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ON A SINGULAR INITIAL-VALUE PROBLEM FOR THE NAVIER–STOKES EQUATIONS

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Dedicated with affection and admiration to Vladimir Maz'ya on the occasion of his 75th birthday

Abstract. This paper presents a recent result for the problem introduced eleven years ago by Fraenkel and McLeod [A diffusing vortex circle in a viscous fluid. In *IUTAM Symposium on Asymptotics, Singularities and Homogenisation in Problems of Mechanics*, Kluwer (2003), 489–500], but described only briefly there. We shall prove the following, as far as space allows. The vorticity ω of a diffusing vortex circle in a viscous fluid has, for small values of a non-dimensional time, a second approximation $\omega_A + \omega_1$ that, although formulated for a fixed, finite Reynolds number λ and exact for $\lambda = 0$ (then $\omega = \omega_A$), tends to a smooth limiting function as $\lambda \uparrow \infty$. In §§1 and 2 the necessary background and apparatus are described; §3 outlines the new result and its proof.

§1. *Introduction.* In a certain weak sense, this paper is a continuation of [1]. However, no knowledge of [1] is required if the reader is willing to accept that a vorticity field in $\mathbb{R}^3$ (subject to mild restrictions, but not required to have any symmetry) has a *centroid of vorticity* moving with a velocity $\mathbf{U}(t)$ that is given by an explicit formula when the vorticity $\boldsymbol{\omega}(\cdot, t)$ throughout $\mathbb{R}^3$ is known. This result is essentially due to Saffman [2]; it was generalized a little (and perhaps clarified and sharpened) in [1].

We seek a solution of the Navier–Stokes equations with the initial condition illustrated in Figure 1: at time zero, vorticity $\boldsymbol{\omega}$ is concentrated on, and is tangential to, a horizontal circle in $\mathbb{R}^3$. This initial vorticity induces an initial velocity field that has infinite kinetic energy. (The circle then diffuses and moves vertically, at first with infinite velocity; at all positive times $t > 0$ the kinetic energy is finite.)

More precisely, consider incompressible fluid occupying all of $\mathbb{R}^3$ and at rest at infinity there; let $x := (x_1, x_2, x_3)$ be such that the frame (Ox_1, Ox_2, Ox_3) moves, relative to the motionless fluid at infinity, with the velocity $(0, 0, U(t))$ of the centroid of vorticity, the axes remaining parallel to their initial positions. The *fluid velocity* relative to this moving frame is written $\mathbf{v}(x, t)$ and the *vorticity* is

$$\boldsymbol{\omega} := \operatorname{curl} \mathbf{v} = \nabla \times \mathbf{v}.$$

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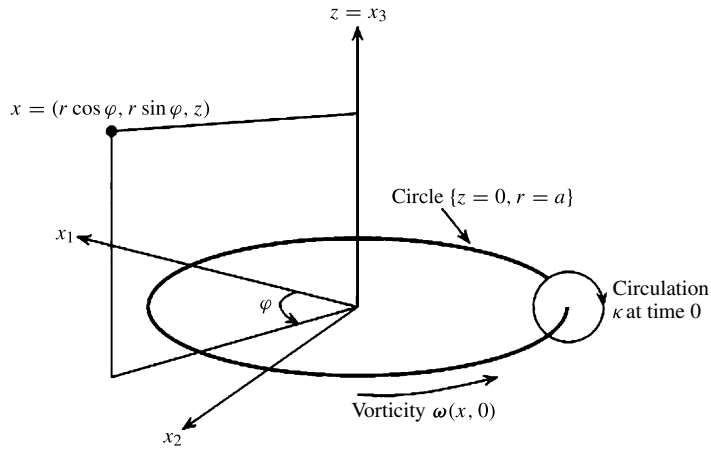


Figure 1: The initial condition.

Our *time variable* is $t = \nu T$, where T denotes physical time and ν is the kinematic viscosity (a given positive constant). This choice of t simplifies the heat operator in (1.3) below and simplifies most subsequent equations.

In writing $\mathbf{U}(t) = (0, 0, U(t))$, we have restricted attention to the cylindrical symmetry implied by the initial condition

$$\boldsymbol{\omega}(x, 0) = \kappa\delta(z)\delta(r - a)\mathbf{e}^{\phi}, \tag{1.1}$$

in which the circulation κ and the radius a are given positive constants, cylindrical co-ordinates (z, r, ϕ) are defined by $x =: (r\cos\phi, r\sin\phi, z)$, the unit vector $\mathbf{e}^{\phi} := (-\sin\phi, \cos\phi, 0)$ and δ denotes the Dirac generalized function.

In terms of the vorticity $\boldsymbol{\omega}$, the fluid velocity (relative to our moving frame) is

$$\mathbf{v}(x, t) = -(0, 0, U(t)) + \nabla \times \int_{\mathbb{R}^3} \frac{1}{4\pi|x - x'|}\boldsymbol{\omega}(x', t)\, dx'. \tag{1.2}$$

With (1.1) and (1.2) understood, we seek $\boldsymbol{\omega}(x, t)$ such that

$$\left(\frac{\partial}{\partial t} - \Delta\right)\boldsymbol{\omega} = -\frac{1}{\nu}((\mathbf{v}\cdot\nabla)\boldsymbol{\omega} - (\boldsymbol{\omega}\cdot\nabla)\mathbf{v}) \quad \text{in } \mathbb{R}^3 \times (0, \bar{t}\,) \tag{1.3}$$

for some small $\bar{t} > 0$.

Of course, it would be better to solve the problem (1.1)–(1.3) for all $t > 0$, but this is beyond us because we seek rather explicit answers. There are two excuses for considering only small t, or, rather, small t/a^2, which is non-dimensional. First, once a solution for $t > 0$ has been established, the general theory of the Navier–Stokes equations implies a continuation of the solution to all time, thanks to finite energy for $t > 0$, cylindrical symmetry and the absence of a swirl velocity (of a velocity component in the direction $\mathbf{e}^{\phi}$). Secondly, if the viscosity ν is small, which may be the case of primary interest, then the requirement that $\nu T/a^2$ be small does not demand that the physical time T be small.

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In view of (1.1), we write

$$\boldsymbol{\omega}(x,t) =: \omega(z,r,t)\mathbf{e}^{\phi},$$

and seek the solution of (1.1)–(1.3) in the scalar form $\omega = \omega_A + \omega_1 + \rho$, where ω_A is to be a first approximation for small t/a^2 and $\omega_A + \omega_1$ is to be a second (improved) approximation; the remainder ρ is to make $\omega_A + \omega_1 + \rho$ an exact solution and is to be $o(\omega_1)$ as $t \downarrow 0$. Here are some details.

(i) The nonlinear terms on the right-hand side of (1.3) are expected to be small for small t/a^2, because $\boldsymbol{\omega}$ should be approximately constant and large on small circles in a meridional plane (ϕ = constant) centred at $(z,r) = (0,a)$, so that $\mathbf{v}$ is approximately tangential to such circles and approximately of constant magnitude on each of them. (If the initial vortex circle were a straight line, then these nonlinear terms would vanish.) If the right-hand member of (1.3) is neglected, there results the formal approximation

$$\omega_A(z,r,t) = \frac{\kappa}{4\pi t}\exp\left(-\frac{s^2}{4t}\right)\left(\frac{a}{r}\right)^{1/2} B\left(\frac{ar}{2t}\right), \tag{1.4}$$

where $s := \{z^2 + (r-a)^2\}^{1/2}$ and B is a known function such that $B(y) \to 1$ as $y \to \infty$; in fact,

$$B(y) := (2\pi y)^{1/2}\mathrm{e}^{-y} I_1(y) \quad (0 \leqslant y < \infty), \tag{1.5}$$

where I_1 is the modified Bessel function of the first kind and of order one (as in [**3**, p. 77]).

