X, \mathcal{A})$ stands for the real line or the disjoint union of two copies of it, endowed with their Borel sets.

If $\mathbb{P}$ is the distribution of an ID point process on $(X, \mathcal{A})$, there exists a unique σ-finite measure Q on $(X^*, \mathcal{A}^*)$ such that $Q(\mu_0) = 0$ (μ_0 denotes the null measure) which characterizes completely the distribution of an ID point process through the formula:

$$\int_{X^*} \exp\left(-\int_X f(x)\nu(dx)\right)\mathbb{P}(d\nu) = \exp\left[\int_{X^*}\left(\exp -\int_X f(x)\nu(dx)\right) - 1\,Q(d\nu)\right]$$

for any non-negative function with compact support f.

If m is a Radon measure on $(X, \mathcal{A})$ and m^* the distribution of the Poisson measure over $(X, \mathcal{A}, m)$, it is easy to see, thanks to the classical formula,

$$\int_{X^*} \exp\left(-\int_X f(x)\nu(dx)\right)m^*(d\nu) = \exp\left[\int_X (e^{-f(x)} - 1)m(dx)\right]$$

that the KLM measure Q_m of m^* is the image measure of m by the application $\Delta : x \to \delta_x$ from $(X, \mathcal{A})$ to $(X_0^*, \mathcal{A}_0^*)$ where $X_0^* \subset X^*$ is the set of Dirac masses. Then we can describe the KLM measure of a Poisson measure as the measure consisting of one Dirac mass 'randomized' according to the measure on the base.

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4.3. *Structure of Poissonian joinings.* We will deduce the structure of a Poissonian joining of distribution $\mathfrak{P}$ of Poisson suspensions built over $(X, \mathcal{A}, \mu, T)$, $(Y, \mathcal{B}, \nu, S)$ where $(X, \mathcal{A}, \mu) = (Y, \mathcal{B}, \nu) = (\mathbb{R}, \mathcal{B}, \lambda)$.

Let $\mathfrak{P}$ be the distribution of a Poissonian joining between $(X, \mathcal{A}, \mu, T)$ and $(Y, \mathcal{B}, \nu, S)$. Call $Q_{\mathfrak{P}}$ the KLM measure of $\mathfrak{P}$ and decompose it along the $T_* \times S_*$-invariant subsets $A = X^* \times \{\mu_0\}$, $B = \{\mu_0\} \times Y^*$ and $C = X^* \times Y^* \setminus (A \cup B)$, thus $Q_{\mathfrak{P}} = Q_1 \otimes \delta_{\mu_0} + \delta_{\mu_0} \otimes Q_2 + \widetilde{Q}$ where the three members of the sum are the restriction to A, B and C of $Q_{\mathfrak{P}}$. Q_1 and Q_2 are the KLM measures of point processes on X^* and Y^* and satisfy $Q_\mu = Q_1 + \widetilde{Q} \circ \pi_X^{-1}$ and $Q_\nu = Q_2 + \widetilde{Q} \circ \pi_Y^{-1}$.

Since $Q_1 \ll Q_\mu$, $Q_2 \ll Q_\nu$, $\widetilde{Q} \circ \pi_X^{-1} \ll Q_\mu$ and $\widetilde{Q} \circ \pi_2^{-1} \ll Q_\nu$, all of these measures are supported on the set $\{\delta_x,\ x \in \mathbb{R}\}$ and $\widetilde{Q}$ is supported on the set $\{\delta_x,\ x \in \mathbb{R}^2\}$. Denote by $\mu' = Q_1 \circ \Delta$, $\nu' = Q_2 \circ \Delta$, $\tilde{\mu} = (\widetilde{Q} \circ \pi_X^{-1}) \circ \Delta$, $\tilde{\nu} = (\widetilde{Q} \circ \pi_Y^{-1}) \circ \Delta$ and $\gamma = \widetilde{Q} \circ (\Delta \times \Delta)$.

If $\widetilde{\gamma^*}$ denotes the image of γ^* by the mapping $\pi_X^* \times \pi_Y^*$, we have

$$\mathfrak{P} = (\mu'^* \otimes \delta_{\mu_0}) * \widetilde{\gamma^*} * (\delta_{\mu_0} \otimes \nu'^*).$$

We have proved the following structure theorem.

THEOREM 4.2. *Let $(X^*, \mathcal{A}^*, \mu^*, T_*)$ and $(Y^*, \mathcal{B}^*, \nu^*, S_*)$ be two Poisson suspensions. Let $(X^* \times Y^*, \mathcal{A}^* \otimes \mathcal{B}^*, \mathfrak{P}, T_* \times S_*)$ be a Poissonian joining between them (i.e. $\mathfrak{P}$ is ID). There exists T-invariant measures μ' and $\tilde{\mu}$, S-invariant measures μ' and $\tilde{\mu}$ such that $\mu = \mu' + \tilde{\mu}$, $\nu = \nu' + \tilde{\nu}$ and γ is a $T_1 \times T_2$-invariant joining of $\tilde{\mu}$ and $\tilde{\nu}$. If $\tilde{\gamma^*}$ denotes the image of γ^* by the mapping $\pi_X^* \times \pi_Y^*$, we have*

$$\mathfrak{P} = (\mu'^* \otimes \delta_{\mu_0}) * \widetilde{\gamma^*} * (\delta_{\mu_0} \otimes \nu'^*).$$

Remark 4.3. We mention that Poissonian joinings of higher order can be defined in an obvious way. It is then easily checked that these joinings satisfy the PID property, i.e. any pairwise independent joining is in fact independent (two vectors $(X_1, X_2, \ldots, X_n)$ and $(Y_1, Y_2, \ldots, Y_k)$ such that their joint distribution is ID are independent if and only if for any choice (i, j), X_i and Y_j are independent).

4.4. *Poissonian joinings and Markov operators.* We present the point of view taken in [**6**].

