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Dynamics of quasiregular mappings with constant complex dilatation

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Abstract. The study of quadratic polynomials is a foundational part of modern complex dynamics. In this article, we study quasiregular counterparts to these in the plane. More specifically, let $h : \mathbb{C} \to \mathbb{C}$ be an $\mathbb{R}$-linear map and consider the quasiregular mapping $H = g \circ h$, where g is a quadratic polynomial. By studying H and via the Böttcher-type coordinate constructed in A. Fletcher and R. Fryer [On Böttcher coordinates and quasiregular maps. *Contemp. Math.* **575** (2012), 53–76], we are able to obtain results on the dynamics of any degree-two mapping of the plane with constant complex dilatation. We show that any such mapping has either one, two or three fixed external rays, that all cases can occur and exhibit how the dynamics changes in each case. We use results from complex dynamics to prove that these mappings are nowhere uniformly quasiregular in a neighbourhood of infinity. We also show that in most cases, two such mappings are not quasiconformally conjugate on a neighbourhood of infinity.

1. *Introduction*

1.1. *Background.*

The modern interest in complex dynamics was initiated by Douady and Hubbard in the 1980s, following on from the work of Fatou, Julia, Montel and others towards the beginning of the twentieth century. For an overview of this theory, see for example [**6**, **16**]. Douady and Hubbard initially focused on the iteration of quadratic polynomials and, with the advent of computer-generated images, showed how the iteration of a simply stated function could have very complicated behaviour. In their further work, they showed how the behaviour of the family $z^2 + c$ is fundamental and that the Mandelbrot set in parameter space can appear in the parameter space of other families of holomorphic functions.

This is essentially because of the straightening theorem, which allows one to quasiconformally conjugate (or, in language common to complex dynamics, find a hybrid equivalence to) a holomorphic function which behaves like a polynomial to a

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genuine polynomial. Quasiconformal techniques have proved essential in modern complex dynamics, for example in Sullivan's proof of the no wandering domains theorem.

We recall that quasiconformal mappings are, informally, homeomorphisms which map infinitesimal circles to infinitesimal ellipses, with a uniform bound on the eccentricity. Dropping the injectivity requirement yields quasiregular mappings. The complex dilatation μ_f of a quasiconformal mapping is defined by $\mu_f(z) = f_{\overline{z}}(z)/f_z(z)$ and satisfies $\|\mu_f\|_\infty \leq k < 1$. Via the measurable Riemann mapping theorem of Ahlfors and Bers, given any measurable function μ with $\|\mu\|_\infty \leq k < 1$, there exists a quasiconformal mapping with this dilatation μ. Moreover, the Stoilow factorization of quasiregular mappings in the plane states that if $f : \mathbb{C} \to \mathbb{C}$ is quasiregular, then f has a decomposition as $f = g \circ h$, where g is holomorphic and h is quasiconformal.

Quasiregular mappings can be defined in $\mathbb{R}^n$ and are the natural generalization of holomorphic mappings to higher dimensions. They share some of the value distribution properties of holomorphic functions, which allows one to construct an iteration theory for them; see for example [**1**]. This has become a topic of research interest, and it also makes sense to restrict to quasiregular mappings in the plane, where the theory has different features compared to higher dimensions. This is largely due to the measurable Riemann mapping theorem.

The iteration of quasiregular mappings goes back to at least [**3**, **14**], where dynamics of mappings of the form $z^2 + b\overline{z} + c$ and $|z|^{2\alpha-2}z^2 + c$ were considered. See also [**4**, **5**, **17**, **18**, **20**] for further results in these directions.

The first iteration of mappings called quasiregular appeared in [**13**], where mappings with a uniform bound on the distortion of the iterates were considered. These are special mappings and, in particular, every uniformly quasiregular mapping of the plane is a quasiconformal conjugate of a holomorphic function [**12**]. This implies that we get no new features when compared to complex dynamics.

Since quadratic iteration is fundamental for complex dynamics, it is desirable to thoroughly study a quasiregular version of quadratic polynomials on which to build the theory in the plane. By the Stoilow decomposition, and as in [**7**], we argue that the simplest quasiregular mapping in the plane is a composition of a quadratic polynomial and an $\mathbb{R}$-linear mapping or, equivalently, a quasiconformal map for which the complex dilatation is constant.

The iteration of such mappings was studied in [**8**], where it was shown that every such mapping is linearly conjugate to a mapping of the form $h(z)^2 + c$, where h is an affine stretch and $c \in \mathbb{C}$. The dynamics of these mappings has certain similarities and differences with respect to quadratic polynomials. For example, Figure 1 gives an example of a mapping for which $\partial I(f)$ has uncountably many components and some of these contain continua. Compare this with the dichotomy for quadratic polynomials, where $J(f)$ is either connected or totally disconnected. It was shown in [**7**] that there is a Böttcher coordinate for these mappings in a neighbourhood of infinity. This coordinate is a quasiconformal map which conjugates $h(z)^2 + c$ to $h(z)^2$ near infinity. As with complex dynamics, the quasiconformal conjugacy class is the interesting one. In future work, we aim to extend the class of mappings for which we have a Böttcher coordinate so that the complex dilatation need not be constant near the fixed point.

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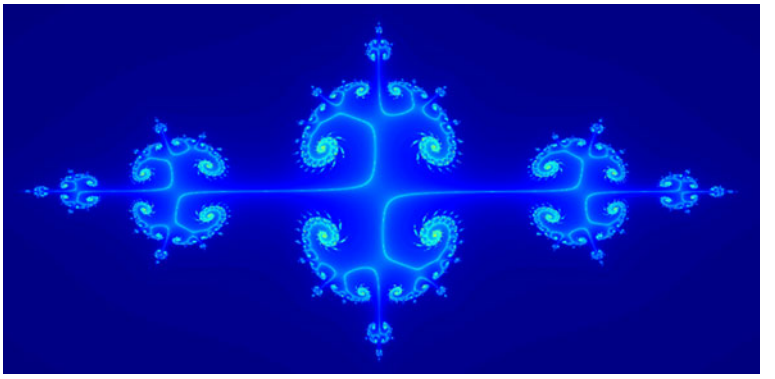


FIGURE 1. The non-escaping set for $K = 1.25$, $\theta = \pi/2$ and $c = -0.7 + 0.002i$.

In the current paper, we aim to focus on the mappings $h(z)^2$, and use the Böttcher coordinate to extend our results to all degree-two mappings of constant complex dilatation.

1.2. *Statement of results.* Let $K > 1$ and $\theta \in (-\pi/2, \pi/2]$. Write $h_{K,\theta}$ for the affine stretch by a factor K in the direction $e^{i\theta}$, that is,

$$h_{K,\theta} := \left(\frac{K+1}{2}\right)z + e^{2i\theta}\left(\frac{K-1}{2}\right)\overline{z}.$$

If it is clear which K, θ we are dealing with, for brevity we will write $h = h_{K,\theta}$. If $K = 1$, then this mapping is the identity and does not depend on θ. If the mapping h is fixed, we will write $H(z) = h(z)^2$.

The justification for studying these mappings in the class of degree-two quasiregular mappings of the plane with constant complex dilatation is given by the following proposition.

PROPOSITION 1.1. *Let $f : \mathbb{C} \to \mathbb{C}$ be quasiregular of degree two and let f have constant complex dilatation that is not identically 0. Then f is linearly conjugate to a unique mapping of the form $h_{K,\theta}(z)^2 + c$ for some $K > 1$, $\theta \in (-\pi/2, \pi/2]$ and $c \in \mathbb{C}$.*

We prove this proposition in the next section. A *ray* is a semi-infinite line $R_\phi = \{te^{i\phi} \mid t \geq 0\}$. It is not hard to see that h maps rays to rays, and so H also maps rays to rays. This means that H induces an increasing mapping $\widetilde{H} : \mathbb{R} \to \mathbb{R}$ that is 2π-periodic, and is given by $\widetilde{H}(\varphi) = \arg[H(re^{i\varphi})]$ for any $r > 0$.

Definition 1.2. We say that a ray R_ϕ which is fixed by H is *locally repelling*, *locally expanding* or *neutral* if the induced mapping satisfies $\widetilde{H}'(\phi) < 1$, $\widetilde{H}'(\phi) > 1$ or $\widetilde{H}'(\phi) = 1$, respectively.

THEOREM 1.3. *Let $\theta \in (-\pi/2, \pi/2)\backslash\{0\}$, $K > 1$ and $H(z) = h_{K,\theta}(z)^2$. Then there exists $K_\theta > 1$ such that:*

- *for $K < K_\theta$, there is one fixed ray that is locally repelling;*

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- *for $K = K_\theta$, there are two fixed rays, one of which is locally repelling and one that is neutral; further, the neutral fixed ray is repelling on one side and attracting on the other;*
- *for $K > K_\theta$, there are three fixed rays, one of which is locally attracting and two that are locally repelling.*

When $\theta = 0$, the first and third statements above hold, but when $K = K_\theta$ there is just one neutral fixed ray which is locally attracting on both sides. When $\theta = \pi/2$, there is only one fixed ray for all $K > 1$ and it is always locally repelling.

We next investigate the pre-images of these fixed rays.

Definition 1.4. If H has two or three fixed rays, let R_ϕ be the fixed ray that is not locally repelling. Define Λ to be the basin of attraction of R_ϕ, that is,

$$\Lambda = \{z \in \mathbb{C} \mid \arg(H^n(z)) \to \phi\}.$$

THEOREM 1.5. *If H has one fixed ray R_ϕ, then $\{H^{-k}(R_\phi)\}_{k=0}^{\infty}$ is dense in $\mathbb{C}$. If H has two or three fixed rays, then Λ is dense in $\mathbb{C}$.*

Via the Böttcher coordinate constructed in [**7**], these theorems have analogues for mappings of the form $h(z)^2 + c$ for $c \in \mathbb{C}$. We first make the following definition in analogy with complex dynamics.

Definition 1.6. Let $f(z) = h(z)^2 + c$. Then the *external ray* E_φ of f with angle $\varphi \in [0, 2\pi)$ is given by the image of the ray R_φ under the quasiconformal Böttcher coordinate $\psi = \psi(K, \theta, c)$ which conjugates f to H. The external ray E_φ is defined only in the range of ψ, that is, a neighbourhood of infinity.

We remark that each E_φ is an asymptotically conformal arc of a quasicircle, since the Böttcher coordinate is asymptotically conformal as $|z| \to \infty$. The collection $\{E_\varphi \mid \varphi \in [0, 2\pi)\}$ foliates a neighbourhood of infinity. We define an external ray E_φ which is fixed by f to be attracting, repelling or neutral if the corresponding fixed ray R_φ of H is attracting, repelling or neutral, respectively. The following corollary is an immediate application of Theorem 1.3.

COROLLARY 1.7. *Let $\theta \in (-\pi/2, \pi/2)\backslash\{0\}$, $K > 1$, $c \in \mathbb{C}$ and $f(z) = h_{K,\theta}(z)^2 + c$. Then, with K_θ as in Theorem 1.3:*

- *for $K < K_\theta$, there is one fixed external ray of f that is locally repelling;*
- *for $K = K_\theta$, there are two fixed rays, one of which is locally repelling and one that is neutral. Further, the neutral fixed ray is repelling on one side and attracting on the other;*
- *for $K > K_\theta$, there are three fixed rays, one of which is locally attracting and two that are locally repelling.*

When $\theta = 0$, the first and third statements above hold, but when $K = K_\theta$ there is just one neutral fixed external ray which is locally attracting on both sides. When $\theta = \pi/2$, there is only one fixed external ray for all $K > 1$ and it is always locally repelling.

In particular, the value of c plays no role in how many fixed external rays f has. Theorem 1.5 also has the following immediate corollary.

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COROLLARY 1.8. *With the notation as above, if f has one fixed external ray E_ϕ, then $\{f^{-k}(E_\phi)\}_{k=0}^{\infty}$ is dense in a neighbourhood of infinity. If f has two or three fixed external rays, then the basin of attraction of the non-repelling fixed external ray is dense in a neighbourhood of infinity.*

In [**2**], the Julia set for quasiregular mappings of polynomial type is defined, but only when the degree is larger than the distortion. For such mappings, $J(f)$ is defined by

$$J(f) = \{x \in \mathbb{R}^n \mid \mathbb{R}^n \setminus O^+(U) \text{ has capacity zero for all neighbourhoods } U \text{ of } x\},$$

where $O^+(U)$ denotes the forward orbit of the set U. We omit the definition of capacity zero here, but remark that such sets must be necessarily of Hausdorff dimension zero [**19**, Corollary VII.1.16]. In our setting, the condition on the distortion implies $K < 2$. We will see in the proof of Theorem 1.3 that $K_\theta \geq 2$. This implies that with the extra assumption of $K < 2$, our mappings always have only one fixed repelling ray, and the pre-images of this ray are dense in a neighbourhood of infinity. In particular, this means that our mappings topologically behave analogously to $z^2 + c$ near infinity. Recall that the escaping set of H is given by

$$I(H) = \{z \in \mathbb{C} \mid |H^n(z)| \to \infty \text{ as } n \to \infty\}.$$

Remark 1.9. There are examples where $J(f)$ and $\partial I(f)$ do not coincide for quasiregular mappings, see [**1**], although this always holds for holomorphic functions. We conjecture that there exist mappings of the form $f(z) = h(z)^2 + c$ for which $J(f)$ is strictly contained inside $\partial I(f)$ based on the fact that f may have saddle fixed points.

For a quasiregular mapping of polynomial type whose degree is larger than the distortion, it was proved in [**10**] that the escaping set is a connected neighbourhood of infinity. However, such mappings can have dynamically undesired behaviour outside the closure of the escaping set; for example in [**1**] a mapping is constructed which locally behaves like a winding mapping. We show that this does not happen for H and use Theorem 1.5 to give a complete decomposition of the plane into dynamically important sets for H.

COROLLARY 1.10. *Let $K > 1$, $\theta \in (-\pi/2, \pi/2]$ and $H(z) = h_{K,\theta}(z)^2$. Then*

$$\mathbb{C} = I(H) \cup \partial I(H) \cup \mathcal{A}(0),$$

where $\mathcal{A}(0)$ is the basin of attraction of the fixed point 0.

The next result is a refinement of a result from [**7**]. We first make the following definition.

Definition 1.11. Given a plane domain U, a quasiregular mapping $f : U \to \mathbb{C}$ is called *nowhere uniformly quasiregular* if, for every open set $V \subset U$, the maximal dilatation of $f^n|_V$ is unbounded.

For example, the quasiconformal mapping $f(x + iy) = Kx + iy$ is easily seen to be nowhere uniformly quasiregular for any $K > 1$.

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THEOREM 1.12. *Let $K > 1$ and $\theta \in (-\pi/2, \pi/2]$. Then the mapping $h_{K,\theta}(z)^2$ is nowhere uniformly quasiregular. Moreover, $h_{K,\theta}(z)^2 + c$ is nowhere uniformly quasiregular in a neighbourhood of infinity.*

This theorem implies a result of [**7**], that $h(z)^2 + c$ cannot be quasiconformally conjugate to $z^2 + c'$ in a neighbourhood of infinity. We will next see that $h_{K_1,\theta_1}(z)^2 + c_1$ is not quasiconformally conjugate to $h_{K_2,\theta_2}(z)^2 + c_2$ for most pairs $(K_1, \theta_1) \neq (K_2, \theta_2)$ in $(1, \infty) \times (-\pi/2, \pi/2]$. We conjecture that this is true for every such pair $(K_1, \theta_1) \neq (K_2, \theta_2)$, but our methods do not show this.