(ii) The exponential in (1.4) prompts us to introduce *inner variables*

$$\sigma := \frac{s}{(4t)^{1/2}}, \qquad \theta := \tan^{-1}\frac{r-a}{z}; \tag{1.6}$$

then the amplitude $\kappa/4\pi t$ in (1.4) prompts us to pose

$$\omega_1(z,r,t) = (4t)^{-1/2}\tilde{\omega}_1(\sigma,\theta). \tag{1.7}$$

It suffices to consider ω_1 in an *inner region*: $t \downarrow 0$ with σ fixed, so that $s \downarrow 0$, because in an *outer region*: $t \downarrow 0$ with $s \geqslant$ constant > 0, not only ω_A but also ω is exponentially small.

The rest of this paper is devoted mainly to description of $\tilde{\omega}_1$; the *Reynolds number*

$$\lambda := \frac{\kappa}{2\pi\nu} \tag{1.8}$$

will be an important parameter.

(iii) The problem for the remainder ρ was sketched in [**1**]; the function $\rho(z,r,t)$ must be shown to exist and to be suitably small on the whole set $\mathbb{R} \times [0,\infty) \times (0,\bar{t}\,]$. Considerable progress has been made with this problem since [**1**] was written, but this analysis (which can only estimate ρ) is too long and too elaborate to be described here.

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§2. *The perturbation ω_1 for fixed Reynolds number λ.* With ω_1 as in (1.7), we adopt the notation

$$\begin{aligned}
&\text{(a)}\quad (\sigma,\theta)\in E:=(0,\infty)\times(-\pi,\pi],\\
&\text{(b)}\quad \Delta_\sigma:=\left(\frac{\partial}{\partial\sigma}\right)^2+\frac{1}{\sigma}\frac{\partial}{\partial\sigma}+\frac{1}{\sigma^2}\left(\frac{\partial}{\partial\theta}\right)^2,\\
&\text{(c)}\quad (A\tilde\omega_1)(\sigma_0,\theta_0):=\frac{1}{2\pi}\iint_E \log\frac{1}{|\sigma e^{i\theta}-\sigma_0 e^{i\theta_0}|}\,\tilde\omega_1(\sigma,\theta)\,\sigma\,d\sigma\,d\theta,\\
&\text{(d)}\quad \omega_{A,0}(\sigma,t):=\frac{\kappa}{4\pi t}e^{-\sigma^2},
\end{aligned}\tag{2.1}$$

in which $A\tilde\omega_1$ is a stream function describing the plane flow induced by vorticity $\tilde\omega_1$; the approximation $\omega_{A,0}$ to ω_A is that appropriate to $t\downarrow 0$ with σ fixed. We seek $\tilde\omega_1(\sigma,\theta)$ by linearizing (1.2) and (1.3) about $\omega_{A,0}$; the problem is then to solve the equation

$$\begin{aligned}
&-\left(\Delta_\sigma+2\sigma\frac{\partial}{\partial\sigma}+2\right)\tilde\omega_1+\lambda\frac{1-e^{-\sigma^2}}{\sigma^2}\frac{\partial}{\partial\theta}\tilde\omega_1-4\lambda e^{-\sigma^2}\frac{\partial}{\partial\theta}(A\tilde\omega_1)\\
&\quad=\frac{\kappa\lambda}{\pi a}g(\sigma)\cos\theta\quad\text{on } E,
\end{aligned}\tag{2.2}$$

with side conditions

$$\tilde\omega_1(\sigma,\theta)\to 0\quad\text{as } \sigma\downarrow 0 \text{ and as } \sigma\uparrow\infty.\tag{2.3}$$

The function g is a known, smooth function such that:

(a) $g(\sigma)=O(\sigma)$ as $\sigma\downarrow 0$; and

(b) $g(\sigma)=O(\sigma\log\sigma\, e^{-\sigma^2})$ as $\sigma\uparrow\infty$;
in fact,

(c)

$$g(\sigma):=\sigma e^{-\sigma^2}\left(\frac{3}{2}\frac{1-e^{-\sigma^2}}{\sigma^2}+\left(\log\frac{1}{\sigma}-\int_\sigma^\infty\frac{e^{-\rho^2}}{\rho}\,d\rho\right)-\frac{1}{2}(\gamma_E+1-\log 2)\right),\tag{2.4}$$

where $\gamma_E=0.5772\ldots$ denotes Euler's constant.

THEOREM 2.1. *For fixed $\lambda\in[0,\infty)$, the problem (2.2)–(2.3) for $\tilde\omega_1$ has a pointwise, unique solution; in particular, $\tilde\omega_1(\cdot,\theta)\in C^\infty[0,\infty)$, $\tilde\omega_1(0,\theta)=0$ and $\tilde\omega_1(\sigma,\theta)=o(e^{-\sigma^2/2})$ as $\sigma\uparrow\infty$.*

Here we have space only to sketch the main steps of the proof.

(i) Under the transformation

$$\tilde\omega_1(\sigma,\theta)=e^{-\sigma^2/2}q(\sigma,\theta),\tag{2.5}$$

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equation (2.2) becomes

$$-(\Delta_\sigma - \sigma^2)q + \lambda \frac{1 - \mathrm{e}^{-\sigma^2}}{\sigma^2} \frac{\partial q}{\partial \theta} - 4\lambda \mathrm{e}^{-\sigma^2/2} T(\mathrm{e}^{-\sigma^2/2} q) = \lambda \mathrm{e}^{\sigma^2/2} f(\sigma, \theta) \quad \text{on } E, \tag{2.6}$$

where the operator $T := (\partial/\partial\theta)A$ and $f(\sigma, \theta) := (\kappa/\pi a) g(\sigma) \cos\theta$. Let

$$(\xi, \eta) := \sigma(\cos\theta, \sin\theta), \qquad q_*(\xi, \eta) = q_*(\sigma\cos\theta, \sigma\sin\theta) := q(\sigma, \theta). \tag{2.7}$$

The condition in (2.3) for $\sigma \downarrow 0$ will be implicit in what follows; it was imposed only to make q_* decent at the origin, because we shall find that q is of form $q_c(\sigma)\cos\theta + q_s(\sigma)\sin\theta$. Henceforth the functions q_* *and* q *will be identified* wherever no confusion is possible. Similarly, the Cartesian coordinate and polar coordinate representations of other functions will be identified.

(ii) In the first instance we establish a weak solution of (2.6). Let the *real Hilbert space* Z be the completion of the set $C_c^\infty(\mathbb{R}^2)$, of real-valued, infinitely differentiable functions on $\mathbb{R}^2$ having compact support, in the norm implied by the inner product

$$\langle u, v\rangle_Z := \iint_{\mathbb{R}^2} (\nabla u \cdot \nabla v + \sigma^2 uv)\, d\xi\, d\eta. \tag{2.8}$$

We shall call q a *weak solution of* (2.6) if (and only if) $q \in Z$ and, for all test functions $u \in Z$,

$$\begin{aligned} B(u, q) &:= \iint_{\mathbb{R}^2} \Big(\nabla u \cdot \nabla q + \sigma^2 uq + \lambda \frac{1 - \mathrm{e}^{-\sigma^2}}{\sigma^2} u \frac{\partial q}{\partial \theta} \\ &\qquad - 4\lambda \mathrm{e}^{-\sigma^2/2} u\, T(\mathrm{e}^{-\sigma^2/2} q) \Big)\, d\xi\, d\eta \\ &= \lambda \iint_{\mathbb{R}^2} \mathrm{e}^{\sigma^2/2} f u\, d\xi\, d\eta. \end{aligned} \tag{2.9}$$

(iii) Here is the key step of the proof.