A sub-Markov operator between $L^2(\mu)$ and $L^2(\nu)$ is a non-negative operator Φ such that $\Phi 1 \leq 1$ and $\Phi^* 1 \leq 1$.

A Poissonian joining in the sense of [**6**] is a joining such that the corresponding Markov operator acting between $L^2(\mu^*)$ and $L^2(\nu^*)$ acts as the exponential of a sub-Markov operator Φ between $L^2(\mu)$ and $L^2(\nu)$ intertwinning U_T and U_S. The author then proves that these joinings have the structure described in Theorem 4.2 thus defining the same family of joinings.

We will show that the hypothesis that Φ is a sub-Markov operator is automatically satisfied if Φ^* is a Markov operator. But let us prove first a slightly more general result which will be needed under this form in the last chapter.

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PROPOSITION 4.4. *Let $(X^*, \mathcal{A}^*, \mu^*)$ and $(Y^*, \mathcal{B}^*, \nu^*)$ be two Poisson measures and Ψ, a Markov operator from $L^2(\mu^*)$ to $L^2(\nu^*)$ which sends $\mathfrak{C}_X$, the first chaos of $L^2(\mu^*)$, to $\mathfrak{C}_Y$, the first chaos of $L^2(\nu^*)$. Then the restriction of Ψ to $\mathfrak{C}_X$ induces a sub-Markov operator Θ from $L^2(\mu)$ to $L^2(\nu)$.*

Proof. Let N_X and N_Y denote the random measures with distribution μ^* and ν^*. M_X and M_Y denote the extended centred random measures, i.e.

$$M_X(g) := \lim_{n\to\infty}\left(\int_Y g_n\, dN_Y - \int_Y g_n\, d\nu\right) \quad \text{in } L^2(\nu^*),$$

$g_n \in L^1(\nu) \cap L^2(\nu)$, $g \in L^2(\nu)$, $\lim_{n\to\infty} g_n = g$ in $L^2(\nu)$.

Θ is defined through the identification between $\mathfrak{C}_X$ and $L^2(\mu)$, and $\mathfrak{C}_Y$ and $L^2(\nu)$. We have to show it is a positive operator. It will be a consequence of the following fact: non-negative random variables belonging to $\mathbb{C} \oplus \mathfrak{C}_Y$ are of the form $\int_Y f\, dN + c$ where $f \in L^1(\nu) \cap L^2(\nu)$, $f \geq 0$ and $c \geq 0$.

First, we recall that the Lévy measure ρ of a positive infinitely divisible integrable random variable U satisfies $\rho(\mathbb{R}_-) = 0$, $\int_{\mathbb{R}_+} u\rho(du) < \infty$,

$$\mathbb{E}[e^{-\lambda U}] = \exp -\left[\lambda a + \int_{\mathbb{R}_+} 1 - e^{-\lambda u}\rho(du)\right] \quad \text{with } a \geq 0$$

and $\mathbb{E}[U] = a + \int_{\mathbb{R}_+} u\rho(du)$ (see [**19**, p. 285]).

Now, an element of $\mathbb{C} \oplus \mathfrak{C}_Y$ is of the form $M_Y(g) + b$ where $M_Y(g)$ is the centred stochastic integral of the function $g \in L^2(\nu)$ and b a constant. It is an ID random variable without Gaussian part and its Lévy measure is $\rho := (\nu \circ g^{-1})_{|\mathbb{R}^\star}$. If now $M_Y(g) + b$ is non-negative, we deduce that $\nu(g^{-1}(\mathbb{R}^\star_-)) = \rho(\mathbb{R}^\star_-) = 0$, thus $g \geq 0$. Since by hypothesis, $M_Y(g) + b \in L^2(\nu^*)$, it is also integrable, thus $\int_{\mathbb{R}_+} u\rho(du) = \int_Y g(y)\mu(dy) < \infty$. Then $g \in L^1(\nu) \cap L^2(\nu)$ and we can deduce that $M_Y(g) = \int_Y g\, dN_Y - \int_Y g\, d\nu$. Finally, since $\mathbb{E}[M_Y(g) + b] = b$, $b \geq \int_Y g\, d\nu$ and the result follows with $f = g$ and $c = b - \int_Y g\, d\nu$.

It is enough to check the positivity of Θ on indicator functions of sets of finite μ-measure. We have, by construction, $\Phi M_X(1_A) = M_Y(\Theta 1_A)$, but since 1_A is in $L^1(\mu) \cap L^2(\mu)$,

$$M_X(1_A) = \int_X 1_A\, dN_X - \int_X 1_A\, d\mu \quad \text{and} \quad \Phi M_X(1_A) = \Phi\int_X 1_A\, dN_X - \Phi\int_X 1_A\, d\mu.$$

$\int_X 1_A\, dN_X$ is a non-negative random variable, then $\Phi\int_X 1_A\, dN_X$ is also non-negative. From the first part of the proof, there exists $f \in L^1(\nu) \cap L^2(\nu)$, $f \geq 0$ and $c \geq 0$ such that $\Phi\int_X 1_A\, dN_X = \int_Y f\, dN_Y + c$.

Finally, since $\mathbb{E}[\Phi\int_X 1_A\, dN_X] = \int_X 1_A\, d\mu = \int_Y f\, d\nu + c$,

$$\Phi M_X(1_A) = \int_Y f\, dN_Y - \int_Y f\, d\nu = M_Y(f).$$

By identification, $\Theta 1_A = f \geq 0$ and since $c \geq 0$, $\int_Y \Theta 1_A\, d\nu \leq \int_X 1_A\, d\mu$.

Note that it is not true in general that, if g is non-negative and in $L^2(\nu)$, there exists b such that $M_Y(g) + b$ is non-negative.

By construction, the adjoint $\Theta^\star$ satisfies the same properties as Θ.

