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Robust Semi-Discrete and Fully Discrete Hybrid Stress Finite Element Methods for Elastodynamic Problems

Xiaojing Xu[1] and Xiaoping Xie[2,*]

[1] *School of Science, Southwest University of Science and Technology, Mianyang, Sichuan 621010, China*
[2] *School of Mathematics, Sichuan University, Chengdu, Sichaun 610064, China*

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Abstract. This paper analyzes semi-discrete and fully discrete hybrid stress quadrilateral finite element methods for 2-dimensional linear elastodynamic problems. The methods use a 4 node hybrid stress quadrilateral element in the space discretization. In the fully discrete scheme, an implicit second-order scheme is adopted in the time discretization. We derive optimal a priori error estimates for the two schemes and an unconditional stability result for the fully discrete scheme. Numerical experiments confirm the theoretical results.

AMS subject classifications: 65N12, 65N15, 65N30

Key words: Elastodynamic problem, hybrid stress finite element, semi-discrete, fully discrete, error estimate.

1 Introduction

Consider the following 2-dimensional linear elastodynamic problems:

$$\begin{cases} \mathbf{u}_{tt}-\mathrm{div}\boldsymbol{\sigma}=\mathbf{f}(\mathbf{x},t), & (\mathbf{x},t)\in\Omega\times[0,T], \\ \boldsymbol{\sigma}=\mathbb{C}\epsilon(\mathbf{u})=2\mu\epsilon(\mathbf{u})+\lambda\mathrm{div}\mathbf{u}\mathbf{I}, & (\mathbf{x},t)\in\Omega\times[0,T], \\ \mathbf{u}=0, & (\mathbf{x},t)\in\Gamma\times[0,T], \\ \mathbf{u}(\mathbf{x},0)=\boldsymbol{\varphi}_0(\mathbf{x}), \quad \mathbf{u}_t(\mathbf{x},0)=\boldsymbol{\varphi}_1(\mathbf{x}), & \mathbf{x}\in\Omega, \end{cases} \tag{1.1}$$

where $\Omega\subset\Re^2$ is a convex and bounded open set with boundary Γ, T a positive constant, $\mathbf{u}=(u_1,u_2)^T$ the displacement field, $\boldsymbol{\sigma}=(\sigma_{ij})_{2\times2}$ the symmetric stress tensor, $\epsilon(\mathbf{u})=(\nabla\mathbf{u}+(\nabla\mathbf{u})^T)/2$ the strain tensor, and $\mathbf{f}(\mathbf{x},t)$ the body force. $\lambda,\mu>0$ are the Lamé parameters. $\boldsymbol{\varphi}_0(\mathbf{x})$, $\boldsymbol{\varphi}_1(\mathbf{x})$ are initial data. We assume that $\mathbf{f}$, $\boldsymbol{\varphi}_0$ and $\boldsymbol{\varphi}_1$ are as regular as necessary.

*Corresponding author.
Email: xuxiaojing0603@126.com (X. J. Xu), xpxie@scu.edu.cn (X. P. Xie)

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Makridakis [1] analyzed semi-discrete and fully discrete mixed finite element methods for linear elastodynamics, where in the fully-discrete scheme rational approximations to the cosine were used for the time discretization. In [2], Bécache, Joly and Tsogka changed the two-order system (1.1) to a first order system, and constructed a new family of mixed finite elements, which allow mass lumping and lead to efficiency computation once explicit time discretization is made. Boulaajine, Farhloul and Paquet [3, 4] proposed a new dual mixed finite element method to solve the linear elastodynamic system with explicit/implicit Newmark schemes for the time discretizaton. In [5], Lai, Huang and Chen established a C^0-continuous time stepping displacement-type finite element method. Hughes and Hulbert [6] developed space-time finite element methods which use the discontinuous Galerkin method in time and incorporate stabilizing terms of least-squares type. Idesman [7] presented space-time finite element methods on structured and unstructured meshes based on time continuous and discontinuous Galerkin methods. In [8], Cheng and Xie considered a space-time nonconforming finite element method.

For static elasticity problems, the hybrid stress finite element method pioneered by Pian [9] is known to be an efficient approach to improve the performance of the standard 4-node compatible displacement quadrilateral (bilinear) element [10–14]. The method is based on the Hellinger-Reissner variational principle which includes unknowns of displacements and stresses, and leads to a displacement-type scheme after locally eliminating stress parameters. In [10] Pian and Sumihara proposed a robust 4-node hybrid stress quadrilateral element (abbr. PS element) by using isoparametric bilinear displacement approximations and a rational choice of 5-parameter stress mode. Yu, Xie and Carstensen [13] proved that the PS element is uniformly convergent on certain meshes as the Lamé constant $\lambda \to \infty$.

By using PS element for the space discretization, Yu and Xie [15] presented semi-discrete and fully discrete hybrid stress quadrilateral finite element methods for the linear elastodynamic problems (1.1), where a second-order center difference is used for the time discretization in the fully discrete scheme. Error estimates of the two schemes, as well as a conditional stability result for the fully discrete scheme, were derived there with the constant factors in all the estimates depending on λ.

In this paper, we shall further study the semi-discrete hybrid stress scheme proposed in [15] and develop a new fully discrete scheme by using an implicit second-order scheme in the time discretization. Our analysis and new scheme improve [15] in the following aspects.

- The constant factors in the derived optimal error estimates of the semi-discrete and fully discrete schemes are uniform with respect to the Lamé constant λ.
- The new fully discrete scheme is unconditionally stable.

The rest of the paper is organized as follows. Section 2 introduces notations and weak formulations. Section 3 is devoted to the error estimation of the semi-discrete hybrid

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stress scheme. Section 4 derives the stability and error estimation of the fully discrete hybrid stress scheme. We provide some numerical results in the last section to verify the theory results.

2 Notations and weak formulations

Throughout this paper, we denote by $H^r(\Omega)$ the standard Sobolev spaces with norm $\|\cdot\|_r$ and semi-norm $|\cdot|_r$ for any nonnegative integer r. In particular, $H^0(\Omega):=L^2(\Omega)$ is the space of square integrable functions with the usual L^2 inner product $(\cdot,\cdot)$.

For a time dependent function $v(\mathbf{x},t)$, sometimes we simply write it as v or $v(t)$. Let $C^k([0,T];H^r(\Omega))$ be the space of k-times continuously differentiable maps of $[0,T]$ into $H^r(\Omega)$. For $1\leq p\leq\infty$, denote

$$L^p([0,T];H^r(\Omega)):=\{v:[0,T]\to H^r(\Omega);\ \|v\|_{L^p(H^r)}<\infty\},$$

where $\|v\|_{L^p(H^r)}:=(\int_0^T\|v(\cdot,t)\|_r^p dt)^{1/p}$, when $p<\infty$, and $\|v\|_{L^\infty(H^r)}=\text{esssup}_{0\leq t\leq T}\|v(\cdot,t)\|_r$, when $p=\infty$. The corresponding semi-norm is given by $|v|_{L^p(H^r)}:=(\int_0^T|v(\cdot,t)|_r^p dt)^{1/p}$, when $p<\infty$, and $|v|_{L^\infty(H^r)}=\text{esssup}_{0\leq t\leq T}|v(\cdot,t)|_r$ when $p=\infty$. We note that all the notations above can be extended to vector-valued/tensor-valued functions naturally.

For convenience, we use the notation $a\lesssim b$ to represent that there exists a generic positive constant C, independent of the space and time mesh parameters, h and Δt, and the Lamé parameter λ, such that $a\leq Cb$.

We recall two Gronwall's inequalities [16]:

Continuous Gronwall's inequality. Let $\phi(\cdot)\geq 0$ be such that $\phi_t(t)\leq\rho\phi(t)+\eta(t)$ for $0\leq t\leq T$, where $\rho\geq 0$ is some constant and $\eta(\cdot)\geq 0$, $\eta\in L^1([0,T])$. Then it holds

$$\phi(t)\leq e^{\rho T}\Big(\phi(0)+\int_0^T\eta(s)ds\Big),\quad\forall t\in[0,T]. \tag{2.1}$$

Discrete Gronwall's inequality. Let $g_0\geq 0$ and two nonnegative sequences $(k_n)_{n\geq 0}$, $(p_n)_{n\geq 0}$ be given. Suppose that the sequence $(\phi_n)_{n\geq 0}$ satisfies

$$\begin{cases}\phi_0\leq g_0,\\ \phi_n\leq g_0+\sum_{s=0}^{n-1}p_s+\sum_{s=0}^{n-1}k_s\phi_s,\quad\forall n\geq 1.\end{cases} \tag{2.2}$$

Then it holds

$$\phi_n\leq\Big(g_0+\sum_{s=0}^{n-1}p_s\Big)\exp\Big(\sum_{s=0}^{n-1}k_s\Big),\quad\forall n\geq 1. \tag{2.3}$$

We define two spaces as follows

$$V:=(H_0^1(\Omega))^2=\{\mathbf{v}\in(H^1(\Omega))^2:\mathbf{v}|_\Gamma=0\},$$
$$\Sigma:=\Big\{\boldsymbol{\tau}\in L^2(\Omega;\mathbb{R}^{2\times 2}_{sym}):\int_\Omega tr\boldsymbol{\tau}d\mathbf{x}=0\Big\},$$

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where $L^2(\Omega;\mathbb{R}^{2\times 2}_{sym})$ denotes the space of square-integrable symmetric tensors with the norm $\|\cdot\|_0$ defined by $\|\boldsymbol{\tau}\|_0^2:=(\boldsymbol{\tau},\boldsymbol{\tau})$, and $tr\boldsymbol{\tau}:=\tau_{11}+\tau_{22}$ represents the trace of tensor $\boldsymbol{\tau}$. Notice that on the space V, the semi-norm $|\cdot|_1$ is equivalent to the norm $\|\cdot\|_1$.

The weak problem for the elastodynamic system (1.1) reads as: find $(\boldsymbol{\sigma},\mathbf{u})\in C^0([0,T];\Sigma)\times C^2([0,T];V)$ such that

$$\begin{cases} a(\boldsymbol{\sigma}(t),\boldsymbol{\tau})+b(\boldsymbol{\tau},\mathbf{u}(t))=0, & \forall\boldsymbol{\tau}\in\Sigma,\\ b(\boldsymbol{\sigma}(t),\mathbf{v})+(\mathbf{f}(t),\mathbf{v})=(\mathbf{u}_{tt}(t),\mathbf{v}), & \forall\mathbf{v}\in V,\\ \mathbf{u}(0)=\boldsymbol{\varphi}_0, \quad \mathbf{u}_t(0)=\boldsymbol{\varphi}_1, \end{cases} \tag{2.4}$$

where

$$a(\boldsymbol{\sigma},\boldsymbol{\tau}):=(\boldsymbol{\sigma},\mathbb{C}^{-1}\boldsymbol{\tau})=\frac{1}{2\mu}(\boldsymbol{\sigma},\boldsymbol{\tau})-\frac{\lambda}{4\mu(\mu+\lambda)}(tr\boldsymbol{\sigma},tr\boldsymbol{\tau}),$$
$$b(\boldsymbol{\tau},\mathbf{v}):=-(\boldsymbol{\tau},\epsilon(\mathbf{v})).$$

Define an energy norm $\|\cdot\|_a$ on Σ as

$$\|\boldsymbol{\tau}\|_a^2:=a(\boldsymbol{\tau},\boldsymbol{\tau})=\frac{1}{2\mu}\|\boldsymbol{\tau}-\frac{1}{2}tr\boldsymbol{\tau}I\|_0^2+\frac{1}{4(\mu+\lambda)}\|tr\boldsymbol{\tau}\|_0^2 \quad \text{for } \boldsymbol{\tau}\in\Sigma.$$

Then it holds

$$\frac{1}{\sqrt{2(\mu+\lambda)}}\|\boldsymbol{\tau}\|_0\le\|\boldsymbol{\tau}\|_a\le\frac{1}{\sqrt{2\mu}}\|\boldsymbol{\tau}\|_0. \tag{2.5}$$

The following continuity conditions are immediate:

$$|a(\boldsymbol{\sigma},\boldsymbol{\tau})|\le\|\boldsymbol{\sigma}\|_a\|\boldsymbol{\tau}\|_a, \quad \boldsymbol{\sigma},\boldsymbol{\tau}\in C^0([0,T];\Sigma), \tag{2.6a}$$
$$|b(\boldsymbol{\tau},\mathbf{v})|\lesssim\|\boldsymbol{\tau}\|_0|\mathbf{v}|_1, \quad \boldsymbol{\tau}\in C^0([0,T];\Sigma), \quad \mathbf{v}\in C^2([0,T];V). \tag{2.6b}$$

3 Semi-discrete hybrid stress finite element method

3.1 Geometric properties of quadrilaterals

In what follows we assume that Ω is a convex polygonal domain. Let $\mathcal{T}_h$ be a conventional quadrilateral mesh of Ω. Let $K\in\mathcal{T}_h$ be an arbitrary quadrilateral with diameter h_K. Denote $h:=\max_{K\in\mathcal{T}_h}\{h_K\}$. Let $Z_i(x_i,y_i)$, $1\le i\le 4$ be the four vertices of K and T_i denotes the sub-triangle of K with vertices Z_{i-1}, Z_i and Z_{i+1} (the index on Z_i is modulo 4). Define

$$\rho_K=\min_{1\le i\le 4}\{\text{diameter of circle inscribed in } T_i\}.$$

Throughout the paper, we assume that the partition $\mathcal{T}_h$ satisfies the following "shape-regularity" hypothesis [17]: there exists a constant $\varrho>2$ independent of h such that for all $K\in\mathcal{T}_h$,

$$h_K\le\varrho\rho_K. \tag{3.1}$$

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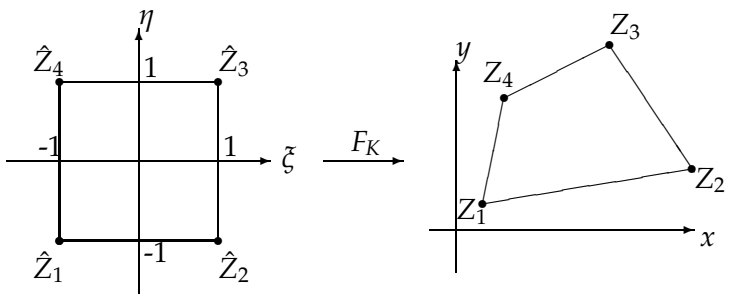


Figure 1: The mapping F_K.

