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GENERIC AUTOMORPHISMS WITH PRESCRIBED FIXED FIELDS

BIJAN AFSHORDEL

Abstract. This article addresses the question which structures occur as fixed structures of stable structures with a generic automorphism. In particular we give a Galois theoretic characterization. Furthermore, we prove that any pseudofinite field is the fixed field of some model of $ACFA$, any one-free pseudo-differentially closed field of characteristic zero is the fixed field of some model of $DCFA$, and that any one-free PAC field of finite degree of imperfection is the fixed field of some model of $SCFA$.

§1. Introduction. A difference field is a field with a distinguished automorphism. Originating from early work of Birkhoff, Ritt, Cohn, and others in the early 20th century, their model-theoretic properties have been thoroughly studied since the 1990s. A difference field is called generic, and its underlying automorphism is called a generic automorphism, if it is existentially closed in all its difference field extensions. Generic difference fields are sometimes called difference closed. It is well-known that the class of generic difference fields is elementary, its theory is denoted by $ACFA$.

The fixed field of a generic automorphism is a pure pseudofinite field, and given a pseudofinite field k one can fairly easily show that there is some model of $ACFA$ whose fixed field is elementarily equivalent to k.

A natural question is whether, given a pseudofinite field k, is there a model of $ACFA$ whose fixed field actually equals k. Note that in characteristic zero any generic difference field has infinite transcendence degree, while there are many pseudofinite subfields of the algebraic closure of $\mathbb{Q}$.

The same question arises in the context of difference-differential fields of characteristic zero, whose common theory admits a model companion called $DCFA$, and in the context of fields with a finite named p-basis endowed with an automorphism fixing the p-basis, whose common theory also admits a model companion that we denote by $SCFA$.

We give a positive answer to each of the above questions working in a more general context. Consider a countable stable L-theory T with elimination of imaginaries and quantifier elimination. Let σ be a new function symbol and $L_\sigma = L \cup \{\sigma\}$. We denote by T_σ the theory whose models are L_σ-structures (M, σ), where $M \models T$ and $\sigma \in \mathrm{Aut}(M)$ is an automorphism of M. We assume that T_σ has a model companion, which we call TA, and that TA has elimination of imaginaries. Given a model (M, σ)

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of TA we write $\mathrm{Fix}(M,\sigma)$ for the subset of M which is fixed by σ pointwise, which is naturally an L-substructure of M. If A is any subset of some model M of T, we denote by $\mathrm{Gal}(A)$ the group of permutations of the algebraic closure of A in M that leave A pointwise fixed and are elementary in the sense of the L-structure M. It is a profinite group in a natural way and we call it the absolute Galois group of A. If K is an L-structure elementarily equivalent to $\mathrm{Fix}(M,\sigma)$, we can embed K into some model of T and assign to K its absolute Galois group as a subset of that model. By virtue of quantifier elimination, this does not depend on the embedding chosen, so $\mathrm{Gal}(K)$ is an invariant of K, thus we speak of the absolute Galois group of K. In this setting we prove:

THEOREM 1.1. *Let (M,σ) be a model of TA and $K \equiv_L \mathrm{Fix}(M,\sigma)$. Then there is some model (N,σ) of TA with $\mathrm{Fix}(N,\sigma) = K$ if and only if $\mathrm{Gal}(K)$ is the profinite completion $\widehat{\mathbb{Z}}$ of $\mathbb{Z}$.*

We note that the only if direction was already known. In the context of fields we obtain the following corollaries.

COROLLARY 1.2. *Every difference field whose fixed field k is pseudofinite embeds into some model of ACFA with fixed field k.*

COROLLARY 1.3. *Every difference-differential field of characteristic zero whose fixed differential field (k,d) is one-free pseudo-differentially closed embeds into some model of DCFA with fixed differential field (k,d).*

COROLLARY 1.4. *In positive characteristic p, every difference field whose fixed field k is one-free PAC and of finite Ershov invariant embeds into some model of SCFA with fixed field k.*

This work assumes a certain familiarity with the tools and notions from stable and simple theories. Section 2 introduces notation and provides preliminaries about stable structures with a generic automorphism and their fixed structures. In section 3 we introduce the notion of conservative embeddings over subsets. We show that if $K \preccurlyeq_L \mathrm{Fix}(M,\sigma\phi)$ and if K and $\mathrm{Fix}(M,\sigma)$ satisfy a certain condition on definable and algebraic closures, then $\mathrm{Fix}(M,\sigma)$ is conservatively embedded over K in (M,σ) for any model (M,σ) of TA. This is used in section 4 in the proof of the above theorem, where we also discuss some variants. The applications to generic automorphisms of various theories of fields are treated in section 5. The last section is independent from the rest of the article and contains a result concerning elimination of imaginaries.

§2. **Preliminaries.** This section introduces notation and recalls some known facts about stable theories with a generic automorphism. Our main reference is [6].

Let T be a (countable) complete stable theory with infinite models having elimination of quantifiers and imaginaries in the language L. We denote by acl_T and dcl_T the algebraic resp. definable closure in models of T. For a subset $A \subset M$ of a model of T, we write $\mathrm{Gal}(A)$ for the group of elementary permutations of $\mathrm{acl}_T(A)$ that leave A pointwise fixed and call it the Galois group of A. For a tuple $\bar{a}$, possibly infinite, and a parameter set A, we denote by $\mathrm{tp}_T(\bar{a}/A)$ the type of $\bar{a}$ over A and by $\mathrm{qftp}_T(\bar{a}/A)$ its quantifier-free type over A.

We let $L_\sigma = L \cup \{\sigma\}$, where σ is a new unary function symbol and T_σ be the theory whose models are those L_σ-structures (M,σ) with $M \models T$ and $\sigma \in \mathrm{Aut}(M)$.

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By abuse of notation we write σ for the function symbol as well as for the automorphism of (M,σ). We always assume that T_σ has a model companion, which we denote by TA. In particular TA is model complete. We write acl_σ and dcl_σ for the algebraic resp. definable closure in models of TA. If A is a subset of some model of TA, then $\mathrm{cl}_\sigma(A)$ denotes the closure of A under σ and σ^{-1}. Although these closures depend on the ambient model of TA, there will be no ambiguity. Likewise, if $\bar{a}$ is a tuple in some model of TA and A is a parameter set, we denote by $\mathrm{tp}_\sigma(\bar{a}/A)$ the type of $\bar{a}$ over A and by $\mathrm{qftp}_\sigma(\bar{a}/A)$ its quantifier-free type over A.

For any L_σ-structure (A,σ), not necessarily a model of T_σ, we denote by

$$\mathrm{Fix}(A,\sigma) = \{x \in A \mid \sigma(x) = x\},$$

the set of elements of A, which are invariant under σ. Let us observe some basic facts:

REMARK 2.1. *Let (M,σ) be a model of T_σ and ϕ be an automorphism of M that is L-definable without parameters.*

1. *(M,σ) is a model of TA if and only if $(M,\sigma\phi)$ is. Moreover, a subset of M is definable in the L_σ-structure (M,σ) using parameters from $A \subset M$ if and only if it is A-definable in the L_σ-structure $(M,\sigma\phi)$.*
2. *$\mathrm{Fix}(M,\sigma)$ is a $\mathrm{dcl_T}$-closed subset of M. If (M,σ) is a model of TA, then $\mathrm{Fix}(M,\sigma)$ is nonempty, and hence we can view it as an L-substructure of M. The same applies to $\mathrm{Fix}(M,\sigma\phi)$.*

PROOF. We only prove that $\mathrm{Fix}(M,\sigma) \neq \emptyset$ in case (M,σ) is a model of TA and leave the rest to the reader. So let $(M,\sigma) \models TA$. Choose an element $a \notin M$ in some elementary extension of M. Then $\mathrm{tp}_T(a/M)$ is stationary, because T is stable. Therefore $\sigma(\mathrm{tp}_T(a/M)) = \mathrm{tp}_T(a/M)$, whence the map $\sigma \cup \{(a,a)\}$ is elementary in the sense of T, so there is a model (N,σ) of T_σ that extends (M,σ) and contains a with $\sigma(a) = a$. As (M,σ) is existentially closed in (N,σ), the assertion follows. ⊣

So $\mathrm{Fix}(M,\sigma)$ is an L-substructure of M for any model (M,σ) of TA; we will not distinguish between the subset $\mathrm{Fix}(M,\sigma)$ of M and the L-substructure $\mathrm{Fix}(M,\sigma)$ of M, and talk about the fixed structure of (M,σ).