LEMMA 2.2. *The bilinear form B satisfies, for all u and v in Z,*

$$B(u, u) = \|u\|^2, \tag{2.10}$$

$$|B(u, v)| \leqslant (1 + k_B\lambda)\|u\|\|v\|, \tag{2.11}$$

where $\|\cdot\| = \|\cdot\|_Z$ and k_B is an absolute constant (independent of the variables, parameters and functions in question).

Partial proof. We shall prove only that (2.10) holds for all functions in $C_c^\infty(\mathbb{R}^2)$. The remainder of the proof is neither trivial nor immediate, but it is of a kind familiar in Sobolev space theory and its application to partial differential equations.

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In view of the definition of B in (2.9), we wish to prove that, for all $\varphi \in C_c^\infty(\mathbb{R}^2)$,

$$\iint_{\mathbb{R}^2} \frac{1-\mathrm{e}^{-\sigma^2}}{\sigma^2}\varphi\frac{\partial\varphi}{\partial\theta}\,d\xi\,d\eta = 0$$

and

$$\iint_{\mathbb{R}^2} \mathrm{e}^{-\sigma^2/2}\varphi\, T(\mathrm{e}^{-\sigma^2/2}\varphi)\,d\xi\,d\eta = 0.$$

The first of these is immediate because $\int_{-\pi}^{\pi}\varphi(\partial\varphi/\partial\theta)\,d\theta = 0$. For the second, let $A(\mathrm{e}^{-\sigma^2/2}\varphi) =: \psi$; then $\mathrm{e}^{-\sigma^2/2}\varphi = -\Delta\psi$ and we wish to prove that

$$-\iint_{\mathbb{R}^2}(\Delta\psi)\frac{\partial\psi}{\partial\theta}\,d\xi\,d\eta = 0.$$

Here it suffices to integrate over an open disc (or ball) $\mathcal{B}(0,R)$ with centre the origin and radius R so large that $\mathcal{B}(0,R)$ contains the compact support of $\Delta\psi$. Thus the integral may be written

$$-\int_{\partial\mathcal{B}(0,R)}\frac{\partial\psi}{\partial\sigma}\frac{\partial\psi}{\partial\theta}R\,d\theta + \iint_{\mathcal{B}(0,R)}\nabla\psi\cdot\frac{\partial}{\partial\theta}\nabla\psi\,d\xi\,d\eta.$$

That this last integral over $\mathcal{B}(0,R)$ vanishes is immediate as before. The boundary integral is now independent of R and vanishes because $\partial\psi/\partial\sigma$ and $\partial\psi/\partial\theta$ are both $O(R^{-1})$ as $R\uparrow\infty$, by the definition (2.1)(c) of the operator A. □

(iv) *Existence and uniqueness of a weak solution.* The forcing function in (2.6) amply satisfies the condition

$$\iint_{\mathbb{R}^2}\frac{\mathrm{e}^{\sigma^2}f(\sigma,\theta)^2}{1+\sigma^2}\,d\xi\,d\eta < \infty, \tag{2.12}$$

because $f(\sigma,\theta) = (\kappa/\pi a)g(\sigma)\cos\theta$ with g as in (2.4). This condition is sufficient to make the forcing integral in (2.9), namely,

$$F(u) := \iint_{\mathbb{R}^2}\mathrm{e}^{\sigma^2/2}f u\,d\xi\,d\eta, \quad u\in Z,$$

a bounded linear functional evaluated at u. In other words, F belongs to the dual space Z^* of Z. The Lax–Milgram lemma now implies the following.

LEMMA 2.3. *Equation (2.6) has a unique weak solution q and*

$$\frac{\lambda}{1+k_B\lambda}\|F\|_{Z^*} \leqslant \|q\|_Z \leqslant \lambda\|F\|_{Z^*}. \tag{2.13}$$

(v) *Regularity theory: pointwise estimates.* We separate the variables σ and θ. Let Y denote the real Hilbert space of functions $y : [0,\infty)\to\mathbb{R}$ such that the

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functions having values $y(\sigma)\cos\theta$ or $y(\sigma)\sin\theta$ belong to Z. It can be proved that, equivalently, Y is the completion of the set

$$D := \{\zeta \in C_c^\infty[0,\infty) \mid \zeta(0) = 0\},$$

where the compact support of ζ may extend to the origin, in the norm implied by the inner product

$$\langle v, w\rangle_Y := \int_0^\infty \left(v'w' + \left(\frac{1}{\sigma^2} + \sigma^2\right)vw\right)\sigma\, d\sigma,$$

where the $(\cdot)'$ denotes differentiation.

It can then be proved that, if

(a)
$$Q(\sigma) := q_c(\sigma) + \mathrm{i}q_s(\sigma) \quad \text{where } (q_c, q_s) \in Y^2; \tag{2.14}$$

(b) the operator T_1 is defined by

$$(T_1 y)(\sigma) := \frac{1}{2}\int_0^\infty \left(\frac{\rho}{\sigma} \wedge \frac{\sigma}{\rho}\right) y(\rho)\,\rho\, d\rho \quad \text{for all } y \in Y, \tag{2.15}$$

where $a \wedge b$ denotes the lower envelope, or lesser, of a and b;

(c) for all test functions $v \in Y$,

$$\begin{aligned}\int_0^\infty \Bigg(v'Q' + \left(\frac{1}{\sigma^2} + \sigma^2\right)vQ \\ - \mathrm{i}\lambda\frac{1 - \mathrm{e}^{-\sigma^2}}{\sigma^2}vQ + \mathrm{i}4\lambda\mathrm{e}^{-\sigma^2/2}vT_1(\mathrm{e}^{-\sigma^2/2}Q)\Bigg)\sigma\, d\sigma \\ = \lambda\int_0^\infty \mathrm{e}^{\sigma^2/2} f_c v\,\sigma\, d\sigma,\end{aligned} \tag{2.16}$$

where $f_c(\sigma) := (\kappa/\pi a)g(\sigma)$; and

(d)
$$q(\sigma,\theta) := q_c(\sigma)\cos\theta + q_s(\sigma)\sin\theta; \tag{2.17}$$

then q satisfies (2.9), so that the right-hand member of (2.17) is the unique weak solution of (2.6). Conversely, equations (2.9), (2.17) and (2.14) imply (2.16).