We define the isoparametric bilinear mapping $F_K:\widehat{K}=[-1,1]^2\longrightarrow K$ (see Fig. 1) as

$$\begin{pmatrix} x \\ y \end{pmatrix}=F_K(\xi,\eta)=\begin{pmatrix} \sum_{i=1}^{4} x_i N_i(\xi,\eta) \\ \sum_{i=1}^{4} y_i N_i(\xi,\eta)) \end{pmatrix}, \tag{3.2}$$

where ξ, η are the local isoparametric coordinates, and

$$N_1=\frac{1}{4}(1-\xi)(1-\eta), \qquad N_2=\frac{1}{4}(1+\xi)(1-\eta),$$
$$N_3=\frac{1}{4}(1+\xi)(1+\eta), \qquad N_4=\frac{1}{4}(1-\xi)(1+\eta).$$

We can rewrite (3.2) as

$$x=a_0+a_1\xi+a_2\eta+a_{12}\xi\eta, \qquad y=b_0+b_1\xi+b_2\eta+b_{12}\xi\eta,$$

where

$$\begin{pmatrix} a_0 & b_0 \\ a_1 & b_1 \\ a_2 & b_2 \\ a_{12} & b_{12} \end{pmatrix}=\frac{1}{4}\begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & -1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix}\begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_4 & y_4 \end{pmatrix}.$$

3.2 Hybrid stress finite element spaces

Let $\Sigma_h\subset\Sigma$ and $V_h\subset V$ be finite dimensional spaces respectively for stress and displacement approximations. For the PS element [10], the isoparametric bilinear interpolation is used for the displacement approximation, i.e., the displacement space V_h is chosen as

$$V_h=\{\mathbf{v}\in V:\hat{\mathbf{v}}=\mathbf{v}|_K\circ F_K\in span\{1,\xi,\eta,\xi\eta\}^2 \ \text{ for all } \ K\in\mathcal{T}_h\}.$$

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For simplicity we abbreviate the symmetric stress tensor $\boldsymbol{\tau}=(\tau_{ij})_{2\times 2}$ as $\boldsymbol{\tau}=(\tau_{11},\tau_{22},\tau_{12})^T$. The 5-parameters stress mode on $\hat{K}$ for PS element takes the form

$$\boldsymbol{\tau}=P\beta \ \text{ for } \ \beta:=(\beta_1,\cdots,\beta_5)^T\in\mathbb{R}^5, \tag{3.3}$$

where

$$P=\begin{pmatrix} 1 & 0 & 0 & \eta & \frac{a_2^2}{b_2^2}\xi \\ 0 & 1 & 0 & \frac{b_1^2}{a_1^2}\eta & \xi \\ 0 & 0 & 1 & \frac{b_1}{a_1}\eta & \frac{a_2}{b_2}\xi \end{pmatrix}.$$

Then the corresponding stress space for PS element is

$$\Sigma_h:=\{\boldsymbol{\tau}\in\Sigma:\hat{\boldsymbol{\tau}}=\boldsymbol{\tau}|_K\circ F_K \text{ is of form } (3.3) \text{ for all } K\in\mathcal{T}_h\}.$$

Introduce the spaces

$$W_h:=\{q\in L^2(\Omega):\hat{q}=q|_K\circ F_K\in span\{1,\xi,\eta\} \text{ for all } K\in\mathcal{T}_h\},$$
$$\bar{W}_h:=\{q\in W_h:\bar{q}|_K=\text{constant for all } K\in\mathcal{T}_h\}.$$

Lemma 3.1 (see [13]). Assume that for any $\bar{q}\in\bar{W}_h$ there exists some $\mathbf{v}\in V_h$ with $\|\bar{q}\|_0^2\lesssim\int_\Omega\bar{q}\mathrm{d}iv\mathbf{v}d\mathbf{x}$, $|\mathbf{v}|_1^2\lesssim\|\bar{q}\|_0^2$. Then for any $\boldsymbol{\tau}_h\in KerB_h:=\{\boldsymbol{\tau}_h\in\Sigma_h:\int_\Omega\boldsymbol{\tau}_h:\epsilon(\mathbf{v}_h)d\mathbf{x}=0 \text{ for all } \mathbf{v}_h\in V_h\}$, it holds

$$\|\boldsymbol{\tau}_h\|_0^2\lesssim a(\boldsymbol{\tau}_h,\boldsymbol{\tau}_h). \tag{3.4}$$

Remark 3.1. This Lemma states that any quadrilateral mesh which is stable for the Stokes element Q_1-P_0 is sufficient for (3.4). As we know, the only unstable case for Q_1-P_0 is checkerboard mode. Thereupon, any quadrilateral mesh which breaks the checkerboard mode is sufficient for the uniform stability (3.4).

Lemma 3.2 (see [13]). For any $\mathbf{v}_h\in V_h$, it holds

$$|\mathbf{v}_h|_1\lesssim\sup_{0\neq\boldsymbol{\tau}_h\in\Sigma_h}\frac{b(\boldsymbol{\tau}_h,\mathbf{v}_h)}{\|\boldsymbol{\tau}_h\|_0}.$$

3.3 Semi-discrete scheme and error estimation

Let $\boldsymbol{\varphi}_{0,h}$ and $\boldsymbol{\varphi}_{1,h}$ be respectively approximations of initial data $\boldsymbol{\varphi}_0$ and $\boldsymbol{\varphi}_1$. The corresponding semi-discrete scheme for the problem (2.4) reads as: find $(\boldsymbol{\sigma}_h,\mathbf{u}_h)\in C^0([0,T];\Sigma_h)\times C^2([0,T];V_h)$, such that

$$\begin{cases} a(\boldsymbol{\sigma}_h(t),\boldsymbol{\tau}_h)+b(\boldsymbol{\tau}_h,\mathbf{u}_h(t))=0, & \forall\boldsymbol{\tau}_h\in\Sigma_h, \\ b(\boldsymbol{\sigma}_h(t),\mathbf{v}_h)+(\mathbf{f}(t),\mathbf{v}_h)=(\mathbf{u}_{h,tt}(t),\mathbf{v}_h), & \forall\mathbf{v}_h\in V_h, \\ \mathbf{u}_h(0)=\boldsymbol{\varphi}_{0,h}, \quad \mathbf{u}_{h,t}(0)=\boldsymbol{\varphi}_{1,h}. \end{cases} \tag{3.5}$$

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Lemma 3.3 (see [15]). The semi-discrete scheme (3.5) admits a unique solution $(\boldsymbol{\sigma}_h(\cdot),\mathbf{u}_h(\cdot))$.

We define two "elliptic projection" operators,

$$\Pi_1:\Sigma\to\Sigma_h \quad \text{and} \quad \Pi_2:V\to V_h,$$

as follows: For $\mathbf{u}(t)\in V$ and $\boldsymbol{\sigma}(t)=\mathbb{C}\epsilon(\mathbf{u}(t))\in\Sigma$ with $t\in[0,T]$, let $(\Pi_1\boldsymbol{\sigma}(t),\Pi_2\mathbf{u}(t)):=(\widehat{\boldsymbol{\sigma}_h}(t),\widehat{\mathbf{u}_h}(t))\in\Sigma_h\times V_h$ satisfy

$$\begin{cases} a(\widehat{\boldsymbol{\sigma}_h}(t),\boldsymbol{\tau}_h)+b(\boldsymbol{\tau}_h,\widehat{\mathbf{u}_h}(t))=0, & \forall\boldsymbol{\tau}_h\in\Sigma_h,\\ b(\widehat{\boldsymbol{\sigma}_h}(t),\mathbf{v}_h)=b(\boldsymbol{\sigma}(t),\mathbf{v}_h), & \forall\mathbf{v}_h\in V_h. \end{cases} \tag{3.6}$$

By standard arguments in the theory of mixed finite element methods [18], we have the following error estimate.

Lemma 3.4. Let $(\boldsymbol{\sigma}(t),\mathbf{u}(t))$ and $(\widehat{\boldsymbol{\sigma}_h}(t),\widehat{\mathbf{u}_h}(t))$ be the solutions to the weak problem (2.4) and the discrete "elliptic projection" problem (3.6), respectively. If $\mathbf{u}(t)\in(H^2(\Omega))^2\cap(H_0^1(\Omega))^2$, $\boldsymbol{\sigma}(t)\in(H^1(\Omega))^{2\times2}$ for $t\in[0,T]$, then, under the same conditions as in Lemma 3.1, it holds the estimate

$$\|\boldsymbol{\sigma}(t)-\widehat{\boldsymbol{\sigma}_h}(t)\|_0+|\mathbf{u}(t)-\widehat{\mathbf{u}_h}(t)|_1\lesssim h(\|\boldsymbol{\sigma}(t)\|_1+\|\mathbf{u}(t)\|_2). \tag{3.7}$$

Lemma 3.5. Under the same conditions as in Lemma 3.4, it holds

$$\|\mathbf{u}(t)-\widehat{\mathbf{u}_h}(t)\|_0\lesssim h^2(\|\boldsymbol{\sigma}(t)\|_1+\|\mathbf{u}(t)\|_2). \tag{3.8}$$

Additionally, if $\mathbf{u}_t(t)\in(H^2(\Omega))^2$, $\boldsymbol{\sigma}_t(t)\in(H^1(\Omega))^{2\times2}$ for $t\in[0,T]$, then it holds

$$\|\mathbf{u}_t(t)-\widehat{\mathbf{u}_{h,t}}(t)\|_0\lesssim h^2(\|\boldsymbol{\sigma}_t(t)\|_1+\|\mathbf{u}_t(t)\|_2). \tag{3.9}$$

If $\mathbf{u}_{tt}(t)\in(H^2(\Omega))^2$, $\boldsymbol{\sigma}_{tt}(t)\in(H^1(\Omega))^{2\times2}$, then it holds

$$\|\mathbf{u}_{tt}(t)-\widehat{\mathbf{u}_{h,tt}}(t)\|_0\lesssim h^2(\|\boldsymbol{\sigma}_{tt}(t)\|_1+\|\mathbf{u}_{tt}(t)\|_2). \tag{3.10}$$

If $\mathbf{u}_{ttt}(t)\in(H^2(\Omega))^2$, $\boldsymbol{\sigma}_{ttt}(t)\in(H^1(\Omega))^{2\times2}$, then it holds

$$\|\mathbf{u}_{ttt}(t)-\widehat{\mathbf{u}_{h,ttt}}(t)\|_0\lesssim h^2(\|\boldsymbol{\sigma}_{ttt}(t)\|_1+\|\mathbf{u}_{ttt}(t)\|_2). \tag{3.11}$$

Proof. Here, we focus on the proof of (3.8), and the remaining will follow naturally.

We use a duality argument. For $t\in[0,T]$, let $\boldsymbol{\phi}$ be the solution of

$$\begin{cases} -\mathrm{div}(\mathbb{C}\epsilon(\boldsymbol{\phi}))=\mathbf{u}(t)-\widehat{\mathbf{u}_h}(t) & \text{in } \Omega,\\ \boldsymbol{\phi}=0 & \text{on } \Gamma. \end{cases} \tag{3.12}$$

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By the regularity theory, we have

$$\|\boldsymbol{\phi}\|_2+\lambda\|\mathrm{div}\boldsymbol{\phi}\|_1\lesssim\|\mathbf{u}(t)-\widehat{\mathbf{u}_h}(t)\|_0. \tag{3.13}$$

Let $P_h:\Sigma\to\Sigma_h$ be the L^2 projection operator and $I_h:V\to V_h$ be some interpolation operator. By (3.6) the following Garlerkin orthogonality holds:

$$(\boldsymbol{\sigma}(t)-\widehat{\boldsymbol{\sigma}_h}(t),\epsilon(\mathbf{v}_h))=0,\quad\forall\mathbf{v}_h\in V_h. \tag{3.14}$$

Then from (3.12) it follows

$$\begin{aligned}
&\|\mathbf{u}(t)-\widehat{\mathbf{u}_h}(t)\|_0^2\\
=&-\int_\Omega(\mathbf{u}(t)-\widehat{\mathbf{u}_h}(t))\cdot\mathrm{div}(\mathbb{C}\epsilon(\boldsymbol{\phi}))\mathrm{d}\mathbf{x}\\
=&\int_\Omega\epsilon(\mathbf{u}(t)-\widehat{\mathbf{u}_h}(t)):\mathbb{C}\epsilon(\boldsymbol{\phi})\mathrm{d}\mathbf{x}\\
=&\int_\Omega\boldsymbol{\sigma}(t):\epsilon(\boldsymbol{\phi})\mathrm{d}\mathbf{x}-\int_\Omega\epsilon(\widehat{\mathbf{u}_h}(t)):\mathbb{C}\epsilon(\boldsymbol{\phi})\mathrm{d}\mathbf{x}\quad\text{(by (2.4))}\\
=&\int_\Omega\boldsymbol{\sigma}(t):\epsilon(\boldsymbol{\phi})\mathrm{d}\mathbf{x}-\int_\Omega\epsilon(\widehat{\mathbf{u}_h}(t)):P_h(\mathbb{C}\epsilon(\boldsymbol{\phi}))\mathrm{d}\mathbf{x}+\int_\Omega\epsilon(\widehat{\mathbf{u}_h}(t)):(P_h(\mathbb{C}\epsilon(\boldsymbol{\phi}))-\mathbb{C}\epsilon(\boldsymbol{\phi}))\mathrm{d}\mathbf{x}\\
=&\int_\Omega\boldsymbol{\sigma}(t):\epsilon(\boldsymbol{\phi})\mathrm{d}\mathbf{x}-\int_\Omega\widehat{\boldsymbol{\sigma}_h}(t):\mathbb{C}^{-1}P_h(\mathbb{C}\epsilon(\boldsymbol{\phi}))\mathrm{d}\mathbf{x}+\int_\Omega\epsilon(\mathbf{u}(t)):(P_h(\mathbb{C}\epsilon(\boldsymbol{\phi}))-\mathbb{C}\epsilon(\boldsymbol{\phi}))\mathrm{d}\mathbf{x}\\
&+\int_\Omega(\epsilon(\widehat{\mathbf{u}_h}(t))-\epsilon(\mathbf{u}(t))):(P_h(\mathbb{C}\epsilon(\boldsymbol{\phi}))-\mathbb{C}\epsilon(\boldsymbol{\phi}))\mathrm{d}\mathbf{x}\quad\text{(by (3.6))}\\
=&\int_\Omega(\boldsymbol{\sigma}(t)-\widehat{\boldsymbol{\sigma}_h}(t)):(\epsilon(\boldsymbol{\phi})-\epsilon(I_h\boldsymbol{\phi}))\mathrm{d}\mathbf{x}+\int_\Omega(\epsilon(\mathbf{u}(t))-P_h(\epsilon(\mathbf{u}(t)))):(P_h(\mathbb{C}\epsilon(\boldsymbol{\phi}))-\mathbb{C}\epsilon(\boldsymbol{\phi}))\mathrm{d}\mathbf{x}\\
&+\int_\Omega(\epsilon(\widehat{\mathbf{u}_h}(t))-\epsilon(\mathbf{u}(t))):(P_h(\mathbb{C}\epsilon(\boldsymbol{\phi}))-\mathbb{C}\epsilon(\boldsymbol{\phi}))\mathrm{d}\mathbf{x}\quad\text{(by (3.14))}\\
\leq&\|\boldsymbol{\sigma}(t)-\widehat{\boldsymbol{\sigma}_h}(t)\|_0|\boldsymbol{\phi}-I_h\boldsymbol{\phi}|_1+(\|\epsilon(\mathbf{u}(t))-P_h(\epsilon(\mathbf{u}(t)))\|_0+|\mathbf{u}(t)-\widehat{\mathbf{u}_h}(t)|_1)\|\mathbb{C}\epsilon(\boldsymbol{\phi})-P_h(\mathbb{C}\epsilon(\boldsymbol{\phi}))\|_0\\
\lesssim&h^2(\|\boldsymbol{\sigma}(t)\|_1+\|\mathbf{u}(t)\|_2)(\|\boldsymbol{\phi}\|_2+\|\mathbb{C}\epsilon(\boldsymbol{\phi})\|_1)\quad\text{(by (3.7))}\\
\lesssim&h^2(\|\boldsymbol{\sigma}(t)\|_1+\|\mathbf{u}(t)\|_2)\|\mathbf{u}(t)-\widehat{\mathbf{u}_h}(t)\|_0,\quad\text{(by (3.13))}
\end{aligned}$$