Let R_ϕ be a fixed ray of H. We recall from [**7**] that the complex dilatation of H^n at $z \in R_\phi$ is given by $\mu_n(z) = A^{n-1}(\mu_1)$, where A is the Möbius transformation

$$A(w) = \frac{\mu_1 + e^{-i\phi} w}{1 + e^{-i\phi}\overline{\mu_1} w}$$

and $\mu_1 = e^{2i\theta}(K-1)/(K+1) \in \mathbb{D}$. The trace of A, denoted $\mathrm{Tr}(A)$, is the trace of the matrix representing A, which is normalized to have determinant 1.

Definition 1.13. Given a fixed ray R_ϕ of H, define the function ξ by $\xi(\phi) = \mathrm{Tr}(A)^2$, where A is the Möbius map constructed above.

Given $K_1, K_2 > 1$ and $\theta_1, \theta_2 \in (-\pi/2, \pi/2]$, let H_1, H_2 be $h^2_{K_1,\theta_1}$, $h^2_{K_2,\theta_2}$, respectively. Denote the fixed rays of H_1 by R_{ϕ_i} and the fixed rays of H_2 by R_{ψ_j}. We prove the following theorem.

THEOREM 1.14. *With the notation above, there is no quasiconformal conjugacy between H_1 and H_2 in any neighbourhood of infinity if any of the following conditions hold:*

(i) *the mappings H_1, H_2 have different numbers of fixed rays;*
(ii) *H_1 and H_2 both have one fixed ray, R_{ϕ_1} and R_{ψ_1}, respectively, and $\xi(\phi_1) \neq \xi(\psi_1)$;*
(iii) *if H_1 and H_2 both have two fixed rays R_{ϕ_i} and R_{ψ_i} for $i = 1, 2$, where $\phi_1 > \phi_2$ and $\psi_1 > \psi_2$, and $\xi(\phi_i) \neq \xi(\psi_i)$ for some i;*
(iv) *if H_1 and H_2 both have three fixed rays R_{ϕ_i} and R_{ψ_j}, $i, j \in \{0, 1, 2\}$, respectively, where $\phi_1 > \phi_0 > \phi_2$ and $\psi_1 > \psi_0 > \psi_2$, and $\xi(\phi_i) \neq \xi(\psi_i)$ for some i.*

We can rule out some further cases in the following corollary.

COROLLARY 1.15. *If $\theta \in [0, \pi/2)$ is fixed and $K_1, K_2 > 1$ with $K_1 \neq K_2$, then $H_{K_1,\theta}$ and $H_{K_2,\theta}$ are not quasiconformally equivalent on any neighbourhood of infinity.*

1.3. *Further work.* We briefly outline some avenues of further research leading from the results of this paper.

- We see from Theorem 1.14 that we are most of the way to showing that the pair (K, θ) parameterizes the quasiconformal conjugacy class of $h_{K,\theta}(z)^2 + c$ near infinity. We conjecture that this should be the case.
- In this paper, we have focused on the case where the degree of the quasiregular mapping is two. It is a natural question to ask what happens for larger degrees $d > 2$.

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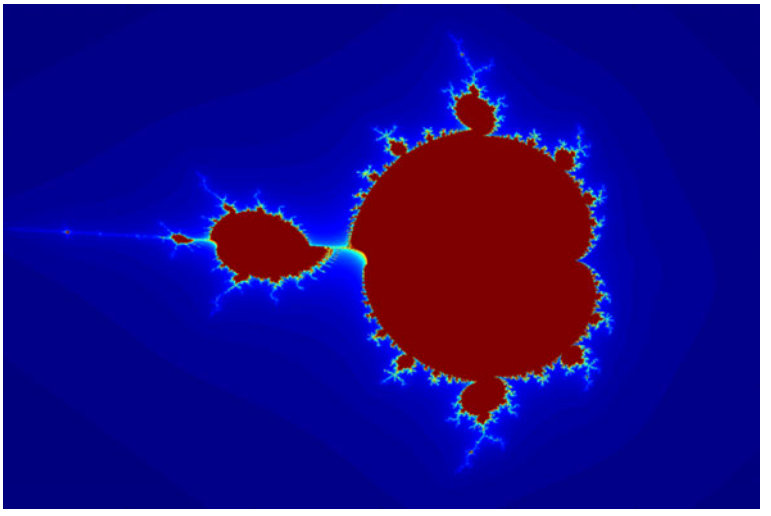


FIGURE 2. The parameter space for $K = 1.25$, $\theta = 1.4$.

To find the fixed rays, which are clearly important to the dynamics, this will require a study of the equation

$$\tan(\phi/d - \theta) = \frac{\tan(\phi - \theta)}{K}.$$

One would then obtain a polynomial P_d in $\tan((\phi - \theta)/d)$ whose roots would give the fixed rays. Questions here include how many fixed rays can occur and whether we only ever have one fixed ray if $K < d$.

- We have restricted to the case where μ is constant. Similar results analogous to the Böttcher coordinate should hold when μ is allowed to vary. Finding conditions on such μ would lead to improved results as corollaries to the main results in this paper.
- Usage of the Böttcher coordinate may lead to a greater understanding of the parameter space of these mappings. This can be defined by

$$\mathcal{M}_{K,\theta} = \{c \in \mathbb{C} \mid 0 \notin I(f_{K,\theta,c})\}.$$

See [**8**] for some basic properties of this set. Figure 2 gives an example of this parameter space.

The paper is organized as follows. In §2, we gather some preliminary material on quasiregular mappings in the plane. In §3, we study the ray structure of the mapping H and prove Theorem 1.3. In §4, we prove Theorem 1.5 and use it in §5 to prove Corollary 1.10 and in §6 to prove Theorem 1.12. Then in §7 we prove Theorem 1.14 and Corollary 1.15.

2. *Preliminaries*

2.1. *Quasiregular mappings.* A *quasiconformal* mapping $f : \mathbb{C} \to \mathbb{C}$ is a homeomorphism such that f is in the Sobolev space $W^1_{2,loc}(\mathbb{C})$ and there exists $k \in [0, 1)$ such that the *complex dilatation* $\mu_f = f_{\overline{z}}/f_z$ satisfies $|\mu_f(z)| < k$ almost everywhere in $\mathbb{C}$. See for

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example [**9**] for more details on quasiconformal mappings. The *distortion* of f at $z \in \mathbb{C}$ is

$$K_f(z) := \frac{1 + |\mu_f(z)|}{1 - |\mu_f(z)|}.$$

A mapping is called K-quasiconformal if $K_f(z) \leq K$ almost everywhere. The smallest such constant is called the *maximal dilatation* and denoted by K_f. If we drop the assumption on injectivity, then f is a *quasiregular mapping*. See for example [**13**, **19**] for the theory of quasiregular mappings. In the plane, every quasiregular mapping has an important decomposition.

THEOREM 2.1. (Stoilow factorization; see for example [**13**, p. 254]) *Let $f : \mathbb{C} \to \mathbb{C}$ be a quasiregular mapping. Then there exist an analytic function g and a quasiconformal mapping h such that $f = g \circ h$.*

We call a quasiregular mapping f *uniformly quasiregular* if there exists $K \geq 1$ such that $K_{f^n}(z) \leq K$ for all $n \in \mathbb{N}$. The dynamics of uniformly quasiregular mappings in the plane is well understood due to the following result of Hinkkanen.

THEOREM 2.2. [**12**] *Every uniformly quasiregular map $f : \mathbb{C} \to \mathbb{C}$ is quasiconformally conjugate to a holomorphic map.*

This means that the dynamics of uniformly quasiregular maps of the plane reduces to complex dynamics, which gives us no new features. To ensure that our study is of independent interest, we require that mappings of the form $h(z)^2 + c$ are not uniformly quasiregular; cf. Theorem 1.12.

2.2. *Definition of H.* Recall that if $K > 1$, $\theta \in (-\pi/2, \pi/2]$, then $h_{K,\theta}$ is the affine mapping given by

$$h_{K,\theta} = \left(\frac{K+1}{2}\right) z + e^{2i\theta}\left(\frac{K-1}{2}\right)\overline{z}.$$

Using the formula for complex dilatation, we see that

$$\mu_{h_{K,\theta}} = e^{2i\theta}\left(\frac{K-1}{K+1}\right) \tag{2.1}$$

and so $\|\mu_{h_{K,\theta}}\|_\infty < 1$, which verifies that $h_{K,\theta}$ is quasiconformal with constant complex dilatation. We now prove Proposition 1.1, which states that every quasiregular mapping in the plane of degree two with constant complex dilatation is linearly conjugate to a unique mapping of the form $h_{K,\theta}(z)^2 + c$ for some $K > 1$, $\theta \in (-\pi/2, \pi/2]$ and $c \in \mathbb{C}$.

Proof of Proposition 1.1. Let f satisfy the hypotheses of the proposition and let $\mu_f \equiv \mu$. By Theorem 2.1, we can write $f = \widetilde{g} \circ \widetilde{h}$ for some quadratic polynomial $\widetilde{g}$ and quasiconformal map $\widetilde{h}$ with constant complex dilatation. We may assume that $\widetilde{h}$ fixes 0.

Let $h = h_{K,\theta}$, where K, θ are chosen such that $((K-1)/(K+1))e^{2i\theta} = \mu$. Then, by the formula for the complex dilatation of a composition (see, for example, [**9**]), we have

$$\mu_{\widetilde{h} \circ h_{K,\theta}^{-1}} \equiv 0.$$

Therefore, there exists a conformal map $A : \mathbb{C} \to \mathbb{C}$ such that $\widetilde{h} = A \circ h_{K,\theta}$.

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We can write $f = g \circ h$, where $g = \widetilde{g} \circ A$ is a quadratic polynomial and $h = h_{K,\theta}$. Finally, applying [**8**, Proposition 3.1] gives the result. □

In this paper we focus on the case where $c = 0$ and we suppress the subscripts K and θ where there will be no confusion. We can restrict ourselves to studying only the $c = 0$ case because of the following theorem from [**7**], which gives a Böttcher coordinate.

THEOREM 2.3. [**7**, Theorem 2.1] *Let $K > 1$, $\theta \in (-\pi/2, \pi/2]$, $h = h_{K,\theta}$ be an affine mapping and $c \in \mathbb{C}$. Then there exist a neighbourhood $U = U(K, \theta, c)$ of infinity and a quasiconformal map $\psi = \psi(K, \theta, c)$ such that*

$$H(\psi(z)) = \psi(f(z))$$

for $z \in U$, where $f(z) = h(z)^2 + c$.

From Proposition 1.1, we know that any degree-two mapping of constant complex dilatation is linearly conjugate to a mapping $f_{K,\theta,c}$ for some K, θ, c. Then Theorem 2.3 tells us that $f_{K,\theta,c}$ is quasiconformally conjugate to $h^2_{K,\theta}$ in a neighbourhood of infinity. We may restrict our attention to the study of dynamics of the mappings $h^2_{K,\theta}$.

We note that for a fixed K, the maps $h^2_{K,\theta}$ and $h^2_{K,-\theta}$ are related.

LEMMA 2.4. *We have $H_{K,-\theta}(z) = \overline{H_{K,\theta}(\overline{z})}$.*

Proof.

$$\begin{aligned}\overline{H_{K,\theta}(\overline{z})} &= \left(\frac{K+1}{2}\right)z + e^{-2i\theta}\left(\frac{K-1}{2}\right)\overline{z} \\ &= H_{K,-\theta}(z).\end{aligned}$$

□

Lemma 2.4 means that we can just study the range $\theta \in [0, \pi/2]$ and then transfer the results using complex conjugation to extend to $\theta \in (-\pi/2, 0)$.

Fix $K > 1$, $\theta \in (-\pi/2, \pi/2]$ and $H = h^2_{K,\theta}$. Contained in the proof of [**8**, Theorem 6.4] is the observation that H takes the form

$$H(re^{i\varphi}) = r^2(1 + (K^2 - 1)\cos^2(\varphi - \theta)) \exp\left[2i\left(\theta + \tan^{-1}\left(\frac{\tan(\varphi - \theta)}{K}\right)\right)\right] \quad (2.2)$$

in polar coordinates. Hence, H maps rays to rays. The mapping H has at least one fixed ray [**8**], and fixed rays correspond to roots of the cubic polynomial

$$P(t) := Kt^3 + (2 - K)\tan(\theta/2)t^2 + (2 - K)t + K\tan(\theta/2), \quad (2.3)$$

where $t = \tan[(\varphi - \theta)/2]$. Since P always has a root $t_0 \in (-1, 1)$, there is always a fixed ray with angle in $(-\pi/2, \pi/2)$. The fact that P is a cubic suggests that there could be one, two or three fixed rays and we will show that all cases are possible.

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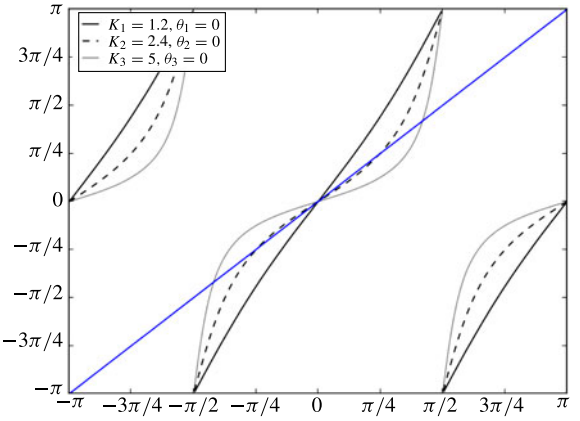


FIGURE 3. $\widetilde{H}$ for $\theta = 0$ and various values of K.

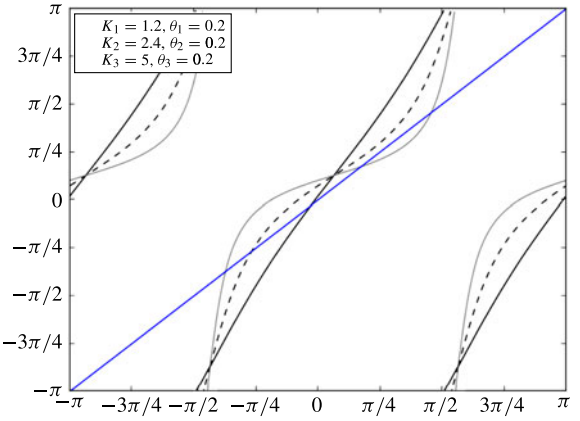


FIGURE 4. $\widetilde{H}$ for $\theta = 0.2$ and various values of K.

3. *Fixed rays of H*

To prove Theorem 1.3, we use the following strategy.