We now choose the test function in (2.16) to be a Green function of the operator

$$-\left(\frac{d}{d\sigma}\right)^2 - \frac{1}{\sigma}\frac{d}{d\sigma} + \left(\frac{1}{\sigma^2} + \sigma^2\right),$$

which results from insertion of (2.17) into (2.6). It is legitimate to choose

$$v(\sigma) = K(\rho,\sigma) := \begin{cases} \dfrac{1}{\sigma}\sinh\dfrac{\sigma^2}{2}\cdot\dfrac{1}{\rho}\exp\left(-\dfrac{\rho^2}{2}\right) & \text{if } \sigma \leqslant \rho, \\ \dfrac{1}{\sigma}\exp\left(-\dfrac{\sigma^2}{2}\right)\cdot\dfrac{1}{\rho}\sinh\dfrac{\rho^2}{2} & \text{if } \sigma \geqslant \rho, \end{cases} \tag{2.18}$$

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because $K(\rho, \cdot) \in Y$ for fixed $\rho \in (0, \infty)$. Then (2.16) yields, after an integration by parts, the integral equation

$$Q(\rho) = \lambda \int_0^\infty K(\rho, \sigma) e^{\sigma^2/2} f_c(\sigma)\, \sigma\, d\sigma + i\lambda \int_0^\infty K(\rho, \sigma) \left(\frac{1 - e^{-\sigma^2}}{\sigma^2} Q(\sigma) - 4e^{-\sigma^2/2} T_1(e^{-\sigma^2/2} Q) \right) \sigma\, d\sigma. \tag{2.19}$$

Since Lemma 2.3 provides bounds for $\|q_c\|_Y$ and $\|q_s\|_Y$, the regularity of Q, and pointwise bounds, can be deduced from (2.19) and from Lemma 3.8 below without great difficulty.

§3. *The perturbation ω_1 as $\lambda \uparrow \infty$.* We return to equations (2.1)–(2.4) and define a stream function $\tilde{\psi}_1 := A\tilde{\omega}_1$. Then $\tilde{\omega}_1 = -\Delta_\sigma \tilde{\psi}_1$ and (2.2) becomes

$$\left(\Delta_\sigma + 2\sigma \frac{\partial}{\partial \sigma} + 2 \right) \Delta_\sigma \tilde{\psi}_1 - \lambda \frac{1 - e^{-\sigma^2}}{\sigma^2} \frac{\partial}{\partial \theta} \Delta_\sigma \tilde{\psi}_1 - 4\lambda e^{-\sigma^2} \frac{\partial \tilde{\psi}_1}{\partial \theta} = \frac{\kappa \lambda}{\pi a} g(\sigma) \cos\theta \quad \text{on } E. \tag{3.1}$$

In view of (2.5) and (2.17), the function $\tilde{\psi}_1$ has the form

$$\tilde{\psi}_1(\sigma, \theta) = \tilde{\psi}_{1c}(\sigma) \cos\theta + \tilde{\psi}_{1s}(\sigma) \sin\theta. \tag{3.2}$$

We divide (3.1) by $\lambda(1 - e^{-\sigma^2})/\sigma^2$, write the $\cos\theta$ and $\sin\theta$ parts as separate equations and define, similarly to (2.14),

$$\Psi(\sigma) = \tilde{\psi}_{1c}(\sigma) + i\tilde{\psi}_{1s}(\sigma). \tag{3.3}$$

With the notation

$$\begin{aligned} &\text{(a)} \quad \Delta_1 := \left(\frac{d}{d\sigma} \right)^2 + \frac{1}{\sigma} \frac{d}{d\sigma} - \frac{1}{\sigma^2}, \\ &\text{(b)} \quad \alpha(\sigma) := \frac{4\sigma^2}{e^{\sigma^2} - 1}, \\ &\text{(c)} \quad \beta(\sigma) := \frac{\sigma^2}{1 - e^{-\sigma^2}}, \\ &\text{(d)} \quad \mathcal{E} := \Delta_1 + 2\sigma \frac{d}{d\sigma} + 2, \end{aligned} \tag{3.4}$$

the problem for $\tilde{\omega}_1$ is to solve the equation

$$-i \frac{\beta(\sigma)}{\lambda} \mathcal{E}(\Delta_1 \Psi) + \{\Delta_1 + \alpha(\sigma)\} \Psi = -i \frac{\kappa}{\pi a} \beta(\sigma) g(\sigma), \quad 0 < \sigma < \infty, \tag{3.5}$$

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with the side conditions

$$\begin{aligned} &\text{as } \sigma \downarrow 0,\ (\Delta_1\Psi)(\sigma) \to 0,\ \Psi'(\sigma) = O(1) \text{ and } \Psi(\sigma) = O(\sigma);\\ &\text{as } \sigma \uparrow \infty,\ (\Delta_1\Psi)(\sigma) \to 0,\ \Psi'(\sigma) = O(\sigma^{-2}) \text{ and } \Psi(\sigma) = O(\sigma^{-1}). \end{aligned} \tag{3.6}$$

Here the conditions on $\Delta_1\Psi$ come from (2.3); the conditions on Ψ' and Ψ are implied by $\Psi = -T_1(\Delta_1\Psi)$, with T_1 as in (2.15), and by conditions on $\Delta_1\Psi$ much weaker than those in Theorem 2.1.

For $\lambda \uparrow \infty$, equation (3.5) with (3.6) seems to form a singular perturbation problem, since a small parameter multiplies the highest derivatives. Surprisingly, this turns out not to be the case; nevertheless there is work to be done.

Apparently, if $\Psi_0(\sigma) := \lim_{\lambda\uparrow\infty} \Psi(\sigma;\lambda)$ exists, then it must satisfy

$$\{\Delta_1 + \alpha(\sigma)\}\Psi_0 = -\mathrm{i}\frac{\kappa}{\pi a}\beta(\sigma)g(\sigma), \quad 0 < \sigma < \infty, \tag{3.7}$$

and the six side conditions (3.6). We proceed to explore this problem.

LEMMA 3.1. *The equation*

$$\{\Delta_1 + \alpha(\sigma)\}u = 0, \quad 0 < \sigma < \infty, \tag{3.8}$$

has solutions

$$U(\sigma) := \frac{1}{\sigma}(1 - \mathrm{e}^{-\sigma^2}) \tag{3.9}$$

and

$$V(\sigma) := \frac{1}{\sigma} - U(\sigma)\log(\mathrm{e}^{\sigma^2} - 1). \tag{3.10}$$

Here U is an eigensolution (with eigenvalue 0) in that it satisfies not only (3.8) but also all six side conditions (3.6).

Proof. This is a matter of direct calculation. □

LEMMA 3.2. *The forcing function in (3.7) is orthogonal to the eigensolution U in the sense that*

$$\int_0^\infty U(\sigma)\beta(\sigma)g(\sigma)\,\sigma\,d\sigma = 0. \tag{3.11}$$

Hence the problem (3.7) and (3.6) has a (non-unique) solution

$$\Psi_0(\sigma) = c_0U(\sigma) + \mathrm{i}\frac{\kappa}{2\pi a}\int_0^\sigma \{U(\rho)V(\sigma) - U(\sigma)V(\rho)\}\beta(\rho)g(\rho)\rho\,d\rho \tag{3.12}$$

for every $c_0 \in \mathbb{C}$.