which yields the desired result. □

Lemma 3.6. Let $(\boldsymbol{\sigma}_h(t),\mathbf{u}_h(t))$ be the solution to the semi-discrete scheme (3.5) with the initial condition

$$\boldsymbol{\varphi}_{0,h}=\widehat{\mathbf{u}_h}(0),\quad\boldsymbol{\varphi}_{1,h}=I_h\boldsymbol{\varphi}_1, \tag{3.15}$$

and $(\widehat{\boldsymbol{\sigma}_h}(t),\widehat{\mathbf{u}_h}(t))$ be the solution to the discrete "elliptic projection" problem (3.6). Suppose $\boldsymbol{\sigma}_{tt}\in L^2([0,T];(H^1(\Omega))^{2\times2})$ and $\mathbf{u}_{tt}\in L^2([0,T];(H^2(\Omega))^2)$, then, under the same conditions as in Lemma 3.1, for $t\in[0,T]$, the following error estimate holds:

$$\begin{aligned}
&\|\boldsymbol{\sigma}_h(t)-\widehat{\boldsymbol{\sigma}_h}(t)\|_a+|\mathbf{u}_h(t)-\widehat{\mathbf{u}_h}(t)|_1+\|\mathbf{u}_{h,t}(t)-\widehat{\mathbf{u}_h}_{,t}(t)\|_0\\
\lesssim&h^2\big(\|\boldsymbol{\sigma}_t(0)\|_1+\|\varphi_1\|_2+\|\boldsymbol{\sigma}_{tt}\|_{L^2(H^1)}+\|\mathbf{u}_{tt}\|_{L^2(H^2)}\big).
\end{aligned}\tag{3.16}$$

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Proof. Denoting $\boldsymbol{\psi}_h(t):=\boldsymbol{\sigma}_h(t)-\widehat{\boldsymbol{\sigma}_h}(t)$, $\mathbf{r}_h(t):=\mathbf{u}_h(t)-\widehat{\mathbf{u}_h}(t)$, from (3.5)-(3.6) we obtain the error system

$$\begin{cases} a(\boldsymbol{\psi}_h(t),\boldsymbol{\tau}_h)+b(\boldsymbol{\tau}_h,\mathbf{r}_h(t))=0, & \forall \boldsymbol{\tau}_h\in\Sigma_h, \\ b(\boldsymbol{\psi}_h(t),\mathbf{v}_h)=(\mathbf{u}_{h,tt}(t)-\mathbf{u}_{tt}(t),\mathbf{v}_h), & \forall \mathbf{v}_h\in V_h. \end{cases} \tag{3.17}$$

Differentiating the first equation of (3.17) with respect to t and taking $\boldsymbol{\tau}_h=\boldsymbol{\psi}_h(t)$, we obtain

$$a(\boldsymbol{\psi}_{h,t}(t),\boldsymbol{\psi}_h(t))+b(\boldsymbol{\psi}_h(t),\mathbf{r}_{h,t}(t))=0. \tag{3.18}$$

Taking $\mathbf{v}_h=\mathbf{r}_{h,t}(t)$ in the second equation of (3.17), we have

$$b(\boldsymbol{\psi}_h(t),\mathbf{r}_{h,t}(t))=(\mathbf{u}_{h,tt}(t)-\mathbf{u}_{tt}(t),\mathbf{r}_{h,t}(t)). \tag{3.19}$$

Combining (3.18) with (3.19) yields

$$a(\boldsymbol{\psi}_{h,t}(t),\boldsymbol{\psi}_h(t))+(\mathbf{u}_{h,tt}(t)-\mathbf{u}_{tt}(t),\mathbf{r}_{h,t}(t))=0.$$

Then it follows

$$\begin{aligned} \frac{d}{dt}\|\boldsymbol{\psi}_h(t)\|_a^2 &= \frac{d}{dt}a(\boldsymbol{\psi}_h(t),\boldsymbol{\psi}_h(t)) \\ &= 2(\mathbf{u}_{tt}(t)-\widehat{\mathbf{u}_{h,tt}}(t),\mathbf{r}_{h,t}(t))-\frac{d}{dt}\|\mathbf{r}_{h,t}(t)\|_0^2 \\ &\le \|\mathbf{u}_{tt}(t)-\widehat{\mathbf{u}_{h,tt}}(t)\|_0^2+\|\mathbf{r}_{h,t}(t)\|_0^2-\frac{d}{dt}\|\mathbf{r}_{h,t}(t)\|_0^2, \end{aligned}$$

which indicates

$$\frac{d}{dt}\left(\|\boldsymbol{\psi}_h(t)\|_a^2+\|\mathbf{r}_{h,t}(t)\|_0^2\right)\le\|\mathbf{u}_{tt}(t)-\widehat{\mathbf{u}_{h,tt}}(t)\|_0^2+\|\mathbf{r}_{h,t}(t)\|_0^2.$$

By (3.15) it holds $\mathbf{r}_h(0)=0$, then $\boldsymbol{\psi}_h(0)=0$. Moreover, it holds

$$\begin{aligned} \|\mathbf{r}_{h,t}(0)\|_0 &= \|\mathbf{u}_{h,t}(0)-\widehat{\mathbf{u}_{h,t}}(0)\|_0 \\ &\le \|\mathbf{u}_t(0)-\mathbf{u}_{h,t}(0)\|_0+\|\mathbf{u}_t(0)-\widehat{\mathbf{u}_{h,t}}(0)\|_0 \\ &\le \|\varphi_1-\varphi_{1,h}\|_0+\|\mathbf{u}_t(0)-\widehat{\mathbf{u}_{h,t}}(0)\|_0 \\ &\lesssim h^2(\|\boldsymbol{\sigma}_t(0)\|_1+\|\varphi_1\|_2). \end{aligned} \tag{3.20}$$

Then, using the continuous Gronwall's inequality (2.1) and the estimate (3.10), we have

$$\begin{aligned} \|\boldsymbol{\psi}_h(t)\|_a^2+\|\mathbf{r}_{h,t}(t)\|_0^2 &\le e^T\left(\|\boldsymbol{\psi}_h(0)\|_a^2+\|\mathbf{r}_{h,t}(0)\|_0^2+\int_0^T\|\mathbf{u}_{tt}(s)-\widehat{\mathbf{u}_{h,tt}}(s)\|_0^2ds\right) \\ &\lesssim h^4\left(\|\boldsymbol{\sigma}_t(0)\|_1^2+\|\varphi_1\|_2^2+\int_0^T\|\boldsymbol{\sigma}_{tt}(s)\|_1^2ds+\int_0^T\|\mathbf{u}_{tt}(s)\|_2^2ds\right), \end{aligned} \tag{3.21}$$

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which yields

$$\begin{aligned}&\|\boldsymbol{\sigma}_h(t)-\widehat{\boldsymbol{\sigma}_h}(t)\|_a+\|\mathbf{u}_{h,t}(t)-\widehat{\mathbf{u}_h}_{,t}(t)\|_0\\ \lesssim& h^2\big(\|\boldsymbol{\sigma}_t(0)\|_1+\|\varphi_1\|_2+\|\boldsymbol{\sigma}_{tt}\|_{L^2(H^1)}+\|\mathbf{u}_{tt}\|_{L^2(H^2)}\big).\end{aligned}\tag{3.22}$$

From Lemma 3.2, the first equation of (3.17), (2.5)-(2.6a) and (3.22), it follows, for $t\in[0,T]$,

$$\begin{aligned}|\mathbf{u}_h(t)-\widehat{\mathbf{u}_h}(t)|_1\lesssim&\sup_{0\neq\boldsymbol{\tau}_h\in\Sigma_h}\frac{-b(\boldsymbol{\tau}_h(t),\mathbf{u}_h(t)-\widehat{\mathbf{u}_h}(t))}{\|\boldsymbol{\tau}_h(t)\|_0}\\ =&\sup_{0\neq\boldsymbol{\tau}_h\in\Sigma_h}\frac{a(\boldsymbol{\psi}_h(t),\boldsymbol{\tau}_h(t))}{\|\boldsymbol{\tau}_h(t)\|_0}\lesssim\|\boldsymbol{\psi}_h(t)\|_a\\ \lesssim& h^2\big(\|\boldsymbol{\sigma}_t(0)\|_1+\|\varphi_1\|_2+\|\boldsymbol{\sigma}_{tt}\|_{L^2(H^1)}+\|\mathbf{u}_{tt}\|_{L^2(H^2)}\big),\end{aligned}$$

which, together with (3.22), indicates the desired estimate. □

Finally, in view of Lemmas 3.4-3.6, triangle inequality, Poincaré inequality and (2.5), we obtain the error estimation for the semi-discrete scheme.

Theorem 3.1. Let $(\boldsymbol{\sigma}(t),\mathbf{u}(t))$ and $(\boldsymbol{\sigma}_h(t),\mathbf{u}_h(t))$ be the solutions to the weak problem (2.4) and the semi-discrete scheme (3.5) with the initial condition (3.15), respectively. Suppose $\boldsymbol{\sigma}\in L^\infty([0,T];(H^1(\Omega))^{2\times2})$, $\boldsymbol{\sigma}_{tt}\in L^2([0,T];(H^1(\Omega))^{2\times2})$ and $\mathbf{u}\in L^\infty([0,T];(H^2(\Omega))^2\cap(H_0^1(\Omega))^2)$, $\mathbf{u}_{tt}\in L^2([0,T];(H^2(\Omega))^2)$, then, under the same conditions as in Lemma 3.1, it holds the following error estimates:

$$\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{L^\infty((L_a^2)^{2\times2})}+|\mathbf{u}-\mathbf{u}_h|_{L^\infty((H^1)^2)}\lesssim\tilde{C}_1(\boldsymbol{\sigma},\mathbf{u})h,\tag{3.23a}$$

$$\|\mathbf{u}-\mathbf{u}_h\|_{L^\infty((L^2)^2)}\lesssim\tilde{C}_1(\boldsymbol{\sigma},\mathbf{u})h^2,\tag{3.23b}$$

where $\|\boldsymbol{\tau}\|_{L^\infty((L_a^2)^{2\times2})}:=esssup_{0\le t\le T}\|\boldsymbol{\tau}(t)\|_a$ and

$$\tilde{C}_1(\boldsymbol{\sigma},\mathbf{u}):=\|\boldsymbol{\sigma}_t(0)\|_1+\|\varphi_1\|_2+\|\boldsymbol{\sigma}\|_{L^\infty(H^1)}+\|\mathbf{u}\|_{L^\infty(H^2)}+\|\boldsymbol{\sigma}_{tt}\|_{L^2(H^1)}+\|\mathbf{u}_{tt}\|_{L^2(H^2)}.$$

Remark 3.1. Under the same conditions as in Theorem 3.1, if $\boldsymbol{\sigma}_t\in L^\infty([0,T];(H^1(\Omega))^{2\times2})$ and $\mathbf{u}_t\in L^\infty([0,T];(H^2(\Omega))^2)$, then it holds

$$\|\mathbf{u}_t-\mathbf{u}_{h,t}\|_{L^\infty((L^2)^2)}\lesssim\tilde{C}_2(\boldsymbol{\sigma},\mathbf{u})h^2,\tag{3.24}$$

where $\tilde{C}_2(\boldsymbol{\sigma},\mathbf{u}):=\|\boldsymbol{\sigma}_t(0)\|_1+\|\varphi_1\|_2+\|\boldsymbol{\sigma}_t\|_{L^\infty(H^1)}+\|\mathbf{u}_t\|_{L^\infty(H^2)}+\|\boldsymbol{\sigma}_{tt}\|_{L^2(H^1)}+\|\mathbf{u}_{tt}\|_{L^2(H^2)}$.

Remark 3.2. We recall that "$a\lesssim b$" represents "$a\le Cb$" with the positive constant C independent of mesh size h and the Lamé parameter λ. Theorem 3.1 shows that the error estimates (3.23a)-(3.23b) are uniform with respect to λ if $\tilde{C}_1(\boldsymbol{\sigma},\mathbf{u})\lesssim1$. Similarly, the estimate (3.24) in Remark 3.1 is uniform if $\tilde{C}_2(\boldsymbol{\sigma},\mathbf{u})\lesssim1$.