$\lambda(A \times B) = \int_B \Theta 1_A\, d\nu$ defines a measure on $(X \times Y, \mathcal{A} \otimes \mathcal{B})$ such that its projections satisfies $\lambda(\cdot \times Y) \leq \mu$ and $\lambda(X \times \cdot) \leq \nu$. Therefore, Θ is a sub-Markov operator. □

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As a direct consequence of the last proposition, we can now formulate a theorem analogous to Gaussian joinings of Gaussian dynamical systems.

THEOREM 4.5. *A joining between Poisson suspensions is Poissonian if and only the associated Markov operator is an exponential.*

Note that, from a structural point of view, it is not true in general that a Poissonian joining is itself (isomorphic to) a Poisson suspension. We are thus led to seek criteria ensuring that the Poissonian joining

$$(X^* \times Y^*, \mathcal{A}^* \otimes \mathcal{B}^*, (\mu'^* \otimes \delta_{\mu_0}) * \widetilde{\gamma^*} * (\delta_{\mu_0} \otimes \nu'^*), T_* \times S_*)$$

is isomorphic to the Poisson suspension

$$((X \times Y)^*, (\mathcal{A} \otimes \mathcal{B})^*, (\mu' \otimes \delta_0 + \gamma + \delta_0 \otimes \nu')^*, (T \times S)^*)$$

and not just a factor. This is obviously the case if $\gamma = 0$ or if γ is a graph joining or a sum of graph joinings. We prove this result in a non-trivial case which is of particular importance.

PROPOSITION 4.6. *In a suspension* $((X \times Y)^*, (\mathcal{A} \otimes \mathcal{B})^*, m^*)$, *if* $\mathcal{A} \times Y \cap X \times \mathcal{B}$ *is non-atomic mod m then* $(\mathcal{A} \times Y)^* \vee (X \times \mathcal{B})^*$ *equals* $(\mathcal{A} \otimes \mathcal{B})^*$ *mod* m^*. *Equivalently, in a suspension* $(X^*, \mathcal{A}^*, \mu^*)$, $\mathcal{C}_1 \subset \mathcal{A}$, $\mathcal{C}_2 \subset \mathcal{A}$: *if* $\mathcal{C}_1 \cap \mathcal{C}_2$ *is non-atomic, then* $\mathcal{C}_1^* \vee \mathcal{C}_2^* = (\mathcal{C}_1 \vee \mathcal{C}_2)^*$.

Proof. The problem can be translated into the following one: we have to reconstruct a Poisson measure on $X \times Y$ knowing its projections on X and Y respectively.

Note that the fact that $\mathcal{F} := \mathcal{A} \times Y \cap X \times \mathcal{B}$ is not trivial implies that $m_{/\mathcal{F}}$ is supported on a graph, say Φ between $(X_{/\mathcal{F}}, \mathcal{A}_{/\mathcal{F}}, \varphi_{1*}(m))$ and $(Y_{/\mathcal{F}}, \mathcal{B}_{/\mathcal{F}}, \varphi_{2*}(m))$, if we denote by φ_1 and φ_2 the factor maps from X to $X_{/\mathcal{F}}$ and from Y to $Y_{/\mathcal{F}}$ respectively (with a slight abuse, we consider $\mathcal{F} \subset \mathcal{A}$ and $\mathcal{F} \subset \mathcal{B}$).

The measure m gives full measure to the set A of points (x, y) such that $\Phi \circ \varphi_1(x) = \varphi_2(y)$, therefore m^* gives full measure to the set of the points $\nu = \sum_{k \in \mathbb{N}} \delta_{(x_k, y_k)} \in (X \times Y)^*$ such that, for any k, $(x_k, y_k) \in A$. Moreover, if $k \neq k'$, then $\varphi_1(x_k) \neq \varphi_1(x_{k'})$ because if it were not the case, the Poisson measure $((X_{/\mathcal{F}})^*, (\mathcal{A}_{/\mathcal{F}})^*, (\varphi_{1*}(m))^*)$ would have multiplicities and this is impossible since $\varphi_{1*}(m)$ is a continuous measure, as $\mathcal{F}$ was assumed non-atomic.

Thus, from the projections $\sum_{i \in \mathbb{N}} \delta_{x_i}$ and $\sum_{j \in \mathbb{N}} \delta_{y_j}$ coming from $\nu = \sum_{k \in \mathbb{N}} \delta_{(x_k, y_k)}$, we know how to reconstruct ν (and this is the only way to do it) since to every point x_i there is exactly one point x_j such that $(x_i, y_j) \in A$ from the preceding discussion. At last, as we've already said, each pair of point (x_k, y_k) of ν satisfy $(x_k, y_k) \in A$, thus there is no ambiguity in the reconstruction of ν from its projections. This shows that $(\mathcal{A} \times Y)^* \vee (X \times \mathcal{B})^*$ is nothing else than the whole σ-algebra $(\mathcal{A} \otimes \mathcal{B})^*$. □

We can now treat the important case of relatively independent joinings.

PROPOSITION 4.7. *Consider two dynamical systems* $(X, \mathcal{A}, \mu, T)$ *and* $(Y, \mathcal{B}, \nu, S)$, *with a common* σ*-finite factor* $(Z, \mathcal{C}, m, R)$, *and consider the relatively independent joining* $(X \times Y, \mathcal{A} \otimes \mathcal{B}, \mu \otimes_{\mathcal{C}} \nu, T \times S)$ *over* $(Z, \mathcal{C}, m, R)$. *Then* $(X^* \times Y^*, \mathcal{A}^* \otimes \mathcal{B}^*, \widetilde{(\mu \otimes_{\mathcal{C}} \nu)}^*, T_* \times S_*)$ *is the relatively independent joining of* $(X^*, \mathcal{A}^*, \mu^*, T_*)$ *and*

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$(Y^*, \mathcal{B}^*, \nu^*, S_*)$ over $(Z^*, \mathcal{C}^*, m^*, R_*)$, i.e. $\widetilde{(\mu \otimes_{\mathcal{C}} \nu)}^* = \mu^* \otimes_{\mathcal{C}^*} \nu^*$. Moreover, if m is a continuous measure, it is isomorphic to $((X \times Y)^*, (\mathcal{A} \otimes \mathcal{B})^*, (\mu \otimes_{\mathcal{C}} \nu)^*, (T \times S)^*)$ (this condition is satisfied for example if T is $\mathbf{II}_\infty$).