REMARK 2.2. *Let (M,σ) be a model of TA. Every automorphism of the L-structure $\mathrm{Fix}(M,\sigma)$ is elementary in the sense of T.*

PROOF. This is valid even if (M,σ) is only a model of T_σ and follows from quantifier elimination of T. ⊣

REMARK 2.3. *One has $\mathrm{Gal}\big(\mathrm{Fix}(M,\sigma)\big) = \widehat{\mathbb{Z}}$ for any model (M,σ) of TA.*

PROOF. We simply sketch a proof, since the result is well-known. As $\mathrm{Fix}(M,\sigma) = \mathrm{Fix}(\mathrm{acl_T}(\mathrm{Fix}(M,\sigma)),\sigma)$, it follows from Galois theory [14] that $\mathrm{Gal}\big(\mathrm{Fix}(M,\sigma)\big)$ is procyclic. So we only have to show that for any $n < \omega$ there is some tuple a with exactly n conjugates over $\mathrm{Fix}(M,\sigma)$. Let to that end $a_0,\ldots,a_{n-1}$ be the beginning of a Morley sequence over M and define a map τ on $M \cup \{a_0,\ldots,a_{n-1}\}$ to be σ on M, $\tau(a_{n-1}) = a_0$ and $\tau(a_\nu) = a_{\nu+1}$ otherwise. As $\{a_0,\ldots,a_{n-1}\}$ is an indiscernible set it follows that τ is elementary in the sense of T, so as in the proof of Remark 2.1 we conclude that there is $a \in M$ with exactly n conjugates over $\mathrm{Fix}(M,\sigma)$. ⊣

The following fact was proved in [6].

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FACT 2.4. 1. *Let* (M_1, σ_1) *and* (M_2, σ_2) *be models of* TA *containing a common substructure* (A, σ), *and let* A *be* $\mathrm{acl_T}$*-closed. Then* $(M_1, \sigma_1) \equiv_A (M_2, \sigma_2)$. *In particular any bijection between* acl_σ*-closed sets that is elementary in the sense of* T *and commutes with* σ *is elementary in the sense of* TA.
2. *If* A *is a subset of a model of* TA, *then* $\mathrm{acl}_\sigma(A) = \mathrm{acl_T}(\mathrm{cl}_\sigma(A))$.
3. *The theory* TA *is simple. Given subsets* A, B, *and* C *of a model of* TA, *the type* $\mathrm{tp}_\sigma(A/BC)$ *does not fork over* B *if and only if* $\mathrm{acl}_\sigma(A)$ *is independent from* $\mathrm{acl}_\sigma(C)$ *over* $\mathrm{acl}_\sigma(B)$ *in the sense of* T.

Quantifier-free Stability. We will be mostly concerned with fields with a generic automorphism: though TA is simple unstable, its quantifier-free fragment is stable. Recall that a first order theory T' is quantifier-free stable if for all cardinals λ with $\lambda^{|T'|} = \lambda$ and any parameter set A of size λ there are at most λ quantifier-free types over A. Equivalently, every quantifier-free formula has finite Δ-rank for all finite sets Δ of quantifier-free formulae.

LEMMA 2.5. *Every completion of* TA *is quantifier-free stable. Furthermore, if* T *is totally transcendental, then* TA *is quantifier-free totally transcendental.*

PROOF. Let $\sigma \in \mathrm{Aut}_T(\mathbb{C})$ such that $(\mathbb{C}, \sigma) \models TA$ and A be a subset of $(\mathbb{C}, \sigma)$ of size λ with $\lambda^{|T|} = \lambda$. Then $\lambda^\omega = \lambda$. We may assume that A is acl_σ-closed. Then A is also $\mathrm{acl_T}$-closed. Consider a tuple $\bar{a} \in (\bar{M}, \sigma)$. Obviously $\mathrm{qftp}_\sigma(\bar{a}/A)$ is uniquely determined by $\mathrm{qftp}_T(\sigma^i(\bar{a})_{i\in\mathbb{Z}}/A)$ by quantifier elimination of T. As T is stable there are only λ-many (quantifier-free) types over A of sequences of length ω. The second part was proved in [2]. ⊣

In particular local ranks for quantifier-free formulae in TA are defined. Quantifier-free stability of TA has the following consequence, which will play an important role in our proof of Theorem 4.2. Recall that for a theory T' and parameter set A in T', the locally isolated quantifier-free types are said to be dense in (the space of) quantifier-free types over A if for any quantifier-free formula $\varphi(\bar{x})$ in the language of T' there is some quantifier-free type π containing φ such that for any finite set Δ of quantifier-free formulae there is some $\delta(\bar{x}) \in \pi$ (in the language of T') such that $\delta(\bar{x}) \vdash \pi|_\Delta$.

PROPOSITION 2.6. *For every completion of* TA *and every parameter set* A *the locally isolated quantifier-free types are dense in the space of quantifier-free types over* A. *Moreover, if* T *is totally transcendental, then the isolated quantifier-free types are dense in the space of quantifier-free types over* A.

PROOF. In a countable stable theory, the locally isolated types are dense over any parameter set (see for example [16], Theorem 11.8). Even if TA is unstable, its quantifier-free fragment is stable by Lemma 2.5. ⊣

Stable embedding. An L_σ-definable subset P of M^n is stably embedded in M if for any m and any set $X \subset M^{mn}$ which is definable using parameters, the trace $X \cap P^m$ is definable using parameters from P.

Recall the following lemma in [4].

LEMMA 2.7 (Chatzidakis-Hrushovski). *Let* M *be sufficiently saturated and* $P \subset M^n$ *be definable over* $\emptyset$. *Let* P^{ind} *be the induced structure on* P *by* M, *i.e., the* $\emptyset$*-definable subsets of* P^m *are those of the form* $P^m \cap X$ *for* $\emptyset$*-definable* $X \subset M^{mn}$.

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Then P is stably embedded in M if and only if any automorphism of P^{ind} lifts to an automorphism of M.

REMARK 2.8. *If TA eliminates imaginaries, then $\mathrm{Fix}(M,\sigma)$ is stably embedded for all models (M,σ).*

§3. Conservative embedding. In the present section we introduce the notion of conservative embedding over a subset and show that for every model (M,σ) of TA, the fixed structure $\mathrm{Fix}(M,\sigma)$ is conservatively embedded over every L-elementary substructure K such that

$$\mathrm{dcl_T}(\,\mathrm{acl_T}(K), \mathrm{Fix}(M,\sigma)\,) = \mathrm{acl_T}(\mathrm{Fix}(M,\sigma))\ .$$

Moreover, the above condition is necessary (see Theorem 4.2). Recall that T stands for a countable complete stable L-theory with quantifier elimination and elimination of imaginaries. We assume that T_σ has a model companion TA that eliminates imaginaries.

DEFINITION 3.1. *Let P be an L-substructure of the L_σ-structure (M,σ) and A be a subset of P. We say that P is conservatively embedded in (M,σ) over A if every subset of (some cartesian power of) P that is L_σ-definable over A in (M,σ) is L-definable over A in P.*

Note that this is different from the notion of conservative extension of theories.

Let us first show that conservative embeddings are preserved in elementary extensions in the following sense.