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4 Fully discrete hybrid stress finite element method

4.1 Fully discrete scheme

Consider a uniform partition of the time domain $[0,T]$, i.e., $0=t_0<t_1<\cdots<t_M=T$, with $t_i=i\Delta t$ $(i=1,2,\cdots,M-1)$ and $\Delta t:=T/M$. Denote $t_{i+\frac{1}{2}}:=\frac{t_i+t_{i+1}}{2}$. For any function φ of t, define

$$\varphi^n:=\varphi(t_n),\qquad \varphi^{n+\frac{1}{2}}:=\frac{\varphi^n+\varphi^{n+1}}{2},\qquad \varphi^{n;\frac{1}{4}}:=\frac{\varphi^{n+1}+2\varphi^n+\varphi^{n-1}}{4},$$
$$\Delta_t\varphi^n:=\frac{\varphi^{n+1}-\varphi^{n-1}}{2\Delta t},\qquad \Delta_t\varphi^{n+\frac{1}{2}}:=\frac{\varphi^{n+1}-\varphi^n}{\Delta t},\qquad \Delta_t^2\varphi^n:=\frac{\varphi^{n+1}-2\varphi^n+\varphi^{n-1}}{(\Delta t)^2}.$$

We can easily see that

$$\Delta_t^2\varphi^n=\frac{\Delta_t\varphi^{n+\frac{1}{2}}-\Delta_t\varphi^{n-\frac{1}{2}}}{\Delta t},\qquad \Delta_t\varphi^n=\frac{\Delta_t\varphi^{n+\frac{1}{2}}+\Delta_t\varphi^{n-\frac{1}{2}}}{2}. \tag{4.1}$$

The corresponding fully discrete hybrid stress finite element scheme for the problem (2.4) reads as: find $(\boldsymbol{\sigma}_h^n,\mathbf{u}_h^n)\in\Sigma_h\times V_h$ for $0\le n\le M$ such that

$$\begin{cases}\left(\frac{2}{\Delta t}\Delta_t\mathbf{u}_h^{\frac{1}{2}},\mathbf{v}_h\right)-b(\boldsymbol{\sigma}_h^{\frac{1}{2}},\mathbf{v}_h)=(\mathbf{f}^{\frac{1}{2}},\mathbf{v}_h)+\left(\frac{2\varphi_1}{\Delta t},\mathbf{v}_h\right), & \forall\mathbf{v}_h\in V_h,\\ a(\boldsymbol{\sigma}_h^{n+1},\boldsymbol{\tau}_h)+b(\boldsymbol{\tau}_h,\mathbf{u}_h^{n+1})=0, & \forall\boldsymbol{\tau}_h\in\Sigma_h,\quad n\ge-1,\\ (\Delta_t^2\mathbf{u}_h^n,\mathbf{v}_h)-b(\boldsymbol{\sigma}_h^{n;\frac{1}{4}},\mathbf{v}_h)=(\mathbf{f}^{n;\frac{1}{4}},\mathbf{v}_h), & \forall\mathbf{v}_h\in V_h,\quad n\ge1,\\ \mathbf{u}_h^0=\widehat{\mathbf{u}_h}(0).\end{cases} \tag{4.2}$$

Lemma 4.1. The fully discrete scheme (4.2) admits a unique solution $(\boldsymbol{\sigma}_h^n,\mathbf{u}_h^n)$ for $n=0,1,2,\cdots,M$.

Proof. Let $\mathbf{u}_h^n|_K=N\alpha^n$, $\boldsymbol{\sigma}_h^n|_K=P\beta^n$ with $\alpha^n:=(\alpha_1^n,\cdots,\alpha_8^n)^T$, $\beta^n:=(\beta_1^n,\cdots,\beta_5^n)^T$ and

$$N:=\begin{pmatrix}N_1&0&N_2&0&N_3&0&N_4&0\\0&N_1&0&N_2&0&N_3&0&N_4\end{pmatrix},$$

where P and N_i $(i=1,\cdots,4)$ are the same as in (3.3) and (3.2), respectively.

Taking $\tau_h|_K=P\tilde{\beta}$ with an arbitrary $\tilde{\beta}\in\mathbb{R}^5$ in the second equation of (4.2), we obtain

$$\int_K P^T\mathbb{C}^{-1}Pd\mathbf{x}\beta^n-\int_K P^T\epsilon(N)d\mathbf{x}\alpha^n=0,$$

where $\epsilon(N)$ is the strain on N. Denote

$$G=\int_K P^T\epsilon(N)d\mathbf{x},\qquad H=\int_K P^T\mathbb{C}^{-1}Pd\mathbf{x},$$

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then the above relation yields

$$\beta^n = H^{-1}G\alpha^n. \tag{4.3}$$

For $n=1$, taking $\mathbf{v}_h|_K = N\tilde{\alpha}$ with an arbitrary $\tilde{\alpha}\in\mathbb{R}^8$ in the first equation of (4.2) and using (4.3), we obtain

$$\begin{aligned}&\sum_K\left(\frac{1}{(\Delta t)^2}\int_K N^T N d\mathbf{x}+\frac{1}{4}\int_K \epsilon(N^T)Pd\mathbf{x}H^{-1}G\right)(\alpha^1)\\ =&\frac{1}{(\Delta t)^2}\sum_K\int_K N^T N d\mathbf{x}(\alpha^0)-\frac{1}{4}\sum_K\int_K\epsilon(N^T)Pd\mathbf{x}(\beta^0)+\frac{1}{2}\sum_K\int_K N^T\left(\mathbf{f}^{\frac{1}{2}}+\frac{2\varphi_1}{\Delta t}\right)d\mathbf{x}.\end{aligned}$$

Denote

$$Q=\sum_K\int_K N^T N d\mathbf{x},\qquad F=\frac{1}{2}\sum_K\int_K N^T\left(\mathbf{f}^{\frac{1}{2}}+\frac{2\varphi_1}{\Delta t}\right)d\mathbf{x},$$

we then have

$$\left(\frac{1}{(\Delta t)^2}Q+\frac{1}{4}G^T H^{-1}G\right)(\alpha^1)=\frac{1}{(\Delta t)^2}Q(\alpha^0)-\frac{1}{4}G^T\beta^0+F. \tag{4.4}$$

For $2\le n\le M$, taking $\mathbf{v}_h|_K = N\tilde{\alpha}$ with an arbitrary $\tilde{\alpha}\in\mathbb{R}^8$ in the third equation of (4.2) and using (4.3), we obtain

$$\begin{aligned}&\sum_K\left(\frac{1}{(\Delta t)^2}\int_K N^T N d\mathbf{x}+\frac{1}{4}\int_K\epsilon(N^T)Pd\mathbf{x}H^{-1}G\right)(\alpha^n)\\ =&\frac{1}{(\Delta t)^2}\sum_K\int_K N^T N d\mathbf{x}(2\alpha^{n-1}-\alpha^{n-2})-\frac{1}{4}\sum_K\int_K\epsilon(N^T)Pd\mathbf{x}(2\beta^{n-1}+\beta^{n-2})+\sum_K\int_K N^T\mathbf{f}^{\frac{1}{4}}d\mathbf{x}.\end{aligned}$$

Denote $\widetilde{F}=\sum_K\int_K N^T\mathbf{f}^{\frac{1}{4}}d\mathbf{x}$, then we have

$$\left(\frac{1}{(\Delta t)^2}Q+\frac{1}{4}G^T H^{-1}G\right)(\alpha^n)=\frac{1}{(\Delta t)^2}Q(2\alpha^{n-1}-\alpha^{n-2})-\frac{1}{4}G^T(2\beta^{n-1}+\beta^{n-2})+\widetilde{F}. \tag{4.5}$$

Since u_h^0 is given, which means α^0 is known, by (4.3) we get β^0. It is easy to verify that matrix $\frac{1}{(\Delta t)^2}Q+\frac{1}{4}G^T H^{-1}G$ is symmetric positive definite, then the linear system (4.4) admits a unique solution α^1. Applying (4.3) again, we get β^1. Analogously, using (4.5) and (4.3), we will get a unique sequence (α^n,β^n) for $2\le n\le M$. This completes the proof. □

4.2 Stability analysis

Lemma 4.2. Let $(\sigma_h^n,\mathbf{u}_h^n)$ $(n=0,1,2,\cdots,M)$ be the solution to the fully discrete scheme (4.2), then, for $J=1,2,\cdots,M-1$, it holds

$$\|\sigma_h^{J+\frac{1}{2}}\|_a^2+\|\Delta_t\mathbf{u}_h^{J+\frac{1}{2}}\|_0^2=2\Delta t\sum_{n=1}^{J}(\mathbf{f}^{n,\frac{1}{4}},\Delta_t\mathbf{u}_h^n)+\|\Delta_t\mathbf{u}_h^{\frac{1}{2}}\|_0^2+\|\sigma_h^{\frac{1}{2}}\|_a^2. \tag{4.6}$$

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Proof. From the second equation of (4.2) it follows

$$a(\boldsymbol{\sigma}_h^{n+1}-\boldsymbol{\sigma}_h^{n-1},\boldsymbol{\tau}_h)+b(\boldsymbol{\tau}_h,\mathbf{u}_h^{n+1}-\mathbf{u}_h^{n-1})=0,\quad \forall\boldsymbol{\tau}_h\in\Sigma_h,$$

which, by taking $\boldsymbol{\tau}_h=\frac{1}{2\Delta t}\boldsymbol{\sigma}_h^{n;\frac{1}{4}}$, yields

$$a(\Delta_t\boldsymbol{\sigma}_h^n,\boldsymbol{\sigma}_h^{n;\frac{1}{4}})+b(\boldsymbol{\sigma}_h^{n;\frac{1}{4}},\Delta_t\mathbf{u}_h^n)=0. \tag{4.7}$$

Taking $\mathbf{v}_h=\Delta_t\mathbf{u}_h^n$ in the third equation of (4.2), we get

$$b(\boldsymbol{\sigma}_h^{n;\frac{1}{4}},\Delta_t\mathbf{u}_h^n)=(\Delta_t^2\mathbf{u}_h^n,\Delta_t\mathbf{u}_h^n)-(\mathbf{f}^{n;\frac{1}{4}},\Delta_t\mathbf{u}_h^n). \tag{4.8}$$

Subtracting (4.8) from (4.7) implies

$$a(\Delta_t\boldsymbol{\sigma}_h^n,\boldsymbol{\sigma}_h^{n;\frac{1}{4}})+(\Delta_t^2\mathbf{u}_h^n,\Delta_t\mathbf{u}_h^n)=(\mathbf{f}^{n;\frac{1}{4}},\Delta_t\mathbf{u}_h^n). \tag{4.9}$$

On the other hand, from (4.1) it follows

$$\begin{aligned}(\Delta_t^2\mathbf{u}_h^n,\Delta_t\mathbf{u}_h^n)&=\Big(\frac{\Delta_t\mathbf{u}_h^{n+\frac{1}{2}}-\Delta_t\mathbf{u}_h^{n-\frac{1}{2}}}{\Delta t},\frac{\Delta_t\mathbf{u}_h^{n+\frac{1}{2}}+\Delta_t\mathbf{u}_h^{n-\frac{1}{2}}}{2}\Big)\\&=\frac{1}{2\Delta t}(\|\Delta_t\mathbf{u}_h^{n+\frac{1}{2}}\|_0^2-\|\Delta_t\mathbf{u}_h^{n-\frac{1}{2}}\|_0^2).\end{aligned} \tag{4.10}$$

Since

$$\begin{aligned}\Delta_t\boldsymbol{\sigma}_h^n&=\frac{\boldsymbol{\sigma}_h^{n+1}-\boldsymbol{\sigma}_h^{n-1}}{2\Delta t}=\frac{\boldsymbol{\sigma}_h^{n+1}+\boldsymbol{\sigma}_h^n-\boldsymbol{\sigma}_h^n-\boldsymbol{\sigma}_h^{n-1}}{2\Delta t}=\frac{\boldsymbol{\sigma}_h^{n+\frac{1}{2}}-\boldsymbol{\sigma}_h^{n-\frac{1}{2}}}{\Delta t},\\\boldsymbol{\sigma}_h^{n;\frac{1}{4}}&=\frac{\boldsymbol{\sigma}_h^{n+1}+\boldsymbol{\sigma}_h^n}{4}+\frac{\boldsymbol{\sigma}_h^n+\boldsymbol{\sigma}_h^{n-1}}{4}=\frac{\boldsymbol{\sigma}_h^{n+\frac{1}{2}}+\boldsymbol{\sigma}_h^{n-\frac{1}{2}}}{2},\end{aligned}$$

we have

$$\begin{aligned}a(\Delta_t\boldsymbol{\sigma}_h^n,\boldsymbol{\sigma}_h^{n;\frac{1}{4}})&=(\Delta_t\boldsymbol{\sigma}_h^n,\mathbb{C}^{-1}\boldsymbol{\sigma}_h^{n;\frac{1}{4}})=\Big(\frac{\boldsymbol{\sigma}_h^{n+\frac{1}{2}}-\boldsymbol{\sigma}_h^{n-\frac{1}{2}}}{\Delta t},\mathbb{C}^{-1}\frac{\boldsymbol{\sigma}_h^{n+\frac{1}{2}}+\boldsymbol{\sigma}_h^{n-\frac{1}{2}}}{2}\Big)\\&=\frac{1}{2\Delta t}(\|\boldsymbol{\sigma}_h^{n+\frac{1}{2}}\|_a^2-\|\boldsymbol{\sigma}_h^{n-\frac{1}{2}}\|_a^2).\end{aligned} \tag{4.11}$$

In light of (4.9)-(4.11), it holds

$$\frac{1}{2\Delta t}(\|\boldsymbol{\sigma}_h^{n+\frac{1}{2}}\|_a^2-\|\boldsymbol{\sigma}_h^{n-\frac{1}{2}}\|_a^2)+\frac{1}{2\Delta t}(\|\Delta_t\mathbf{u}_h^{n+\frac{1}{2}}\|_0^2-\|\Delta_t\mathbf{u}_h^{n-\frac{1}{2}}\|_0^2)=(\mathbf{f}^{n;\frac{1}{4}},\Delta_t\mathbf{u}_h^n). \tag{4.12}$$

Then (4.6) is obtained by summing (4.12) from $n=1,\cdots,J$. □

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Theorem 4.1. Assuming $\Delta t<1$, then the implicit scheme (4.2) is unconditionally stable in the following sense: for $J=1,2,\cdots M-1$, it holds

$$\begin{aligned}&\|\boldsymbol{\sigma}_h^{J+\frac{1}{2}}\|_a^2+|\mathbf{u}_h^{J+\frac{1}{2}}|_1^2+\|\Delta_t\mathbf{u}_h^{J+\frac{1}{2}}\|_0^2\\ \lesssim&\Big[\|\Delta_t\mathbf{u}_h^{\frac{1}{2}}\|_0^2+\|\boldsymbol{\sigma}_h^{\frac{1}{2}}\|_a^2+T\Big(\max_{t\in[0,T]}\|\mathbf{f}(t)\|_0\Big)^2\Big]\times\exp(2T).\end{aligned}\tag{4.13}$$