Proof. Again this is a matter of direct calculation, but the calculation is not short. With β defined by (3.4)(c) and g by (2.4)(c), the analytic proof of the orthogonality condition (3.11) seems to require a page. (However, with any machine capable of numerical integration, numerical verification of (3.11) is

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quick and easy.) We note that Liouville's formula for Wronskians yields

$$U(\sigma)V'(\sigma) - U'(\sigma)V(\sigma) = -\frac{2}{\sigma}, \quad 0 < \sigma < \infty. \tag{3.13}$$

The following lemma is also relevant. □

LEMMA 3.3. *Define, for suitable functions* f,

$$(\mathcal{G}f)(\sigma) := \frac{1}{2}\int_0^\sigma \{U(\rho)V(\sigma) - U(\sigma)V(\rho)\}f(\rho)\rho\,d\rho, \quad 0 < \sigma < \infty, \tag{3.14}$$

and

$$J(f) := \int_0^\infty V(\rho)f(\rho)\rho\,d\rho. \tag{3.15}$$

Assume that $f \in C[0,\infty)$, *that* $\int_0^\infty U(\rho)f(\rho)\rho\,d\rho = 0$, *that* $f(\sigma) = O(\sigma)$ *as* $\sigma \downarrow 0$ *and that* $f(\sigma) = O(\sigma^m \mathrm{e}^{-\sigma^2})$, *with* $m \geqslant 1$, *as* $\sigma \uparrow \infty$. *Then*

$$\{\Delta_1 + \alpha(\sigma)\}(\mathcal{G}f)(\sigma) = -f(\sigma) \quad \textit{in } (0,\infty); \tag{3.16}$$

as $\sigma \downarrow 0$,

$$(\mathcal{G}f)(\sigma) = O(\sigma^3) \quad \textit{and} \quad (\Delta_1\mathcal{G}f)(\sigma) = O(\sigma); \tag{3.17}$$

as $\sigma \uparrow \infty$,

$$(\mathcal{G}f)(\sigma) = -\tfrac{1}{2}J(f)\sigma^{-1} + O(\sigma^m \mathrm{e}^{-\sigma^2}), \tag{3.18}$$

$$(\mathcal{G}f)'(\sigma) = \tfrac{1}{2}J(f)\sigma^{-2} + O(\sigma^{m-1}\mathrm{e}^{-\sigma^2}), \tag{3.19}$$

$$(\mathcal{G}f)''(\sigma) = -J(f)\sigma^{-3} + O(\sigma^m \mathrm{e}^{-\sigma^2}), \tag{3.20}$$

and

$$(\Delta_1\mathcal{G}f)(\sigma) \quad \textit{and} \quad (\mathcal{E}\mathcal{G}f)(\sigma) \quad \textit{are } O(\sigma^m \mathrm{e}^{-\sigma^2}). \tag{3.21}$$

Proof. Equation (3.16) follows from the definition of $\mathcal{G}f$ and two differentiations. That $(\mathcal{G}f)(\sigma) = O(\sigma^3)$ as $\sigma \downarrow 0$ also follows from the definition; then the differential equation (3.16) shows that $(\Delta_1\mathcal{G}f)(\sigma) = O(\sigma)$ as $\sigma \downarrow 0$. In order to prove (3.18) and (3.19), we note that $U(\sigma) \sim 1/\sigma$ and $V(\sigma) \sim -\sigma$ as $\sigma \uparrow \infty$, whence

$$\int_0^\sigma U(\rho)f(\rho)\rho\,d\rho = \left(\int_0^\infty - \int_\sigma^\infty\right)U(\rho)f(\rho)\rho\,d\rho = 0 + O(\sigma^{m-1}\mathrm{e}^{-\sigma^2}),$$
$$\int_0^\sigma V(\rho)f(\rho)\rho\,d\rho = \left(\int_0^\infty - \int_\sigma^\infty\right)V(\rho)f(\rho)\rho\,d\rho = J(f) + O(\sigma^{m+1}\mathrm{e}^{-\sigma^2}),$$

from which (3.18) and (3.19) follow.

The differential equation (3.16) and the estimate (3.18) of $\mathcal{G}f$ imply that $(\Delta_1\mathcal{G}f)(\sigma) = O(\sigma^m\mathrm{e}^{-\sigma^2})$ as $\sigma \uparrow \infty$. What has been proved now implies the estimates of $(\mathcal{G}f)''(\sigma)$ and $(\mathcal{E}\mathcal{G}f)(\sigma)$ for $\sigma \uparrow \infty$. □

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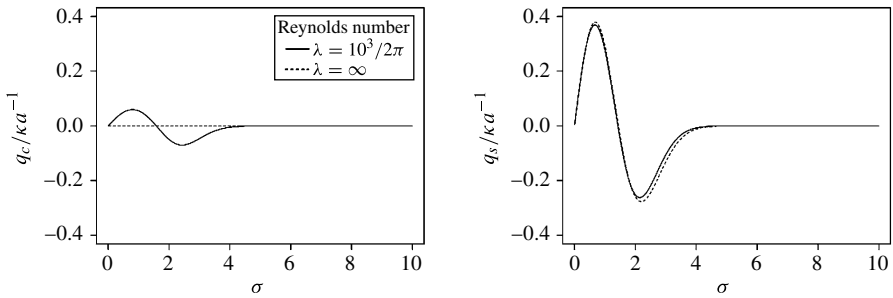


Figure 2: The perturbations for $q_c(\alpha)$ and $q_s(\alpha)$ for $\lambda = 10^3/2\pi$ and $\lambda = \infty$.

The result (3.12) prompts two questions. How (if at all) is c_0 to be evaluated? How smooth is Ψ_0? Analogues of both of these questions will have to be answered more generally for each function Ψ_n in an identity

$$\Psi(\sigma;\lambda) = \sum_{n=0}^{N} \lambda^{-n}\Psi_n(\sigma) + R_N(\sigma;\lambda).$$

Here we anticipate later results and note that, with J as in (3.15),

$$c_0 = \mathrm{i}\frac{\kappa}{2\pi a}J(\beta g) = \mathrm{i}\frac{\kappa}{a}(0.11527\ldots). \tag{3.22}$$

This follows from the equation governing Ψ_1, which requires an orthogonality condition involving Ψ_0.

Because of the function $Q =: Q(\cdot;\lambda)$ defined by (2.14) and (2.16), we now define $Q_0 = q_{0c} + \mathrm{i}q_{0s}$ by

$$\begin{aligned} Q_0(\sigma) &:= -\mathrm{e}^{\sigma^2/2}(\Delta_1\Psi_0)(\sigma) \\ &= \mathrm{e}^{\sigma^2/2}\big(4c_0\sigma\mathrm{e}^{-\sigma^2} - \alpha(\sigma)(\mathcal{G}h_0)(\sigma) - h_0(\sigma)\big), \end{aligned} \tag{3.23}$$

where $h_0 := -\mathrm{i}(\kappa/\pi a)\beta g$. Evidently $q_{0c} = 0$.

It will emerge from Theorem 3.5 that Q_0 is the limit of $Q(\cdot;\lambda)$ as $\lambda \uparrow \infty$. In Figures 2 and 3, Q_0 is compared with $Q(\cdot;\lambda)$ for large λ; these values of $Q(\cdot;\lambda)$ were obtained by numerical solution of the equation

$$-(\Delta_1 - \sigma^2)Q - \mathrm{i}\lambda\frac{1-\mathrm{e}^{-\sigma^2}}{\sigma^2}Q + \mathrm{i}4\lambda\mathrm{e}^{-\sigma^2/2}T_1(\mathrm{e}^{-\sigma^2/2}Q) = \frac{\kappa\lambda}{\pi a}\mathrm{e}^{\sigma^2/2}g(\sigma). \tag{3.24}$$

This equation is equivalent to (2.6), because of (2.17). It is also the pointwise form of (2.16); with the condition that $Q(\sigma) \to 0$ as $\sigma \downarrow 0$ and as $\sigma \uparrow \infty$, it has a pointwise, unique solution. Figures 2 and 3 are consistent with the result of Theorem 3.5 that, as $\lambda \uparrow \infty$,

$$q_c(\cdot;\lambda) = O(\lambda^{-1}) \quad \text{and} \quad q_s(\cdot;\lambda) - q_{0s} = O(\lambda^{-2}).$$

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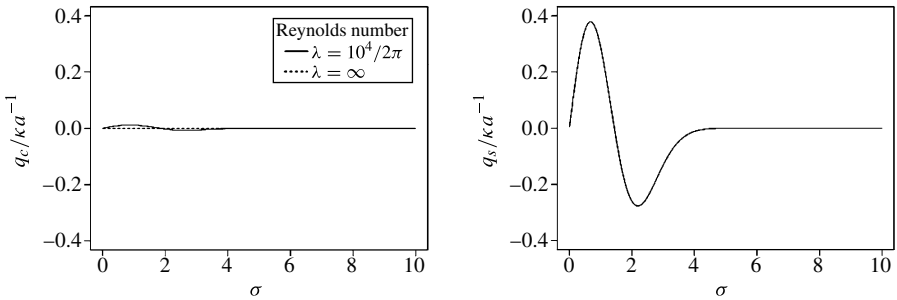


Figure 3: The perturbations for $q_c(\alpha)$ and $q_s(\alpha)$ for $\lambda = 10^4/2\pi$ and $\lambda = \infty$.