- Given K, θ, show that the argument of $H = H_{K,\theta}$ induces a map $\widetilde{H} : S^1 \to S^1$.
- Determine the possible locations of fixed points of $\widetilde{H}$.
- When $\theta = 0$, show that if $K \leq 2$, then $\widetilde{H}$ has one repelling fixed point, or a neutral one in the case $K = 2$, and if $K > 2$, then $\widetilde{H}$ has three fixed points, two repelling and one attracting.
- When $\theta = \pi/2$, show that $\widetilde{H}$ only ever has one repelling fixed point.
- For $\theta \in (0, \pi/2)$, show that there exists $K_\theta > 2$ such that $\widetilde{H}$ has two fixed points, one repelling and one neutral. If $K < K_\theta$, then $\widetilde{H}$ has one repelling fixed point. If $K > K_\theta$, then $\widetilde{H}$ has three fixed points, one attracting and two repelling.

We see in Figure 3 how one or three fixed rays may occur when $\theta = 0$ and in Figure 4 how one, two or three fixed rays may occur when $\theta \neq 0$.

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3.1. *The induced map $\widetilde{H}$ of S^1.* Given K, θ, the map $H = H_{K,\theta}$ induces a map $S^1 \to S^1$ as follows.

Definition 3.1. Define $\widetilde{H} : S^1 \to S^1$ by

$$\widetilde{H}(e^{i\varphi}) = e^{i\psi} \quad \text{where } \arg[H(re^{i\varphi})] = \psi$$

for any $r > 0$.

By lifting $\widetilde{H}$ to $\mathbb{R}$, we obtain a 2π-periodic mapping $\mathbb{R} \to \mathbb{R}$. We will often use $\widetilde{H}$ to denote both the mapping on S^1 and its lift to $\mathbb{R}$, and the usage should be clear from context. We also remark that by the definition of H, $\widetilde{H}$ is actually a π-periodic mapping. We have that

$$\widetilde{H}(\varphi) = 2\theta + 2\tan^{-1}\left(\frac{\tan(\varphi - \theta)}{K}\right) \tag{3.1}$$

when $\widetilde{H}$ is viewed as the mapping lifted to $\mathbb{R}$.

Viewed as a mapping on S^1, $\widetilde{H}$ is two-to-one. Points in S^1 correspond to rays in $\mathbb{C}$ and so fixed points of $\widetilde{H}$ correspond to fixed rays of H. In this way, we reduce our study of fixed rays of H to fixed points of the circle endomorphism $\widetilde{H}$. Given a sector $S \subset \mathbb{C}$, we will denote by $\widetilde{S}$ the corresponding subset of S^1 or interval in $\mathbb{R}/2\pi\mathbb{Z}$.

Definition 3.2. Define the map $\widetilde{h} : \mathbb{R} \to \mathbb{R}$ by

$$\widetilde{h}(\varphi) = \psi \quad \text{where } \arg[h(re^{i\varphi})] = \psi$$

for any $r > 0$.

Recall from Lemma 2.4 that

$$H_{K,-\theta}(z) = \overline{H_{K,\theta}(\overline{z})}.$$

Hence, we only need to consider the case $\theta \in [0, \pi/2]$. The ray R_φ is a fixed ray of $H_{K,\theta}$ if and only if $R_{-\varphi}$ is a fixed ray of $H_{K,-\theta}$ and they have the same behaviour. For the rest of this section, we assume $\theta \in [0, \pi/2]$.

3.2. *Locations of fixed rays of H.* By results in [**8**], we know that H has at least one fixed ray. First, we will narrow down the sectors where these fixed rays can be.

LEMMA 3.3. *If $\theta > 0$, then any fixed ray R_ϕ of H lies in one of the sectors*

$$\mathbb{F}_\theta^+ = \{R_\varphi \mid 2\theta < \varphi < \theta + \pi/2\}$$

or

$$\mathbb{F}_\theta^- = \{R_\varphi \mid \theta - \pi/2 < \varphi < 0\}.$$

If $\theta = 0$, any fixed ray is in $\mathbb{F}_\theta^\pm \cup \{R_0\}$. If $\theta = \pi/2$, then R_0 is the only fixed ray.

Proof. Recall that our normalization for θ requires $\theta \in (-\pi/2, \pi/2]$ and that by Lemma 2.4 we need only consider $\theta \geq 0$. Define the following quadrants of $\mathbb{C}$:

$$\begin{aligned} Q_1 &= \{R_\varphi \mid 0 < \varphi - \theta < \pi/2\}, \\ Q_2 &= \{R_\varphi \mid -\pi/2 < \varphi - \theta < 0\}, \\ Q_3 &= \{R_\varphi \mid -\pi/2 < \varphi - \theta < -\pi\}, \\ Q_4 &= \{R_\varphi \mid \pi/2 < \varphi - \theta < \pi\}. \end{aligned}$$

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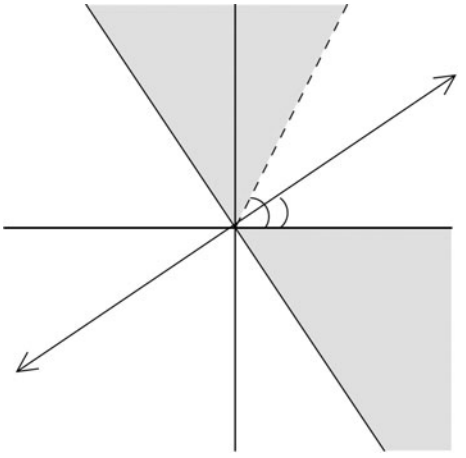


FIGURE 5. Diagram showing the regions $\mathbb{F}_\theta^{\pm}$.

Let $0 < \theta < \pi/2$. Using the fact that H doubles angles, it is elementary to check that there can be no fixed ray in Q_3 or Q_4, and further that any fixed ray in Q_1 must lie in $\mathbb{F}_\theta^+$ and any fixed ray in Q_2 must lie in $\mathbb{F}_\theta^-$.

When $\theta = 0$, the above holds with the addition that the ray R_0 is always fixed. It is easy to see that R_0 is fixed when $\theta = \pi/2$ and that this is the only possible fixed ray. □

See Figure 5 for the situation considered in this section.

3.3. *Local expansion and contraction.* In this subsection, we will study $\widetilde{H}'$, as this determines whether a small neighbourhood of φ is contracted or expanded under $\widetilde{H}$. Since $\widetilde{H}$ is sense preserving, $\widetilde{H}' > 0$. Let us make this more precise.

Definition 3.4. An interval $I \subset \mathbb{R}/2\pi\mathbb{Z}$ is *expanded* by $\widetilde{H}$ if

$$|\widetilde{H}(I)| > |I|$$

and is *contracted* by $\widetilde{H}$ if

$$|\widetilde{H}(I)| < |I|.$$

It is easy to see that if $\widetilde{H}'(\varphi) < 1$ or $\widetilde{H}'(\varphi) > 1$, then there exists some neighbourhood V of φ that is contracted or expanded respectively by $\widetilde{H}$. Further, if there is some closed interval I such that $\widetilde{H}'(\varphi) < 1$ or $\widetilde{H}'(\varphi) > 1$ for all $\varphi \in I$, then it follows that I is contracted or expanded respectively by $\widetilde{H}$.

LEMMA 3.5. *For $K < 2$ and any θ, $\widetilde{H}'(\varphi) > 1$ for all $\varphi \in \mathbb{R}/2\pi\mathbb{Z}$. When $K > 2$, there is a single interval $J \subset (\theta - \pi/2, \theta + \pi/2)$, where we have $\widetilde{H}'(\varphi) < 1$ and, further, θ is the mid point of J.*

Proof. This lemma follows by considering

$$\widetilde{H}'(\varphi) = \frac{2K}{1 + (K^2 - 1)\cos^2(\varphi - \theta)}. \tag{3.2}$$

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The reader may check that if $K < 2$, then

$$2K > K^2 \geq 1 + (K^2 - 1)\cos^2(\varphi - \theta). \tag{3.3}$$

Then (3.3) implies $\widetilde{H}'(\varphi) > 1$ when $K < 2$, proving the first part of the lemma. When $K = 2$, we observe that $\widetilde{H}'(\theta) = 1$ and, from (3.2), that, if $\varphi \neq \theta$, then $\widetilde{H}'(\varphi) > 1$.

When $K > 2$, we have $\widetilde{H}'(\theta) < 1$, by (3.2). By continuity, there exists an interval J containing θ such that $\widetilde{H}'(\varphi) < 1$ for $\varphi \in J$. Since

$$\widetilde{H}'(\varphi) \to 2K \quad \text{as } \varphi - \theta \to \pm\pi/2,$$

it follows that $J \neq (\theta - \pi/2, \theta + \pi/2)$. By differentiating the expression for $\widetilde{H}$ again, the only critical point of $\widetilde{H}'$ is at $\phi = \theta$. Using this and the fact that $\widetilde{H}'$ is symmetric about θ, the result follows. □

Definition 3.6. Given $K > 2$ and θ, denote by $J = J_K$ the interval $(\theta - \eta, \theta + \eta)$ for $\eta = \eta_K$, where $\widetilde{H}'(\theta) < 1$. Note that η does not depend on θ.

We remark that as $\widetilde{H}$ is π-periodic, the translate of J by π is a second interval where $\widetilde{H}'(\varphi) < 1$. However, there can be no fixed points here from Lemma 3.3 and so we are not concerned with this other interval in this section.

3.4. *Special cases.* We will now investigate the fixed points of $\widetilde{H}$. The cases where $\theta = 0$ and $\theta = \pi/2$ are special cases. We first show that if $\theta = 0$, then $\widetilde{H}$ can never have a neutral fixed point which is not 0.

LEMMA 3.7. *Let $\theta = 0$. Suppose $\varphi \neq 0$ and $\widetilde{H}'(\varphi) = 1$; then φ cannot be fixed.*

Proof. As $\widetilde{H}'(\varphi) = 1$, (3.2) implies

$$\varphi = \cos^{-1}\left[\left(\frac{2K - 1}{K^2 - 1}\right)^{1/2}\right],$$

where we take the positive square root since $|\varphi| < \pi/2$. Suppose that φ is fixed, so that $\widetilde{H}(\varphi) = \varphi$. Then (3.1) implies

$$\cos^{-1}\left[\left(\frac{2K - 1}{K^2 - 1}\right)^{1/2}\right] = 2\tan^{-1}\left[\frac{\tan(\cos^{-1}[((2K - 1)/(K^2 - 1))^{1/2}])}{K}\right]. \tag{3.4}$$

An analysis of this equation using elementary trigonometric techniques shows that the only solutions are $K = 0, 1, 2$. Since $K = 0$ and $K = 1$ are not permissible values, the only valid solution is $K = 2$, which implies $\varphi = 0$. This completes the proof. □

LEMMA 3.8. *If $\theta = 0$, then $\widetilde{H}$ has one repelling fixed point $\phi_0 = 0$ when $K < 2$, has one neutral fixed point when $K = 2$ and has three fixed points $\pi/2 < \phi_2 < \phi_0 = 0 < \phi_1 < \pi/2$ when $K > 2$, where ϕ_1 and ϕ_2 are repelling and ϕ_0 is attracting. Further, $\phi_2 = -\phi_1$.*

Proof. First substitute $\theta = 0$ into (3.1) to obtain

$$\widetilde{H}(\varphi) = 2\tan^{-1}[(\tan\varphi)/K].$$

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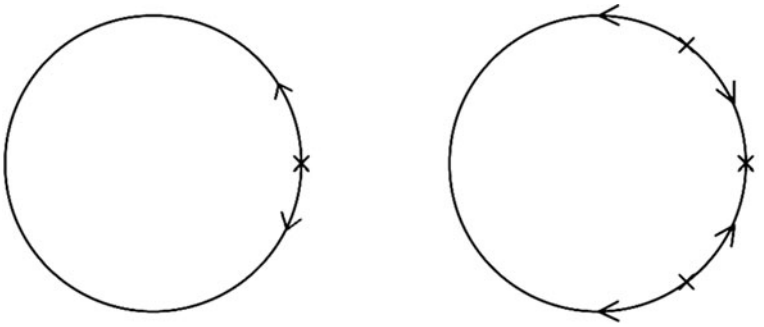


FIGURE 6. Diagram showing the dynamics of $\widetilde{H}$ with respect to the fixed points ϕ_i for $\theta = 0$ when $K < 2$ (left) and when $K > 2$ (right).

Then any fixed point ϕ must satisfy the equation

$$K \tan(\phi/2) = \tan \phi. \tag{3.5}$$

Since $\phi_0 = 0$ satisfies (3.5), it is always a fixed point when $\theta = 0$. Lemma 3.5 implies that for $K < 2$, $\widetilde{H}'(\phi_0) > 1$, so ϕ_0 is repelling. When $K > 2$, we see that $\widetilde{H}'(0) < 1$ and $\phi_0 = 0$ is attracting.

Let $K < 2$ and suppose we had some other fixed point ϕ. Without loss of generality, assume $\phi > \phi_0$; then the interval $[\phi_0, \phi]$ is fixed under $\widetilde{H}$, since the image of $[0, \pi]$ under $\widetilde{H}$ is $[0, 2\pi]$. However, this interval must also be expanded, as all $\varphi \in [\phi_0, \phi]$ satisfy $\widetilde{H}'(\varphi) > 1$, which is a contradiction.

Now let $K > 2$. Recall Definition 3.6 and the interval J. Write $J = (-\eta, \eta)$ and recall that Lemma 3.7 implies that neither $\pm\eta$ can be fixed for $K > 2$ and hence

$$|\widetilde{H}(J)| < |J|.$$

Since $\widetilde{H}(\pi/2) > \pi/2$ and $\widetilde{H}(\eta) < \eta$, by continuity there exists a fixed point $\phi_1 \in (\eta, \pi/2)$. Similarly, there is a fixed point $\phi_2 \in (-\pi/2, -\eta)$. Further, $\widetilde{H}'(\phi_i) > 1$ for $i = 1, 2$, so they are repelling. As ϕ_1 and ϕ_2 must satisfy (3.5) and, since tan is odd, we must have $\phi_2 = -\phi_1$. Since $\widetilde{H}$ can have at most three fixed points in S^1, these account for them all.

Finally, we deal with the case when $K = 2$. Here we have from Lemma 3.5 that $\widetilde{H}'(0) = 1$ and so ϕ_0 is a neutral fixed point. However, $\widetilde{H}'(\varphi) > 1$ for $\varphi \in (-\pi/2, \pi/2)\backslash\{0\}$, so any interval with one end point 0 is expanded by $\widetilde{H}$. This implies that there are no other fixed points. □

See Figure 6 for the situation considered in this Lemma 3.8.

LEMMA 3.9. *If $\theta = \pi/2$, then $\phi_0 = 0$ is the only fixed point of $\widetilde{H}$ for all $K > 1$ and it is always repelling.*

Proof. By Lemma 3.3, we know $\phi_0 = 0$ is the only fixed point of $\widetilde{H}$. Substituting $\varphi = 0$ and $\theta = \pi/2$ into (3.2), we see that

$$\widetilde{H}'(\phi_0) = 2K.$$

As $K > 1$, we have that ϕ_0 is repelling. □

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3.5. *The general case $\theta \in (0, \pi/2)$.* Recall the sectors $F_\theta^{\pm} \subset \mathbb{C}$ from Lemma 3.3, and the corresponding intervals $\widetilde{F}_\theta^{\pm} \subset S^1$. We first show that we can always locate one fixed point of $\widetilde{H}$.