If a Poissonian joining is a relatively independent joining over a factor $\mathcal{F}$, then $\mathcal{F}$ is a Poissonian factor.

Proof. The Markov operator that describes a relatively independent joining over a σ-finite factor is the conditional-expectation operator on this factor (see [**7**, p. 129]). It is now sufficient to recall that the exponential of the conditional expectation of a factor on the base is the conditional expectation of the corresponding Poissonian factor (see §2.2). The second point is a direct application of Proposition 4.6.

For the last result, consider a Poissonian joining which is a relatively independent joining over a factor $\mathcal{F}$. Given the structure result of the distribution of any Poissonian joining (Theorem 4.2), we can assume that μ' and ν' are zero. Now, we only have to remark that a joining (in finite or infinite measure) is a relatively independent joining if and only if the associated Markov operator is an orthogonal projection. The Markov operator $E_{\mathcal{F}}$ associated to the Poissonian joining is the exponential of an operator p associated to the joining on the base. But $E_{\mathcal{F}}$ is an orthogonal projection and preserves the first chaos, thus its restriction to it, which is identified to p is also an orthogonal projection and a Markov operator as well. Thus, there exists a factor $\mathcal{B}$ such that $p = E_{\mathcal{B}}$, but, as we have already seen, the exponential of a conditional expectation is the conditional expectation on the corresponding Poissonian factor, and we deduce $\mathcal{F} = \mathcal{B}^*$, i.e., $\mathcal{F}$ is a Poissonian factor. □

Remark. It can be shown that Poissonian self-joinings of weakly mixing (= ergodic, here) (respectively mildly mixing, respectively mixing, respectively K) suspensions are also weakly mixing (respectively mildly mixing, respectively mixing, respectively K). From the last proposition, we can conclude that a weakly mixing Poisson suspension is always relatively weakly mixing over its Poissonian factors.

4.5. *An example.* Let us give a concrete example on a particularly simple system. Aaronson and Nadkarni in [**2**] have proved the existence of a system $(X, \mathcal{A}, \mu, T)$ (said to have *minimal self joinings*) whose only ergodic self-joinings are of the form $\mu(A \cap T^{-k}B)$. It is easy to describe all Poissonian self-joinings of the corresponding Poisson suspension. They are of the form:

$$\mathfrak{P} = ((a\mu)^* \otimes (a\mu)^*) * \cdots * (c_{-k}\nu_{-k})^* * \cdots * (c_0\nu_0)^* * \cdots * (c_k\nu_k)^* * \cdots$$

where $\nu_k(A \times B) = \mu(A \cap T^{-k}B)$ and $\sum_{k\in\mathbb{Z}} c_k = 1 - a$ with $a \geq 0$, $c_k \geq 0$, $k \in \mathbb{Z}$.

In this case, Poissonian joinings are reduced to the minimum possible. In this example, the Poissonian centralizer (i.e. Poissonian automorphisms commuting with T_*) is trivial; that is, it consists only in the powers of T_*.

This example shows that an ergodic Poisson suspension (which always has the aforementioned self-joinings) is never simple in the ergodic sense; indeed, by taking $c_1 > 0$ and $c_2 > 0$, the corresponding Poissonian joining is ergodic and not a graph joining nor the product joining.

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4.6. *Strong disjointness and ID-disjointness.* We recall (see [**1**]) that a (c_1, c_2)-joining between $(X, \mathcal{A}, \mu, T)$ and $(Y, \mathcal{B}, \nu, S)$ is a joining between $(X, \mathcal{A}, c_1\mu, T)$ and $(Y, \mathcal{B}, c_2\nu, S)$. Two systems are strongly disjoint if there is no (c_1, c_2)-joining possible.

We have the following links between strong disjointness and ID-disjointness.

THEOREM 4.8. *Two dynamical systems are strongly disjoint if their Poisson suspensions are ID-disjoint.*

Two ergodic dynamical systems are strongly disjoint if and only if their associated Poisson suspensions are ID-disjoint.

Proof. Assume there exists a (c_1, c_2)-joining γ between $(X, \mathcal{A}, \mu, T)$ and $(Y, \mathcal{B}, \nu, S)$. We can assume that $c_1 \geq c_2$ and form the non-trivial Poissonian joining with distribution

$$\widetilde{\left(\frac{1}{c_1}\gamma\right)}^* * \left(\delta_{\mu_0} \otimes \left(\left(1 - \frac{c_2}{c_1}\right)\nu\right)^*\right).$$

If the base systems are ergodic and their suspensions not ID-disjoint there exists a non-trivial Poissonian joining of the form $(\mu'^* \otimes \delta_{\mu_0}) * \widetilde{\gamma^*} * (\delta_{\mu_0} \otimes \nu'^*)$. By ergodicity, there exists a_1 and a_2 such that $\mu' = a_1\mu$ and $\nu' = a_2\nu$, therefore γ is a $((1 - a_1), (1 - a_2))$-joining between $(X, \mathcal{A}, \mu, T)$ and $(Y, \mathcal{B}, \nu, S)$. □

Since disjointness implies ID-disjointness, a useful immediate corollary is the following.