Definition. We shall say that a function $\varphi : [0, \infty) \to \mathbb{C}$ is *satisfactory on* $[0, k)$ if (and only if) there exist coefficients b_n and a number $k > 0$ such that

$$\varphi(\sigma) = \sum_{n=0}^{\infty} b_n \sigma^{2n+1} \quad \text{for } 0 \leqslant \sigma < k.$$

LEMMA 3.4.
(i) *The function βg is satisfactory on* $[0, (2\pi)^{1/2})$.
(ii) *If f is satisfactory on* $[0, (2\pi)^{1/2})$, *then so is $\mathcal{G}f$.*

Proof. (i) We note that

$$\beta(\sigma)g(\sigma) = \sigma \mathrm{e}^{-\sigma^2}\left(\frac{3}{2} + \frac{\sigma^2}{1-\mathrm{e}^{-\sigma^2}}\left(\int_\sigma^1 \frac{1-\mathrm{e}^{-\rho^2}}{\rho}\,d\rho + C_0 + C_1\right)\right),$$

where

$$C_0 := -\int_1^\infty \frac{\mathrm{e}^{-\rho^2}}{\rho}\,d\rho, \qquad C_1 := -\frac{1}{2}(\gamma_E + 1 - \log 2),$$

and that the function w defined by

$$w(z) = \frac{z}{1-\mathrm{e}^{-z}} \quad \text{if } z \in \mathbb{C} \setminus \{0\} \setminus \{\text{poles}\} \quad \text{and} \quad w(0) = 1,$$

is holomorphic for $|z| < 2\pi$.

(ii) Let

$$W(\sigma) := \frac{1}{\sigma} - U(\sigma)\log\frac{\mathrm{e}^{\sigma^2}-1}{\sigma^2},$$

where the limiting value of $(\mathrm{e}^{\sigma^2}-1)/\sigma^2$ is taken at $\sigma = 0$. Then

$$V(\sigma) = W(\sigma) - 2U(\sigma)\log\sigma$$

and

$$(\mathcal{G}f)(\sigma) = \frac{1}{2}W(\sigma)\int_0^\sigma U(\rho)f(\rho)\,\rho\,d\rho - \frac{1}{2}U(\sigma)\int_0^\sigma W(\rho)f(\rho)\,\rho\,d\rho + \sigma^2 U(\sigma)\int_0^1 (\log t)U(\sigma t)f(\sigma t)\,t\,dt.$$

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The functions with values $\sigma^2 W(\sigma)$, $U(\sigma)$ and $f(\sigma)$ are all satisfactory on $[0, (2\pi)^{1/2})$, so that $\mathcal{G}f$ inherits this property. □

THEOREM 3.5. *The perturbation $\tilde{\omega}_1$ has a representation*

$$\begin{aligned}\tilde{\omega}_1(\sigma, \theta; \lambda) = &\cos\theta\{\lambda^{-1}\zeta_1(\sigma) + \lambda^{-3}\zeta_3(\sigma) + \zeta_5(\sigma; \lambda)\} \\ &+ \sin\theta\{\zeta_0(\sigma) + \lambda^{-2}\zeta_2(\sigma) + \zeta_4(\sigma; \lambda)\} \end{aligned} \tag{3.25}$$

in which, for $m = 0, 1, 2, 3$ and $n = 4, 5$:

(a) *the functions ζ_m and $\zeta_n(\cdot; \lambda)$ belong to $C^\infty[0, \infty)$ and are satisfactory on $[0, (2\pi)^{1/2})$;*

(b) *as $\sigma \uparrow \infty$, $\zeta_m(\sigma) = O(\sigma^{2m+4}\mathrm{e}^{-\sigma^2})$ and $\zeta_n(\sigma; \lambda) = o(\mathrm{e}^{-\sigma^2/2})$ for fixed λ; and*

(c) *as $\lambda \uparrow \infty$, $\zeta_n(\sigma; \lambda) = O(\lambda^{-n})$, uniformly over $\sigma \in [0, \infty)$.*

The proof will be by means of further lemmas. Let $\Psi := \tilde{\psi}_{1c} + \mathrm{i}\tilde{\psi}_{1s}$, as before, and let $\Omega := -\Delta_1\Psi$, so that $\Omega = \tilde{\omega}_{1c} + \mathrm{i}\tilde{\omega}_{1s}$. Our plan is to construct identities

$$\Psi(\sigma; \lambda) = \sum_{n=0}^{N} \lambda^{-n}\Psi_n(\sigma) + R_N(\sigma; \lambda), \tag{3.26}$$

$$\Omega(\sigma; \lambda) = \sum_{n=0}^{N} \lambda^{-n}\Omega_n(\sigma) + r_N(\sigma; \lambda), \tag{3.27}$$

in which estimates of the remainders R_N and r_N can be crude. In fact, we shall prove only that R_N and r_N are $O(\lambda^{1-N})$, but this is sufficient for (3.25) if $N \geqslant 6$.

The terms of the expansion of Ψ are to satisfy

$$\{\Delta_1 + \alpha(\sigma)\}\Psi_n = h_n, \quad n = 0, 1, \ldots, N, \tag{3.28}$$

where

$$h_0(\sigma) = -\mathrm{i}\frac{\kappa}{\pi a}\beta(\sigma)g(\sigma), \tag{3.29}$$

$$h_n := \mathrm{i}\beta\mathcal{E}(\Delta_1\Psi_{n-1}) \quad \text{for } n = 1, \ldots, N, \tag{3.30}$$

and

$$-\mathrm{i}\lambda^{-1}\beta(\sigma)\mathcal{E}(\Delta_1 R_N) + \{\Delta_1 + \alpha(\sigma)\}R_N = \mathrm{i}\lambda^{-N-1}\beta(\sigma)\mathcal{E}(\Delta_1\Psi_N); \tag{3.31}$$

then the right-hand member of (3.26) will satisfy the equation (3.5) governing Ψ.

Since $\Omega = -\Delta_1\Psi$ and $\Psi = T_1\Omega$, this scheme corresponds to

$$-\Omega_n + \alpha(\sigma)T_1\Omega_n = h_n, \quad n = 0, 1, \ldots, N, \tag{3.32}$$

$$\mathrm{i}\lambda^{-1}\beta(\sigma)\mathcal{E}r_N - r_N + \alpha(\sigma)T_1 r_N = -\mathrm{i}\lambda^{-N-1}\beta(\sigma)\mathcal{E}\Omega_N, \tag{3.33}$$

where $h_n = -\mathrm{i}\beta(\sigma)\mathcal{E}\Omega_{n-1}$ for $n = 1, \ldots, N$.