LEMMA 3.10. *There is exactly one fixed point $\phi \in \widetilde{\mathbb{F}}_\theta^-$ of $\widetilde{H}$ for all $K > 1$. Further, it is a repelling fixed point.*

Proof. Recalling the notation of Lemma 3.3, we know that any fixed point of $\widetilde{H}$ in $\widetilde{Q}_2$ must lie in $\widetilde{\mathbb{F}}_\theta^-$. We also have that $\widetilde{Q}_2 \subset \widetilde{H}(\widetilde{Q}_2)$. Recall that $\widetilde{H}$ is orientation preserving, injective when restricted to $\widetilde{Q}_2$ and continuous. Hence, there must be a fixed point $\phi \in \widetilde{\mathbb{F}}_\theta^-$. We will see that ϕ is the only fixed point in $\widetilde{\mathbb{F}}_\theta^-$.

From Lemma 3.5 and Definition 3.6, $J \subset \widetilde{Q}_1 \cup \widetilde{Q}_2$ when $K > 2$. If $K \leq 2$, then $J = \emptyset$ and every interval with end point ϕ is expanded by $\widetilde{H}$. Therefore, $\widetilde{H}$ has no other fixed points.

Finally, suppose $K > 2$ and $J \neq \emptyset$. Then, since $\phi < 0$ is fixed and $\theta > 0$, the interval $[\phi, \theta]$ is expanded by $\widetilde{H}$ and so $\phi \notin J$. Suppose that there is some other fixed point $\phi' \in \widetilde{\mathbb{F}}_\theta^-$. Without loss of generality, assume $\phi < \phi'$. Then the interval $I = [\phi, \phi']$ satisfies $\widetilde{H}(I) = I$ and $I \cap J = \emptyset$. Therefore, I is expanded by $\widetilde{H}$, which is a contradiction. Therefore, there can only be one fixed point in $\widetilde{\mathbb{F}}_\theta^-$. □

To find whether there are other fixed points, in the next step we will see that given $\theta \in (0, \pi/2)$, we can choose K so that there are exactly two fixed points of $\widetilde{H}$.

LEMMA 3.11. *Let $\theta \in (0, \pi/2)$. There exists one $K_\theta > 2$ such that $\widetilde{H}_{K_\theta,\theta}$ has a neutral fixed point ϕ_{K_θ}.*

Proof. Note that (3.2) implies that if $\widetilde{H}'_{K,\theta}(\phi_{K_\theta}) = 1$, then

$$\phi_{K_\theta} = \cos^{-1}\left[\left(\frac{2K-1}{K^2-1}\right)^{1/2}\right] + \theta.$$

From (3.1) and the assumption ϕ_{K_θ} is fixed, we must have

$$\cos^{-1}\left[\left(\frac{2K-1}{K^2-1}\right)^{1/2}\right] + \theta = 2\tan^{-1}\left[\tan\left(\cos^{-1}\left[\left(\frac{2K-1}{K^2-1}\right)^{1/2}\right]\right) \Big/ K\right] + 2\theta. \tag{3.6}$$

Using elementary trigonometric techniques, one finds that

$$\theta = \cos^{-1}\left[\left(\frac{2K-1}{K^2-1}\right)^{3/2}(K-1)\right], \tag{3.7}$$

since $\theta > 0$. Writing (3.7) as $\theta = \cos^{-1}[F(K)]$, and viewing F as a function $(2, \infty) \to \mathbb{R}$, one can calculate that $\cos^{-1} \circ F : (2, \infty) \to \mathbb{R}$ is an increasing function, which is therefore injective. Further, $F(2) = 1$ and $F(K) > 0$; hence,

$$\cos^{-1} \circ F : (2, \infty) \to (0, \pi/2)$$

is bijective. By observing that given $\theta \in (0, \pi/2)$ we can find exactly one $K_\theta > 2$ satisfying (3.7), this completes the proof. □

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We next show that for this value K_θ, the corresponding $\widetilde{H}$ has only the two fixed points constructed thus far.

LEMMA 3.12. *Let $\theta \in (0, \pi/2)$ and let $K = K_\theta$. Then $\widetilde{H}$ has two fixed points, one of which is the neutral fixed point of Lemma 3.11, $\phi_{K_\theta} \in \widetilde{\mathbb{F}}_\theta^+$, and one of which is the repelling fixed point, $\phi \in \widetilde{\mathbb{F}}_\theta^-$.*

Proof. Fix $\theta \in (0, \pi/2)$. First, by Lemma 3.10, there is always exactly one repelling fixed point of $\widetilde{H}$ in $\widetilde{\mathbb{F}}_\theta^-$, and so any remaining fixed points will lie in $\widetilde{\mathbb{F}}_\theta^+$. We know from Lemma 3.11 that for a neutral fixed point we require $K > 2$ for any $\theta \in (0, \pi/2)$. Recall the interval $J = J_K = (\theta - \eta_K, \theta + \eta_K)$ from Definition 3.6, which is non-empty for $K > 2$, and consider the subinterval

$$J_K^+ = (\theta, \theta + \eta_K).$$

Writing $\varphi_K^+ = \theta + \eta_K$ and $\widetilde{H}_K$ to emphasize dependence on K, we see that

$$\widetilde{H}_K(\varphi_K^+) = 2\tan^{-1}\left[\left(\frac{K-2}{K(2K-1)}\right)^{1/2}\right] + 2\theta.$$

For K just above 2, we see that $\widetilde{H}_K(\varphi_K^+) \approx 2\theta > \varphi_K^+$. Now, as $K \to \infty$, we have both $\varphi_K^+ \to \pi/2 + \theta$ and $\widetilde{H}_K(\varphi_K^+) \to 2\theta$. Since $\theta < \pi/2$, we have $\widetilde{H}_K(\varphi_K^+) < \varphi_K^+$ for all large enough K.

By continuity, there exists some K_θ such that $\widetilde{H}_{K_\theta}(\varphi_{K_\theta}^+) = \varphi_{K_\theta}^+$. By construction, $\widetilde{H}'_{K_\theta}(\varphi_{K_\theta}^+) = 1$ and hence it is a neutral fixed point. By Lemma 3.11, we know that it is the only neutral fixed point for our given θ. We take ϕ_{K_θ} to be $\varphi_{K_\theta}^+$.

To see that this is the only fixed point in $\widetilde{\mathbb{F}}_\theta^+$, consider any interval contained in $\widetilde{\mathbb{F}}_\theta^+$ with one end point at $\phi_{K_\theta}^+$. Then the interior of the interval is either contained in J and the interval is contracted, or it is contained in the complement of J and the interval is expanded. In either case, the other end point of the interval cannot be a fixed point. □

For $K < K_\theta$, the next lemma shows that we only have one fixed point.

LEMMA 3.13. *Let $\theta \in (0, \pi/2)$. For $K < K_\theta$, there exists only one fixed point ϕ of $\widetilde{H}$. Further, $\phi \in \widetilde{\mathbb{F}}_\theta^-$ and ϕ is repelling.*

Proof. From Lemma 3.10, we know that there must be exactly one repelling fixed point in $\widetilde{\mathbb{F}}_\theta^-$. By Lemma 3.3, we are left to show that there are no fixed points in $\widetilde{\mathbb{F}}_\theta^+$. Using the idea of Lemma 3.12 and that when $K < K_\theta$ we know that $\widetilde{H}_K(\varphi_K^+) > \varphi_K^+$, the lemma follows. □

For $K > K_\theta$, we get three fixed points.

LEMMA 3.14. *Let $\theta \in (0, \pi/2)$. For $K > K_\theta$, there exist three fixed points of $\widetilde{H}$. There are fixed points ϕ_0, ϕ_1 and ϕ_2 such that $\phi_2 < \phi_0 < \phi_1$, ϕ_1 and ϕ_2 are repelling and ϕ_0 is attracting. Further, we have $\phi_1, \phi_0 \in \widetilde{\mathbb{F}}_\theta^+$ and $\phi_2 \in \widetilde{\mathbb{F}}_\theta^-$.*

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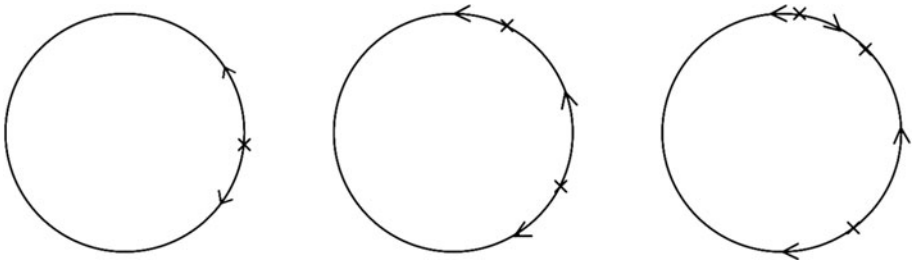


FIGURE 7. Diagram showing the local dynamics of $\widetilde{H}$ for $\theta \neq 0$ and $K < K_\theta$ (left), $K = K_\theta$ (centre) and $K > K_\theta$ (right).

Proof. From Lemma 3.10, we know that there must be exactly one repelling fixed point $\phi_2 \in \widetilde{\mathbb{F}}_\theta^-$. We are left to show that there are two fixed points in $\widetilde{\mathbb{F}}_\theta^+$, by Lemma 3.3.

By the methods of Lemma 3.12, we see that for $K > K_\theta$ we have $\widetilde{H}(\phi_{K_\theta}) < \phi_{K_\theta}$. Since $\widetilde{H}(\theta) = 2\theta > \theta$, by continuity there exists some $\phi_0 \in (\theta, \phi_{K_\theta})$ which is fixed by $\widetilde{H} = \widetilde{H}_K$. Similarly, as $\widetilde{H}(\pi/2 + \theta) = \pi + 2\theta > \pi/2 + \theta$, there exists $\phi_1 \in (\phi_{K_\theta}, \pi/2 + \theta)$ that is fixed by $\widetilde{H}$. Hence, for $K > K_\theta$, we have three fixed points. Note that we can have at most three fixed points, since the fixed points of $\widetilde{H}$ correspond to roots of the cubic P given in (2.3).

Finally, we have that $\phi_0 \in J$ and $\phi_1, \phi_2 \notin J$ by construction, and so ϕ_0 is attracting and ϕ_1, ϕ_2 are repelling. □

See Figure 7 for the situation considered in this section.

The preceding lemmas prove Theorem 1.3.

4. *Pre-images of fixed rays and basins of attraction*

In this section, we will prove Theorem 1.5 by studying the backward orbits of the fixed points of $\widetilde{H}$ and any basins of attraction. We will see that $\widetilde{H}$ restricted to S^1 is actually a Blaschke product. We take advantage of this fact, and use properties of Julia sets and Fatou sets of rational functions. In view of Lemma 2.4, throughout this section we assume $\theta \in [0, \pi/2]$.

4.1. *Basin of attraction.*

In the case where $\widetilde{H}$ has two or three fixed points on S^1, we will see that the non-repelling fixed point has a basin of attraction. When $\widetilde{H}$ has three fixed points, the basin is formed by a union of open intervals, whereas when $\widetilde{H}$ has two fixed points, the basin is formed by a union of half-open intervals.

Definition 4.1. The *basin of attraction* $\widetilde{\Lambda}$ of a non-repelling fixed point ϕ of $\widetilde{H}$ is given by

$$\widetilde{\Lambda} := \{\varphi \in S^1 \mid \widetilde{H}^n(\varphi) \to \phi \text{ as } n \to \infty\}.$$

The *immediate basin of attraction* $\widetilde{\Lambda}^*$ is the component of $\widetilde{\Lambda}$ containing ϕ.

Recall that we use a tilde to denote sets in S^1 and that the basin of attraction of the non-repelling fixed ray R_ϕ of H in $\mathbb{C}$ will be $\Lambda = \{R_\varphi \mid \varphi \in \widetilde{\Lambda}\}$.

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LEMMA 4.2. *Recalling the notation of Lemma 3.11, suppose that $\widetilde{H}$ has two fixed points. Then the neutral fixed point ϕ_{K_θ} has an immediate basin of attraction $\widetilde{\Lambda}^* = (\phi, \phi_{K_\theta}]$ that is the interval bounded by ϕ_{K_θ} and the repelling fixed point $\phi \in \widetilde{\mathbb{F}}_\theta^-$.*

Proof. This follows from Lemma 3.12 and a simple analysis of intervals that are expanded and contracted by $\widetilde{H}$. □

LEMMA 4.3. *When $\widetilde{H}$ has three fixed points $\phi_2 < \phi_0 < \phi_1$ as in Lemma 3.14, the attracting fixed point ϕ_0 has an immediate basin of attraction*

$$\widetilde{\Lambda}^* = (\phi_2, \phi_1).$$

Proof. This lemma follows from Lemma 3.14. □

4.2. *Writing $\widetilde{H}$ as a Blaschke product.* We will consider the case $\theta = 0$, since $h_{K,\theta}$ can be obtained from $h_{K,0}$ by pre-composing and post-composing by the corresponding rotations. Let $h = h_{K,0}$. The induced map $\widetilde{h}$ on S^1 cannot be a Möbius map; however, it is π-periodic and so we can renormalize $\widetilde{h} : (-\pi/2, \pi/2] \to (-\pi/2, \pi/2]$ to a map $\hat{h} : (-\pi, \pi] \to (-\pi, \pi]$ by defining

$$\hat{h}(\varphi) = 2\widetilde{h}(\varphi/2). \tag{4.1}$$

This map $\hat{h}$ has an attracting fixed point at $\varphi = 0$ and a repelling fixed point at $\varphi = \pi$.

LEMMA 4.4. *The map $\hat{h}$ agrees with the Möbius map*

$$A_K(z) = \frac{z + \alpha}{1 + \overline{\alpha} z}$$

on S^1, where $\alpha = (K - 1)/(K + 1)$.

Proof. Recall that the induced map is given by $\widetilde{h}(\varphi) = \tan^{-1}(\tan(\varphi)/K)$ and so

$$\tan(\hat{h}(\varphi)) = \frac{2K \sin \varphi}{(K^2 - 1) + (K^2 + 1)\cos \varphi}.$$

Define A_K to be the Möbius map

$$A_K(z) = \frac{z + \alpha}{1 + \overline{\alpha} z}, \tag{4.2}$$

where $\alpha = (K - 1)/(K + 1)$. By construction, $A_K(S^1) = S^1$. One can calculate that if $\widetilde{A}_K$ denotes the map that A_K induces on S^1, we see that

$$\widetilde{A}_K(\varphi) = \frac{2K \sin \varphi}{(K^2 - 1) + \cos \varphi (K^2 + 1)} = \hat{h}(\varphi).$$

This shows that $\hat{h}$ is a Möbius map of S^1. □

LEMMA 4.5. *Let $H = H_{K,\theta}$. Then $\widetilde{H} : S^1 \to S^1$ agrees with the Blaschke product B on S^1 given by*

$$B(z) = \frac{z^2 + \mu}{1 + \overline{\mu} z^2} = \left(\frac{z - a}{1 - \overline{a} z}\right)\left(\frac{z + a}{1 + \overline{a} z}\right),$$

where $\mu = e^{2i\theta}(K - 1)/(K + 1)$ is the complex dilatation of H and $a = e^{i(\theta - \pi/2)}((K - 1)/(K + 1))^{1/2}$.