LEMMA 3.2. *Let $(M_1,\sigma_1) \preccurlyeq (M_2,\sigma_2)$ be models of T_σ and φ be an L_σ-formula without parameters. Assume that $\varphi(M_1)$ is an L-substructure of M_1 and $A \subset \varphi(M_1)$. Then $\varphi(M_1)$ is conservatively embedded in (M_1,σ_1) over A if and only if $\varphi(M_2)$ is so in (M_2,σ_2).*

PROOF. The proof is straightforward using relativization of quantifiers. ⊣

The above connection between the definable and the algebraic closure is preserved by taking elementary extensions.

PROPOSITION 3.3. *Let $(N,\sigma) \preccurlyeq (M,\sigma)$ be models of T_σ. Then*

$$\mathrm{dcl_T}\big(\,\mathrm{acl_T}(\mathrm{Fix}(N,\sigma))\,,\ \mathrm{Fix}(M,\sigma)\,\big) = \mathrm{acl_T}\big(\,\mathrm{Fix}(M,\sigma)\,\big)\ .$$

PROOF. We denote $F_M = \mathrm{Fix}(M,\sigma)$ and $F_N = \mathrm{Fix}(N,\sigma)$. We will show that $\mathrm{Aut}_T\big(\mathrm{acl_T}(F_M)/\mathrm{acl_T}(F_N), F_M\big)$ is trivial. By Galois theory (see [14]) the automorphism σ (of M) restricts to topological generators of $\mathrm{Gal}(F_M)$ and $\mathrm{Gal}(F_N)$, respectively, because $F_M = \mathrm{Fix}(\mathrm{acl_T}(F_M),\sigma)$ and $F_N = \mathrm{Fix}(\mathrm{acl_T}(F_N),\sigma)$. Hence both $\mathrm{Gal}(F_M)$ and $\mathrm{Gal}(F_N)$ are procyclic and the natural restriction map

$$\mathrm{Gal}(F_M) \xrightarrow{\ \mathrm{res}\ } \mathrm{Gal}(F_N)$$

is surjective. Thus $\mathrm{Aut}_T\big(\mathrm{acl_T}(F_M)/\mathrm{acl_T}(F_N), F_M\big)$ is trivial if and only if res is an isomorphism. Again by Galois theory (see [14]) it suffices to show that for all $n \in \mathbb{N}$ and $\bar{b} \in \mathrm{acl_T}(F_M)$ of degree n over F_M there is $\bar{c} \in \mathrm{acl_T}(F_N)$ of degree n over F_N. To see this, note that as σ generates $\mathrm{Gal}(F_M)$ and $\bar{b}$ has degree n over F_M, we have

$$(M,\sigma) \models\ \exists \bar{x}\ \ \sigma^n(\bar{x}) = \bar{x} \bigwedge_{i=1}^{n-1} \sigma^i(\bar{x}) \neq \bar{x}$$

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and thus

$$(N,\sigma) \models \exists \bar{x} \;\; \sigma^n(\bar{x}) = \bar{x} \bigwedge_{i=1}^{n-1} \sigma^i(\bar{x}) \neq \bar{x} \ ,$$

which yields the desired result. ⊣

The next two lemmata are easy observations.

LEMMA 3.4. *Let (M,σ) be a model of T_σ and $K \preccurlyeq_L \mathrm{Fix}(M,\sigma)$. Then*

$$\mathrm{acl_T}(K) \cap \mathrm{Fix}(M,\sigma) = K \ .$$

PROOF. The proof is straightforward using quantifier elimination of T. ⊣

LEMMA 3.5. *Let A be a set of parameters and $p \in S(\mathrm{acl_T}(A))$, both in the sense of T. If $\beta \in \mathrm{Aut}(\mathbb{C}/A)$ is an automorphism of the monster model of T over A such that $\beta(p) = p$, then $p|_{\mathrm{acl}(A)\cap \mathrm{Fix}(\beta)}$ is stationary.*

PROOF. By elimination of imaginaries, the canonical base c of p is contained in $\mathrm{acl_T}(A)$. As $\beta(p) = p$, it follows that c is also contained in $\mathrm{Fix}(\beta)$. ⊣

LEMMA 3.6. *Let (M,σ) be a sufficiently saturated model of TA and $K \preccurlyeq_L \mathrm{Fix}(M,\sigma)$. If*

$$\mathrm{dcl_T}\big(\mathrm{acl_T}(K), \mathrm{Fix}(M,\sigma)\big) = \mathrm{acl_T}\big(\mathrm{Fix}(M,\sigma)\big) \ ,$$

then the natural restriction map

$$\mathrm{res} : \mathrm{Aut}_{L_\sigma}\big((M,\sigma)/K\big) \longrightarrow \mathrm{Aut}_L\big(\mathrm{Fix}(M,\sigma)/K\big)$$

is surjective. Here $\mathrm{Aut}_L\big(\mathrm{Fix}(M,\sigma)/K\big)$ denotes the group of automorphisms over K of the L-structure $\mathrm{Fix}(M,\sigma)$.

Note that if ϕ is an automorphism of T that is L-definable over $\emptyset$, and if $K \preccurlyeq_L \mathrm{Fix}(M,\sigma\phi)$ with

$$\mathrm{dcl_T}\big(\mathrm{acl_T}(K), \mathrm{Fix}(M,\sigma\phi)\big) = \mathrm{acl_T}\big(\mathrm{Fix}(M,\sigma\phi)\big) \ ,$$

then any $\alpha \in \mathrm{Aut}_L\big(\mathrm{Fix}(M,\sigma\phi)/K\big)$ lifts to an automorphism of $(M,\sigma\phi)$ over K and hence to an automorphism of (M,σ) over K.

PROOF OF LEMMA 3.6. Note first that any automorphism α of (M,σ) restricts to an automorphism of the L-structure $\mathrm{Fix}(M,\sigma)$, because the latter is definable in (M,σ) without parameters.

Now if α is in $\mathrm{Aut}_L\big(\mathrm{Fix}(M,\sigma)/K\big)$, then it is a partial elementary map in the sense of T, by Remark 2.2. It commutes with $\sigma|_{\mathrm{Fix}(M,\sigma)}$, because σ is the identity on $\mathrm{Fix}(M,\sigma)$. We are going to lift α to a permutation $\tilde{\alpha}$ of $\mathrm{acl_T}(\mathrm{Fix}(M,\sigma))$ that is elementary in the sense of T and commutes with σ. As $\mathrm{acl_T}(\mathrm{Fix}(M,\sigma)) = \mathrm{acl}_\sigma(\mathrm{Fix}(M,\sigma))$ by Facts 2.4.2 and 2.4.1 then implies that $\tilde{\alpha}$ is elementary in the sense of TA. In particular α is an automorphism of the induced structure from (M,σ) on $\mathrm{Fix}(M,\sigma)$. By saturation, Lemma 2.7 implies that α lifts to an automorphism of (M,σ) over K.

To lift α we work entirely in the L-structure M. In particular, types, independence and so on refer to the theory T.

Let $\bar{a} \in \mathrm{Fix}(M,\sigma)$ and $\bar{b} \in \mathrm{acl_T}(K)$. As $\mathrm{acl_T}(K) \cap \mathrm{Fix}(M,\sigma) = K$ by Lemma 3.4 and because α leaves K pointwise fixed, it follows from Lemma 3.5 that $\mathrm{tp}_T(\bar{a}/K)$ is stationary. So $\mathrm{tp}_T(\alpha(\bar{a})/K)$ is stationary as well and equals $\mathrm{tp}_T(\bar{a}/K)$.