COROLLARY 4.9. *Two dynamical systems are strongly disjoint if their Poisson suspensions are disjoint.*

This corollary is a powerful tool to prove strong disjointness results. Let us illustrate it by an example. In [**9**], Poisson entropy of a transformation T is defined as the Kolmogorov entropy of its Poisson suspension. Poisson entropy is shown to coincide, for quasi-finite systems (see [**10**] for the definition), with Krengel's and Parry's definition of entropy (see [**16**]). Moreover it is proved a dichotomy for these systems: T, quasi-finite and ergodic, is either a remotely infinite system, or it possesses a Pinsker (σ-finite)-factor. It is conjectured that this dichotomy holds for any system; quasi-finite systems constitutes a very large class anyway.

We now prove a strong disjointness result for a large class of systems which includes quasi-finite ones in particular. By Theorem 3.4, if T is remotely infinite then T_* is K and by a well-known result, in finite measure, K-systems are disjoint from zero entropy ones. Now, using the fact that to be not strongly disjoint is an equivalence relation, Corollary 4.9 implies the following.

THEOREM 4.10. *If T is a remotely infinite system and if S has a zero Poisson entropy (σ-finite) factor, then they are strongly disjoint.*

4.7. *ID-disjointness and spectral theory.* The definition of ID-disjointness can be extended to other infinitely divisible systems and notably stationary ID processes as studied in [**13**]. A Poisson suspension $(X^*, \mathcal{A}^*, \mu^*, T_*)$ and a stationary ID process

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$(\mathbb{R}^{\mathbb{Z}}, \mathcal{B}^{\otimes\mathbb{Z}}, \mathbb{P}, S)$ are ID-disjoint if the only way to join them with an ID distribution (on $X^* \times \mathbb{R}^{\mathbb{Z}}$) is the product measure.

PROPOSITION 4.11. *A Poisson suspension and a stationary Gaussian processes are ID-disjoint.*

Proof. Assume there exists a joining between a Poisson suspension N and a stationary Gaussian process $\{X_n\}_{n\in\mathbb{Z}}$ such that the pair $(N, \{X_n\}_{n\in\mathbb{Z}})$ has an infinitely divisible distribution on $X^* \times \mathbb{R}^{\mathbb{Z}}$. It is clear that, for all finite family $A_1, \ldots, A_k \in \mathcal{A}$, $0 < \mu(A_i) < \infty$, and all finite subset $\{i_1, \ldots, i_l\}$ of $\mathbb{Z}$, the random vector (in $\mathbb{R}^{k+l}$) $\{N(A_1), \ldots, N(A_k), X_1, \ldots, X_l\}$ is ID. From the Lévy-Khintchine formula, such a vector is always the independent sum $\{Y_1, \ldots, Y_{k+l}\} + \{Z_1, \ldots, Z_{k+l}\}$ of an ID random vector without Gaussian part and a Gaussian random vector. The non-negativity of $N(A_i)$, $i \le k$ implies that $Z_i = 0$, $i \le k$ and Cramér theorem implies that Y_i, $i > k$ is Gaussian and hence zero. This implies that $\{N(A_1), \ldots, N(A_k)\}$ and $\{X_1, \ldots, X_l\}$ are independent. It follows that N and $\{X_n\}_{n\in\mathbb{Z}}$ are independent, thus proving ID-disjointness. □

Surprisingly, ID-disjointness with Gaussian systems will serve to prove a general result on spectral theory of infinite measure preserving systems.

Gaussian systems are spectral in nature. They are fully described by a symmetric measure on the circle and it may happen that some measures on the circle, appearing as spectral measures of a dynamical system, forces the Gaussian character of the associated stationary process. These measures are said to possess the 'Foïaş and Strătilă' property.

Definition 4.12. (See [**12**]) A symmetric measure σ on $[-\pi, \pi[$, possesses the 'Foïaş and Strătilă' (FS) property if, for any ergodic probability preserving dynamical system $(Y, \mathcal{C}, \nu, S)$, every $f \in L^2_0(\nu, S)$ with spectral measure σ, is a Gaussian random variable.

Gaussian processes associated to these measures have led to deep and beautiful results on their structure (see [**12**]).

THEOREM 4.13. *A measure with the (FS) property is never the spectral measure of a vector in a $\mathbf{II}_\infty$ system.*

Proof. Assume there exists a $\mathbf{II}_\infty$ system $(X, \mathcal{A}, \mu, T)$ and a vector f such that σ_f has the (FS) property. Consider now $(M_X, \mathcal{M}_\mathcal{A}, \mathcal{P}_\mu, \widetilde{T})$; it is ergodic by Theorem 3.4. The centred stochastic integral $M(f) \in \mathfrak{C}$ has spectral measure σ_f. The stationary process $\{M(f) \circ \widetilde{T}^n\}_{n\in\mathbb{Z}}$ is ergodic and then Gaussian since σ_f has the (FS) property. But from Lemma 4.11, $M(f)$ is necessarily zero. □

The following corollary can be seen as a slight extension of the Foïaş and Strătilă property.

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COROLLARY 4.14. *Let* $(\Omega, \mathcal{F}, \mu, T)$ *be a conservative dynamical system and* $f \in L^2(\mu)$ *with spectral measure* σ *that possesses the* (FS) *property, then* $f = f1_{\mathfrak{P}}$ *mod* μ *where* $\mathfrak{P}$ *is the part of* $(\Omega, \mathcal{F}, \mu, T)$ *of type* $\mathbf{II}_1$.

Proof. Let $\mu_\infty := \mu_{|\mathfrak{P}^c}$ and $\mu_f := \mu_{|\mathfrak{P}}$. We have

$$\hat{\sigma}(k) = \int_\Omega ff \circ T^k \, d\mu$$
$$= \int_\Omega (f1_{\mathfrak{P}})(f1_{\mathfrak{P}}) \circ T^k \, d\mu_f + \int_\Omega (f1_{\mathfrak{P}^c})(f1_{\mathfrak{P}^c}) \circ T^k \, d\mu_\infty = \hat{\sigma}_f(k) + \hat{\sigma}_\infty(k)$$

where $\sigma_f + \sigma_\infty = \sigma$.