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LEMMA 3.6. *In order that equation (3.28), with the side conditions (3.6), have a solution, it is necessary that*

$$\int_0^\infty U(\sigma)h_n(\sigma)\sigma\, d\sigma = 0, \quad n = 0, 1, \ldots, N; \tag{3.34}$$

equivalently, that

$$\int_0^\infty \sigma^2 g(\sigma)\, d\sigma = 0 \quad \text{if } n = 0, \tag{3.35}$$

$$\int_0^\infty \sigma^2(\mathcal{E}\Omega_{n-1})(\sigma)\, d\sigma = 0 \quad \text{if } n = 1, \ldots, N. \tag{3.36}$$

Proof. Let $\mathcal{M} := \Delta_1 + \alpha(\sigma)$. Assume that u and v are in $C^2(0, \infty)$, that $\sigma u(\sigma)v'(\sigma) \to 0$ as $\sigma \downarrow 0$ and as $\sigma \uparrow \infty$ and that $\sigma u'(\sigma)v(\sigma) \to 0$ as $\sigma \downarrow 0$ and as $\sigma \uparrow \infty$. Then integration by parts yields

$$\int_0^\infty u(\mathcal{M}v)\,\sigma\, d\sigma = \int_0^\infty (\mathcal{M}u)v\,\sigma\, d\sigma. \tag{3.37}$$

Now let $u = U$ and $v = \Psi_n$. If Ψ_n satisfies (3.6), then the foregoing hypotheses are satisfied. If, also, $\mathcal{M}\Psi_n = h_n$, then

$$\int_0^\infty U h_n\,\sigma\, d\sigma = \int_0^\infty U(\mathcal{M}\Psi_n)\,\sigma\, d\sigma = \int_0^\infty (\mathcal{M}U)\Psi_n\,\sigma\, d\sigma = 0. \tag{3.38}$$

Equations (3.35) and (3.36) follow from the identity $U(\sigma)\beta(\sigma) = \sigma$ and from the definitions of h_n. □

If h_n satisfies not only the orthogonality condition (3.34) but also the other hypotheses on f in Lemma 3.3 (and this will be the case), then the differential equation (3.28), with side conditions (3.6), has solutions

$$\Psi_n = c_n U - \mathcal{G}h_n, \quad n = 0, 1, \ldots, N, \tag{3.39}$$

whence

$$\Omega_n(\sigma) = -(\Delta_1\Psi_n)(\sigma) = 4c_n\sigma e^{-\sigma^2} - \alpha(\sigma)(\mathcal{G}h_n)(\sigma) - h_n(\sigma), \tag{3.40}$$

for every $c_n \in \mathbb{C}$.

In order to evaluate $c_0, \ldots, c_N$ and in order to discuss r_N, we extend the definition (3.30) to h_{N+1}. Recall from Lemma 3.2 that for $n = 0$ the orthogonality condition (3.34) has already been established.

LEMMA 3.7. *For $n = 0, 1, \ldots, N$, the necessary condition $\int_0^\infty U h_{n+1}\sigma\, d\sigma = 0$ implies that $c_n = -\frac{1}{2}J(h_n)$, where $J(\cdot)$ is defined by (3.15).*

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Proof. Extended to Ω_N, the orthogonality condition (3.36) states that, for $n = 0, 1, \ldots, N$,

$$0 = \int_0^\infty \sigma^2(\mathcal{E}\Omega_n)\, d\sigma = -4\int_0^\infty \sigma^2\Omega_n\, d\sigma,$$

by an integration by parts for which it suffices that $\Omega_n \in C^2(0,\infty)$, that $\Omega_n'(\sigma) = o(\sigma^{-2})$ both as $\sigma \downarrow 0$ and as $\sigma \uparrow \infty$, that $\Omega_n = o(\sigma^{-1})$ as $\sigma \downarrow 0$ and that $\Omega_n = o(\sigma^{-3})$ as $\sigma \uparrow \infty$.

Next, we observe that, if $\Psi_n \in C^2(0,\infty)$ and if both $\sigma^2\Psi_n'(\sigma)$ and $\sigma\Psi_n(\sigma)$ have limits both as $\sigma \downarrow 0$ and as $\sigma \uparrow \infty$, then

$$0 = \int_0^\infty \sigma^2(\Delta_1\Psi_n)\, d\sigma = [\sigma^2\Psi_n' - \sigma\Psi_n]_0^\infty,$$

in which limiting values are implied on the right-hand side. In view of (3.39), the orthogonality condition is now

$$c_n[\sigma^2U' - \sigma U]_0^\infty = [\sigma^2(\mathcal{G}h_n)' - \sigma(\mathcal{G}h_n)]_0^\infty.$$

Referring to the definition of U in (3.9) and to the description of $\mathcal{G}f$ in Lemma 3.3, we are led to $c_n = -\frac{1}{2}J(h_n)$. □

It is time to relate the ζ_m and $\zeta_n(\cdot;\lambda)$ in Theorem 3.5 to the Ω_n and r_N in (3.27). We noted after (3.23) that Q_0 is imaginary. Since $\Omega(\sigma) = \exp(-\sigma^2/2)Q(\sigma)$, the function Ω_0 is imaginary. Then, since $h_1 = -\mathrm{i}\beta\mathcal{E}\Omega_0$, the function h_1 and the coefficient c_1 are real. Equation (3.40) shows that Ω_1 is real. An easy induction now shows that Ω_n *is imaginary if* n *is even and* Ω_n *is real if* n *is odd.*

Accordingly, if N is odd, then

$$\begin{aligned}
&\zeta_1 = \Omega_1,\ \zeta_3 = \Omega_3 \quad \text{and} \quad \zeta_5(\cdot;\lambda) = \lambda^{-5}\Omega_5 + \cdots + \lambda^{-N}\Omega_N + \operatorname{Re} r_N(\cdot;\lambda),\\
&\zeta_0 = -\mathrm{i}\Omega_0,\ \zeta_2 = -\mathrm{i}\Omega_2 \quad \text{and} \quad \zeta_4(\cdot;\lambda) = -\mathrm{i}(\lambda^{-4}\Omega_4 + \cdots + \lambda^{-N+1}\Omega_{N-1})\\
&\qquad + \operatorname{Im} r_N(\cdot;\lambda).
\end{aligned} \tag{3.41}$$

If N is even, then there is a similar array.

Because of the explicit formula (3.40) for Ω_n (with $h_n = -\mathrm{i}\beta\mathcal{E}\Omega_{n-1}$, with $c_n = -\frac{1}{2}J(h_n)$ and with the operator $\mathcal{G}$ described by Lemmas 3.3 and 3.4), enough may have been said about Ω_n to justify the claims made for ζ_0 to ζ_3 in Theorem 3.5. For example, the result

$$\zeta_m(\sigma) = O(\sigma^{2m+4}\mathrm{e}^{-\sigma^2}) \quad \text{as } \sigma \uparrow \infty \tag{3.42}$$

follows for $m = 0$ from $\beta(\sigma) \sim \sigma^2$ and from the overestimate $g(\sigma) = O(\sigma^2\mathrm{e}^{-\sigma^2})$, which imply that h_0 and Ω_0 are $O(\sigma^4\mathrm{e}^{-\sigma^2})$. Then repeated use of $h_{n+1} = -\mathrm{i}\beta\mathcal{E}\Omega_n$ leads to (3.42).