Proof. By (4.6) and Cauchy-Schwarz inequality, we have

$$\|\boldsymbol{\sigma}_h^{J+\frac{1}{2}}\|_a^2+\|\Delta_t\mathbf{u}_h^{J+\frac{1}{2}}\|_0^2\le\Delta t\sum_{n=1}^{J}\|\mathbf{f}^{n;\frac{1}{4}}\|_0^2+\Delta t\sum_{n=1}^{J}\|\Delta_t\mathbf{u}_h^n\|_0^2+\|\Delta_t\mathbf{u}_h^{\frac{1}{2}}\|_0^2+\|\boldsymbol{\sigma}_h^{\frac{1}{2}}\|_a^2.$$

Moreover, it holds

$$\begin{aligned}\sum_{n=1}^{J}\|\Delta_t\mathbf{u}_h^n\|_0^2&=\sum_{n=1}^{J}\Big\|\frac{\Delta_t\mathbf{u}_h^{n+\frac{1}{2}}+\Delta_t\mathbf{u}_h^{n-\frac{1}{2}}}{2}\Big\|_0^2\\&\le\frac{1}{2}\Big(\sum_{n=1}^{J}\|\Delta_t\mathbf{u}_h^{n+\frac{1}{2}}\|_0^2+\sum_{n=1}^{J}\|\Delta_t\mathbf{u}_h^{n-\frac{1}{2}}\|_0^2\Big)\\&\le\frac{1}{2}\Big(2\sum_{n=0}^{J-1}\|\Delta_t\mathbf{u}_h^{n+\frac{1}{2}}\|_0^2+\|\Delta_t\mathbf{u}_h^{J+\frac{1}{2}}\|_0^2\Big)\\&\le\sum_{n=0}^{J-1}\|\Delta_t\mathbf{u}_h^{n+\frac{1}{2}}\|_0^2+\frac{1}{2}\|\Delta_t\mathbf{u}_h^{J+\frac{1}{2}}\|_0^2.\end{aligned}$$

Since $\Delta t<1$, the above two estimates indicate

$$\|\boldsymbol{\sigma}_h^{J+\frac{1}{2}}\|_a^2+\frac{1}{2}\|\Delta_t\mathbf{u}_h^{J+\frac{1}{2}}\|_0^2\le\Delta t\sum_{n=1}^{J}\|\mathbf{f}^{n;\frac{1}{4}}\|_0^2+\Delta t\sum_{n=0}^{J-1}\|\Delta_t\mathbf{u}_h^{n+\frac{1}{2}}\|_0^2+\|\Delta_t\mathbf{u}_h^{\frac{1}{2}}\|_0^2+\|\boldsymbol{\sigma}_h^{\frac{1}{2}}\|_a^2,$$

which, together with the discrete Gronwall's inequality (2.3), yields

$$\|\boldsymbol{\sigma}_h^{J+\frac{1}{2}}\|_a^2+\|\Delta_t\mathbf{u}_h^{J+\frac{1}{2}}\|_0^2\le2\Big[\|\Delta_t\mathbf{u}_h^{\frac{1}{2}}\|_0^2+\|\boldsymbol{\sigma}_h^{\frac{1}{2}}\|_a^2+T\Big(\max_{t\in[0,T]}\|\mathbf{f}(t)\|_0\Big)^2\Big]\times\exp(2T).\tag{4.14}$$

On the other hand, from Lemma 3.2, (2.5) and the second equation of (4.2) it follows

$$|\mathbf{u}_h^{J+\frac{1}{2}}|_1\lesssim\sup_{0\neq\boldsymbol{\tau}_h\in\Sigma_h}\frac{\int_\Omega\boldsymbol{\tau}_h:\epsilon(\mathbf{u}_h^{J+\frac{1}{2}})d\mathbf{x}}{\|\boldsymbol{\tau}_h\|_0}=\sup_{0\neq\boldsymbol{\tau}_h\in\Sigma_h}\frac{a(\boldsymbol{\sigma}_h^{J+\frac{1}{2}},\boldsymbol{\tau}_h)}{\|\boldsymbol{\tau}_h\|_0}\lesssim\|\boldsymbol{\sigma}_h^{J+\frac{1}{2}}\|_a,$$

which, together with (4.14), implies the desired result (4.13). □

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4.3 Error analysis

Lemma 4.3. Let $(\boldsymbol{\sigma}_h^n,\mathbf{u}_h^n)$ and $(\widehat{\boldsymbol{\sigma}_h},\widehat{\mathbf{u}_h})$ be respectively the solutions to the fully discrete scheme (4.2) and the discrete "elliptic projection" problem (3.6), then it holds

$$\begin{aligned}&\max_{0\le n\le M-1}\|\boldsymbol{\sigma}_h^{n+\frac{1}{2}}-\widehat{\boldsymbol{\sigma}_h}^{n+\frac{1}{2}}\|_a+\max_{0\le n\le M-1}|\mathbf{u}_h^{n+\frac{1}{2}}-\widehat{\mathbf{u}_h}^{n+\frac{1}{2}}|_1+\max_{0\le n\le M-1}\|\Delta_t\mathbf{u}_h^{n+\frac{1}{2}}-\Delta_t\widehat{\mathbf{u}_h}^{n+\frac{1}{2}}\|_0\\ \lesssim&\|\boldsymbol{\sigma}_h^{\frac{1}{2}}-\widehat{\boldsymbol{\sigma}}_h^{\frac{1}{2}}\|_a+\|\Delta_t\mathbf{u}_h^{\frac{1}{2}}-\Delta_t\widehat{\mathbf{u}}_h^{\frac{1}{2}}\|_0+\Delta t\sum_{j=1}^{M-1}\|(\mathbf{u}_{tt})^{j;\frac{1}{4}}-\Delta_t^2\widehat{\mathbf{u}_h}(t_j)\|_0.\end{aligned}\tag{4.15}$$

Proof. Setting $\boldsymbol{\psi}_h^{n+1}=\boldsymbol{\sigma}_h^{n+1}-\widehat{\boldsymbol{\sigma}_h}(t_{n+1})$, $\mathbf{r}_h^{n+1}=\mathbf{u}_h^{n+1}-\widehat{\mathbf{u}_h}(t_{n+1})$ and combining (3.6) and (4.2), we have

$$\begin{cases}a(\boldsymbol{\psi}_h^{n+1},\boldsymbol{\tau}_h)+b(\boldsymbol{\tau}_h,\mathbf{r}_h^{n+1})=0, & \forall\boldsymbol{\tau}_h\in\Sigma_h,\quad n\ge-1,\\ b(\boldsymbol{\psi}_h^{n;\frac{1}{4}},\mathbf{v}_h)=(\Delta_t^2\mathbf{u}_h^n-(\mathbf{u}_{tt})^{n;\frac{1}{4}},\mathbf{v}_h), & \forall\mathbf{v}_h\in V_h,\quad n\ge1.\end{cases}\tag{4.16}$$

Apply the difference operator Δ_t to the first equation of (4.16) and take $\boldsymbol{\tau}_h=\boldsymbol{\psi}_h^{n;\frac{1}{4}}$, we then obtain

$$a(\Delta_t\boldsymbol{\psi}_h^n,\boldsymbol{\psi}_h^{n;\frac{1}{4}})+b(\boldsymbol{\psi}_h^{n;\frac{1}{4}},\Delta_t\mathbf{r}_h^n)=0,\quad n\ge1.\tag{4.17}$$

Taking $\mathbf{v}_h=\Delta_t\mathbf{r}_h^n$ in the second equation of (4.16), we get

$$b(\boldsymbol{\psi}_h^{n;\frac{1}{4}},\Delta_t\mathbf{r}_h^n)=(\Delta_t^2\mathbf{u}_h^n-(\mathbf{u}_{tt})^{n;\frac{1}{4}},\Delta_t\mathbf{r}_h^n),\quad n\ge1.\tag{4.18}$$

Subtracting (4.18) from (4.17), we arrive at

$$\begin{aligned}a(\Delta_t\boldsymbol{\psi}_h^n,\boldsymbol{\psi}_h^{n;\frac{1}{4}})=&-(\Delta_t^2\mathbf{u}_h^n-(\mathbf{u}_{tt})^{n;\frac{1}{4}},\Delta_t\mathbf{r}_h^n)\\ =&-(\Delta_t^2\mathbf{r}_h^n,\Delta_t\mathbf{r}_h^n)+((\mathbf{u}_{tt})^{n;\frac{1}{4}}-\Delta_t^2\widehat{\mathbf{u}_h}^n,\Delta_t\mathbf{r}_h^n),\end{aligned}$$

which indicates

$$\begin{aligned}&\Big(\frac{\boldsymbol{\psi}_h^{n+\frac{1}{2}}-\boldsymbol{\psi}_h^{n-\frac{1}{2}}}{\Delta t},\mathbb{C}^{-1}\frac{\boldsymbol{\psi}_h^{n+\frac{1}{2}}+\boldsymbol{\psi}_h^{n-\frac{1}{2}}}{2}\Big)+\Big(\frac{\Delta_t\mathbf{r}_h^{n+\frac{1}{2}}-\Delta_t\mathbf{r}_h^{n-\frac{1}{2}}}{\Delta t},\frac{\Delta_t\mathbf{r}_h^{n+\frac{1}{2}}+\Delta_t\mathbf{r}_h^{n-\frac{1}{2}}}{2}\Big)\\ =&\Big((\mathbf{u}_{tt})^{n;\frac{1}{4}}-\Delta_t^2\widehat{\mathbf{u}_h}^n,\frac{\Delta_t\mathbf{r}_h^{n+\frac{1}{2}}+\Delta_t\mathbf{r}_h^{n-\frac{1}{2}}}{2}\Big).\end{aligned}$$

This equality shows

$$\begin{aligned}&\frac{1}{2\Delta t}(\|\boldsymbol{\psi}_h^{n+\frac{1}{2}}\|_a^2-\|\boldsymbol{\psi}_h^{n-\frac{1}{2}}\|_a^2)+\frac{1}{2\Delta t}(\|\Delta_t\mathbf{r}_h^{n+\frac{1}{2}}\|_0^2-\|\Delta_t\mathbf{r}_h^{n-\frac{1}{2}}\|_0^2)\\ =&\Big((\mathbf{u}_{tt})^{n;\frac{1}{4}}-\Delta_t^2\widehat{\mathbf{u}_h}^n,\frac{\Delta_t\mathbf{r}_h^{n+\frac{1}{2}}+\Delta_t\mathbf{r}_h^{n-\frac{1}{2}}}{2}\Big).\end{aligned}\tag{4.19}$$

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Multiplying (4.19) by $2\Delta t$, replacing n by j and summing these equations from $j=1$ to n, we have

$$\begin{aligned}
&\|\boldsymbol{\psi}_h^{n+\frac{1}{2}}\|_a^2+\|\Delta_t\mathbf{r}_h^{n+\frac{1}{2}}\|_0^2=\|\boldsymbol{\psi}_h^{\frac{1}{2}}\|_a^2+\|\Delta_t\mathbf{r}_h^{\frac{1}{2}}\|_0+\Delta t\sum_{j=1}^{n}((\mathbf{u}_{tt})^{j;\frac{1}{4}}-\Delta_t^2\widehat{\mathbf{u}_h}(t_j),\Delta_t\mathbf{r}_h^{j+\frac{1}{2}}+\Delta_t\mathbf{r}_h^{j-\frac{1}{2}})\\
\leq&\|\boldsymbol{\psi}_h^{\frac{1}{2}}\|_a^2+\|\Delta_t\mathbf{r}_h^{\frac{1}{2}}\|_0^2+\Delta t\sum_{j=1}^{n}\|(\mathbf{u}_{tt})^{j;\frac{1}{4}}-\Delta_t^2\widehat{\mathbf{u}_h}(t_j)\|_0\|\Delta_t\mathbf{r}_h^{j+\frac{1}{2}}+\Delta_t\mathbf{r}_h^{j-\frac{1}{2}})\|_0\\
\leq&\|\boldsymbol{\psi}_h^{\frac{1}{2}}\|_a^2+\|\Delta_t\mathbf{r}_h^{\frac{1}{2}}\|_0^2+2\Delta t\max_{0\leq i\leq M-1}\|\Delta_t\mathbf{r}_h^{i+\frac{1}{2}}\|_0\sum_{j=1}^{M-1}\|(\mathbf{u}_{tt})^{j;\frac{1}{4}}-\Delta_t^2\widehat{\mathbf{u}_h}(t_j)\|_0\\
\leq&\|\boldsymbol{\psi}_h^{\frac{1}{2}}\|_a^2+\|\Delta_t\mathbf{r}_h^{\frac{1}{2}}\|_0^2+\frac{1}{2}\max_{0\leq i\leq M-1}\|\Delta_t\mathbf{r}_h^{i+\frac{1}{2}}\|_0^2+2(\Delta t)^2\Big(\sum_{j=1}^{M-1}\|(\mathbf{u}_{tt})^{j;\frac{1}{4}}-\Delta_t^2\widehat{\mathbf{u}_h}(t_j)\|_0\Big)^2. \qquad (4.20)
\end{aligned}$$

Taking the supremum on n on the left-hand side of (4.20), we get

$$\begin{aligned}
&\max_{0\leq n\leq M-1}\|\boldsymbol{\psi}_h^{n+\frac{1}{2}}\|_a^2+\frac{1}{2}\max_{0\leq n\leq M-1}\|\Delta_t\mathbf{r}_h^{n+\frac{1}{2}}\|_0^2\\
\leq&\|\boldsymbol{\psi}_h^{\frac{1}{2}}\|_a^2+\|\Delta_t\mathbf{r}_h^{\frac{1}{2}}\|_0^2+2(\Delta t)^2\Big(\sum_{j=1}^{M-1}\|(\mathbf{u}_{tt})^{j;\frac{1}{4}}-\Delta_t^2\widehat{\mathbf{u}_h}(t_j)\|_0\Big)^2. \qquad (4.21)
\end{aligned}$$

From Lemma 3.2, (2.5) and the first equality of (4.16) it follows

$$|\mathbf{r}_h^{n+\frac{1}{2}}|_1\lesssim\sup_{0\neq\boldsymbol{\tau}_h\in\Sigma_h}\frac{\int_\Omega\boldsymbol{\tau}_h:\epsilon(\mathbf{r}_h^{n+\frac{1}{2}})d\mathbf{x}}{\|\boldsymbol{\tau}_h\|_0}=\sup_{0\neq\boldsymbol{\tau}_h\in\Sigma_h}\frac{a(\boldsymbol{\psi}_h^{n+\frac{1}{2}},\boldsymbol{\tau}_h)}{\|\boldsymbol{\tau}_h\|_0}\lesssim\|\boldsymbol{\psi}_h^{n+\frac{1}{2}}\|_a,$$

which leads to

$$\max_{0\leq n\leq M-1}|\mathbf{r}_h^{n+\frac{1}{2}}|_1\lesssim\max_{0\leq n\leq M-1}\|\boldsymbol{\psi}_h^{n+\frac{1}{2}}\|_a. \qquad (4.22)$$

Combining (4.21) with (4.22) completes the proof. □

The following two lemmas, i.e., Lemmas 4.4-4.5, are to further estimate the right-hand side terms of (4.15) in Lemma 4.3.