This yields to the relation of absolute continuity $\sigma_\infty \ll \sigma$ and as such, σ_∞ possesses the (FS) property too (see [**12**]) unless it is the zero measure. But the system $(\Omega, \mathcal{F}, \mu_\infty, T)$, of type $\mathbf{II}_\infty$, would have a vector, $f1_{\mathfrak{P}^c}$, with a spectral measure that possesses the (FS) property. This is not possible from Theorem 4.13. Hence σ_∞ is zero and so is $f1_{\mathfrak{P}^c}$, that is $f = f1_{\mathfrak{P}}$ mod μ. □

5. *'Simple' Poisson suspensions*

At first sight, a Poisson suspension seems to be a much more complex object than the base on which it is built. The transformation T sends one point to another while T_* maps a countable set of points to another. The aim of this chapter is to focus on Poisson suspensions were the gap of complexity is considerably reduced.

First, spectrally, the complexity will be equivalent if both the suspension and the base have simple spectrum. We will see that simple spectrum (indeed, slightly less restrictive hypothesis will suffice) will force the centralizer of both transformations to be the same and force the natural factors of a Poisson suspension we can think of to be Poissonian factors. Moreover, the particularly clear structure will enable us to compare this systems to the apparently similar looking Gaussian systems.

It is easily deduced that a Poisson suspension will have simple spectrum if and only if it has simple spectrum on each chaos and if, for all $m \neq n$, $\sigma^{*m} \perp \sigma^{*n}$. We thus are led to seek criteria under which these conditions are satisfied.

Ageev in [**3**] (see also [**4**]) has proved the following result (we present a reduced version of it).

THEOREM 5.1. *Let* U *be a unitary operator acting on a complex Hilbert space* H. *Assume* U *has simple and continuous spectrum and moreover, that, for countably many* θ, $0 < \theta < 1$, $\theta \mathrm{Id} + (1-\theta)U$ *belongs to the weak closure of powers of* U.

Then, for all $n \geq 1$ $U^{\otimes n}$ *has simple spectrum on* $\mathfrak{S}^n H$ *(the nth symmetric tensor power of* H*) and the corresponding maximal spectral types are all pairwise mutually singular.*

Therefore, if T is such that U_T satisfies assumptions of Theorem 5.1 then T_* has simple spectrum.

It is easy to construct an appropriate rank-one transformation T in infinite measure such that U_T satisfies assumptions of Theorem 5.1; the existence of a simple spectrum suspension is thus ensured.

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In what follows, we will often consider a base $(X, \mathcal{A}, \mu, T)$ whose maximal spectral type σ satisfies

$$\sigma \perp \sum_{k=2}^{+\infty} \frac{1}{k!}\sigma^{*k} \tag{5.1}$$

and will refer to it as satisfying property (5.1).

It concerns, of course, Poisson suspension with simple spectrum.

5.1. *Centralizer.* We begin with a result similar to the Gaussian case.

PROPOSITION 5.2. *Let T and S have property (5.1). Assume there exists an automorphism φ such that $T_*\varphi = \varphi S_*$, then φ is indeed a Poissonian automorphism. In particular $C(T_*) \simeq C(T)$.*

Proof. Take f in the first chaos; $f \circ \varphi$ has the same spectral measure under S_* as f under T_*, thus, since the latter is absolutely continuous with respect to σ and that $\sigma \perp \sum_{k=2}^{+\infty}(1/k!)\sigma^{*k}$, $f \circ \varphi$ also belongs to the first chaos. This proves that φ, seen as a Markov operator on $L^2(\mu^*)$, preserves the first chaos; therefore, thanks to Proposition 4.4 φ induces a sub-Markov operator Φ on $L^2(\mu)$ that intertwines T and S. But φ is also an isometry, thus Φ is an isometry of $L^2(\mu)$ and this implies, since Φ is also a sub-Markov operator, that Φ is indeed a Markov operator which comes from an automorphism, say R. We have thus proved that, for each, $A \in \mathcal{A}$, $0 < \mu(A) < \infty$:

$$N(A) \circ \varphi = N(R^{-1}A).$$

Under the graph measure on $X^* \times X^*$ given by $\mu^*(A \cap \varphi^{-1}B)$, the bivariate Poisson measure takes the form $(N(\cdot), N(\cdot) \circ \varphi) = (N(\cdot), N(R^{-1}(\cdot)))$ and this proves that this graph measure is ID. The graph joining given by φ between T_* and S_* is therefore Poissonian and we can deduce that φ, as a Markov operator, is the exponential of Φ which is in turn corresponds to R^*, that is $\varphi = R^*$. □

5.2. *Comparison with Gaussian systems.* So far, it was not clear that Poisson suspensions are not (isomorphic to) Gaussian systems. The next two propositions will show, hopefully, that we are not reasoning on the same objects. The first one has been proved by François Parreau (unpublished).