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Proof. For $\varphi \in (-\pi/2+\theta, \pi/2+\theta]$, we have $\widetilde{h}_{K,\theta}(\varphi) = \widetilde{h}_{K,0}(\varphi-\theta)+\theta$. Using Lemma 4.4, we see that

$$\begin{aligned}\widetilde{h}_{K,\theta}(\varphi) &= \frac{\hat{h}(2\varphi-2\theta)}{2}+\theta \\ &= \frac{A_K(2\varphi-2\theta)}{2}+\theta.\end{aligned}$$

As $\widetilde{H}_{K,\theta} = 2\widetilde{h}_{K,\theta}$, we obtain

$$\widetilde{H}_{K,\theta}(\varphi) = 2\theta + A_K(2\varphi-2\theta)$$

for $\varphi \in (-\pi/2+\theta, \pi/2+\theta]$ and, by π-periodicity, for the remaining values of φ, we have

$$\begin{aligned}\widetilde{H}_{K,\theta}(\varphi) &= 2\theta + 2\pi + A_K(2\varphi+2\pi-2\theta) \\ &= 2\theta + A_K(2\varphi-2\theta).\end{aligned}$$

Letting $z = e^{i\varphi}$, recalling (4.2) we see that

$$\begin{aligned}\widetilde{H}_{K,\theta}(e^{i\varphi}) &= e^{2i\theta}\frac{e^{i(2\varphi-2\theta)}+\alpha}{1+\alpha e^{i(2\varphi-2\theta)}} \\ &= \frac{e^{2i\varphi}+\alpha e^{2i\theta}}{1+\alpha e^{-2i\theta}e^{2i\varphi}} \\ &= \frac{z^2+\mu}{1+\overline{\mu}z^2}, \qquad (4.3)\end{aligned}$$

where $\mu = e^{2i\theta}(K-1)/(K+1)$ is the complex dilatation of H. □

4.3. *Proof of Theorem 1.5.* We can now use the standard results on the iteration theory of Blaschke products as given in the following proposition. See for example [**6**]. Here we write $J(B)$ and $F(B)$ for the Julia and Fatou sets of B, respectively.

PROPOSITION 4.6. *Let B be a Blaschke product of degree two. Then the Julia set $J(B)$ is contained in S^1 and we have the following cases.*

- *If B has one fixed point in S^1, one fixed point in $\mathbb{D}$ and one fixed point in $\mathbb{C}\backslash\overline{\mathbb{D}}$, then $J(B) = S^1$.*
- *If B has one fixed point in S^1 of multiplicity three, and no other fixed points, then $J(B) = S^1$.*
- *If B has one repelling and one neutral fixed point in S^1, then $J(B)$ is a Cantor subset of S^1.*
- *If B has three fixed points in S^1, then $J(B)$ is a Cantor subset of S^1.*

Our Blaschke product B has three fixed points, counting multiplicity. Note that Theorem 1.3 tells us how many fixed points B has on S^1.

Suppose that $\widetilde{H}$ has one fixed point $\phi \in S^1$. We know from Lemma 4.5 and Proposition 4.6 that the Julia set of B is S^1 and so $\phi \in J$. Since $J(B) = \overline{O^-(z)}$ for any $z \in J(B)$, we immediately have that $\{\widetilde{H}^{-k}(\phi)\}_{k=0}^{\infty}$ is dense in S^1.

Suppose that $\widetilde{H}$ has more than one fixed point. Let ϕ be the non-repelling fixed point. By Lemma 4.6, we know that the Julia set $J(B)$ of B is a Cantor subset of S^1.

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This implies $E = F(B) \cap S^1$ is a dense subset of S^1. Consider a point $z \in E$; then any neighbourhood $U \subset F(B)$ of z contains points in $U \cap \mathbb{D}$. By the Denjoy–Wolff theorem, we have $B^n(w) \to \phi$ as $n \to \infty$ for every $w \in U \cap \mathbb{D}$. As U is contained in the Fatou set, the iterates $\{B^n\}$ are a normal family on U and so $B^n(z) \to \phi$ as $n \to \infty$. This implies $z \in \widetilde{\Lambda}$ and hence $\widetilde{\Lambda}$ is dense in S^1.

This completes the proof of Theorem 1.5.

5. *Proof of Corollary 1.10*

Fix $K > 1$ and $\theta \in (-\pi/2, \pi/2]$. By [**8**, Theorem 4.3], the escaping set $I(H)$ is a connected, completely invariant, open neighbourhood of infinity and $\partial I(H)$ is a completely invariant closed set. The point 0 is clearly fixed by H and, since

$$|z|^2 \leq |H(z)| \leq K^2|z|^2,$$

there is a neighbourhood of 0 contained in the basin of attraction $\mathcal{A}(0)$. It is therefore clear that $\mathcal{A}(0)$ is completely invariant and open.

Let R_ϕ be a fixed ray of H. Then, on R_ϕ, we have

$$H(re^{i\phi}) = \alpha r^2 e^{i\phi},$$

where $\alpha = (1 + (K^2 - 1)\cos^2(\phi - \theta))$ by the polar form (2.2) of H. For $r = 1/\alpha$, this point is fixed, for $r > 1/\alpha$ the point is in $I(H)$ and for $r < 1/\alpha$ the point is in $\mathcal{A}(0)$. By complete invariance, any pre-image of R_ϕ breaks up into $\mathcal{A}(0)$, $I(H)$ and $\partial I(H)$ in the same way.

Assume $K < K_\theta$; then, by Theorem 1.3, H has one fixed ray and by Theorem 1.5 the set $\{H^{-k}(R_\phi) : k \geq 0\}$ is dense in $\mathbb{C}$. Since $\mathcal{A}(0)$ is open, this proves the result in this case.

On the other hand, if $K \geq K_\theta$, then, by Theorem 1.3, write Λ for the basin of attraction of the non-repelling fixed ray R_ϕ. By Theorem 1.5, Λ is dense in $\mathbb{C}$. Suppose $R_\varphi \in \Lambda$. Then $H^n(R_\varphi) \to R_\phi$. Since $\mathcal{A}(0)$ and $I(H)$ are open, it is not hard to see that R_φ breaks up in the same way that R_ϕ does. Since Λ is dense in $\mathbb{C}$, the openness of $\mathcal{A}(0)$ again implies the result in this case, which proves the corollary.

6. *Nowhere uniformly quasiregular mappings*

We now prove Theorem 1.12, that H is nowhere uniformly quasiregular. The second part of the theorem follows immediately from applying the Böttcher coordinate. We will use Theorem 1.5. In [**7**], it was shown that H was not uniformly quasiregular by considering points on a fixed ray. Here, we will use density of pre-images of the fixed ray in the one fixed ray case, and density of the basin of attraction in the remaining cases. Let us first define what we mean by nowhere uniformly quasiregular.

Recalling Definition 1.11, the notion of being nowhere uniformly quasiregular is equivalent to the following definition.

Definition 6.1. We define the distortion of a function $f : \mathbb{C} \to \mathbb{C}$ at a point $z \in \mathbb{C}$ as

$$K_z(f) := \limsup_{\operatorname{diam}(U) \to 0} K(f|U),$$

where U is any neighbourhood of z and the lim sup is taken as the diameters of these neighbourhoods tend to zero.

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Definition 6.2. A function $f : \mathbb{C} \to \mathbb{C}$ is *nowhere uniformly quasiregular* if

$$K_z(f^n) \to \infty \quad \text{as } n \to \infty \text{ for all } z \in \mathbb{C}.$$

Note the difference between $K_z(f)$ and $K_f(z)$. The distortion at a point $K_f(z)$ is only defined almost everywhere, whereas $K_z(f)$ is defined everywhere.

We start by recalling the following proposition [**7**, Proposition 7.3].

PROPOSITION 6.3. *Let R_ϕ be a fixed ray of H. Then the complex dilatation $\mu_{H^n}(z)$ of H^n for $z \in R_\phi$ does not depend on z and satisfies $\mu_{H^n} = A^{n-1}(\mu_H)$, where A is the hyperbolic Möbius map*

$$A(w) = \frac{\mu_H + e^{-i\phi} w}{1 + e^{-i\phi}\overline{\mu_H} w}. \tag{6.1}$$

In particular,

$$|\mu_{H^n}(z)| \to 1 \quad \text{as } n \to \infty.$$

LEMMA 6.4. *If H has one fixed ray R_ϕ, then H is nowhere uniformly quasiregular.*

Proof. Fix $z \in \mathbb{C}$. Theorem 1.5 tells us that $\mathcal{P} = \{H^{-k}(R_\phi)\}$ is dense in $\mathbb{C}$. If z lies on a ray $R_\varphi \in \mathcal{P}$, then there must exist some m such that $H^m(R_\varphi) = R_\phi$. That is, $H^m(z)$ lies on the ray R_ϕ. We can apply the formula for the complex dilatation of the composition of functions, from for example [**9**], to obtain

$$\mu_{H^n \circ H^m}(z) = \frac{\mu_{H^m}(z) + r_{H^m}(z)\mu_{H^n}(H^m(z))}{1 + r_{H^m}(z)\overline{\mu_{H^m}(z)}\mu_{H^n}(H^m(z))}, \tag{6.2}$$

where $r_{H^m}(z) = \overline{(H^m)_z(z)}/(H^m)_z(z)$. Notice that $|r_{H^m}(z)| = 1$ and that if we define

$$B(w) := r_{H^m}(z)\left(\frac{w + \mu_{H^m}(z)\overline{r_{H^m}(z)}}{1 + \overline{[r_{H^m}(z)\mu_{H^m}(z)]}w}\right), \tag{6.3}$$

then B is a Möbius map of the disc and further we see that

$$B[\mu_{H^n}(H^m(z))] = \mu_{H^n \circ H^m}(z)$$

for $n \geq 1$. Using the fact that $H^{n+m}(z) \in R_\phi$ for $n \geq 0$, (6.3), Proposition 6.3 and the remark afterwards, we see that (6.2) becomes

$$\mu_{H^n \circ H^m}(z) = B(A^{n-1}(\mu_{H^m}(z))) \tag{6.4}$$

for $n \geq 1$. We know that $|A^n(w)| \to 1$ as $n \to \infty$ for any $w \in \mathbb{D}$ and that $B(\partial\mathbb{D}) = \partial\mathbb{D}$, and so we have

$$|\mu_{H^\ell(z)}(z)| \to 1 \quad \text{as } \ell \to \infty.$$

Any neighbourhood $U \ni z$ trivially contains z and so $K_z(H^\ell)$ is unbounded as $\ell \to \infty$.

Next, suppose that z lies on a ray not in $\mathcal{P}$. As $\mathcal{P}$ is dense in $\mathbb{C}$, any neighbourhood $U \ni z$ must intersect a ray $R_\varphi \in \mathcal{P}$. Picking one such ray, there must exist m (depending on the neighbourhood U) such that $H^m(R_\varphi) = R_\phi$ and we can apply the same argument above to conclude that $K_z(H^\ell)$ is unbounded as $\ell \to \infty$ for any $z \in \mathbb{C}$. □

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In the cases of more than one fixed ray, it is no longer true that any neighbourhood of a point contains the pre-image of a fixed ray. However, we can take advantage of the fact that $H^n(z)$ either ends up on a fixed ray, or the argument of $H^n(z)$ tends to the argument of the non-repelling fixed ray.

LEMMA 6.5. *If H has more than one fixed ray, then H is nowhere uniformly quasiregular.*

Proof. Fix $z \in \mathbb{C}$. From Theorem 1.5, we know that either z lies on the pre-image of a fixed ray, or $z \in \Lambda$. In the first case, the result follows from the methods of the previous lemma. In the second case, we know that the argument of $H^n(z)$ tends to the argument of the non-repelling fixed ray ϕ as $n \to \infty$.

We define the sequence $\phi_n \in S^1$ by $H^n(z) \in R_{\phi_n}$. Then $\phi_n \to \phi$ as $n \to \infty$, where ϕ is the non-repelling fixed point of $\widetilde{H}$. Again we use the formula for the complex dilatation of composition of functions from [**9**] reformulated slightly differently than in (6.2) to see that

$$\mu_{H^n}(z) = \mu_{H^{n-1}\circ H}(z) = \frac{\mu_H(z) + r_1(z)\mu_{H^{n-1}}(H(z))}{1 + r_1(z)\overline{\mu_H(z)}\mu_{H^{n-1}}(H(z))}.$$

Recalling that μ_H is constant, we can write

$$\mu_{H^n}(z) = A_1(\mu_{H^{n-1}}(H(z))),$$

where A_1 is the Möbius map

$$A_1(w) = \frac{\mu_H + r_H(z)w}{1 + r_H(z)\overline{\mu_H}w}.$$

Using the same method, we may write

$$\mu_{H^{n-1}}(H(z)) = A_2(\mu_{H^{n-2}}(H^2(z)),$$

where A_2 is the Möbius map

$$A_2(w) = \frac{\mu_H + r_H(H(z))w}{1 + r_H(H(z))\overline{\mu_H}w}.$$

By induction, we may write

$$\mu_{H^n}(z) = A_1 \circ A_2 \circ \cdots \circ A_{n-1}(\mu_H(H^{n-1}(z))),$$

where each A_i is a Möbius map given by

$$A_i(w) = \frac{\mu_H + r_H(H^{i-1}(z))w}{1 + r_H(H^{i-1}(z))\overline{\mu_H}w}.$$

It is not hard to see that $H_z(z) = (K+1)h(z)$, and so

$$r_H(H^{j-1}(z)) = \exp(-2i\arg[h(H^{j-1}(z))]).$$

As $j \to \infty$, we have $\arg[h(H^{j-1}(z))] \to \arg[h(re^{i\phi})]$ for any $r > 0$. Since ϕ is a fixed ray of H, $\arg[h(re^{i\phi})] = \phi/2$. In particular, we have that the Möbius maps A_i converge to the Möbius map

$$A(w) = \frac{\mu_H + e^{-i\phi}w}{1 + e^{-i\phi}\overline{\mu_H}w}.$$

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By Proposition 6.3, A is a hyperbolic Möbius map with fixed point $\alpha \in \partial\mathbb{D}$. Recalling that $\mu_H(z) = e^{2i\theta}(K-1)/(K+1)$ for all $z \in \mathbb{C}$, we can write

$$\mu_{H^n}(z) = A_1 \circ A_2 \circ \cdots \circ A_{n-1}(\mu_H) =: t_{n-1}(\mu_H). \tag{6.5}$$

To finish the proof, we need the following result on sequences of hyperbolic Möbius transformations. This is a combination of results from [**11**, **15**].