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As $\bar{b}$ is independent from $\mathrm{Fix}(M,\sigma)$ over K, it follows that $\mathrm{tp}_T(\bar{a}/K\bar{b})$ equals $\mathrm{tp}_T(\alpha(\bar{a})/K\bar{b})$, which in turn implies that $\mathrm{tp}_T(\bar{a},\bar{b}) = \mathrm{tp}_T(\alpha(\bar{a}),\bar{b})$. So the map $\mathrm{id}|_{\mathrm{acl_T}(K)} \cup \alpha$ is elementary in the sense of T. It lifts uniquely to an elementary permutation $\tilde{\alpha}$ of $\mathrm{dcl_T}\big(\mathrm{acl_T}(K), \mathrm{Fix}(M,\sigma)\big)$, which is $\mathrm{acl_T}\big(\mathrm{Fix}(M,\sigma)\big)$ by assumption. Thus $\tilde{\alpha}$ commutes with σ by construction, which finishes the proof. ⊣

PROPOSITION 3.7. *Let (M,σ) be a model of TA and $K \preccurlyeq_L \mathrm{Fix}(M,\sigma\phi)$, where ϕ is an automorphism of T that is L-definable over $\emptyset$. If*

$$\mathrm{dcl_T}\big(\mathrm{acl_T}(K), \mathrm{Fix}(M,\sigma\phi)\big) = \mathrm{acl_T}\big(\mathrm{Fix}(M,\sigma\phi)\big),$$

then $\mathrm{Fix}(M,\sigma\phi)$ is conservatively embedded over K in (M,σ).

PROOF. First note that if (N,σ) is an elementary extension of (M,σ), then $\mathrm{Fix}(M,\sigma\phi)$ is conservatively embedded over K in (M,σ) if and only if $\mathrm{Fix}(N,\sigma\phi)$ is so in (N,σ) by Lemma 3.2. Furthermore it follows from Proposition 3.3 that $\mathrm{dcl_T}\big(\mathrm{acl_T}(K), \mathrm{Fix}(N,\sigma\phi)\big) = \mathrm{acl_T}\big(\mathrm{Fix}(N,\sigma\phi)\big)$. So we may assume that (M,σ) is sufficiently saturated.

Let X be a subset of (some cartesian power of) $\mathrm{Fix}(M,\sigma\phi)$ that is K-definable in (M,σ). As we have noted above, it follows from Lemma 3.6 that we can lift any automorphism of $\mathrm{Fix}(M,\sigma\phi)$ over K to an automorphism of $(M,\sigma\phi)$ over K, which is an automorphism of (M,σ) over K as well by Remark 2.1.1. This shows that for any tuple $a \in \mathrm{Fix}(M,\sigma\phi)$ the type of a over K in the sense of $\mathrm{Fix}(M,\sigma\phi)$ (relativized to $\sigma\phi(x) = x$) implies modulo the theory of (M,σ) the type of a over K in (M,σ). Hence by compactness, the set X is definable in the L-structure $\mathrm{Fix}(M,\sigma\phi)$ using parameters from K. ⊣

As $\mathrm{Fix}(M,\sigma\phi)$ is conservatively embedded over itself in (M,σ), we obtain the following result. It was proved for $ACFA$ in [5], for separably closed fields with generic automorphism in [3], when T is DCF_0 in [2] and for strongly minimal T in [12].

COROLLARY 3.8. *Let (M,σ) be a model of TA. Then every subset X of (some cartesian power of) $\mathrm{Fix}(M,\sigma\phi)$ definable in (M,σ) using parameters is definable in the L-structure $\mathrm{Fix}(M,\sigma\phi)$ using parameters.*

§4. Prescribed fixed structures.

We prove our main theorem and discuss some variants. Recall that T stands for a countable complete stable L-theory with quantifier elimination and elimination of imaginaries. We assume that T_σ has a model companion TA that eliminates imaginaries.

PROPOSITION 4.1. *Let (M,σ) be a model of TA and A be an acl_σ-closed subset of (M,σ) such that $\mathrm{Fix}(A,\sigma) \preccurlyeq_L \mathrm{Fix}(M,\sigma)$. If $\mathrm{Fix}(M,\sigma)$ is conservatively embedded over $\mathrm{Fix}(A,\sigma)$ in (M,σ), then*

$$\mathrm{acl}_\sigma(A,\bar{a}) \cap \mathrm{Fix}(M,\sigma) = \mathrm{Fix}(A,\sigma)$$

for every tuple $\bar{a} \in M$ whose quantifier-free L_σ-type $\mathrm{qftp}_\sigma(\bar{a}/A)$ over A is locally isolated.

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PROOF. Let $\bar{a}$ be a tuple in M whose quantifier-free L_σ-type over A is locally isolated. We abbreviate $\mathrm{Fix}(A,\sigma)$ by K and show first that

$$\mathrm{dcl}_T(\mathrm{cl}_\sigma(A,\bar{a})) \cap \mathrm{Fix}(M,\sigma) = K\ .$$

Let $b \in \mathrm{dcl}_T(\mathrm{cl}_\sigma(A,\bar{a}))$ be fixed by σ. Then b is L-definable over

$$A, \sigma^{-n}(\bar{a}), \ldots, \sigma^{-1}(\bar{a}), \bar{a}, \sigma(\bar{a}), \ldots, \sigma^n(\bar{a})$$

for some $n \in \mathbb{N}$, and, as $\sigma(b) = b$, applying σ^n we see that b is already L-definable over $A, \bar{a}, \ldots, \sigma^n(\bar{a})$ for some $n \in \mathbb{N}$, since σ is an L_σ-automorphism. So there is some L-formula

$$\psi(\bar{x}_0, \bar{x}_1, \ldots, \bar{x}_n; z, \bar{y})$$

and $\bar{e} \in A$ such that $\psi(\bar{a}, \sigma(\bar{a}), \ldots, \sigma^n(\bar{a}); z, \bar{e})$ defines b in M. As T eliminates quantifiers, we may assume that ψ is quantifier-free. Let

$$\Delta = \{\, \psi(\bar{x}, \sigma(\bar{x}), \ldots, \sigma^n(\bar{x}); z, \bar{y})\ ,\ \sigma(z) = z \,\}$$

and choose an $L_\sigma(A)$-formula $\delta(\bar{x}) \in \mathrm{qftp}_\sigma(\bar{a}/A)$ isolating $\mathrm{qftp}_\sigma(\bar{a}/A)|_\Delta$. Consider the L_σ-formula

$$\Phi(z) \ = \ \exists \bar{x}\ \ \Big(\delta(\bar{x})\ \psi(\bar{x}, \sigma(\bar{x}), \ldots, \sigma^n(\bar{x}); z, \bar{e})\ \sigma(z) = z\Big)$$

with parameters from A and let X be the subset of $\mathrm{Fix}(M,\sigma)$ defined by Φ.
We claim that X is L_σ-definable in (M,σ) over $A \cap \mathrm{Fix}(M,\sigma)$. To see this, note that both A and $\mathrm{Fix}(M,\sigma)$ are dcl_σ-closed. By assumption TA eliminates imaginaries, whence $\mathrm{Fix}(M,\sigma)$ is stably embedded by Remark 2.8, so the canonical parameter of X is in $\mathrm{Fix}(M,\sigma)$. On the other hand the canonical parameter of X is also in A. Thus X is L_σ-definable in (M,σ) over $A \cap \mathrm{Fix}(M,\sigma) = K$.

By assumption $\mathrm{Fix}(M,\sigma)$ is conservatively embedded over K in (M,σ), so it follows that X is L-definable over K in the L-structure $\mathrm{Fix}(M,\sigma)$. It is nonempty, because $b \in X$, hence $\mathrm{Fix}(M,\sigma) \models \exists z\ z \in X$. But $K \preccurlyeq_L \mathrm{Fix}(M,\sigma)$, so $K \models \exists z\ z \in X$, whence there is $\lambda \in X(K)$.

We have shown that the $L_\sigma(A)$-formula

$$\psi(\bar{x}, \sigma(\bar{x}), \ldots, \sigma^n(\bar{x}); \lambda, \bar{e})$$

is consistent with $\delta(\bar{x})$ for some $\lambda \in K$. Thus

$$\psi(\bar{x}, \sigma(\bar{x}), \ldots, \sigma^n(\bar{x}); \lambda, \bar{e}) \in \mathrm{qftp}_\sigma(\bar{a}/A)$$

and therefore $b = \lambda \in K$, because $\psi(\bar{a}, \ldots, \sigma^n(\bar{a}); z, \bar{e})$ has exactly one solution in M. This shows that $\mathrm{dcl}_T(\mathrm{cl}_\sigma(A,\bar{a})) \cap \mathrm{Fix}(M,\sigma) = K$.