PROPOSITION 5.3. *A suspension $(X^*, \mathcal{A}^*, \mu^*, T_*)$ with T satisfying property (5.1) is not isomorphic to a Gaussian system.*

Proof. Suppose there exists a Gaussian system $(Y, \mathcal{Y}, \nu, S)$ isomorphic to $(X^*, \mathcal{A}^*, \mu^*, T_*)$ by an isomorphism φ. Call m the maximal spectral type of the first chaos of $L^2(\nu)$. Due to the assumptions on the spectral structure of $L^2(\mu^*)$, we cannot have $m \perp \sigma$. So, consider ρ such that $\rho \ll m$ and $\rho \ll \sigma$. Take f to be a vector in the first chaos of $L^2(\nu)$ with spectral measure ρ; f is thus a Gaussian random variable. Now the vector $f \circ \varphi^{-1}$ belongs to $L^2(\mu^*)$, has spectral measure ρ and $\{f \circ \varphi^{-1} \circ T_*^n\}_{n\in\mathbb{Z}}$ is a Gaussian stationary process under μ^*. But since $\rho \ll \sigma$, we have $\rho \perp \sum_{k=2}^{+\infty}(1/k!)\sigma^{*k}$ which implies that, for all $n \in \mathbb{Z}$,

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$f \circ \varphi^{-1} \circ T_*^n$ belongs to the first chaos of $L^2(\mu^*)$ which implies that the graph joining is ID, which is impossible by ID-disjointness (Proposition 4.11). □

It is well known that Gaussian systems are always isomorphic to an arbitrary number of direct products of transformations. We will show in the next proposition, that it is no longer the case for some Poisson suspensions. This establishes a drastic difference with Gaussian systems.

PROPOSITION 5.4. *Let T have property (5.1). $(X, \mathcal{A}, \mu, T)$ is ergodic if and only if there do not exist two non-trivial T_*-invariant and independent sub-σ-algebras $\mathcal{B}$ and $\mathcal{C}$ such that $\mathcal{A}^* = \mathcal{B} \vee \mathcal{C}$, i.e. $(X^*, \mathcal{A}^*, \mu^*, T_*)$ is not isomorphic to a non-trivial direct product.*

Proof. Assume $(X, \mathcal{A}, \mu, T)$ to be ergodic. Let $\mathcal{B}$ be a non-trivial T_*-invariant sub-σ-algebra of $\mathcal{A}^*$ and suppose there exists another T_*-invariant sub-σ-algebra $\mathcal{C}$ independent of $\mathcal{B}$ and such that $\mathcal{A}^* = \mathcal{B} \vee \mathcal{C}$. Denote by ν_a (respectively ν_b) the (reduced) maximal spectral type of the quotient action $T_{*/\mathcal{B}}$ (respectively $T_{*/\mathcal{C}}$). We must have, up to equivalence of measures,

$$\sum_{k=1}^{+\infty} \frac{1}{k!} \sigma^{*k} = \nu_a + \nu_b + \nu_a * \nu_b.$$

Assume $\sigma \perp \nu_b$; we have thus

$$\nu_b \ll \sum_{k=2}^{+\infty} \frac{1}{k!} \sigma^{*k}$$

which implies that

$$\nu_a * \nu_b \ll \sum_{k=2}^{+\infty} \frac{1}{k!} \sigma^{*k}$$

and then $\sigma \ll \nu_a$ since $\sigma \perp \sum_{k=2}^{+\infty} (1/k!)\sigma^{*k}$. This means that all vectors with spectral measure σ are measurable with respect to $\mathcal{B}$. But these vectors are exactly those of the first chaos and these vectors generate the whole σ-algebra $\mathcal{A}^*$, that is $\mathcal{A}^* = \mathcal{B}$, which is a contradiction with the initial assumptions. By symmetry, we deduce that $\sigma \not\perp \nu_a$ and $\sigma \not\perp \nu_b$, and this implies the existence of two centred stochastic integrals in the first chaos $M(f_a)$ and $M(f_b)$ measurable with respect to $\mathcal{B}$ and $\mathcal{C}$ respectively. Then $\{M(f_a) \circ T_*^n\}_{n\in\mathbb{Z}}$ and $\{M(f_b) \circ T_*^n\}_{n\in\mathbb{Z}}$ are independent, which is in fact impossible because, by using a result of Maruyama (see [**13**]), this would imply the disjointness of the support of the functions $\{f_a \circ T^n\}_{n\in\mathbb{Z}}$ with the support of the functions $\{f_b \circ T^n\}_{n\in\mathbb{Z}}$ and these two non-trivial T-invariant sets cannot be disjoint since $(X, \mathcal{A}, \mu, T)$ is ergodic.

The converse is obvious. □

5.3. *Poissonian factors and IDp processes.* We will study factors of a Poisson suspension generated by processes which are stochastic integrals. Our aim is to show that they have the expected form, that is, they are Poissonian factors. We will need to combine Proposition 4.4 with the following lemma.

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LEMMA 5.5. *Consider a factor $\mathcal{B}$ of $(X^*, \mathcal{A}^*, \mu^*, T_*)$ and the corresponding conditional expectation $E_{\mathcal{B}}$. Assume $E_{\mathcal{B}}$ preserves the first chaos $\mathfrak{C}$ and consider $\mathcal{C}$, the factor of $(X, \mathcal{A}, \mu, T)$ generated by those $g \in L^2(\mu)$ such that $I(g) \in E_{\mathcal{B}}(\mathfrak{C})$. Then $\mathcal{C}^* \subset \mathcal{B}$.*

Proof. We denote by V the sub-Markov operator (thanks to Proposition 4.4) induced by $E_{\mathcal{B}}$ on $L^2(\mu)$; V is an orthogonal projector. From $\int_X V1_A \, d\mu \leq \mu(A)$, the T-invariant measure $A \mapsto \int_X V1_A \, d\mu$ is absolutely continuous with respect to μ. Thus we have $\int_X V1_A \, d\mu = \int_X 1_A p \, d\mu$ for a function $p \leq 1$. Take $\{f_n\}_{n\in\mathbb{N}}$ a linearly dense family of positive functions, and form $f = \sum_{n\in\mathbb{N}} a_n f_n$ such that, $a_n > 0$ and $f \in L^2(\mu)$. Let S be the support of Vf (it is a T-invariant set), since V is a projector:

$$\int_X Vf \, d\mu = \int_X V(Vf) \, d\mu = \int_X Vfp \, d\mu.$$

Thus $p \equiv 1$ on S. It is immediate to see that $Vg = 0$ if g is zero on S. Take $g > 0$. As $gVf = 0$, we have $\langle Vg, f\rangle = \langle g, Vf\rangle = \int_X gVf \, d\mu = 0$ and, by positivity, we deduce that $\langle Vg, f_n\rangle = 0$ for any $n \in \mathbb{N}$; thus $Vg = 0$.

