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A universal asymptotic algorithm for elastic thin shells

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This work presents an asymptotic algorithm for the derivation of equations of thin elastic shells. The algorithm is based on the analysis of a boundary value problem for the Navier system in a thin region. The analysis covers both the membrane theory and the moment theory of elastic shells, including the eigenvalue problems.

1 Introduction

The modern theory of elastic plates and shells is well-developed in the engineering and mathematics literature. We refer to the monographs by Donnell (1976) and Novozhilov (1970), presenting a classical engineering approach to the derivation of lower-dimensional models of thin plates and shells.

In the recent work by Ciarlet (1997), one can find a combination of a variational approach and asymptotic analysis used for problems associated with elastic plates and elastic multi-structures. The asymptotic analysis of fields in multi-structures, based on the method of compound asymptotic expansions, was developed in Kozlov, Maz'ya & Movchan (1999).

For particular examples of engineering structures, appropriate physical assumptions enable one to derive simple lower-dimensional models. The books by Ciarlet (1997) and Kozlov, Maz'ya & Movchan (1999) illustrated alternative mathematical algorithms for the derivation of lower-dimensional models. These algorithms give high-order approximation for solutions in, for example, the theory of degenerate pile structures or the theory of adhesive joints represented by 'imperfect interfaces'.

Substantial work on justification of theories of elastic shells based on a variational approach was published by Berdichevski (1983), Ciarlet & Lods (1996a, 1996b) and Ciarlet, Lods & Miara (1996). The authors investigate static problems of linear elasticity posed in thin curvilinear layers (thin shells) of small thickness $2\varepsilon h$, with ε being a small positive non-dimensional parameter. As $\varepsilon \to 0$, lower-dimensional models incorporating *ad hoc* approximations were derived, and the accuracy of these approximations was estimated. Estimates for the eigenvalues and a spectral analysis for shells were presented by Kesavan & Sabu (2000).

The motivation for the work, presented in this paper, came from the analysis of multi-structures including two and three dimensional limit subsets. Consider an example of an

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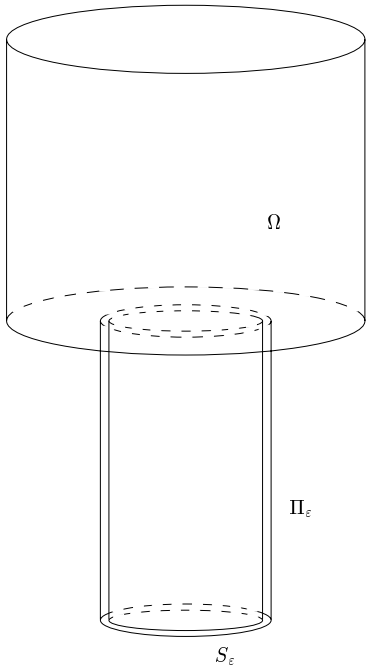


FIGURE 1. 2D–3D multi-structure.

elastic structure shown in Figure 1; let the base region S_ε of the hollow cylinder Π_ε be clamped, and the remaining surface of the structure be free of tractions; ε stands for a small thickness of the cylinder wall. We would like to know a spectrum of eigenfrequencies and a set of corresponding eigenfunctions for the elasticity problem associated with this multi-structure. Which model should we use to describe the cylindrical foundation: hollow rod, membrane model for shells or moment theory of shells? Of course, the approximation depends on the geometry of the structure, elastic parameters and the vibration mode. However, it is hardly possible to use physical intuition to provide a recipe for all possible situations.

The present paper complements the results by Ciarlet & Lods (1996a, 1996b) and Ciarlet, Lods & Miara (1996), who applied a variational method to the analysis of mebrane and flexural theories of elastic shells. Our asymptotic analysis enables one to classify, in the frequency region, the limit models of vibration of elastic shells and to construct high-order asymptotic approximations for the displacement fields.

We study a boundary value problem as well as an eigenvalue problem of linear elasticity in a thin domain corresponding to a shell. The structure of the paper can be described as follows:

- §2 includes the governing equations and the definition of local coordinates.
- §3 presents a formal asymptotic algorithm that leads to the membrane theory of elastic shells. The leading ansatz for the displacement has the form

$$\mathbf{u} \sim \mathbf{u}^{(0)}\left(\alpha, \beta, \frac{\gamma}{\varepsilon}\right) + \varepsilon \mathbf{u}^{(1)}\left(\alpha, \beta, \frac{\gamma}{\varepsilon}\right) + \varepsilon^2 \mathbf{U}\left(\alpha, \beta, \frac{\gamma}{\varepsilon}\right),$$