Finally we show that every tuple $\bar{a}$ with $\mathrm{dcl}_T(\mathrm{cl}_\sigma(A,\bar{a})) \cap \mathrm{Fix}(M,\sigma) = K$ has the property that

$$\mathrm{acl}_\sigma(A,\bar{a}) \cap \mathrm{Fix}(M,\sigma) = K\ .$$

$\mathrm{acl}_T(\mathrm{dcl}_T(\mathrm{cl}_\sigma(A,\bar{a})))$. Note that for every first order theory with elimination of imaginaries, every definably closed set B and every automorphism α of a sufficiently saturated model mapping B into itself, every element b algebraic over B that is fixed by α, is algebraic over $B \cap \mathrm{Fix}(\alpha)$. To see this, choose a formula ψ isolating b over B. As b is fixed by α, it follows that ψ is invariant under α and thus the canonical parameter for ψ is in $B \cap \mathrm{Fix}(\alpha)$.

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This implies that every $b \in \mathrm{acl}_\sigma(A, \bar{a})$ which is fixed by σ is in the acl_T-closure of

$$\mathrm{dcl}_\mathrm{T}(\mathrm{cl}_\sigma(A, \bar{a})) \cap \mathrm{Fix}(M, \sigma) = K\,.$$

It follows that $b \in \mathrm{acl}_\mathrm{T}(K)$. As A is algebraically closed and contains K, we conclude that $b \in A$ and hence $b \in K$, since $\sigma(b) = b$. ⊣

THEOREM 4.2. *Let T be a countable complete stable L-theory with quantifier elimination and elimination of imaginaries. Assume that TA exists and has elimination of imaginaries. Let (M, σ) be a model of TA and $K \preccurlyeq_L \mathrm{Fix}(M, \sigma)$. Then the following are equivalent*:

(1) *There is some $(N, \sigma) \equiv (M, \sigma)$ with $\mathrm{Fix}(N, \sigma) = K$.*
(2) $\mathrm{Gal}(K) = \widehat{\mathbb{Z}}$.
(3) $\mathrm{dcl}_\mathrm{T}\big(\mathrm{acl}_\mathrm{T}(K), \mathrm{Fix}(M, \sigma)\big) = \mathrm{acl}\big(\mathrm{Fix}(M, \sigma)\big)$.
(4) *$\mathrm{Fix}(M, \sigma)$ is conservatively embedded over K in (M, σ).*

PROOF. (1) implies (2) is Remark 2.3.

To see that (2) implies (3), note that $K = \mathrm{acl}_\mathrm{T}(K) \cap \mathrm{Fix}(M, \sigma)$ by Lemma 3.4, so by Galois theory ([14]) we have $K = \mathrm{Fix}(\mathrm{acl}_\mathrm{T}(K), \sigma|_{\mathrm{acl}_\mathrm{T}(K)})$. Hence the natural restriction map res : $\mathrm{Gal}(\mathrm{Fix}(M, \sigma)) \longrightarrow \mathrm{Gal}(K)$ is surjective. As both Galois groups equal $\widehat{\mathbb{Z}}$, res must be an isomorphism, because any surjective endomorphism of bounded[1] profinite groups is an automorphism. The assertion follows as in the proof of Proposition 3.3.

That (3) implies (4) is the content of Proposition 3.7.

We finally prove that (4) implies (1). In view of Lemma 3.2 we may assume that (M, σ) is sufficiently[2] saturated. We construct (N, σ) with the aid of the following standard chain argument. Starting with $N_0 = \mathrm{acl}_\sigma(K)$, whose set of elements fixed by σ is precisely K by Lemma 3.4, we build an ascending chain $(N_\nu)_{\nu<\omega}$ of acl_σ-closed subsets of (M, σ) with the property that every consistent quantifier-free $L_\sigma(N_\nu)$-formula has a realization in $N_{\nu+1}$ and furthermore that $\mathrm{Fix}(M, \sigma) \cap N_\nu = K$ for all $\nu < \omega$. Then we let $N = \bigcup N_\nu$. By model completeness of TA, any L_σ-formula is equivalent modulo TA to an existential formula. Tarski's Test yields that $(N, \sigma|_N)$ is an elementary substructure of (M, σ) and $\mathrm{Fix}(N, \sigma) = K$ by construction.

The only delicate point is to ensure $\mathrm{Fix}(N_\nu, \sigma) = K$ for all $\nu < \omega$. So let A be an acl_σ-closed subset of M such that $\mathrm{Fix}(A, \sigma) = K$ and $\varphi(\bar{x})$ be a consistent quantifier-free $L_\sigma(A)$-formula. The theory TA is quantifier-free stable by Lemma 2.5 and countable because T is. By Proposition 2.6 and by saturation of (M, σ) we can choose a tuple $\bar{a} \in M$ satisfying φ whose quantifier-free type $\mathrm{qftp}_\sigma(\bar{a}/A)$ over A in the sense of (M, σ) is locally isolated. Then Proposition 4.1 implies that $\mathrm{acl}_\sigma(A, \bar{a}) \cap \mathrm{Fix}(M, \sigma\phi) = \mathrm{Fix}(A, \sigma)$. ⊣

For the applications in section 5, the following corollary will be suitable.

COROLLARY 4.3. *Let (M, σ) be a model of TA and A be an acl_σ-closed subset of M. Denote $K = \mathrm{Fix}(A, \sigma)$. If $\mathrm{Gal}(K) = \widehat{\mathbb{Z}}$ and $K \preccurlyeq_L \mathrm{Fix}(M, \sigma)$, then (A, σ) embeds into a model (N, σ) of TA such that $\mathrm{Fix}(N, \sigma) = K$.*

[1] Recall that a profinite group G is called bounded if for every natural number n there are up to isomorphism only finitely many homomorphic images of order n.

[2] In fact we only need that (M, σ) is $|K|^+$-saturated.

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We now discuss two variants of Theorem 4.2. The first is a strengthening in case T is totally transcendental. Let us examine once more the proof of Theorem 4.2. We use saturation of (M,σ) only in order to realize a locally isolated quantifier-free type containing the formula $\varphi(\bar{x})$. As mentioned earlier in Lemma 2.5, we have that TA is quantifier-free totally transcendental if T is totally transcendental. In this case, Proposition 2.6 allows us to choose a tuple $\bar{a}$ realizing $\varphi(\bar{x})$ whose quantifier-free type is isolated. The proof of Proposition 4.1 yields now that $\mathrm{acl}_\sigma(A\bar{a})\cap\mathrm{Fix}(M,\sigma) = K$, without using that T is countable.[3] We have proved the following theorem.