Lemma 4.4. Suppose $\sigma_t,\sigma_{ttt}\in L^\infty([0,T];(H^1(\Omega))^{2\times2})$ and $\mathbf{u}_t,\mathbf{u}_{ttt}\in L^\infty([0,T];(H^2(\Omega))^2)$, then, under the same conditions as in Lemma 3.1, the following error estimate holds:

$$\begin{aligned}
&\|\sigma_h^{\frac{1}{2}}-\widehat{\sigma}_h^{\frac{1}{2}}\|_a+\|\Delta_t\mathbf{u}_h^{\frac{1}{2}}-\Delta_t\widehat{\mathbf{u}}_h^{\frac{1}{2}}\|_0\\
\lesssim&h^2(\|\sigma_t\|_{L^\infty(H^1)}+\|\mathbf{u}_t\|_{L^\infty(H^2)})+(\Delta t)^2(\|\sigma_{ttt}\|_{L^\infty(H^1)}+\|\mathbf{u}_{ttt}\|_{L^\infty(H^2)}). \qquad (4.23)
\end{aligned}$$

Proof. Set $\boldsymbol{\psi}_h^n=\sigma_h^n-\widehat{\sigma_h}(t_n)$, $\mathbf{r}_h^n=\mathbf{u}_h^n-\widehat{\mathbf{u}_h}(t_n)$, combine (3.6) and (4.2), then we have

$$\begin{cases}
a(\Delta_t\boldsymbol{\psi}_h^{\frac{1}{2}},\boldsymbol{\tau}_h)+b(\boldsymbol{\tau}_h,\Delta_t\mathbf{r}_h^{\frac{1}{2}})=0, & \forall\boldsymbol{\tau}_h\in\Sigma_h,\\
b(\boldsymbol{\psi}_h^{\frac{1}{2}},\mathbf{v}_h)=\Big(\dfrac{2}{\Delta t}\Delta_t\mathbf{u}_h^{\frac{1}{2}}-\dfrac{2}{\Delta t}\varphi_1-\mathbf{u}_{tt}^{\frac{1}{2}},\mathbf{v}_h\Big), & \forall\mathbf{v}_h\in V_h.
\end{cases} \qquad (4.24)$$

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Taking $\boldsymbol{\tau}_h=\boldsymbol{\psi}_h^{\frac{1}{2}}$, $\mathbf{v}_h=\Delta_t\mathbf{r}_h^{\frac{1}{2}}$, subtracting the second equation of (4.24) from the first, and noting that $\varphi_1=\mathbf{u}_t(0)$, we get

$$\begin{aligned}a(\Delta_t\boldsymbol{\psi}_h^{\frac{1}{2}},\boldsymbol{\psi}_h^{\frac{1}{2}})&=-\Big(\frac{2}{\Delta t}\Delta_t\mathbf{u}_h^{\frac{1}{2}}-\frac{2}{\Delta t}\varphi_1-\mathbf{u}_{tt}^{\frac{1}{2}},\Delta_t\mathbf{r}_h^{\frac{1}{2}}\Big)\\&=-\frac{2}{\Delta t}(\Delta_t\mathbf{u}_h^{\frac{1}{2}}-\Delta_t\widehat{\mathbf{u}}_h^{\frac{1}{2}},\Delta_t\mathbf{r}_h^{\frac{1}{2}})-\Big(\frac{2}{\Delta t}\Delta_t\widehat{\mathbf{u}}_h^{\frac{1}{2}}-\frac{2}{\Delta t}\mathbf{u}_t(0)-\mathbf{u}_{tt}^{\frac{1}{2}},\Delta_t\mathbf{r}_h^{\frac{1}{2}}\Big),\end{aligned}$$

which shows

$$\begin{aligned}&\frac{1}{2\Delta t}(\|\boldsymbol{\psi}_h^1\|_a^2-\|\boldsymbol{\psi}_h^0\|_a^2)+\frac{2}{\Delta t}\|\Delta_t\mathbf{r}_h^{\frac{1}{2}}\|_0^2\\&=\Big(\mathbf{u}_{tt}^{\frac{1}{2}}+\frac{2}{\Delta t}\mathbf{u}_t(0)-\frac{2}{\Delta t}\Delta_t\widehat{\mathbf{u}}_h^{\frac{1}{2}},\Delta_t\mathbf{r}_h^{\frac{1}{2}}\Big)\\&=\Big(\mathbf{u}_{tt}^{\frac{1}{2}}+\frac{2}{\Delta t}\mathbf{u}_t(0)-\frac{2}{\Delta t}\Delta_t\mathbf{u}^{\frac{1}{2}},\Delta_t\mathbf{r}_h^{\frac{1}{2}}\Big)+\Big(\frac{2}{\Delta t}\Delta_t\mathbf{u}^{\frac{1}{2}}-\frac{2}{\Delta t}\Delta_t\widehat{\mathbf{u}}_h^{\frac{1}{2}},\Delta_t\mathbf{r}_h^{\frac{1}{2}}\Big)\\&\le\Big\|\mathbf{u}_{tt}^{\frac{1}{2}}+\frac{2}{\Delta t}\mathbf{u}_t(0)-\frac{2}{\Delta t}\Delta_t\mathbf{u}^{\frac{1}{2}}\Big\|_0\|\Delta_t\mathbf{r}_h^{\frac{1}{2}}\|_0+\frac{2}{\Delta t}\|\Delta_t\mathbf{u}^{\frac{1}{2}}-\Delta_t\widehat{\mathbf{u}}_h^{\frac{1}{2}}\|_0\|\Delta_t\mathbf{r}_h^{\frac{1}{2}}\|_0.\end{aligned}\tag{4.25}$$

Multiplying (4.25) by $2\Delta t$ and noting that $\|\boldsymbol{\psi}_h^0\|_a^2=0$, we easily obtain

$$\begin{aligned}\|\boldsymbol{\psi}_h^1\|_a^2+4\|\Delta_t\mathbf{r}_h^{\frac{1}{2}}\|_0^2&\le2\Delta t\|\mathbf{u}_{tt}^{\frac{1}{2}}+\frac{2}{\Delta t}\mathbf{u}_t(0)-\frac{2}{\Delta t}\Delta_t\mathbf{u}^{\frac{1}{2}}\|_0\|\Delta_t\mathbf{r}_h^{\frac{1}{2}}\|_0+4\|\Delta_t\mathbf{u}^{\frac{1}{2}}-\Delta_t\widehat{\mathbf{u}}_h^{\frac{1}{2}}\|_0\|\Delta_t\mathbf{r}_h^{\frac{1}{2}}\|_0\\&\le(\Delta t)^2\Big\|\mathbf{u}_{tt}^{\frac{1}{2}}+\frac{2}{\Delta t}\mathbf{u}_t(0)-\frac{2}{\Delta t}\Delta_t\mathbf{u}^{\frac{1}{2}}\Big\|_0^2+2\|\Delta_t\mathbf{u}^{\frac{1}{2}}-\Delta_t\widehat{\mathbf{u}}_h^{\frac{1}{2}}\|_0^2+3\|\Delta_t\mathbf{r}_h^{\frac{1}{2}}\|_0^2,\end{aligned}$$

which implies

$$\begin{aligned}&\|\boldsymbol{\psi}_h^1\|_a^2+\|\Delta_t\mathbf{r}_h^{\frac{1}{2}}\|_0^2\\&\le(\Delta t)^2\Big\|\mathbf{u}_{tt}^{\frac{1}{2}}+\frac{2}{\Delta t}\mathbf{u}_t(0)-\frac{2}{\Delta t}\Delta_t\mathbf{u}^{\frac{1}{2}}\Big\|_0^2+2\|\Delta_t\mathbf{u}^{\frac{1}{2}}-\Delta_t\widehat{\mathbf{u}}_h^{\frac{1}{2}}\|_0^2.\end{aligned}\tag{4.26}$$

The thing left is to estimate the right-hand side terms of (4.26). On one hand, by Taylor expansion with integral remainder, we get

$$\mathbf{u}_{tt}^{\frac{1}{2}}+\frac{2}{\Delta t}\mathbf{u}_t(0)-\frac{2}{\Delta t}\Delta_t\mathbf{u}^{\frac{1}{2}}=\frac{1}{2}\int_0^{\Delta t}\mathbf{u}_{ttt}(t)dt-\frac{1}{(\Delta t)^2}\int_0^{\Delta t}(t-\Delta t)^2\mathbf{u}_{ttt}(t)dt,$$

which yields

$$\Big\|\mathbf{u}_{tt}^{\frac{1}{2}}+\frac{2}{\Delta t}\mathbf{u}_t(0)-\frac{2}{\Delta t}\Delta_t\mathbf{u}^{\frac{1}{2}}\Big\|_0\lesssim\Delta t\|\mathbf{u}_{ttt}\|_{L^\infty(L^2)}.\tag{4.27}$$

On the other hand, since

$$\Delta_t\widehat{\mathbf{u}}_h^{\frac{1}{2}}=\Delta_t\Pi_2\mathbf{u}^{\frac{1}{2}}=\Pi_2\Delta_t\mathbf{u}^{\frac{1}{2}},$$

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it holds

$$\begin{aligned}
&\|\Delta_t \mathbf{u}^{\frac{1}{2}} - \Delta_t \widehat{\mathbf{u}}_h^{\frac{1}{2}}\|_0 \\
=&\|(I-\Pi_2)\Delta_t \mathbf{u}^{\frac{1}{2}}\|_0 = \Big\|(I-\Pi_2)\frac{\mathbf{u}(\Delta t)-\mathbf{u}(0)}{\Delta t}\Big\|_0 \\
=&\Big\|(I-\Pi_2)\mathbf{u}_t\Big(\frac{\Delta t}{2}\Big)+\frac{1}{2\Delta t}\int_{\frac{\Delta t}{2}}^{\Delta t}(t-\Delta t)^2(I-\Pi_2)\mathbf{u}_{ttt}(t)dt+\frac{1}{2\Delta t}\int_0^{\frac{\Delta t}{2}} t^2(I-\Pi_2)\mathbf{u}_{ttt}(t)dt\Big\|_0 \\
\leq&\Big\|(I-\Pi_2)\mathbf{u}_t\Big(\frac{\Delta t}{2}\Big)\Big\|_0+\frac{1}{2\Delta t}\int_{\frac{\Delta t}{2}}^{\Delta t}(t-\Delta t)^2\|(I-\Pi_2)\mathbf{u}_{ttt}(t)\|_0 dt+\frac{1}{2\Delta t}\int_0^{\frac{\Delta t}{2}} t^2\|(I-\Pi_2)\mathbf{u}_{ttt}(t)\|_0 dt \\
\lesssim& h^2(\|\boldsymbol{\sigma}_t\|_{L^\infty(H^1)}+\|\mathbf{u}_t\|_{L^\infty(H^2)})+(\Delta t)^2(\|\boldsymbol{\sigma}_{ttt}\|_{L^\infty(H^1)}+\|\mathbf{u}_{ttt}\|_{L^\infty(H^2)}),
\end{aligned} \tag{4.28}$$

which, together with (4.26)-(4.27), complete the proof. □

Lemma 4.5. Suppose $\boldsymbol{\sigma}_{tt},\boldsymbol{\sigma}_{ttt}\in L^\infty([0,T];(H^1(\Omega))^{2\times 2})$ and $\mathbf{u}_{tt},\mathbf{u}_{ttt}\in L^\infty([0,T];(H^2(\Omega))^2)$, $\mathbf{u}_{tttt}\in L^\infty([0,T];(L^2(\Omega))^2)$, then, under the same conditions as in Lemma 3.1, the following error estimate holds:

$$\begin{aligned}
&\Delta t\sum_{j=1}^{M-1}\|(\mathbf{u}_{tt})^{j;\frac{1}{4}}-\Delta_t^2\widehat{\mathbf{u}_h}(t_j)\|_0 \\
\lesssim& h^2(\|\boldsymbol{\sigma}_{tt}\|_{L^\infty(H^1)}+\|\boldsymbol{\sigma}_{ttt}\|_{L^\infty(H^1)}+\|\mathbf{u}_{tt}\|_{L^\infty(H^2)}+\|\mathbf{u}_{ttt}\|_{L^\infty(H^2)})+(\Delta t)^2\|\mathbf{u}_{tttt}\|_{L^\infty(L^2)}.
\end{aligned} \tag{4.29}$$

Proof. By noticing that (3.10)-(3.11) hold, the remaining proof follows from the same lines as in [4]. □

We are now in a position to show the following main error estimates for the proposed fully discrete hybrid stress finite element method.