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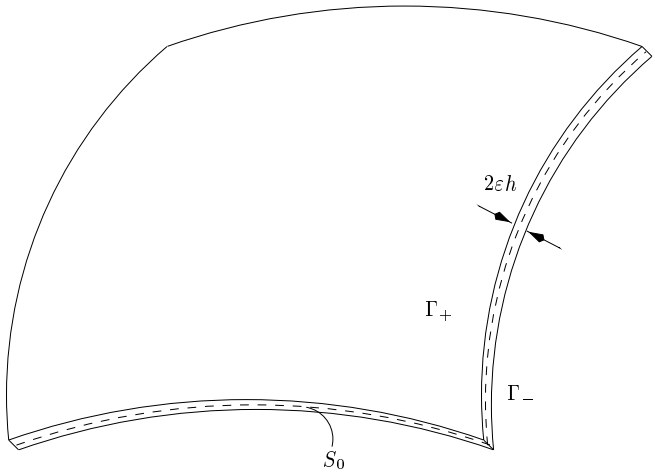


FIGURE 2. The thin shell Ω_ε.

where (α, β, γ) is an orthogonal system of canonical curvilinear coordinates on the limit surface; α, β correspond to the tangential directions, and γ corresponds to the transverse direction.

- §4 deals with the analysis of the eigenvalue problems that correspond to the moment theory of elastic shells. We give an accurate asymptotic derivation which shows that the amplitude of oscillations corresponding to this model has the form

$$\mathbf{u} = \mathrm{diag}\{\sqrt{\varepsilon}, \sqrt{\varepsilon}, 1\}\mathbf{v}\left(\frac{\alpha}{\sqrt{\varepsilon}}, \frac{\beta}{\sqrt{\varepsilon}}, \frac{\gamma}{\varepsilon}; \varepsilon\right).$$

  It is shown that the resulting lower-dimensional system of equations corresponds to Vlasov's theory of elastic shells (see Vlasov (1958) and Aslanyan & Lidskii (1970)).
- §5 includes simple illustrative examples for cylindrical shells.
- Finally, in §6 we remark on generalisation of the asymptotic algorithm to geometrically nonlinear problems and give simple illustrations associated with the von Kármán theory of elastic plates.

We note that for the evaluation of first several eigenfrequencies of the multi-structure (Figure 1) one can replace the three-dimensional solid Ω by a rigid body, use the membrane theory for eigensolutions associated with smooth variation in the space variables, and use Vlasov's moment theory for eigensolutions associated with large gradients (as described in §4) of displacement components. We also emphasise that the approach given in this work is different from the one developed by Ciarlet, Lods and Miara; it works equally well for static problems and for problems of vibration of thin shells and provides the limit equations as well as the higher-order approximation for the displacement field. The analysis presented in §4 for a moment theory (flexural theory) of vibrating shells gives an example which was not covered by the work of Ciarlet, Lods & Miara (1996b) on the flexural shell equations.

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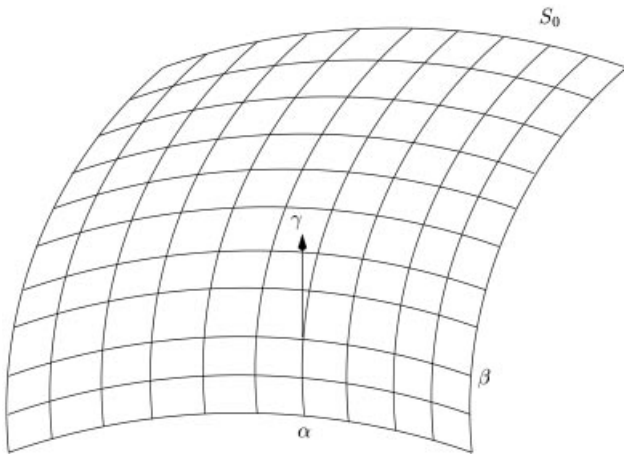


FIGURE 3. The orthogonal system of canonical curvilinear coordinates.

2 Formulation of the problem

Let Ω_ε be a thin set defined by

$$\Omega_\varepsilon = S_0 \times (-\varepsilon h, \varepsilon h), \tag{2.1}$$

where S_0 is a smooth surface; ε is a small positive parameter. In the text below, we normalise the displacement components by h; equivalently we can use $h = 1$ in all the derivations. The upper and lower surfaces of Ω_ε are denoted by Γ_+ and Γ_-, respectively (see Figure 2).

The displacement vector $\mathbf{u}$ satisfies the system of Navier equations

$$\mathbf{Lu} + \mathbf{F} = \mathbf{0} \quad \text{in} \quad \Omega_\varepsilon \tag{2.2}$$

and traction boundary conditions

$$\boldsymbol{\sigma}^{(\mathbf{n})}(\mathbf{u}) = \varepsilon \mathbf{P}^{\pm} \quad \text{on} \quad \Gamma_+ \quad \text{and} \quad \Gamma_-. \tag{2.3}$$

The purpose is to develop a formal asymptotic algorithm and to derive an accurate lower-dimensional model for a thin elastic solid Ω_ε. Here,

$$\mathbf{Lu} = \mu \nabla^2 \mathbf{u} + (\lambda + \mu) \nabla \nabla \cdot \mathbf{u} \tag{2.4}$$

and

$$\boldsymbol{\sigma}^{(\mathbf{n})}(\mathbf{u}) = \sigma(\mathbf{u})\mathbf{n}, \tag{2.5}$$

where $\boldsymbol{\sigma}$ is the stress tensor, and $\mathbf{n}$ is the unit outward normal; λ and μ are the Lamé elastic moduli.