THEOREM 4.4. *Let T be a complete totally transcendental L-theory with quantifier elimination and elimination of imaginaries. Assume that TA exists and has elimination of imaginaries.*

Let (M,σ) be a model of TA and $K \preccurlyeq_L \mathrm{Fix}(M,\sigma)$. Then the following are equivalent:

(0) *There is some $(N,\sigma) \preccurlyeq (M,\sigma)$ with $\mathrm{Fix}(N,\sigma) = K$.*
(1) *There is some $(N,\sigma) \equiv (M,\sigma)$ with $\mathrm{Fix}(N,\sigma) = K$.*
(2) $\mathrm{Gal}(K) = \widehat{\mathbb{Z}}$.
(3) $\mathrm{dcl}_T\big(\mathrm{acl}_T(K), \mathrm{Fix}(M,\sigma)\big) = \mathrm{acl}_T\big(\mathrm{Fix}(M,\sigma)\big)$.
(4) *$\mathrm{Fix}(M,\sigma)$ is conservatively embedded over K in (M,σ).*

For the second variant of Theorem 4.2, let T and TA be as in that theorem. Let (M,σ) be a model of TA and $A = \mathrm{acl}_\sigma(A)$ be a subset of M. Consider an automorphism ϕ of T which is L-definable over $\emptyset$. Then $(M,\sigma\phi)$ is a model of TA by Remark 2.1.1 and A is acl_σ-closed in $(M,\sigma\phi)$. Furthermore, by the same remark, the quantifier-free type in the sense of (M,σ) of a tuple $\bar{a} \in M$ is locally isolated in (M,σ) if and only if the quantifier-free type in the sense of $(M,\sigma\phi)$ of $\bar{a}$ is locally isolated in $(M,\sigma\phi)$. Hence if $K = \mathrm{Fix}(A,\sigma\phi) \preccurlyeq_L \mathrm{Fix}(M,\sigma\phi)$ and $\mathrm{dcl}_T(\mathrm{acl}_T(K), \mathrm{Fix}(M,\sigma\phi)) = \mathrm{acl}_T(\mathrm{Fix}(M,\sigma\phi))$, then $\mathrm{Fix}(M,\sigma\phi)$ is conservatively embedded over K in $(M,\sigma\phi)$ by Proposition 3.7, and so by Proposition 4.1 we get

$$\mathrm{acl}_\sigma(A,\bar{a}) \cap \mathrm{Fix}(M,\sigma\phi) = \mathrm{Fix}(A,\sigma\phi)\,.$$

The same argument applies if we consider several ϕs at once, so the proof of Theorem 4.2 shows also the following corollary.

COROLLARY 4.5. *Let $(\phi_i)_{i\in I}$ be a family of L-automorphisms of T, with each ϕ_i L-definable over $\emptyset$, and (M,σ) be a model of TA. If A is an acl_σ-closed subset of (M,σ) such that $\mathrm{Fix}(A,\sigma\phi_i) \preccurlyeq_L \mathrm{Fix}(M,\sigma\phi_i)$ and*

$$\mathrm{dcl}_T\big(\mathrm{acl}_T(\mathrm{Fix}(A,\sigma\phi_i)), \mathrm{Fix}(M,\sigma\phi_i)\big) = \mathrm{acl}_T\big(\mathrm{Fix}(M,\sigma\phi_i)\big)$$

for all $i \in I$, then there is a model $(N,\sigma) \equiv (M,\sigma)$ of TA such that for all $i \in I$

$$\mathrm{Fix}(N,\sigma\phi_i) = \mathrm{Fix}(A,\sigma\phi_i)\,.$$

As before, if T is totally transcendental, we can choose (N,σ) to be an elementary submodel of (M,σ).

§5. Fixed fields of generic automorphisms. We now apply the results of the previous section to the case of generic automorphisms of fields. For a field k we denote

[3] The proofs of Lemma 2.5 and Proposition 2.6 remain true for arbitrary cardinality if T is totally transcendental.

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by $\mathrm{abs}(k)$ the absolute part of k, which is the subfield of elements algebraic over the prime field. Frob denotes the Frobenius homomorphism in positive characteristic. Recall that a field is called one-free if its absolute Galois group is the profinite completion $\widehat{\mathbb{Z}}$ of $\mathbb{Z}$. Recall also that a difference field (F, σ) is a field F endowed with an automorphism σ and a difference-differential field (F, d, σ) is a differential field (F, d) endowed with a field automorphism σ that commutes with the derivation d. We call $\mathrm{Fix}(F, d, \sigma)$ the fixed differential field of (F, d, σ), which is the differential subfield of (F, d) whose elements are fixed by σ. Our reference for algebra is [10], and [7] for difference fields. For the model theory of difference fields we refer to [4], and for difference-differential fields to [2].

We will need the following algebraic lemma.

LEMMA 5.1. *Any difference-differential field (F, d, σ) whose fixed differential field (k, d) has procyclic absolute Galois group admits an extension $(F^{\mathrm{sep}}, \bar{d}, \bar{\sigma})$ with $\mathrm{Fix}(F^{\mathrm{sep}}, \bar{d}, \bar{\sigma}) = (k, d)$.*

PROOF. Recall from [11] that the derivation d on F extends uniquely to a derivation $\bar{d}$ on F^{sep}. If $\alpha \in F^{\mathrm{sep}}$ and $p(X) \in F[X]$ is the normalized minimal polynomial of α over F, then $\bar{d}(\alpha) = -\frac{p^d(\alpha)}{p'(\alpha)}$, where p' is the formal derivative of p and p^d is the polynomial obtained by applying d to the coefficients of p. A simple computation shows that any extension of σ to F^{sep} commutes with $\bar{d}$.

Thus we only have to show that there is some extension $\bar{\sigma}$ of σ to F^{sep} such that $\mathrm{Fix}(F^{\mathrm{sep}}, \bar{\sigma}) = \mathrm{Fix}(F, \sigma)$. Let to that end $L = F \cap k^{\mathrm{sep}}$ be the elements of F separably algebraic over k. Since k has procyclic absolute Galois group, any element $x \in L$, which generates the unique (in k^{sep}) algebraic extension of k of degree $[k[x] : k]$, is mapped into L by all automorphisms of k^{sep} over k. Therefore L is invariant under $\mathrm{Gal}(k^{\mathrm{sep}}/k)$. As σ transforms algebraic elements into algebraic ones, it restricts to an automorphism of L with fixed field k. Thus L/k is Galois and $\sigma|_L$ generates $\mathrm{Gal}(L/k)$.

Now $\sigma|_L$ extends to a topological generator τ of $\mathrm{Gal}(k^{\mathrm{sep}}/k)$, because $\mathrm{Gal}(k^{\mathrm{sep}}/k)$ is procyclic and $\mathrm{res} : \mathrm{Gal}(k^{\mathrm{sep}}/k) \to \mathrm{Gal}(L/k)$ is surjective. Of course k^{sep}/L is a Galois extension and thus by choice of L, the fields F and k^{sep} are linearly disjoint over L. So as σ and τ agree on L we can find a (unique) automorphism of Fk^{sep} extending σ and τ. We lift it to an automorphism $\bar{\sigma} \in \mathrm{Aut}(F^{\mathrm{sep}})$ and claim that $\mathrm{Fix}(F^{\mathrm{sep}}, \bar{\sigma}) = \mathrm{Fix}(F, \sigma) = k$. To see this, note first that $\mathrm{Fix}(F^{\mathrm{sep}}, \overline{\sigma}) \subset \mathrm{Fix}(F, \sigma)^{\mathrm{sep}}$. Indeed, let $p \in F[X]$ be the normalized minimal polynomial of some $x \in \mathrm{Fix}(F^{\mathrm{sep}}, \bar{\sigma})$. Because $\bar{\sigma}(x) = x$, one calculates that $p^\sigma(x) = p^\sigma(\bar{\sigma}(x)) = \bar{\sigma}(p(x)) = 0$, where p^σ denotes the polynomial obtained by applying σ to the coefficients of p. This implies $p = p^\sigma$ and thus that all coefficients of p are contained in $\mathrm{Fix}(F, \sigma)$, so $x \in k^{\mathrm{sep}}$. Finally, as τ is a topological generator of $\mathrm{Gal}(k^{\mathrm{sep}}/k)$, it follows that $x \in k$. ⊣

5.1. Pseudofinite fields and models of $ACFA$. Let L be the language of rings and ACF be the theory of algebraically closed fields (of a fixed characteristic), which is strongly minimal and eliminates quantifiers and imaginaries. The model companion $ACFA$ of ACF_σ exists and eliminates imaginaries. Given any pseudofinite field k, there is some model of $ACFA$ whose fixed field is elementarily equivalent to k. For more details and proofs about $ACFA$ we refer to [4], and to [1] for pseudofinite fields.