Theorem 4.2. Let $(\boldsymbol{\sigma}(t),\mathbf{u}(t))$ be the solution to the weak problem (2.4) such that $\boldsymbol{\sigma},\boldsymbol{\sigma}_t,\boldsymbol{\sigma}_{tt},\boldsymbol{\sigma}_{ttt}\in L^\infty([0,T];(H^1(\Omega))^{2\times 2})$, $\boldsymbol{\sigma}_{tt}\in L^\infty([0,T];(L^2(\Omega))^{2\times 2})$, $\mathbf{u}\in L^\infty([0,T];(H^2(\Omega))^2\cap(H_0^1(\Omega))^2)$, $\mathbf{u}_t,\mathbf{u}_{tt},\mathbf{u}_{ttt}\in L^\infty([0,T];(H^2(\Omega))^2)$ and $\mathbf{u}_{tttt}\in L^\infty([0,T];(L^2(\Omega))^2)$. Let $(\boldsymbol{\sigma}_h^n,\mathbf{u}_h^n)$ $(n=0,1,\cdots,M)$ be the solution to the fully discrete scheme (4.2). Then, under the same conditions as in Lemma 3.1, it holds the following error estimates:

$$\begin{aligned}
&\max_{0\leq n\leq M-1}\|\boldsymbol{\sigma}(t_{n+\frac{1}{2}})-\boldsymbol{\sigma}_h^{n+\frac{1}{2}}\|_a+\max_{0\leq n\leq M-1}|\mathbf{u}(t_{n+\frac{1}{2}})-\mathbf{u}_h^{n+\frac{1}{2}}|_1 \\
&\lesssim C_1(\boldsymbol{\sigma},\mathbf{u})h+C_2(\boldsymbol{\sigma},\mathbf{u})(\Delta t)^2,
\end{aligned} \tag{4.30a}$$

$$\max_{0\leq n\leq M-1}\|\mathbf{u}(t_{n+\frac{1}{2}})-\mathbf{u}_h^{n+\frac{1}{2}}\|_0\lesssim C_3(\boldsymbol{\sigma},\mathbf{u})h^2+C_4(\boldsymbol{\sigma},\mathbf{u})(\Delta t)^2, \tag{4.30b}$$

where

$$\begin{aligned}
C_1(\boldsymbol{\sigma},\mathbf{u}):=&\|\boldsymbol{\sigma}\|_{L^\infty(H^1)}+\|\mathbf{u}\|_{L^\infty(H^2)}+\|\boldsymbol{\sigma}_t\|_{L^\infty(H^1)}+\|\mathbf{u}_t\|_{L^\infty(H^2)}+\|\boldsymbol{\sigma}_{tt}\|_{L^\infty(H^1)} \\
&+\|\mathbf{u}_{tt}\|_{L^\infty(H^2)}+\|\boldsymbol{\sigma}_{ttt}\|_{L^\infty(H^1)}+\|\mathbf{u}_{ttt}\|_{L^\infty(H^2)}, \\
C_2(\boldsymbol{\sigma},\mathbf{u}):=&\|\boldsymbol{\sigma}_{tt}\|_{L^\infty(L^2)}+|\mathbf{u}_{tt}|_{L^\infty(H^1)}+\|\boldsymbol{\sigma}_{ttt}\|_{L^\infty(H^1)}+\|\mathbf{u}_{ttt}\|_{L^\infty(H^2)}+\|\mathbf{u}_{tttt}\|_{L^\infty(L^2)},
\end{aligned}$$

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$$C_3(\sigma,\mathbf{u}) := \|\sigma\|_{L^\infty(H^1)} + \|\mathbf{u}\|_{L^\infty(H^2)} + \|\sigma_t\|_{L^\infty(H^1)} + \|\mathbf{u}_t\|_{L^\infty(H^2)} + \|\sigma_{tt}\|_{L^\infty(H^1)} + \|\mathbf{u}_{tt}\|_{L^\infty(H^2)} + \|\sigma_{ttt}\|_{L^\infty(H^1)} + \|\mathbf{u}_{ttt}\|_{L^\infty(H^2)},$$
$$C_4(\sigma,\mathbf{u}) := \|\mathbf{u}_{tt}\|_{L^\infty(L^2)} + \|\sigma_{ttt}\|_{L^\infty(H^1)} + \|\mathbf{u}_{ttt}\|_{L^\infty(H^2)} + \|\mathbf{u}_{tttt}\|_{L^\infty(L^2)}.$$

Proof. By triangle inequality and Taylor expansion, we obtain

$$\begin{aligned}
&\|\sigma(t_{n+\frac{1}{2}}) - \sigma_h^{n+\frac{1}{2}}\|_a + |\mathbf{u}(t_{n+\frac{1}{2}}) - \mathbf{u}_h^{n+\frac{1}{2}}|_1 \\
&\le \|\sigma(t_{n+\frac{1}{2}}) - \sigma^{n+\frac{1}{2}}\|_a + \|\sigma^{n+\frac{1}{2}} - \sigma_h^{n+\frac{1}{2}}\|_a + |\mathbf{u}(t_{n+\frac{1}{2}}) - \mathbf{u}^{n+\frac{1}{2}}|_1 + |\mathbf{u}^{n+\frac{1}{2}} - \mathbf{u}_h^{n+\frac{1}{2}}|_1 \\
&\lesssim (\Delta t)^2(\|\sigma_{tt}\|_{L^\infty(L^2)} + |\mathbf{u}_{tt}|_{L^\infty(H^1)}) + \|\sigma^{n+\frac{1}{2}} - \sigma_h^{n+\frac{1}{2}}\|_a + |\mathbf{u}^{n+\frac{1}{2}} - \mathbf{u}_h^{n+\frac{1}{2}}|_1, \\
&\|\mathbf{u}(t_{n+\frac{1}{2}}) - \mathbf{u}_h^{n+\frac{1}{2}}\|_0 \le \|\mathbf{u}(t_{n+\frac{1}{2}}) - \mathbf{u}^{n+\frac{1}{2}}\|_0 + \|\mathbf{u}^{n+\frac{1}{2}} - \mathbf{u}_h^{n+\frac{1}{2}}\|_0 \\
&\lesssim (\Delta t)^2\|\mathbf{u}_{tt}\|_{L^\infty(L^2)} + \|\mathbf{u}^{n+\frac{1}{2}} - \mathbf{u}_h^{n+\frac{1}{2}}\|_0.
\end{aligned}$$

Then the desired results follow from triangle inequality, poincaré inequality, Lemmas 3.4-3.5 and Lemmas 4.3-4.5. □

Remark 4.1. Under the same conditions as in Theorem 4.2, it also holds

$$\max_{0\le n\le M-1} \|\mathbf{u}_t(t_{n+\frac{1}{2}}) - \Delta_t \mathbf{u}_h^{n+\frac{1}{2}}\|_0 \lesssim C_5(\sigma,\mathbf{u})h^2 + C_6(\sigma,\mathbf{u})(\Delta t)^2, \tag{4.31}$$

where

$$C_5(\sigma,\mathbf{u}) := \|\sigma_t\|_{L^\infty(H^1)} + \|\mathbf{u}_t\|_{L^\infty(H^2)} + \|\sigma_{tt}\|_{L^\infty(H^1)} + \|\mathbf{u}_{tt}\|_{L^\infty(H^2)} + \|\sigma_{ttt}\|_{L^\infty(H^1)} + \|\mathbf{u}_{ttt}\|_{L^\infty(H^2)},$$
$$C_6(\sigma,\mathbf{u}) := \|\sigma_{ttt}\|_{L^\infty(H^1)} + \|\mathbf{u}_{ttt}\|_{L^\infty(H^2)} + \|\mathbf{u}_{tttt}\|_{L^\infty(L^2)}.$$

Theorem 4.3. Under the same conditions as in Theorem 4.2, it holds the estimate

$$\max_{1\le n\le M-1} \|\sigma(t_n) - \sigma_h^{n;\frac{1}{4}}\|_0 \lesssim C_7(\sigma,\mathbf{u})\frac{h^2}{\Delta t} + C_8(\sigma,\mathbf{u})\Delta t, \tag{4.32}$$

where

$$\begin{aligned}
C_7(\sigma,\mathbf{u}) &:= \|\sigma\|_{L^\infty(H^1)} + \|\mathbf{u}\|_{L^\infty(H^2)} + \|\sigma_t\|_{L^\infty(H^1)} + \|\mathbf{u}_t\|_{L^\infty(H^2)} + \|\sigma_{tt}\|_{L^\infty(H^1)} \\
&\quad + \|\mathbf{u}_{tt}\|_{L^\infty(H^2)} + \|\sigma_{ttt}\|_{L^\infty(H^1)} + \|\mathbf{u}_{ttt}\|_{L^\infty(H^2)}, \\
C_8(\sigma,\mathbf{u}) &:= \|\sigma_{tt}\|_{L^\infty(L^2)} + |\mathbf{u}_{tt}|_{L^\infty(H^1)} + \|\sigma_{ttt}\|_{L^\infty(H^1)} + \|\mathbf{u}_{ttt}\|_{L^\infty(H^2)} \\
&\quad + \|\mathbf{u}_{tttt}\|_{L^\infty(L^2)} + \|\mathbf{f}_{tt}\|_{L^\infty(L^2)}.
\end{aligned}$$

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Proof. For any $\gamma_h \in H := \{\gamma_h \in \Sigma_h : b(\gamma_h, \mathbf{v}_h) = (\Delta_t^2 \mathbf{u}_h^n - \mathbf{f}^{n;\frac{1}{4}}, \mathbf{v}_h),\ \forall \mathbf{v}_h \in V_h\}$, it holds

$$\|\boldsymbol{\sigma}(t_n) - \boldsymbol{\sigma}_h^{n;\frac{1}{4}}\|_0 \le \|\boldsymbol{\sigma}(t_n) - \gamma_h\|_0 + \|\gamma_h - \boldsymbol{\sigma}_h^{n;\frac{1}{4}}\|_0. \tag{4.33}$$

Since $\gamma_h - \boldsymbol{\sigma}_h^{n;\frac{1}{4}} \in KerB_h$, from Lemma 3.1 it follows

$$\begin{aligned}
\|\gamma_h - \boldsymbol{\sigma}_h^{n;\frac{1}{4}}\|_0 &\lesssim \sup_{0 \neq \boldsymbol{\tau}_h \in KerB_h} \frac{a(\gamma_h - \boldsymbol{\sigma}_h^{n;\frac{1}{4}}, \boldsymbol{\tau}_h)}{\|\boldsymbol{\tau}_h\|_0} \\
&= \sup_{0 \neq \boldsymbol{\tau}_h \in KerB_h} \frac{a(\gamma_h - \boldsymbol{\sigma}(t_n), \boldsymbol{\tau}_h) + a(\boldsymbol{\sigma}(t_n) - \boldsymbol{\sigma}_h^{n;\frac{1}{4}}, \boldsymbol{\tau}_h)}{\|\boldsymbol{\tau}_h\|_0} \\
&= \sup_{0 \neq \boldsymbol{\tau}_h \in KerB_h} \frac{a(\gamma_h - \boldsymbol{\sigma}(t_n), \boldsymbol{\tau}_h) - b(\boldsymbol{\tau}_h, \mathbf{u}(t_n) - \mathbf{u}_h^{n;\frac{1}{4}})}{\|\boldsymbol{\tau}_h\|_0} \\
&\lesssim \|\boldsymbol{\sigma}(t_n) - \gamma_h\|_0 + |\mathbf{u}(t_n) - \mathbf{u}_h^{n;\frac{1}{4}}|_1.
\end{aligned} \tag{4.34}$$

Now we aim to estimate $\|\boldsymbol{\sigma}(t_n) - \gamma_h\|_0$. By Lemma 3.2, for any $\boldsymbol{\tau}_h \in \Sigma_h$ there exists $\mathbf{w}_h \in \Sigma_h$ such that

$$b(\mathbf{w}_h, \mathbf{v}_h) = b(\boldsymbol{\sigma}(t_n) - \boldsymbol{\tau}_h, \mathbf{v}_h) + (\mathbf{f}(t_n) - \mathbf{f}^{n;\frac{1}{4}}, \mathbf{v}_h) + (\Delta_t^2 \mathbf{u}_h^n - \mathbf{u}_{tt}(t_n), \mathbf{v}_h), \quad \forall \mathbf{v}_h \in V_h, \tag{4.35}$$

with

$$\|\mathbf{w}_h\|_0 \lesssim \|\boldsymbol{\sigma}(t_n) - \boldsymbol{\tau}_h\|_0 + \|\mathbf{f}(t_n) - \mathbf{f}^{n;\frac{1}{4}}\|_0 + \|\mathbf{u}_{tt}(t_n) - \Delta_t^2 \mathbf{u}_h^n\|_0.$$

From (4.35), we also know that $\gamma_h = \boldsymbol{\tau}_h + \mathbf{w}_h \in H$, so

$$\begin{aligned}
\|\boldsymbol{\sigma}(t_n) - \gamma_h\|_0 &= \|\boldsymbol{\sigma}(t_n) - \boldsymbol{\tau}_h - \mathbf{w}_h\|_0 \le \|\boldsymbol{\sigma}(t_n) - \boldsymbol{\tau}_h\|_0 + \|\mathbf{w}_h\|_0 \\
&\lesssim \|\boldsymbol{\sigma}(t_n) - \boldsymbol{\tau}_h\|_0 + \|\mathbf{f}(t_n) - \mathbf{f}^{n;\frac{1}{4}}\|_0 + \|\mathbf{u}_{tt}(t_n) - \Delta_t^2 \mathbf{u}_h^n\|_0.
\end{aligned} \tag{4.36}$$

Noting that $\boldsymbol{\tau}_h$ is arbitrary in Σ_h, we combine (4.33), (4.34) and (4.36) to get

$$\begin{aligned}
\|\boldsymbol{\sigma}(t_n) - \boldsymbol{\sigma}_h^{n;\frac{1}{4}}\|_0 &\lesssim \inf_{\boldsymbol{\tau}_h \in \Sigma_h} \|\boldsymbol{\sigma}(t_n) - \boldsymbol{\tau}_h\|_0 + |\mathbf{u}(t_n) - \mathbf{u}_h^{n;\frac{1}{4}}|_1 + \|\mathbf{f}(t_n) - \mathbf{f}^{n;\frac{1}{4}}\|_0 + \|\mathbf{u}_{tt}(t_n) - \Delta_t^2 \mathbf{u}_h^n\|_0 \\
&\lesssim h\|\boldsymbol{\sigma}\|_{L^\infty(H^1)} + (\Delta t)^2 \|\mathbf{f}_{tt}\|_{L^\infty(L^2)} + |\mathbf{u}(t_n) - \mathbf{u}_h^{n;\frac{1}{4}}|_1 + \|\mathbf{u}_{tt}(t_n) - \Delta_t^2 \mathbf{u}_h^n\|_0.
\end{aligned} \tag{4.37}$$

On the other hand, it is easy to derive

$$\begin{aligned}
|\mathbf{u}(t_n) - \mathbf{u}_h^{n;\frac{1}{4}}|_1 &\le \Big|\mathbf{u}(t_n) - \frac{\mathbf{u}(t_{n-\frac{1}{2}}) + \mathbf{u}(t_{n+\frac{1}{2}})}{2}\Big|_1 + \Big|\frac{\mathbf{u}(t_{n-\frac{1}{2}}) + \mathbf{u}(t_{n+\frac{1}{2}})}{2} - \mathbf{u}_h^{n;\frac{1}{4}}\Big|_1 \\
&\lesssim (\Delta t)^2 |\mathbf{u}_{tt}|_{L^\infty(H^1)} + |\mathbf{u}(t_{n+\frac{1}{2}}) - \mathbf{u}_h^{n+\frac{1}{2}}|_1 + |\mathbf{u}(t_{n-\frac{1}{2}}) - \mathbf{u}_h^{n-\frac{1}{2}}|_1, \\
\|\mathbf{u}_{tt}(t_n) - \Delta_t^2 \mathbf{u}_h^n\|_0 &\le \Big\|\mathbf{u}_{tt}(t_n) - \frac{\mathbf{u}_t(t_{n+\frac{1}{2}}) - \mathbf{u}_t(t_{n-\frac{1}{2}})}{\Delta t}\Big\|_0 + \Big\|\frac{\mathbf{u}_t(t_{n+\frac{1}{2}}) - \mathbf{u}_t(t_{n-\frac{1}{2}})}{\Delta t} - \Delta_t^2 \mathbf{u}_h^n\Big\|_0 \\
&\lesssim (\Delta t)^2 \|\mathbf{u}_{tttt}\|_{L^\infty(L^2)} + \Big\|\frac{\mathbf{u}_t(t_{n+\frac{1}{2}}) - \Delta_t \mathbf{u}_h^{n+\frac{1}{2}}}{\Delta t}\Big\|_0 + \Big\|\frac{\mathbf{u}_t(t_{n-\frac{1}{2}}) - \Delta_t \mathbf{u}_h^{n-\frac{1}{2}}}{\Delta t}\Big\|_0.
\end{aligned}$$

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These two inequalities, together with (4.37), Theorem 4.2 and Remark 4.1, indicate the desired result. □

Remark 4.2. We recall that "$a \lesssim b$" represents "$a \leq Cb$" with the positive constant C independent of mesh parameters h, Δt, and the Lamé parameter λ. Theorems 4.2-4.3 and Remark 4.1 show that the corresponding error estimates are uniform with respect to λ if $C_i(\sigma,\mathbf{u}) \lesssim 1$ for $i=1,2,\cdots,8$.