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On the other hand, the remainder r_N requires further discussion. Under the transformations

$$r_N(\sigma) = \mathrm{e}^{-\sigma^2/2} P_N(\sigma) = \mathrm{e}^{-\sigma^2/2}\{p_{Nc}(\sigma) + \mathrm{i} p_{Ns}(\sigma)\}, \tag{3.43}$$

$$p_N(\sigma, \theta) := p_{Nc}(\sigma)\cos\theta + p_{Ns}(\sigma)\sin\theta, \tag{3.44}$$

equation (3.33) becomes

$$-(\Delta_\sigma - \sigma^2)p_N + \frac{\lambda}{\beta(\sigma)}\frac{\partial}{\partial\theta}p_N - 4\lambda \mathrm{e}^{-\sigma^2/2} T(\mathrm{e}^{-\sigma^2/2} p_N) = \lambda^{-N}\mathrm{e}^{\sigma^2/2} f_N(\sigma, \theta), \tag{3.45}$$

where

$$f_N(\sigma, \theta) = \mathrm{Re}\{(\mathcal{E}\Omega_N)(\sigma)\mathrm{e}^{-\mathrm{i}\theta}\}. \tag{3.46}$$

The operator on the left-hand side of (3.45) is that in (2.6); as in §2, it follows that equation (3.45) has a unique weak solution bounded by

$$\|p_N\|_Z \leqslant \frac{\kappa}{a} A_N \lambda^{-N}, \tag{3.47}$$

where A_N depends only on N.

Choosing the test function in the definition of weak solution as in (2.18), we obtain the equation

$$P_N(\rho) = \lambda^{-N}\int_0^\infty K(\rho, \sigma)\,\mathrm{e}^{\sigma^2/2}(\mathcal{E}\Omega_N)(\sigma)\,\sigma\,d\sigma + \mathrm{i}\lambda\int_0^\infty K(\rho, \sigma)\left(\frac{P_N(\sigma)}{\beta(\sigma)} - 4\mathrm{e}^{-\sigma^2/2} T_1(\mathrm{e}^{-\sigma^2/2} P_N)\right)\sigma\,d\sigma. \tag{3.48}$$

This leads without difficulty to a pointwise solution $P_N \in C^2[0, \infty)$ such that $P_N(0) = 0$ and such that $P_N(\sigma) \to 0$ as $\sigma \uparrow \infty$. (Correspondingly, $r_N(\sigma)$ is $o(\exp(-\sigma^2/2))$ as $\sigma \uparrow \infty$.) Equation (3.48) and the bound (3.47) now imply that $P_N(\sigma; \lambda)$ is $O(\lambda^{-N+1})$ uniformly over $\sigma \in [0, \infty)$. Moreover, the equation

$$-(\Delta_1 - \sigma^2)P_N - \frac{\mathrm{i}\lambda}{\beta(\sigma)}P_N + 4\mathrm{i}\lambda\mathrm{e}^{-\sigma^2/2} T_1(\mathrm{e}^{-\sigma^2/2} P_N) = \lambda^{-N}\mathrm{e}^{\sigma^2/2}\mathcal{E}\Omega_N$$

may be written as

$$P_N'' = -\frac{1}{\sigma}P_N' + \left(\frac{1}{\sigma^2} + \sigma^2\right)P_N - \cdots - \lambda^{-N}\mathrm{e}^{\sigma^2/2}\mathcal{E}\Omega_N.$$

The right-hand member of this is in $C^1[k, \infty)$ for any $k > 0$, say in $C^1[1, \infty)$. Therefore $P_N'' \in C^1[1, \infty)$. Repetition of this step shows that $P_N \in C^\infty[1, \infty)$.

It remains to prove that P_N is better than C^2 at and near the origin. We return to equation (3.33) for r_N and to the equation

$$\mathcal{E}\Omega + \frac{\mathrm{i}\lambda}{\beta(\sigma)}\Omega - 4\mathrm{i}\lambda\mathrm{e}^{-\sigma^2} T_1\Omega = -\frac{\kappa\lambda}{\pi a} g$$

for $\Omega = \tilde{\omega}_{1c} + \mathrm{i}\tilde{\omega}_{1s}$; our final lemma applies to both r_N and Ω.

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LEMMA 3.8. *Assume that the equation*

$$\mathcal{E}u + \frac{\mathrm{i}\lambda}{\beta(\sigma)}u - 4\mathrm{i}\lambda \mathrm{e}^{-\sigma^2} T_1 u = \lambda f \tag{3.49}$$

has a solution $u \in C^2[0,\infty)$ *such that* $u(0) = 0$ *and such that* $u(\sigma)$ *is* $o(\exp(-\sigma^2/2))$ *as* $\sigma \uparrow \infty$. *Assume also that* u *is unique because it is the transformed version of a solution in the Hilbert space* Z.

Then u *is satisfactory on* $[0, (2\pi)^{1/2})$ *whenever* f *has this property.*

Proof. We shall prove that there are coefficients a_n such that

$$u(\sigma) = \sum_{n=0}^{\infty} a_n \sigma^{2n+1} \quad \text{for } 0 \leqslant \sigma < b$$

if $b \in (0, (2\pi)^{1/2})$. Here we are not constructing a series solution *ab initio* in the usual way; rather, we are establishing a regularity property of a known, unique solution. Therefore we may regard $a_0 = u'(0)$ and $\int_0^\infty u\, d\sigma$ as known; we proceed to calculate the other coefficients in terms of these. The equation is satisfied, subject to convergence of the series, if for $n = 0, 1, 2, \ldots,$

$$4(n+1)(n+2)a_{n+1} = -4(n+1)a_n - \sum_{j=0}^{n}(B_{n-j}a_j + A_{n-j}\tau_j) + \lambda f_n, \tag{3.50}$$

where

$$B_m = \frac{\mathrm{i}\lambda(-1)^m}{(m+1)!}, \qquad A_m = \frac{4\mathrm{i}\lambda(-1)^{m+1}}{m!},$$
$$\tau_0 = \frac{1}{2}\int_0^\infty u\, d\sigma, \qquad \tau_m = -\frac{1}{4}\frac{a_{m-1}}{m(m+1)} \quad \text{for } m \geqslant 1,$$

and

$$f(\sigma) = \sum_{n=0}^{\infty} f_n \sigma^{2n+1} \quad \text{for } 0 \leqslant \sigma < (2\pi)^{1/2}.$$

Hence there is a constant $C = C(b)$ such that $|f_n| \leqslant C b^{-2n}$.

Now, for every $p \in \{1, 2, 3, \ldots\}$ there is a number $\Gamma_p = \Gamma_p(b, \lambda)$ such that $|a_n| \leqslant \Gamma_p b^{-2n}$ for $n = 0, 1, \ldots, p$. We may suppose that $\Gamma_p \geqslant \kappa/a$. Then (3.50) implies that

$$|a_{n+1}| \leqslant \Gamma_p b^{-2n-2}\varphi(n, \lambda) \quad \text{for } n \leqslant p,$$

where

$$\varphi(n,\lambda) := \frac{2\pi}{4(n+1)(n+2)}\left(4(n+1) + \lambda(1+\pi)\mathrm{e}^{2\pi} + \frac{4\lambda|\tau_0|}{\kappa a^{-1}}\frac{(2\pi)^n}{n!} + \frac{\lambda C}{\kappa a^{-1}}\right).$$

For fixed λ, we choose p so large that $\varphi(p, \lambda) \leqslant 1$ and so large that $\varphi(n, \lambda)$ decreases for $n \geqslant p$. Then $|a_m| \leqslant \Gamma_p b^{-2m}$ not merely for $m \leqslant p$, but also for $m \geqslant p + 1$. □

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