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THEOREM 5.2. *Any difference field whose fixed field k is pseudofinite embeds into some model of ACFA with fixed field k.*

PROOF. Obviously any difference field can be considered a difference-differential field by adding the trivial derivation. So by Lemma 5.1 we may assume that our difference field (F,σ) is separably closed. Further we may assume that (F,σ) is algebraically closed, because σ lifts uniquely to an automorphism $\bar{\sigma}$ of F^{alg}, and one has $\mathrm{Fix}(F^{\mathrm{alg}},\bar{\sigma}) = \mathrm{Fix}(F,\sigma)^{\mathrm{perf}} = \mathrm{Fix}(F,\sigma)$.

The difference field (F,σ) embeds into some model (Ω,σ) of $ACFA$, and is acl_σ-closed therein. We have $\mathrm{abs}(k) = \mathrm{abs}(\mathrm{Fix}(\Omega,\sigma))$ and thus $k \preccurlyeq \mathrm{Fix}(\Omega,\sigma)$, both fields being pseudofinite. Hence Corollary 4.3 finishes the proof. ⊣

COROLLARY 5.3. *Any pseudofinite field k is isomorphic to the fixed field of some model of ACFA. In positive characteristic, if $\mathrm{abs}(k) = \mathrm{Fix}(\mathbb{F}_p^{\mathrm{alg}}, \sigma \circ \mathrm{Frob}^n)$ for some $\sigma \in \mathrm{Gal}(\mathbb{F}_p)$ and $n \in \mathbb{Z}$, then there is some model (K,σ) of ACFA such that $\mathrm{Fix}(K,\sigma\circ\mathrm{Frob}^n) = k$.*

PROOF. To show the first assertion, apply Theorem 5.2 to the difference field (k,id). For the second let $\mathrm{char}(k) = p > 0$. Again because $\mathrm{Gal}(\mathrm{abs}(k))$ is procyclic and the natural restriction map $\mathrm{res} : \mathrm{Gal}(k) \longrightarrow \mathrm{Gal}(\mathrm{abs}(k))$ is surjective, one has that σ lifts to an automorphism $\bar{\sigma}$ of k^{alg} such that $\bar{\sigma}\circ\mathrm{Frob}^n$ is a topological generator of $\mathrm{Gal}(k)$. By Theorem 5.2 the difference field $(k^{\mathrm{alg}}, \sigma\circ\mathrm{Frob}^n)$ embeds into some model (Ω,τ) of $ACFA$ with $\mathrm{Fix}(\Omega,\tau) = k$, and setting $\sigma = \tau\circ\mathrm{Frob}^{-n}$, we obtain a model (Ω,σ) of $ACFA$ with $\mathrm{Fix}(\Omega,\sigma\circ\mathrm{Frob}^n) = k$. ⊣

Note that different choices of the topological generator of $\mathrm{Gal}(k)$ yield non-elementarily equivalent models of $ACFA$ with fixed field k by applying Theorem 5.2 to the difference field $(k^{\mathrm{alg}},\sigma)$.

As an application of Corollary 4.5 we obtain in positive characteristic the following theorem, which was pointed out to us and proved independently by Chatzidakis (unpublished).

THEOREM 5.4. *Let (K,σ) be an algebraically closed difference field and $\Sigma \subseteq \mathbb{Z}$ such that for all $n \in \Sigma$ the fixed fields $\mathrm{Fix}(K,\sigma\circ\mathrm{Frob}^n)$ are pseudofinite. Then there is some model (Ω,σ) of ACFA such that for all $n \in \Sigma$*

$$\mathrm{Fix}(\Omega,\sigma\circ\mathrm{Frob}^n) = \mathrm{Fix}(K,\sigma\circ\mathrm{Frob}^n)\ .$$

5.2. One-free pseudo-differentially closed fields and models of $DCFA$. Let L be the language of differential fields, which is the ring language together with a unary function symbol d, and let T be the theory DCF_0 of differentially closed fields. The theory DCF_0 is ω-stable and eliminates quantifiers and imaginaries. Hrushovski showed, in an unpublished work, that $DCFA$ exists. Elimination of imaginaries for $DCFA$ was proved in [2]. There it is also shown that if (k,d) is a differential field, then $(k,d) \equiv \mathrm{Fix}(\Omega,d,\sigma)$ for some model $\mathrm{Fix}(\Omega,d,\sigma)$ of $DCFA$ if and only if

1. k is pseudofinite of characteristic zero, and
2. (k,d) satisfies the geometric axioms of differentially closed fields. Namely for every (absolutely irreducible) variety V over k and (abs. irred.) subvariety W of the torsor $\tau(V)$ of V projecting generically onto V, there is $a \in V(k)$ such that $(a,d(a)) \in W$.

We refer to [2] for more details and proofs. We call a differential field pseudo-differentially closed if it is pseudofinite of characteristic 0 and satisfies 2. Note

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that (k, d) is one-free[4] pseudo-differentially closed if and only if it is elementarily equivalent to the fixed differential field of some model of $DCFA$.

THEOREM 5.5. *An difference-differential field whose fixed differential field (k, d) is one-free pseudo-differentially closed embeds into a model (Ω, d, σ) of $DCFA$ such that* $\mathrm{Fix}(\Omega, d, \sigma) = (k, d)$.

PROOF. By Lemma 5.1, since the characteristic is 0, we may assume that our difference-differential field (F, d, σ) is algebraically closed. We embed it into some model (Ω, d, σ) of $DCFA$. Then k and $\mathrm{Fix}(\Omega, d, \sigma)$ are elementarily equivalent as pure fields, and hence by Proposition 5.8 of [13], the differential fields (k, d) and $\mathrm{Fix}(\Omega, d, \sigma)$ are elementarily equivalent, because both are pseudo-differentially closed. As F is algebraically closed in the sense of DCF_0 and σ is an automorphism of (F, d), we see that F is acl_σ-closed. Furthermore k^{alg} is the algebraic closure of k in the sense of DCF_0 and any field automorphism of k^{alg} commutes with the derivation on k^{alg}. Hence $\mathrm{Gal}(k) = \widehat{\mathbb{Z}}$, so Corollary 4.3 yields the desired result. ⊣

COROLLARY 5.6. *Any one-free pseudo-differentially closed field is isomorphic to the fixed differential field of some model of $DCFA$.*

PROOF. Apply Theorem 5.5 to the difference-differential field (k, d, id). ⊣

Note that different choices of the topological generator of the absolute Galois group of k yield nonelementarily equivalent models of $DCFA$ having fixed differential field (k, d).

5.3. One-free PAC fields and models of $SCFA$. We fix a prime number p and a positive natural number e. Let L be the ring language augmented by e constant symbols $b_1, \ldots, b_e$ for a p-basis. Denote by $SCF_{e,b}$ the L-theory of separably closed fields of Ershov invariant e. It is model complete, stable and has elimination of imaginaries. Our reference for separably closed fields is [8]. In [3] it is shown that the model companion of $(SCF_{e,b})_\sigma$ exists, which we denote by $SCFA_e$, and that $SCFA_e$ has elimination of imaginaries. It is clear from the description of the completions of $SCFA_e$ given in [3] that for every one-free PAC field k of degree of imperfection e there is some model of $SCFA_e$ whose fixed field is elementarily equivalent to k. For details and proofs about one-free PAC fields we refer to [9].

THEOREM 5.7. *Any difference field whose fixed field k is one-free PAC and of finite Ershov invariant e embeds into some model (Ω, σ) of $SCFA_e$ such that* $\mathrm{Fix}(\Omega, \sigma) = k$.

Let us make the following simple observations. If (F, σ) is any difference field, and if $\bar{c}$ is a tuple fixed by σ such that there is some p-monomial in $\bar{c}$ which can be expressed as a minimal nontrivial linear combination of other p-monomials in $\bar{c}$ with coefficients in F^p, then these coefficients must also be fixed by σ. So every set of p-independent elements in $\mathrm{Fix}(F, \sigma)$ is p-independent in F, or equivalently the field extension $F/\mathrm{Fix}(F, \sigma)$ is separable. Finally, if (Ω, σ) is a model of $SCFA_e$, the p-basis $\bar{b}$ is fixed by σ, so it is a p-basis of $\mathrm{Fix}(\Omega, \sigma)$.