THEOREM 6.6. [**11**, **15**] *Let A, A_j be hyperbolic Möbius maps of $\mathbb{D}$ such that $A^n(z) \to \alpha \in \partial\mathbb{D}$ as $n \to \infty$ and $A_j \to A$ locally uniformly as $j \to \infty$. Suppose that we have sequences t_n, s_n of hyperbolic Möbius maps of $\mathbb{D}$ defined by*

$$t_n(z) = A_1 \circ A_2 \circ \cdots \circ A_n(z),$$
$$s_n(z) = A_n \circ A_{n-1} \circ \cdots \circ A_1(z).$$

Then both $t_n(z) \to \alpha$ and $s_n(z) \to \alpha$ as $n \to \infty$ for all $z \in \mathbb{D}$.

Applying Theorem 6.6 to the sequence t_n given in (6.5), we obtain that $\mu_{H^n}(z) \to \alpha$ and in particular $|\mu_{H^n}(z)| \to 1$. This proves the lemma. □

Note that we fixed z at the beginning of the proof and that a different choice of z will give rise to different Möbius maps A_i. Together Lemmas 6.4 and 6.5 prove Theorem 1.12.

7. *Failure of quasiconformal equivalence*

7.1. *Outline.* Our aim in this section is to prove Theorem 1.14 and show that for most pairs of values of K_i, θ_i, the corresponding maps H_i are not quasiconformally equivalent on any neighbourhood of infinity, and so neither are $H_i + c_i$ for any choice of $c_i \in \mathbb{C}$ via the Böttcher coordinate.

The outline of our strategy is as follows.

- Each fixed ray R_ϕ of H has a corresponding hyperbolic Möbius automorphism of $\mathbb{D}$ which encodes how the complex dilatation of the iterates H^n behaves on R_ϕ.
- If there is a quasiconformal equivalence Ψ between H_1 and H_2 such that $\Psi \circ H_1 = H_2 \circ \Psi$ on a neighbourhood U of infinity, then $1/C \le K_{H_1^n}(z)/K_{H_2^n}(\Psi(z)) \le C$ for some constant $C > 0$, all $n \in \mathbb{N}$ and all $z \in U$.
- We show that in the various cases of different numbers of fixed rays, if there is a quasiconformal equivalence Ψ, then the image of a fixed ray of H_1 under Ψ will be a fixed ray of H_2, intersect a fixed ray of H_2 or converge to a fixed ray of H_2.
- In each case, by comparing the behaviour of the corresponding Möbius maps for the respective fixed rays, we show that if the corresponding traces are different, then there can be no quasiconformal equivalence by contradicting the double inequality above.

7.2. *Consequences of a quasiconformal equivalence.* Throughout this section, we will consider two maps H_1, H_2 associated with K_i, θ_i for $i = 1, 2$.

Recall that two maps $f_1, f_2 : \mathbb{C} \to \mathbb{C}$ are quasiconformally equivalent on a neighbourhood U of infinity if there exist a neighbourhood V of infinity and a quasiconformal map $\Psi : U \to V$ such that

$$\Psi^{-1} \circ f_2 \circ \Psi(z) = f_1(z)$$

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for all $z \in U$.

Using the Böttcher coordinate, we know that $H(z)$ and $H(z) + c$ are quasiconformally equivalent on a neighbourhood of infinity. Therefore, if we are interested in knowing when $H_i + c_i$ are quasiconformally equivalent for $i = 1, 2$, we may reduce to the situation where $c_i = 0$.

If H_1, H_2 are quasiconformally equivalent on a neighbourhood of infinity, then

$$(H_2)^n(\Psi(z)) = \Psi((H_1)^n(z)) \tag{7.1}$$

for all $n \in \mathbb{N}$ and $z \in U$.

LEMMA 7.1. *If H_1 and H_2 are quasiconformally equivalent on a neighbourhood U of infinity, then there exists $C > 0$ such that*

$$\frac{1}{C} \leq \frac{K_{H_1^n}(z)}{K_{H_2^n}(\Psi(z))} \leq C, \tag{7.2}$$

for all $n \in \mathbb{N}$ and $z \in U$.

Proof. This follows immediately from (7.1) and the fact that distortion is sub-multiplicative with respect to composition; see for example [**9**]. We may take $C = (K_\Psi)^2$. □

7.3. *Möbius maps and fixed rays.* Recall that a Möbius map $A : \mathbb{D} \to \mathbb{D}$ is called *hyperbolic* if $\mathrm{Tr}^2(A) > 4$, where Tr denotes the trace of the normalized matrix representing A. The connection here is that in [**7**] it was proved that the Möbius map given in (6.1) is always hyperbolic.

LEMMA 7.2. *Let $A : \mathbb{D} \to \mathbb{D}$ be a hyperbolic Möbius map. Then A is conjugate to a Möbius map $\widehat{A} : \mathbb{H} \to \mathbb{H}$ given by*

$$\widehat{A}(z) = kz,$$

where

$$k = (T - 2 - (T^2 - 4T)^{1/2})/2 < 1$$

and $T := \mathrm{Tr}^2(A)$.

Proof. We know that $\mathrm{Tr}\, A^2 > 4$ and so by standard hyperbolic geometry we can lift to $\widetilde{A} : \mathbb{H} \to \mathbb{H}$. We see that $\widetilde{A}$ has the same trace as A and is of hyperbolic type and so it must be conjugate to $\widehat{A}(z) = kz$ for some $k > 0$. Conjugation preserves trace; hence,

$$k + 1/k + 2 = \mathrm{Tr}^2(\widehat{A}) = \mathrm{Tr}^2(A) = T.$$

Solving this for k and taking the negative square root gives the required equation and also that $k < 1$. Taking the positive square root would give the reciprocal. □

We will need the following result.

THEOREM 7.3. *Let A, $A_j : \mathbb{D} \to \mathbb{D}$ be hyperbolic Möbius maps such that $A^n(z) \to \alpha \in \partial\mathbb{D}$ as $n \to \infty$ and $A_j \to A$ locally uniformly as $j \to \infty$. Let*

$$t_n(z) = A_1 \circ A_2 \circ \cdots \circ A_n(z).$$

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Then

$$d_h(0, t_n(z)) = \log\left[\frac{1}{\prod_{j=1}^{n} k_j}\right] + O(1)$$

for large n, where d_h denotes the hyperbolic metric on $\mathbb{D}$, $k_j < 1$ for all j and $k_j \to k$, where k_j, k are the quantities defined in Lemma 7.2 associated with A_j, A.

In particular, if $A_j = A$ for every $j \in \mathbb{N}$, then

$$d_h(0, A^n(z)) = \log[1/k^n] + O(1)$$

as $n \to \infty$.

Proof. First, if $A^n(z) \to \alpha$ for $z \in \mathbb{D}$, then $t_n(z) \to \alpha$, by Theorem 6.6. Now let $B = A^{-1}$ and $B_j = A_j^{-1}$. Then, if $\alpha, \beta \in \partial\mathbb{D}$ are the attracting and repelling fixed points of A, respectively, then β is the attracting fixed point and α is the repelling fixed point of B. Similarly, if $\alpha_j, \beta_j \in \partial\mathbb{D}$ are the attracting and repelling fixed points of A_j, respectively, then β_j is the attracting fixed point and α_j is the repelling fixed point of B_j. Further, we have $B_j \to B$ and so $\alpha_j \to \alpha$ and $\beta_j \to \beta$ as $j \to \infty$.

We write $\widetilde{B}$ for the lift of B to $\mathbb{H}$ via $\gamma : \mathbb{D} \to \mathbb{H}$ so that $\widetilde{B} = \gamma \circ B \circ \gamma^{-1}$. We choose γ so that $\gamma(\alpha) = \infty$ and $\gamma(0) = i$. This then means that $\gamma(\beta) = X \in \mathbb{R}$ and $\gamma(\beta_j) = X_j \in \mathbb{R}$, where $X_j \to X$ as $j \to \infty$.

We can also conjugate by the maps Φ, $\Phi_i : \mathbb{H} \to \mathbb{H}$, where $\Phi(z) = z - X$ and $\Phi_i(z) = z - X_i$. This means that

$$\widetilde{B}(z) = \Phi^{-1} \circ \widehat{B} \circ \Phi(z) \quad \text{and} \quad \widetilde{B}_j(z) = \Phi_j^{-1} \circ \widehat{B}_j \circ \Phi_j(z),$$

where $\widehat{B}(z) = kz$ and $\widehat{B}_j(z) = k_j z$. The factors k and k_j are determined as in Lemma 7.2 so that $k, k_j < 1$ and $k_j \to k$ as $j \to \infty$. Let

$$s_n := t_n^{-1} = B_n \circ \cdots \circ B_1.$$

Writing $\rho_{\mathbb{H}}$ for the hyperbolic metric on $\mathbb{H}$, by conformal invariance we have

$$\begin{aligned} \rho_h(0, t_n(z)) &= \rho_{\mathbb{H}}(i, \gamma(t_n(z))) \\ &= \rho_{\mathbb{H}}(i, \widetilde{t_n}(\gamma(z)) \\ &= \rho_{\mathbb{H}}(\widetilde{s_n}(i), \gamma(z)). \end{aligned}$$

We can rewrite $\widetilde{s_n}(i)$ as

$$\widetilde{s_n}(i) = \Phi_n^{-1} \circ \widehat{B}_n \circ \Phi_n \circ \Phi_{n-1}^{-1} \circ \widehat{B}_{n-1} \circ \Phi_{n-1} \circ \cdots \circ \Phi_1^{-1} \circ \widehat{B}_1 \circ \Phi_1(i).$$

It is not hard to see that

$$\widetilde{s_n}(i) = \sum_{j=1}^{n} \left((X_{j-1} - X_j) \left(\prod_{m=j}^{n} k_m \right) \right) + X_n + i \prod_{m=1}^{n} k_m,$$

where we use the convention $X_0 = 0$. Writing

$$P_n = \prod_{m=1}^{n} k_m, \quad R_n = \sum_{j=1}^{n} \left((X_{j-1} - X_j) \left(\prod_{m=j}^{n} k_m \right) \right)$$

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and $\gamma(z) = x + iy \in \mathbb{H}$, we see that since $X_n \to X$ and $\widetilde{s_n}(i) \to X$ by Lemma 6.6, we have $R_n \to 0$ as $n \to \infty$. By the formula for the hyperbolic metric in $\mathbb{H}$,

$$\begin{aligned}\rho_{\mathbb{H}}(\widetilde{s_n}(i), \gamma(z)) &= \cosh^{-1}\left(\frac{(R_n + X_n - x)^2 + (P_n - y)^2}{2P_n y}\right) \\ &= \cosh^{-1}\left(\frac{P_n}{2y} + \frac{R_n^2 + X_n^2 + x^2 - 2xR_n - 2xX_n + 2X_nR_n - y}{2P_n}\right).\end{aligned} \tag{7.3}$$

Since $R_n \to 0$, $X_n \to X$ and x, y are fixed,

$$(R_n^2 + X_n^2 + x^2 - 2xR_n - 2xX_n + 2X_nR_n - y) \longrightarrow (X^2 + x^2 - 2xX - y) \tag{7.4}$$

as $n \to \infty$. This expression is bounded. We also have

$$\frac{P_n}{2y} \to 0 \quad \text{as } n \to \infty. \tag{7.5}$$

Hence, from (7.3), (7.4), (7.5) and using the identity $\cosh^{-1}(z) = \log(z + \sqrt{z^2 + 1})$, we can write

$$\begin{aligned}\rho_{\mathbb{H}}(\widetilde{s_n}(i), \gamma(z)) &= \cosh^{-1}\left[O\left(\frac{1}{P_n}\right)\right] \\ &= \log\left(\frac{1}{P_n}\right) + O(1),\end{aligned} \tag{7.6}$$

which proves the lemma. □

Recall from the proof of Lemma 6.4 that the complex dilatation μ_n of H^n on a fixed ray R_ϕ is given by $\mu_n = A^{n-1}(\mu_H)$. Here μ_H is the complex dilatation of H, given by $\mu_H = e^{2i\theta}(K-1)/(K+1)$, and A is the Möbius automorphism of $\mathbb{D}$, given by

$$A(z) := \frac{\mu_H + e^{-i\phi}z}{1 + e^{-i\phi}\overline{\mu_H}z}.$$

It was shown in [**7**] that A is a hyperbolic Möbius map, i.e. $\mathrm{Tr}(A)^2 > 4$. Given a fixed ray R_ϕ of H, the squared trace of A is given by

$$\xi(\phi) := \mathrm{Tr}(A)^2 = \frac{(K+1)^2}{2K}(1 + \cos\phi). \tag{7.7}$$

We will use A_i to denote the Möbius map corresponding to the fixed ray R_{ϕ_i} of H_1 and B_i for the Möbius map corresponding to the fixed ray R_{ψ_j} of H_2.

LEMMA 7.4. *Let H_1 and H_2 be quasiconformally equivalent on a neighbourhood U of infinity, with $\Psi \circ H_1 = H_2 \circ \Psi$. Then, if R_{ϕ_i} and R_{ψ_j} are fixed rays of H_1 and H_2 respectively and*

$$\arg[H_2^n(\Psi(z))] \to \psi_j$$

for some $z \in R_{\phi_i}$,

$$\xi(\phi_i) = \xi(\psi_j).$$

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Note that this lemma takes care of the cases where $\Psi(R_{\phi_i})$ is a fixed ray, intersects a fixed ray in one point (which means that it must intersect in infinitely many) or is a curve which converges to a fixed ray of H_2. This lemma is our key tool in this section.