V restricted to $L^2(\mu_{|S})$ is now a Markov operator and since it is also an orthogonal projection, it is a conditional expectation on a σ-finite factor $\widetilde{\mathcal{C}}$ of $(S, \mathcal{A}_{|S}, \mu_{|S}, T_{|S})$ and then induces the desired (non-σ-finite if $S \neq X$) factor $\mathcal{C}$ (i.e. $\mathcal{C} = \sigma(\widetilde{\mathcal{C}})$) of $(X, \mathcal{A}, \mu, T)$. The only non-zero square-integrable functions measurable with respect to $\mathcal{C}$ are supported on S and their restriction to S are exactly the functions measurable with respect to $\widetilde{\mathcal{C}}$. Thus $\operatorname{Im} V = L^2(X, \mathcal{C}, \mu, T)$ and it implies that $\mathcal{C}^* \subset \mathcal{B}$. □

The assumption that $\Phi(\mathfrak{C}) \subset \mathfrak{C}$ is automatically satisfied if T satisfies property (5.1) since it has already been noticed in [**12**] that, for spectral measures, we have $\sigma_{\Phi f} \ll \sigma_f$.

Our main application of the preceding lemma concerns *infinitely divisible stationary processes without Gaussian part* (abbreviated IDp) (this includes, in particular, all α-stable stationary processes). We recall that these processes are exactly those that can be obtained as stochastic integrals with respect to a Poisson measure (see [**13**] and [**18**]).

THEOREM 5.6. *Assume that $(X, \mathcal{A}, \mu, T)$ has simple spectrum, satisfying property (5.1). Let $I(f)$ be a stochastic integral of $(X^*, \mathcal{A}^*, \mu^*, T_*)$. Then, the factor $\mathcal{B}$ generated by the process $\{I(f) \circ T_*^n\}_{n\in\mathbb{Z}}$ is indeed the Poissonian factor $\mathcal{C}^*$, where $\mathcal{C}$ is the factor of the base generated by $\{f \circ T^n\}_{n\in\mathbb{Z}}$. If $f \in L^2(\mu)$, then we can remove the hypothesis on the simplicity of the spectrum.*

Proof. We consider vectors of the kind $\exp i\langle a, I(f)\rangle - \mathbb{E}[\exp i\langle a, I(f)\rangle]$ where $\langle a, I(f)\rangle$ is some finite linear combination of the $I(f) \circ T_*^n$. In [**18**], we have evaluated the spectral measure of such vectors: it is the measure $|\mathbb{E}[\exp i\langle a, I(f)\rangle]|^2 \sum_{k=1}^{+\infty} (1/k!)\sigma_a^{*k}$, where σ_a is the spectral measure of $\exp i\langle a, f\rangle - 1$ in $(X, \mathcal{A}, \mu, T)$.

Call C_a the cyclic space associated to $\exp i\langle a, I(f)\rangle - \mathbb{E}[\exp i\langle a, I(f)\rangle]$, since

$$\sigma_a \ll \sum_{k=1}^{+\infty} \frac{1}{k!} \sigma_a^{*k},$$

this cyclic space contains vectors with σ_a as spectral measure. Thanks to the mutual singularity of σ^{*n} with σ, these vectors necessarily belong to the first chaos, moreover,

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since $(X, \mathcal{A}, \mu, T)$ has simple spectrum, $I(\exp i\langle a, f\rangle - 1)$ and its iterates stand among them. We have shown that $I(\exp i\langle a, f\rangle - 1)$ are measurable with respect to $\mathcal{B}$. The factor generated by $\exp i\langle a, f\rangle - 1$, for any a is exactly $\mathcal{C}$. Thanks to Lemma 5.5, we have $\mathcal{C}^* \subset \mathcal{B}$ and the result follows as we also have the other inclusion.

The last statement is a direct application of Lemma 5.5. □

This theorem proves that many stationary IDp processes are isomorphic to the Poisson suspension associated to their Lévy measure and not just a factor. In particular, it reduces the study of a large class of stationary IDp process, which may have very bad integrability properties, to that, much more tractable, of a Poisson suspension. For example, it solves, for this class, the following open question.

Is the maximal spectral type of any stationary IDp process of the form $\sum_{k=1}^{+\infty}(1/k!)\sigma^{*k}$?

It is thus maybe worthwhile to give an autonomous version of the last theorem for IDp processes.

THEOREM 5.7. *Let $\{X_n\}_{n\in\mathbb{Z}}$ be a stationary IDp process with Lévy measure Q. Assume that $(\mathbb{R}^{\mathbb{Z}}, \mathcal{B}^{\otimes\mathbb{Z}}, Q, S)$ has simple spectrum and S has property (5.1). Then $\{X_n\}_{n\in\mathbb{Z}}$ is isomorphic to $((\mathbb{R}^{\mathbb{Z}})^*, (\mathcal{B}^{\otimes\mathbb{Z}})^*, Q^*, S_*)$. If X_0 is square integrable, then we can remove the hypothesis on the simplicity of the spectrum.*

Let us finish by giving one last application of Lemma 5.5 that shows that Poissonian factors behave well under classical operations. We already have pointed out (see the discussion preceding Proposition 4.6) that, in general, we do not have $\mathcal{C}_1^* \vee \mathcal{C}_2^* = (\mathcal{C}_1 \vee \mathcal{C}_2)^*$, however, by Lemma 5.5, it is easy to see that the equality holds if T has property (5.1).

Acknowledgements. The author would like to thank the anonymous referee for useful comments and Mariusz Lemańczyk, François Parreau and Jean-Paul Thouvenot for fruitful discussions.

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