5 Numerical tests

Two numerical experiments, Examples 5.1-5.2, are used to test the performance of the proposed fully discrete hybrid stress quadrilateral finite element method (4.2). In both examples, the space domain is $\Omega=[0,1]\times[0,1]$ and the time domain is $[0,T]$ with $T=2$. Different meshes of Ω are shown in Fig. 2.

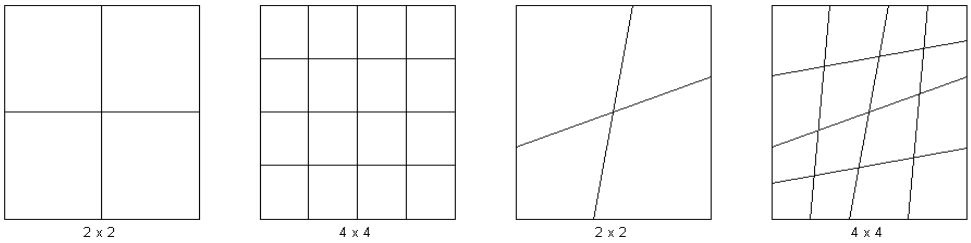


Figure 2: Finite element meshes of the space domain.

Example 5.1 (Plane Stress Test)**.** Consider the problem (1.1) in a plane stress case, where

$$\mu=\frac{E}{2(1+\nu)},\quad \lambda=\frac{E\nu}{(1+\nu)(1-\nu)},$$

with $0<\nu<0.5$ the Poisson ratio and E the Young's modulus. We take $E=1$ and $\nu=0.25$. The initial displacement and velocity are respectively given by

$$\varphi_0=[(1-\nu^2)\sin(\pi x)\sin(\pi y),\nu(1+\nu)\sin(\pi x)\sin(\pi y)],$$
$$\varphi_1=[-(1-\nu^2)\sin(\pi x)\sin(\pi y),-\nu(1+\nu)\sin(\pi x)\sin(\pi y)].$$

The exact displacement solution, $\mathbf{u}=(u_1,u_2)^T$, to this problem is of the form

$$u_1=(1-\nu^2)e^{-t}\sin(\pi x)\sin(\pi y),\quad u_2=\nu(1+\nu)e^{-t}\sin(\pi x)\sin(\pi y).$$

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Table 1: Results of $E^0_{\mathbf{u}}$, $E^1_{\mathbf{u}}$ and E^a_{σ}: Example 5.1 with $\Delta t=0.05$.

	$N\times N$	square mesh	rate	irregular mesh	rate
$E^0_{\mathbf{u}}$	2×2	0.1956	-	0.2483	-
	4×4	0.0479	2.03	0.0592	2.07
	8×8	0.0118	2.02	0.0145	2.03
	16×16	0.0028	2.08	0.0035	2.05
	32×32	0.0006	2.22	0.0007	2.32
$E^1_{\mathbf{u}}$	2×2	0.4435	-	0.5125	-
	4×4	0.2207	1.00	0.2543	1.01
	8×8	0.1105	1.00	0.1266	1.01
	16×16	0.0553	1.00	0.0632	1.00
	32×32	0.0276	1.00	0.0316	1.00
E^a_{σ}	2×2	0.4880	-	0.5256	-
	4×4	0.2510	0.96	0.2687	0.97
	8×8	0.1266	0.99	0.1351	0.99
	16×16	0.0634	1.00	0.0677	1.00
	32×32	0.0317	1.00	0.0339	1.00

We compute the following relative errors for the displacement and stress approximations:

$$E^0_{\mathbf{u}}:=\frac{\max\limits_{0\le n\le M-1}\|\mathbf{u}(t_{n+\frac{1}{2}})-\mathbf{u}_h^{n+\frac{1}{2}}\|_0}{\|\mathbf{u}(t_0)\|_0},$$

$$E^1_{\mathbf{u}}:=\frac{\max\limits_{0\le n\le M-1}|\mathbf{u}(t_{n+\frac{1}{2}})-\mathbf{u}_h^{n+\frac{1}{2}}|_1}{|\mathbf{u}(t_0)|_1},$$

$$E^a_{\sigma}:=\frac{\max\limits_{0\le n\le M-1}\|\sigma(t_{n+\frac{1}{2}})-\sigma_h^{n+\frac{1}{2}}\|_a}{\|\sigma(t_0)\|_a}.$$

Numerical results are listed in Tables 1-2. Table 1 gives results at a fixed time step $\Delta t=0.05$, which show that $E^0_{\mathbf{u}}$ is of $\mathcal{O}(h^2)$-order, and $E^1_{\mathbf{u}}$, E^a_{σ} are both of $\mathcal{O}(h)$-order in the space direction.

Table 2 gives results of $E^0_{\mathbf{u}}$, $E^1_{\mathbf{u}}$ and E^a_{σ} at varying time steps. More precisely, for $E^0_{\mathbf{u}}$, the meshes $(N\times N,M)$ of space and time domains with $M=2N$ ensure $\Delta t=\mathcal{O}(h)$, while for $E^1_{\mathbf{u}}$ and E^a_{σ}, the choices of N and M ensure $\Delta t=\mathcal{O}(h^{1/2})$. Numerical results show that $E^0_{\mathbf{u}}$ is of $\mathcal{O}(h^2+(\Delta t)^2)$-order, and $E^1_{\mathbf{u}}$, E^a_{σ} are both of $\mathcal{O}(h+(\Delta t)^2)$-order.

All the numerical results in Tables 1-2 are conformable to the error estimates in Theorem 4.2.

Example 5.2 (Plane Strain Test)**.** Consider the problem (1.1) in a plane strain case, where

$$\mu=\frac{E}{2(1+\nu)},\quad \lambda=\frac{E\nu}{(1+\nu)(1-2\nu)}.$$

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Table 2: Results of $E^0_{\mathbf{u}}$, $E^1_{\mathbf{u}}$ and E^a_σ: Example 5.1.

	$(N\times N, M)$	square mesh	rate	irregular mesh	rate
$E^0_{\mathbf{u}}$	$(2\times 2, 4)$	0.1503	-	0.1938	-
	$(4\times 4, 8)$	0.0409	1.88	0.0515	1.91
	$(8\times 8, 16)$	0.0106	1.95	0.0132	1.96
	$(16\times 16, 32)$	0.0027	1.97	0.0033	2.00
	$(32\times 32, 64)$	0.0007	1.95	0.0008	2.04
$E^1_{\mathbf{u}}$	$(2\times 2, 10)$	0.4110	-	0.4746	-
	$(4\times 4, 14)$	0.2108	0.96	0.2428	0.97
	$(8\times 8, 20)$	0.1078	0.97	0.1235	0.98
	$(16\times 16, 28)$	0.0547	0.98	0.0626	0.98
	$(32\times 32, 40)$	0.0276	0.99	0.0316	0.99
E^a_σ	$(2\times 2, 10)$	0.4524	-	0.4873	-
	$(4\times 4, 14)$	0.2397	0.92	0.2566	0.93
	$(8\times 8, 20)$	0.1235	0.96	0.1318	0.96
	$(16\times 16, 28)$	0.0628	0.98	0.0670	0.98
	$(32\times 32, 40)$	0.0317	0.99	0.0339	0.98

Table 3: Results of $E^0_{\mathbf{u}}$, $E^1_{\mathbf{u}}$, E^a_σ and E^0_σ: Example 5.2 at square meshes.

	$(N\times N, M)$	$\nu=0.25$	$\nu=0.49$	$\nu=0.499$	$\nu=0.4999$	$\nu=0.49999$
$E^0_{\mathbf{u}}$	$(8\times 8, 16)$	0.0723	0.0837	0.0844	0.0844	0.0844
	$(16\times 16, 32)$	0.0185	0.0215	0.0217	0.0217	0.0217
	$(32\times 32, 64)$	0.0047	0.0054	0.0055	0.0055	0.0055
	rate	1.97	1.98	1.97	1.97	1.97
$E^1_{\mathbf{u}}$	$(8\times 8, 20)$	0.2563	0.2573	0.2574	0.2574	0.2574
	$(16\times 16, 28)$	0.1302	0.1303	0.1303	0.1303	0.1303
	$(32\times 32, 40)$	0.0658	0.0658	0.0658	0.0658	0.0658
	rate	0.98	0.98	0.98	0.98	0.98
E^a_σ	$(8\times 8, 20)$	0.2414	0.2164	0.2150	0.2148	0.2148
	$(16\times 16, 28)$	0.1227	0.1080	0.1071	0.1070	0.1070
	$(32\times 32, 40)$	0.0621	0.0544	0.0539	0.0538	0.0538
	rate	0.98	1.00	1.00	1.00	1.00
E^0_σ	$(8\times 8, 16)$	0.2406	0.2691	0.2713	0.2715	0.2715
	$(16\times 16, 32)$	0.1289	0.1447	0.1459	0.1460	0.1460
	$(32\times 32, 64)$	0.0666	0.0748	0.0754	0.0755	0.0755
	rate	0.93	0.92	0.92	0.92	0.92

We take $E=1$. The initial displacement and velocity are respectively given by

$$\varphi_0=[(x^2-x)^2(4y^3-6y^2+2y),-(y^2-y)^2(4x^3-6x^2+2x)]^T,$$
$$\varphi_1=[-(x^2-x)^2(4y^3-6y^2+2y),(y^2-y)^2(4x^3-6x^2+2x)]^T.$$

The exact displacement solution, $\mathbf{u}=(u_1,u_2)^T$, to this problem is of the form

$$u_1=e^{-t}(x^2-x)^2(4y^3-6y^2+2y),\qquad u_2=-e^{-t}(y^2-y)^2(4x^3-6x^2+2x).$$

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Table 4: Results of $E^0_{\mathbf{u}}$, $E^1_{\mathbf{u}}$, E^a_{σ} and E^0_{σ}: Example 5.2 at irregular meshes.

	$(N\times N, M)$	$\nu=0.25$	$\nu=0.49$	$\nu=0.499$	$\nu=0.4999$	$\nu=0.49999$
$E^0_{\mathbf{u}}$	$(8\times 8, 16)$	0.0814	0.0942	0.0950	0.0951	0.0952
	$(16\times 16, 32)$	0.0209	0.0244	0.0246	0.0246	0.0246
	$(32\times 32, 64)$	0.0053	0.0062	0.0062	0.0062	0.0062
	rate	1.97	1.96	1.97	1.97	1.97
$E^1_{\mathbf{u}}$	$(8\times 8, 20)$	0.2751	0.2761	0.2762	0.2762	0.2763
	$(16\times 16, 28)$	0.1400	0.1401	0.1401	0.1401	0.1401
	$(32\times 32, 40)$	0.0708	0.0708	0.0708	0.0708	0.0708
	rate	0.98	0.98	0.98	0.98	0.98
E^a_{σ}	$(8\times 8, 20)$	0.2562	0.2322	0.2309	0.2309	0.2310
	$(16\times 16, 28)$	0.1303	0.1156	0.1148	0.1147	0.1147
	$(32\times 32, 40)$	0.0659	0.0581	0.0577	0.0576	0.0576
	rate	0.98	1.00	1.00	1.00	1.00
E^0_{σ}	$(8\times 8, 16)$	0.2544	0.2838	0.2935	0.4563	0.7006
	$(16\times 16, 32)$	0.1364	0.1526	0.1539	0.1573	0.1779
	$(32\times 32, 64)$	0.0705	0.0789	0.0796	0.0796	0.0805
	rate	0.93	0.92	0.94	1.26	1.56

Tables 3-Table 4 give results of the relative errors $E^0_{\mathbf{u}}$, $E^1_{\mathbf{u}}$, E^a_{σ} and

$$E^0_{\sigma}:=\frac{\max_{0\le n\le M-1}\|\boldsymbol{\sigma}(t_n)-\boldsymbol{\sigma}_h^{n;\frac{1}{4}}\|_0}{\|\boldsymbol{\sigma}(t_0)\|_0}$$

at different values of Poisson ratio ν.

Similar to Example 5.1, for $E^0_{\mathbf{u}}$ and E^0_{σ}, the meshes $(N\times N,M)$ with $M=2N$ ensure $\Delta t=\mathcal{O}(h)$, while for $E^1_{\mathbf{u}}$ and E^a_{σ}, the choices of N and M ensure $\Delta t=\mathcal{O}(h^{1/2})$. We can see from Tables 3-Table 4 that $E^0_{\mathbf{u}}$, $E^1_{\mathbf{u}}$, E^a_{σ} and E^0_{σ} are respectively of $\mathcal{O}(h^2+(\Delta t)^2)$, $\mathcal{O}(h+(\Delta t)^2)$, $\mathcal{O}(h+(\Delta t)^2)$ and $\mathcal{O}(\frac{h^2}{\Delta t}+\Delta t)$-orders, which are uniform as $\nu\to 0.5$ or $\lambda\to\infty$. This is conformable to the estimates in Theorems 4.2-4.3.

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