Proof. Suppose for a contradiction that $\arg[H_2^n(\Psi(z))] \to \psi_j$ for some $z \in R_{\phi_i}$, but $\xi(\phi_i) \neq \xi(\psi_j)$. By Lemma 7.3,

$$d_h(0,\, A_i^n(\mu_{H_1})) = \log[O(1/k^n)], \tag{7.8}$$

where

$$k = (\mathrm{Tr}(A_i)^2 - 2 - (\mathrm{Tr}(A_i)^4 - 4\,\mathrm{Tr}(A_i)^2)^{1/2})/2.$$

Then, by Proposition 6.3, we see that

$$\begin{aligned} K_{H_1^n}(z) &= \frac{1 + |\mu_{H_1^n}(z)|}{1 - |\mu_{H_1^n}(z)|} \\ &= \exp(d_h(0,\, A_i^{n-1}(\mu_{H_1}))) \\ &= O(1/k^{n-1}). \end{aligned}$$

By hypothesis, $H_2^n(\Psi(z)) \in R_{\gamma_n}$ for some sequence of rays R_{γ_n}, where $\gamma_n \to \psi_j$. As in Lemma 6.5, we may write

$$\mu_{H_2^n}(\Psi(z)) = B_1 \circ \cdots \circ B_{n-1}[\mu_{H_2}(H_2^{n-1}(\Psi(z)))],$$

where μ_{H_2} is constant and each B_m is a Möbius map given by

$$B_m(w) = \frac{\mu_{H_2} + r_{H_2}(H_2^{m-1}(\Psi(z)))w}{1 + r_{H_2}(H_2^{m-1}(\Psi(z)))\overline{\mu_{H_2}}w}.$$

We have $r_{H_2}(H_2^{m-1}(\Psi(z))) \to e^{-i\psi_j}$. Hence, $B_m \to B_j$ as $m \to \infty$. Let $t_n = B_1 \circ \cdots \circ B_{n-1}$. Then, by Lemma 7.3,

$$d_h(0,\, t_n(\mu_{H_2}(\Psi(z)))) = \log\left[\frac{1}{\prod_{j=1}^{n-1} \ell^j}\right] + O(1),$$

where $\ell_m \to \ell$ as $m \to \infty$. Here ℓ, ℓ_m are the quantities from Lemma 7.2 involving the trace squared of the Möbius map B_j corresponding to the fixed ray R_{ψ_j}. By our hypothesis, $k \neq \ell$. By Lemma 7.3, we have

$$\begin{aligned} K_{H_2^n}(\Psi(z)) &= \frac{1 + |\mu_{H_2^n}(\Psi(z))|}{1 - |\mu_{H_2^n}(\Psi(z))|} \\ &= \exp(d_h(0,\, t_{n-1}(\mu_{H_2}(\Psi(z))))) \\ &= O\left(\frac{1}{\prod_{j=1}^{n-1} \ell_j}\right). \end{aligned} \tag{7.9}$$

As $\ell_j \to \ell \neq k$, then, for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that if $k < \ell$ then $\ell_j/k \geq \alpha > 1$ for all $j > N$ and if $k > \ell$ then $\ell_j/k \leq \beta < 1$ for all $j > N$. So, first, if $k > \ell$,

$$\frac{1}{k^n} \Big/ \frac{1}{\prod_{i=1}^n \ell_j} \leq \left(\frac{\prod_{i=1}^N \ell_j}{k^N}\right)\beta^{n-N} \to 0 \quad \text{as } n \to \infty$$

and, if $k < \ell$, then

$$\frac{1}{k^n} \Big/ \frac{1}{\prod_{i=1}^n \ell_j} \geq \left(\frac{\prod_{i=1}^N \ell_j}{k^N}\right)\alpha^{n-N} \to \infty \quad \text{as } n \to \infty.$$

In either case, we contradict Lemma 7.1. □

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7.4. *The case where H_2 has one fixed ray.* We next show that if one of our mappings has one fixed ray, then a quasiconformal equivalence implies that the other mapping must have one fixed ray.

LEMMA 7.5. *Suppose that H_2 has one fixed ray, and H_1 has more than one fixed ray. Then H_1 and H_2 are not quasiconformally equivalent.*

Proof. Suppose that H_2 has one fixed ray R_ψ, H_1 has two or three fixed rays R_{ϕ_0}, R_{ϕ_1} and possibly R_{ϕ_2} and there is a quasiconformal equivalence Ψ between them. If $\Psi(R_{\phi_i})$ is a ray, then (7.1) implies that it must be fixed by H_2. However, as there is only one fixed ray R_ψ of H_2, this implies $\Psi(R_{\phi_i}) = R_\psi$ and hence $\Psi(R_{\phi_j})$ cannot be a ray for $j \neq i$. Since the image $\Psi(R_{\phi_j})$ is not a ray, then, by Theorem 1.5, it must intersect $\{H_2^{-k}(R_\psi)\}$ and so by (7.1) it must intersect R_ψ, contradicting Ψ being injective.

Therefore, $\Psi(R_{\phi_i})$ is not a ray for any i and hence again, by Theorem 1.5, there exists $z_i \in R_{\phi_i}$ such that $\Psi(z_i) \in R_\psi$. We can apply Lemma 7.4 to see that $\xi(\phi_i) = \xi(\phi_j)$ for each i, j. Recall from (7.7) that

$$\operatorname{Tr}(A_i)^2 = \frac{(K_1+1)^2(1+\cos\phi_i)}{2K_1}.$$

As K_1 is fixed, this just depends on $\cos\phi$. To finish the proof, we need to show that $\cos\phi_i \neq \cos\phi_j$ for some pair of fixed rays of H_1, to give us a contradiction. Equivalently, we need to show that $\phi_i \neq -\phi_j$ for some pair of fixed rays of H_1. We will do this in Lemmas 7.6, 7.7 and 7.8 below. □

We have already seen in Lemma 3.8 that when $\theta = 0$, the repelling fixed points satisfy $\phi_1 = -\phi_2$ and the attracting fixed point is always $\phi_0 = 0$. We now consider the remaining cases when $\theta \neq 0$. The following result tells us that if a point of S^1 is moved by a given amount by the map $\widetilde{h}$ induced by h, then there are only two possibilities.

LEMMA 7.6. *Define G by $G(\varphi) := \varphi - \widetilde{h}(\varphi)$ for $\varphi - \theta \in (0, \pi/2)$. Then $G(\varphi_1) = G(\varphi_2)$ implies that either*

$$\varphi_1 = \varphi_2 \quad \textit{or} \quad \varphi_1 = \tan^{-1}(K/\tan(\varphi_2 - \theta)) + \theta.$$

Proof. Suppose $G(\varphi_1) = G(\varphi_2)$. Then this implies

$$\varphi_1 - \tan^{-1}\left(\frac{\tan(\varphi_1-\theta)}{K}\right) + \theta = \varphi_2 - \tan^{-1}\left(\frac{\tan(\varphi_2-\theta)}{K}\right) + \theta,$$

which we rearrange and use the addition formula for $\tan^{-1}$ to yield

$$\varphi_1 - \varphi_2 = \tan^{-1}\left(\frac{K(\tan(\varphi_1-\theta) - \tan(\varphi_2-\theta))}{K^2 + \tan(\varphi_1-\theta)\tan(\varphi_2-\theta)}\right). \tag{7.10}$$

Using elementary trigonometric techniques, we see that this has solutions given by $\tan(\varphi_1 - \theta) = \tan(\varphi_2 - \theta)$, which for our range of possible values implies $\varphi_1 = \varphi_2$. The other solutions are given by

$$\frac{K^2 - K}{K - 1} = \tan(\varphi_1 - \theta)\tan(\varphi_2 - \theta). \tag{7.11}$$

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We know that $K \neq 1$; hence, we obtain

$$\varphi_1 = \tan^{-1}(K/\tan(\varphi_2 - \theta)) + \theta. \tag{7.12}$$

□

We show that if H has two fixed rays, then they cannot be symmetric about the real axis.

LEMMA 7.7. *Let $\theta \neq 0$ and $K > 1$. If the corresponding map H has two fixed rays R_{ϕ_1} and R_{ϕ_2}, then*

$$\phi_1 \neq -\phi_2.$$

Proof. Assume to the contrary $\phi_1 > 0$ and $\phi_2 = -\phi_1$. As ϕ_1 is a neutral fixed point from Lemma 3.12, we have that $\widetilde{H}'(\phi_1) = 1$. From (3.2), this implies

$$\phi_1 = \cos^{-1}\left[\left(\frac{2K-1}{K^2-1}\right)^{1/2}\right] + \theta. \tag{7.13}$$

Since the ϕ_i are fixed by H, it follows that

$$\widetilde{h}(\phi_1) = -\widetilde{h}(\phi_2) \tag{7.14}$$

and

$$G(\phi_1) = -G(\phi_2). \tag{7.15}$$

Since $G(\varphi) = -G(-\varphi + \theta)$, (7.15) implies

$$G(\phi_1) = G(\phi_1 + \theta). \tag{7.16}$$

We can apply Lemma 7.6 with $\varphi_1 = \phi_1 + \theta$ and $\varphi_2 = \phi_1$ to see that

$$\phi_1 + \theta = \tan^{-1}(K/\tan(\phi_1 - \theta)) + \theta. \tag{7.17}$$

Substituting (7.13) into (7.17), and applying more trigonometric manipulations that we leave to the reader, shows that the only solutions of this equation are $K = 0, -1$, which are not valid values of K; hence, (7.13), (7.14) and (7.15) are never satisfied simultaneously, contradicting $\phi_1 = -\phi_2$. □

We have to deal with the case where H has three fixed rays. It is clear that it is not possible for $\cos\phi_i$ to be the same for all three fixed rays, but we find a condition under which they are all different.

LEMMA 7.8. *Let $\theta \neq 0$ and $K > 1$. If H has three fixed rays R_{ϕ_i} satisfying $\phi_2 < \phi_0 < \phi_1$, recalling Lemma 3.14, then*

$$\phi_1 \neq -\phi_0.$$

Further, if

$$\theta \geq \pi/6, \quad \text{then } \phi_i \neq -\phi_j \quad \text{for all } i \neq j.$$

However, if $\theta < \pi/6$, then there exists some K such that

$$\phi_i = -\phi_2$$

for $i = 0$ or $i = 1$.

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Proof. As $\phi_1, \phi_0 > 0$, we must have $\phi_1 \neq -\phi_0$. Suppose $\phi_2 = -\phi_0$. Recall from Lemma 3.3 that for this to be possible

$$\widetilde{\mathbb{F}}_\theta^+ \cap -\widetilde{\mathbb{F}}_\theta^- \neq \emptyset.$$

This implies that ϕ_1 must satisfy the two inequalities

$$2\theta < \phi_1 < \pi/2 + \theta \quad \text{and} \quad 0 < \phi_1 < \pi/2 - \theta,$$

which implies

$$0 < 2\theta < \pi/2 - \theta \Rightarrow 0 < \theta < \pi/6.$$

If $\theta < \pi/6$, then, by Lemma 7.7, when $K = K_\theta$ (recall Lemma 3.12), we know that $\phi_i \neq \phi_2$ for $i = 0, 1$. For $K > K_\theta$, the neutral fixed point splits into two fixed points ϕ_0 and ϕ_1. Further, $\phi_1 \to \pi/2 + \theta$ and $\phi_0 \to 2\theta$ as $K \to \infty$; also, $\phi_2 \to -\pi/2 + \theta$ as $K \to \infty$. Hence, by continuity, there must exist some $K > K_\theta$ such that $\phi_2 = -\phi_i$ for $i = 0$ or $i = 1$. □

The previous lemmas show that if H_2 has one fixed ray and H_1 has two or three fixed rays, then H_1 and H_2 cannot be quasiconformally equivalent on a neighbourhood of infinity.

7.5. *The case where H_2 has two fixed rays.* We move on to the case where both H_1 and H_2 have more than one fixed ray. To start, we will show that if there is a quasiconformal equivalence between H_1 and H_2, it must map the immediate basin of attraction of the non-repelling fixed ray of H_1 into the immediate basin of attraction of the non-repelling fixed ray of H_2.

LEMMA 7.9. *If H_1 and H_2 have immediate basins of attraction Λ_1^* and Λ_2^* respectively for the respective non-repelling fixed rays, and are quasiconformally equivalent in a neighbourhood U of infinity via the map Ψ, then $\Psi(\overline{\Lambda_1^*} \cap U) = \overline{\Lambda_2^*} \cap \Psi(U)$.*

Proof. Since $\overline{\Lambda_1^*}$ is fixed by H_1, we have

$$H_1(\overline{\Lambda_1^*} \cap U) = \overline{\Lambda_1^*} \cap H_1(U).$$

Since Ψ is injective,

$$\Psi(H_1(\overline{\Lambda_1^*} \cap U)) = \Psi(\overline{\Lambda_1^*}) \cap \Psi(H_1(U)).$$

Using the quasiconformal equivalence,

$$H_2(\Psi(\overline{\Lambda_1^*} \cap U)) = \Psi(\overline{\Lambda_1^*}) \cap H_2(\Psi(U)),$$

but we also have

$$H_2(\Psi(\overline{\Lambda_1^*} \cap U)) \subset H_2(\Psi(\overline{\Lambda_1^*})) \cap H_2(\Psi(U)).$$

Therefore, in a neighbourhood U' of infinity, we have

$$\Psi(\overline{\Lambda_1^*}) \cap U' \subset H_2(\Psi(\overline{\Lambda_1^*})) \cap U'.$$

This argument also shows that in a neighbourhood U_n' of infinity, we have

$$\Psi(\overline{\Lambda_1^*}) \cap U_n' \subset H_2^n(\Psi(\overline{\Lambda_1^*})) \cap U_n' \tag{7.18}$$

for any $n \in \mathbb{N}$.

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Now $\Psi(\overline{\Lambda_1^*})$ cannot spiral, as in that case it would intersect all fixed rays of H_2. Then we can apply Lemma 7.4 to see that $\xi(\phi_i)$ and $\xi(\psi_j)$ must all be equal, where R_{ϕ_i} and R_{ψ_j} are fixed rays of H_1 and H_2, respectively. However, Lemmas 7.7 and 7.8 show that this cannot be the case. Therefore, $\Psi(\overline{\Lambda_1^*})$ must be contained in some sector. By (7.18) and Theorem 1.5, we must have that

$$\Psi(\overline{\Lambda_1^*} \cap U) \subset \overline{\Lambda_2^*} \cap \Psi(U)$$

in a neighbourhood of infinity. The same argument applied to Ψ^{-1} shows that

$$\Psi^{-1}(\overline{\Lambda_2^*} \cap \Psi(U)) \subset \overline{\Lambda_1^*} \cap U,$$

and the lemma is proved. □

We will next show that if H_2 has two fixed rays and H_1 has three fixed rays, then there cannot be a quasiconformal equivalence between them.

LEMMA 7.10. *Let H_1, H_2 have three and two fixed rays, respectively. Then there cannot be a quasiconformal equivalence between them in any neighbourhood of infinity.*

Proof. Suppose that H_2 has fixed rays R_{ψ_1}, R_{ψ_2} with $\psi_1 > \psi_2$ and H_1 has fixed rays R_{ϕ_i}, $i = 1, 2, 3$, with $\phi_2 < 0 < \phi_0 < \phi_1$. Suppose for a contradiction that there is a quasiconformal equivalence Ψ between them. Then, by Lemma 7.9, we have $\Psi(\overline{\Lambda_1^*} \cap U) = \overline{\Lambda_2^*} \cap \Psi(U)$, where $\overline{\Lambda_1^*}$ is the closed sector bounded by the rays R_{ϕ_2} and R_{ϕ_1}, and $\overline{\Lambda_2^*}$ is the closed sector bounded by the rays R_{ψ_1} and R_{ψ_2}.

We can lift these sectors to the strip

$$\mathbb{S} := \{z \in \mathbb{C} \mid -\pi/2 \leq \operatorname{Im}(z) \leq \pi/2\}$$

via the quasiconformal maps $F_i : \overline{\Lambda_i^*} \to \mathbb{S}$ given by

$$F_1(re^{is}) = \begin{cases} \log r + i\left(\dfrac{\pi(s-\phi_0)}{2(\phi_1-\phi_0)}\right) & \text{if } \phi_0 \leq s \leq \phi_1, \\ \log r - i\left(\dfrac{\pi(s-\phi_0)}{2(\phi_2-\phi_0)}\right) & \text{if } \phi_2 \leq s < \phi_0 \end{cases}$$

and

$$F_2(re^{is}) = \log r + i\left(\frac{\pi(2s-(\psi_1+\psi_2))}{2(\psi_1-\psi_2)}\right).$$

Then $\Psi : \overline{\Lambda_1^*} \cap U \to \overline{\Lambda_2^*} \cap \Psi(U)$ lifts to a quasiconformal map $P : \Omega_1 \to \Omega_2$, where $\Omega_i = F_i(\overline{\Lambda_i^*}) \subset \mathbb{S}$ for $i = 1, 2$ and satisfies $P \circ F_1 = F_2 \circ \Psi$. Note that Ω_i is a connected subset of $\mathbb{S}$ whose boundary consists of two semi-infinite lines contained in the boundary of $\mathbb{S}$, and a curve γ_i in $\mathbb{S}$ connecting them. See Figure 8.