PROOF. We may assume that our difference field (F, σ) is separably closed by Lemma 5.1. We choose a p-basis $\bar{b} = b_1, \ldots, b_e$ of k. Clearly $\bar{b}$ is also a p-basis of k^{sep}, and we expand k^{sep} to an L-structure. As mentioned above, one has that $\bar{b}$ is p-independent in F.

[4] i.e., $\mathrm{Gal}(k) = \widehat{\mathbb{Z}}$.

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Now Lemma 2.2 of [3] states that if (K,σ) is a difference field with finite p-basis B and $K(\bar{a})_\sigma$ is a finitely generated separable difference field extension of (K,σ), then $K(\bar{a})_\sigma$ embeds into some difference field with p-basis B. This implies that for every finite tuple $\bar{a} \in F$, the difference field $k^{\text{sep}}(\bar{a})_\sigma$ embeds into some separably closed difference field with p-basis $\bar{b}$, or in other words into some model of $(SCF_{e,b})_\sigma$, as $\bar{b}$ is fixed by σ. Thus by compactness (F,σ) embeds into some model of $(SCF_{e,b})_\sigma$, and consequently into some model (Ω,σ) of $SCFA_e$ by model completeness.

By the above remarks, the tuple $\bar{b}$ is a p-basis of $\text{Fix}(\Omega,\sigma)$, whence $\text{Fix}(\Omega,\sigma)/k$ is a separable field extension. As k is algebraically closed in $\text{Fix}(\Omega,\sigma)$, the extension $\text{Fix}(\Omega,\sigma)/k$ is regular, and hence elementary, see [9]. We have $\text{Gal}(k) = \widehat{\mathbb{Z}}$ by assumption and F is algebraically closed in the sense in $SCF_{e,b}$, so we apply Theorem 4.2 to finish the proof. ⊣

COROLLARY 5.8. *Any one-free PAC field with finite Ershov invariant e is isomorphic to the fixed field of some model of $SCFA_e$.*

PROOF. Apply Theorem 5.7 to the difference field (k,id). ⊣

Note again that different choices of the topological generator of $\text{Gal}(k)$ yields nonelementarily equivalent models of $SCFA_e$ with fixed field k.

§6. Elimination of imaginaries. In this section we prove a result concerning elimination of imaginaries for TA.

Every simple theory satisfies the Independence Theorem over models, see [15]. The theory TA satisfies the Independence Theorem also over models of T_σ, see [6]. We say that TA satisfies the Independence Theorem over algebraically closed sets if it satisfies the Independence Theorem over algebraically closed sets of the home sort: Let A be an algebraically closed set of the home sort and B and C be arbitrary sets independent over A. Let further $\bar{a}_1$ and $\bar{a}_2$ be arbitrary tuples having the same type over A such that $\bar{a}_1$ is independent from B over A and $\bar{a}_2$ is independent from C over A. Then there is some tuple $\bar{a}_3$ independent from BC realizing $\text{tp}_\sigma(\bar{a}_1/B) \cup \text{tp}_\sigma(\bar{a}_2/C)$.

PROPOSITION 6.1. *Let T be stable and eliminate imaginaries and let TA satisfy the Independence Theorem over algebraically closed sets. Then TA eliminates imaginaries (i.e., every completion of TA does).*

The result is proved for ACF in [4] and for strongly minimal T with $\text{acl}_T(\emptyset)$ infinite in [6]. We step here along the proof line in [12] where the strongly minimal case is done.

PROOF. Let (M,σ) be a model of TA and e be an imaginary element. We may assume that (M,σ) is sufficiently saturated. To distinguish, we write acl_σ and dcl_σ if we compute the respective closure in the home sort, and $\text{acl}_\sigma^{\text{eq}}$ and $\text{dcl}_\sigma^{\text{eq}}$ if we compute the respective closure in $(M,\sigma)^{\text{eq}}$. Choose a $\emptyset$-definable function f and a tuple $\bar{a} \in M$ such that $e = f(\bar{a})$. Let $\bar{b}$ realize $\text{tp}_\sigma(\bar{a}/e)$ and be independent from $\bar{a}$ over e. Furthermore, let $\bar{c}$ realize $\text{tp}_\sigma(\bar{a}/e)$ and be independent from $\text{acl}_\sigma(\bar{a},\bar{b})$ over e.

Because T is stable and eliminates imaginaries, there is a unique smallest acl_σ-closed subset $A \subset M$ (not M^{eq}) such that $\bar{c}$ is independent from $\text{acl}_\sigma^{eq}(\bar{a},\bar{b})$

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over A. Namely, take for A the acl_T-closure of the canonical base of $p = \mathrm{tp}(\sigma^i(\bar{c})_{i\in\mathbb{Z}}/\mathrm{acl}_\sigma(\bar{a},\bar{b}))$ in the sense of T. Clearly $\sigma(p) = p$, hence A is invariant under σ and σ^{-1} and thus it is acl_σ-closed by Fact 2.4.2. By the definition of independence in TA, see Fact 2.4.3, one sees that A is smallest possible.

By transitivity we have $A \subseteq \mathrm{acl}_\sigma(\bar{a})$ and $A \subseteq \mathrm{acl}_\sigma(\bar{b})$, whence we obtain that $A \subseteq \mathrm{acl}_\sigma(\bar{a}) \cap \mathrm{acl}_\sigma(\bar{b})$. As $\mathrm{acl}^{\mathrm{eq}}_\sigma(\bar{a}) \cap \mathrm{acl}^{\mathrm{eq}}_\sigma(\bar{b}) = \mathrm{acl}^{\mathrm{eq}}_\sigma(e)$ we conclude $A \subseteq \mathrm{acl}^{\mathrm{eq}}_\sigma(e)$. Since $\bar{c}$ is independent from $\mathrm{acl}^{\mathrm{eq}}_\sigma(\bar{a},\bar{b})$ it follows again by transitivity that $\bar{c}$ is independent from e over A, and thus that e is independent from $\bar{c}$ over A. But e is algebraic over $\bar{c}$, so it must be algebraic already over A.

We want to show that e is even definable over A. To that end choose any realization $\bar{a}'$ of $\mathrm{tp}_\sigma(\bar{a}/A)$. There is some $\bar{b}'$ realizing $\mathrm{tp}_\sigma(\bar{a}'/A)$ which is independent from $\bar{a}'$ over A, with $f(\bar{b}') = f(\bar{a}')$ and which is independent from $\bar{b}$ over A. As A is acl_σ-closed we can apply the Independence Theorem over algebraically closed sets to find some realization $\bar{d}$ of $\mathrm{tp}_\sigma(\bar{a}/A\bar{b}) \cup \mathrm{tp}_\sigma(\bar{a}'/A\bar{b}')$. Since $\bar{d}$ realizes $\mathrm{tp}_\sigma(\bar{a}/A\bar{b})$, we see that $f(\bar{d}) = f(\bar{b}) = f(\bar{a})$, and as $\bar{d}$ realizes $\mathrm{tp}_\sigma(\bar{a}'/A\bar{b}')$, one has $f(\bar{d}) = f(\bar{b}') = f(\bar{a}')$. But then $f(\bar{a}) = f(\bar{a}')$, so it follows that e is definable over A, because every A-conjugate e' of e is of the form $e' = f(\bar{a}')$ for some realization $\bar{a}'$ of $\mathrm{tp}_\sigma(\bar{a}/A)$.

What we have shown is that there is some real tuple $\bar{a}'$ with $\bar{a}' \in \mathrm{acl}^{eq}_\sigma(e)$ and $e \in \mathrm{dcl}^{eq}_\sigma(\bar{a}')$. As T eliminates imaginaries, the set of e-conjugates of $\bar{a}'$ is interdefinable with a real tuple, and thus so is e. ⊣

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MATHEMATISCHES INSTITUT
ALBERT-LUDWIGS-UNIVERSITÄT FREIBURG
ECKERSTRAßE 1, D-79104 FREIBURG
GERMANY

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