We want to extend P to a quasiconformal map from $\mathbb{S}$ to itself. There are many ways to do this, and we outline one here. Let T_i be the triangle $\mathbb{S}\backslash\Omega_i$ with vertices at the end points of γ_i and at $-\infty$. Define $q : \partial T_1 \to \partial T_2$ by translation on the respective horizontal semi-infinite lines, and agreeing with P on γ_1.

Let $g_i : T_i \to \mathbb{D}$ be conformal maps of the triangles onto the disc, sending the respective vertices to -1, i, 1, respectively. Then $\widetilde{q} = g_2 \circ q \circ g_1^{-1}$ from S^1 to itself is

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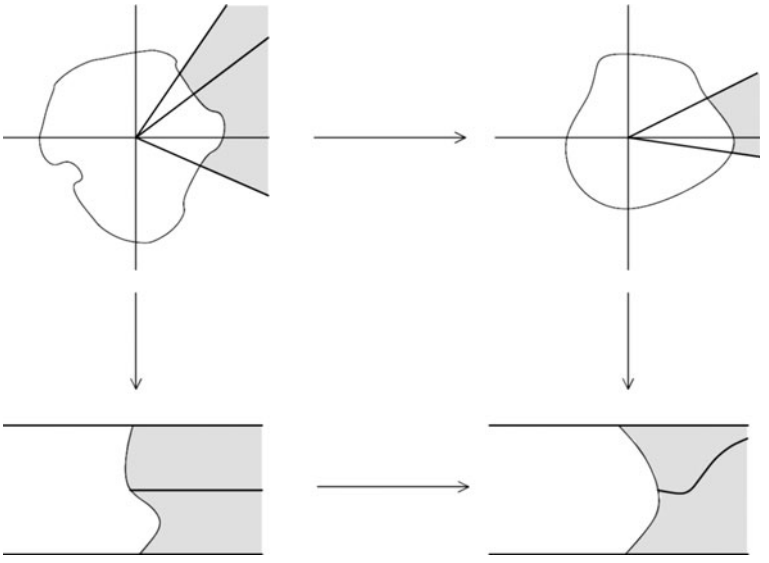


FIGURE 8. Diagram showing how P is induced from the action of Ψ on the sector $\overline{\Lambda_1^*}$.

a quasisymmetric map by construction. Extend to a quasiconformal map $\widetilde{q} : \mathbb{D} \to \mathbb{D}$ via, for example, the Douady–Earle extension (see for example [**9**]). Then we may extend P on the strip $\mathbb{S}$ by setting $P = g_2^{-1} \circ \widetilde{q} \circ g_1$ on T_1. This extension of P is a quasiconformal map by construction.

Now, consider the attracting fixed ray R_{ϕ_0} of H_1 which is contained in the interior of the region $\Lambda_1^* \cap U$. The image of R_{ϕ_0} under Ψ must be contained in $\overline{\Lambda_2^*}$, by Lemma 7.9. Then, by (7.1),

$$\arg[H_2^n(\Psi(z))] \to \psi_1 \tag{7.19}$$

as $n \to \infty$ for $z \in R_{\phi_0}$, since all points in Λ_2^* converge to the neutral fixed ray R_{ψ_1} of H_2. In particular, by lifting to the strip, $F_1(R_{\phi_0})$ is contained in the real line, but $P(F_1(R_{\phi_0})) = F_2(\Psi(R_{\phi_0}))$ is a curve which converges to the upper boundary component $\{\operatorname{Im} z = \pi/2\}$ of $\mathbb{S}$.

This contradicts the lemma below applied to P, completing the proof. □

LEMMA 7.11. *Let $f : \mathbb{S} \to \mathbb{S}$ be a K-quasiconformal map which sends $\pm\infty$ to $\pm\infty$, respectively. Then there exists $\delta < \pi/2$ such that $f(\mathbb{R})$ is contained in the sub-strip $\{z : |\operatorname{Im} z| < \delta\}$ of $\mathbb{S}$.*

Proof. This is a strip version of a well-known result in the disc, and uses the fact that $\mathbb{R}$ is a geodesic in $\mathbb{S}$. More specifically, by [**9**, Theorem 4.3.2], if $f : \mathbb{D} \to \mathbb{D}$ is K-quasiconformal, there exists some a, depending on K, such that f is a (K, a)-quasi-isometry. Then, by [**9**, Lemma 4.3.1], given a geodesic $\gamma \subset \mathbb{D}$, there exists $C > 0$ depending on K such that $f(\gamma)$ is contained in a C-neighbourhood of some geodesic γ'. Lifting to the strip, $\gamma = \mathbb{R}$ and the corresponding γ' is also $\mathbb{R}$. This proves the lemma. □

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7.6. *Proof of Theorem 1.14.* By Lemmas 7.5 and 7.10, we know that if H_1 and H_2 are quasiconformally equivalent in a neighbourhood of infinity, then they must have the same number of fixed rays. In the next two lemmas, we show that under a quasiconformal equivalence, the image of a fixed ray of H_1 must either intersect or approach a fixed ray of H_2.

LEMMA 7.12. *Suppose that H_1 and H_2 are quasiconformally equivalent on a neighbourhood of infinity and both have one fixed ray R_ϕ and R_ψ, respectively. Then there exists $z \in R_\phi \cap U$ such that $\Psi(z) \in R_\psi$.*

Proof. If $\Psi(R_\phi)$ is a ray, then the result follows from (7.1), using the same argument as in the proof of Lemma 7.5. Suppose that $\Psi(R_\phi)$ is not a ray; then it must intersect a sector Δ. By Theorem 1.5,

$$\Delta \cap \{H_2^{-k}(R_\psi)\} \neq \emptyset;$$

hence, there exist some ray $R \subset \Delta$ and $n \in \mathbb{N}$ such that $(H_2)^n(R) = R_\psi$. We can then choose $w \in R_\phi$ such that $\Psi(w) \in R$. From (7.1), we know that

$$(H_2)^n(\Psi(w)) = \Psi((H_1)^n(w)).$$

Choosing $z = (H_1)^n(w)$ completes the proof. □

LEMMA 7.13. *Suppose that H_1 and H_2 are quasiconformally equivalent on a neighbourhood of infinity and have three fixed rays. Let R_{ϕ_0} and R_{ψ_0} be the attracting fixed rays of H_1 and H_2, respectively. Then, for $z \in R_{\phi_0}$,*

$$\arg[H_2^n(\Psi(z))] \to \psi_0 \quad \text{as } n \to \infty.$$

The remaining fixed rays R_{ϕ_i} for $i = 1, 2$ of H_1 and R_{ψ_i} for $i = 1, 2$ of H_2 such that $\phi_2 < \phi_0 < \phi_1$ and $\psi_2 < \psi_0 < \psi_1$ must satisfy $\Psi(R_{\phi_i}) = R_{\psi_i}$ for $i = 1, 2$.

Proof. We know from Lemma 7.9 that

$$\Psi(\overline{\Lambda_1^*} \cap U) = \overline{\Lambda_2^*} \cap U.$$

By Lemma 4.3, we know that $\widetilde{\Lambda}_1^* = (\phi_1, \phi_2)$ and $\widetilde{\Lambda}_2^* = (\psi_1, \psi_2)$, proving the final part of the lemma.

If $\Psi(R_{\phi_0})$ is a ray, then $\Psi(R_{\phi_0}) = R_{\psi_0}$ from (7.1), by using the same argument as in Lemma 7.5 and the fact that we already know that $\Psi(R_{\phi_i}) = R_{\psi_i}$ for $i, j = 1, 2$. Assume that $\Psi(R_{\phi_0})$ is not a ray; then, by Theorem 1.5,

$$\Psi(R_{\phi_0}) \cap \Lambda_2 \neq \emptyset.$$

Choosing $w \in \Psi(R_{\phi_0}) \cap \Lambda_2$ implies $\arg[H_2^n(w)] \to \psi_0$; choosing $z = \Psi^{-1}(w)$ proves the lemma. □

Now finally we piece everything together.

THEOREM 1.14. *There is no quasiconformal conjugacy between H_1 and H_2 in any neighbourhood of infinity if any of the following conditions hold:*

(i) *the mappings H_1, H_2 have different numbers of fixed rays;*

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(ii) *H_1 and H_2 both have one fixed ray, R_{ϕ_1} and R_{ψ_1}, respectively, and $\xi(\phi_1) \neq \xi(\psi_1)$;*
(iii) *if H_1 and H_2 both have two fixed rays R_{ϕ_i} and R_{ψ_i} for $i = 1, 2$, where $\phi_1 > \phi_2$ and $\psi_1 > \psi_2$, and $\xi(\phi_i) \neq \xi(\psi_i)$ for some i;*
(iv) *if H_1 and H_2 both have three fixed rays R_{ϕ_i} and R_{ψ_j}, $i, j \in \{0, 1, 2\}$, respectively, where $\phi_1 > \phi_0 > \phi_2$ and $\psi_1 > \psi_0 > \psi_2$, and $\xi(\phi_i) \neq \xi(\psi_i)$ for some i.*

Proof of Theorem 1.14. Suppose that there is a quasiconformal equivalence Ψ between H_1 and H_2 on some neighbourhood U of infinity. First notice that any neighbourhood of infinity intersects every ray. In particular, it intersects fixed rays of H_1 and H_2. By Lemmas 7.5 and 7.10, H_1 and H_2 must have the same number of fixed rays. Each fixed ray ϕ_i and ψ_j of H_1 and H_2 respectively has a corresponding Möbius map A_i, B_j, respectively.

Suppose that H_1 and H_2 have one fixed ray. Then Lemma 7.12 tells us that we contradict Lemma 7.4 unless $\xi(\phi_1) = \xi(\psi_1)$.

Suppose that H_1 and H_2 have two fixed rays R_{ϕ_i} and R_{ψ_j}, respectively, where $\phi_2 < \phi_1$ and $\psi_2 < \psi_1$. Lemma 7.9 implies $\Psi(R_{\phi_i}) = R_{\psi_i}$ for $i = 1, 2$, and so we contradict Lemma 7.4 unless $\xi(\phi_i) = \xi(\psi_j)$ for both $i = 1, 2$.

Finally, suppose that H_1 and H_2 have three fixed rays. Again, by Lemma 7.13, we contradict Lemma 7.4 unless $\xi(\phi_i) = \xi(\psi_j)$ for $i = 1, 2, 3$. □

We can do better than this and rule out further possibilities.

COROLLARY 1.15. *If $\theta \in [0, \pi/2)$ is fixed and $K_1, K_2 > 1$ with $K_1 \neq K_2$, then $H_{K_1,\theta}$ and $H_{K_2,\theta}$ are not quasiconformally equivalent on any neighbourhood of infinity.*

Proof of Corollary 1.15. Suppose $\theta \in (0, \pi/2)$ is fixed. Recall from Lemma 3.10 that there always exists a fixed point $\phi \in \widetilde{\mathbb{F}}_\theta^-$ of $\widetilde{H}$, that is, $0 > \phi > \theta - \pi/2$. Let $K_1, K_2 > 1$, $\widetilde{h}_1 := \widetilde{h}_{K_1,\theta}$, $\widetilde{h}_2 := \widetilde{h}_{K_2,\theta}$ and $0 > \varphi > \theta - \pi/2$. Then

$$\widetilde{h}_1(\varphi) - \widetilde{h}_2(\varphi) = \tan^{-1}\left(\frac{\tan(\varphi - \theta)}{K_1}\right) + \theta - \tan^{-1}\left(\frac{\tan(\varphi - \theta)}{K_2}\right) - \theta. \tag{7.20}$$

Using the addition formula for $\tan^{-1}$ and simplifying, (7.20) becomes

$$\widetilde{h}_1(\varphi) - \widetilde{h}_2(\varphi) = \tan^{-1}\left(\frac{(K_2 - K_1)\tan(\varphi - \theta)}{K_1 K_2 + \tan^2(\varphi - \theta)}\right). \tag{7.21}$$

Here $\tan(\varphi - \theta) < 0$ is fixed. Hence, if $K_1 > K_2$, then (7.21) implies $\widetilde{h}_1(\varphi) > \widetilde{h}_2(\varphi)$. Let ϕ_K be the fixed point of $\widetilde{h}_{K,\theta}$ in $\widetilde{\mathbb{F}}_\theta^-$. By Lemma 3.13, we know that $\widetilde{h}_1(\phi_{K_1}) = \phi_{K_1}/2$ and $\widetilde{h}_1(\varphi) < \widetilde{h}_1(\varphi)/2$ for $0 < \varphi < \phi_{K_1}$. By (7.21), if $K_2 > K_1$, then $\widetilde{h}_2(\phi_{K_1}) < \phi_{K_1}/2$ and $\widetilde{h}_2(\varphi) < \widetilde{h}_2(\varphi)/2$ for $0 < \varphi < \phi_{K_1}$. Hence, it must be the case that $\phi_{K_2} < \phi_{K_1}$. This shows that ϕ_K decreases as K increases.

Suppose that $H_{K_1,\theta}$ is quasiconformally conjugate to $H_{K_2,\theta}$. Then, by Theorem 1.14, they must have the same number of fixed rays and the corresponding traces squared, because the fixed points $\phi_{K_1}, \phi_{K_2} \in \widetilde{\mathbb{F}}_\theta^-$ of $\widetilde{H}_{K_1,\theta}$ and $\widetilde{H}_{K_2,\theta}$, respectively, must be equal. This implies

$$\frac{(K_1 + 1)^2}{2K_1}(1 + \cos\phi_{K_1}) = \frac{(K_2 + 1)^2}{2K_2}(1 + \cos\phi_{K_2}).$$

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Rearranging, we obtain

$$\frac{K_2(K_1+1)^2}{K_1(K_2+1)^2} = \frac{1+\cos\phi_{K_1}}{1+\cos\phi_{K_2}}. \tag{7.22}$$

Suppose $K_1 < K_2$; then we know that $\phi_{K_1} > \phi_{K_2}$, which implies that the right-hand side of (7.22) is greater than 1; however,

$$K_2(K_1+1)^2 - K_1(K_2+1)^2 = (K_2-K_1)(1-K_1K_2) < 0.$$

Hence, the left-hand side of (7.22) is less than 1, which is a contradiction.

If $\theta = 0$, then $\phi_0 = 0$ is always a fixed point of $\widetilde{H}_{K_1,0}$ and $\widetilde{H}_{K_2,0}$ for any K_1, $K_2 > 1$. Hence, the right-hand side of (7.22) is always equal to 1, but the left-hand side is only equal to 1 if $K_1 = K_2$. □

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