Local coordinates

To characterise the limit surface S_0, we introduce an orthogonal system of canonical curvilinear coordinates (α, β, γ) (see Figure 3). The value $\gamma = 0$ corresponds to points on S_0 and within Ω_ε, $|\gamma| < \varepsilon$. Let $R_1(\alpha, \beta)$ and $R_2(\alpha, \beta)$ denote the principal radii of curvature

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for the surface S_0. Also, assume that $A(\alpha,\beta)$ and $B(\alpha,\beta)$ are the metric coefficients associated with S_0, so that

$$ds^2 = A^2 d\alpha^2 + B^2 d\beta^2,$$

where ds is an elementary length on S_0. Next, we introduce (see Novozhilov, 1970) the metric coefficients H_1, H_2 and H_3 corresponding to the coordinate system (α,β,γ)

$$H_1 = A\Big(1+\frac{\gamma}{R_1}\Big), \qquad H_2 = B\Big(1+\frac{\gamma}{R_2}\Big), \qquad H_3 = 1. \tag{2.6}$$

Also, we shall use a scaled coordinate

$$\rho = \frac{\gamma}{\varepsilon}, \qquad |\rho| < 1. \tag{2.7}$$

Governing equations in curvilinear coordinates with scaling

The system (2.2) represented in coordinates (α,β,γ) has the form

$$\begin{aligned}
&\frac{1}{\varepsilon^2}\Big\{\frac{1}{H_2}\frac{\partial}{\partial\rho}\Big[\frac{H_2}{H_1}\frac{\partial}{\partial\rho}(H_1u_\alpha)\Big]\Big\}\\
&+\frac{1}{\varepsilon}\Big\{\frac{2(1-\nu)}{1-2\nu}\frac{1}{H_1}\frac{\partial}{\partial\alpha}\Big[\frac{1}{H_1H_2}\frac{\partial}{\partial\rho}(H_1H_2u_\gamma)\Big]-\frac{1}{H_2}\frac{\partial}{\partial\rho}\Big(\frac{H_2}{H_1}\frac{\partial u_\gamma}{\partial\alpha}\Big)\Big\}\\
&+\frac{2(1-\nu)}{1-2\nu}\frac{1}{H_1}\frac{\partial}{\partial\alpha}\Big[\frac{1}{H_1H_2}\Big(\frac{\partial}{\partial\alpha}(H_2u_\alpha)+\frac{\partial}{\partial\beta}(H_1u_\beta)\Big)\Big]\\
&-\frac{1}{H_2}\frac{\partial}{\partial\beta}\Big[\frac{1}{H_1H_2}\Big(\frac{\partial(H_2u_\beta)}{\partial\alpha}-\frac{\partial(H_1u_\alpha)}{\partial\beta}\Big)\Big]+\frac{F_\alpha}{\mu}=0.
\end{aligned} \tag{2.8}$$

The second equation of the system (2.2) has the same form as (2.8) and is obtained by a circular permutation of indices (α,β):

$$\begin{aligned}
&\frac{1}{\varepsilon^2}\Big\{\frac{1}{H_1}\frac{\partial}{\partial\rho}\Big[\frac{H_1}{H_2}\frac{\partial}{\partial\rho}(H_2u_\beta)\Big]\Big\}\\
&+\frac{1}{\varepsilon}\Big\{\frac{2(1-\nu)}{1-2\nu}\frac{1}{H_2}\frac{\partial}{\partial\beta}\Big[\frac{1}{H_1H_2}\frac{\partial}{\partial\rho}(H_1H_2u_\gamma)\Big]-\frac{1}{H_1}\frac{\partial}{\partial\rho}\Big(\frac{H_1}{H_2}\frac{\partial u_\gamma}{\partial\beta}\Big)\Big\}\\
&+\frac{2(1-\nu)}{1-2\nu}\frac{1}{H_2}\frac{\partial}{\partial\beta}\Big[\frac{1}{H_1H_2}\Big(\frac{\partial}{\partial\beta}(H_1u_\beta)+\frac{\partial}{\partial\alpha}(H_2u_\alpha)\Big)\Big]\\
&-\frac{1}{H_1}\frac{\partial}{\partial\alpha}\Big[\frac{1}{H_1H_2}\Big(\frac{\partial(H_1u_\alpha)}{\partial\beta}-\frac{\partial(H_2u_\beta)}{\partial\alpha}\Big)\Big]+\frac{F_\beta}{\mu}=0.
\end{aligned} \tag{2.9}$$

Finally, the third equation of the system (2.2) is given by

$$\begin{aligned}
&\frac{1}{\varepsilon^2}\Big\{\frac{2(1-\nu)}{1-2\nu}\frac{\partial}{\partial\rho}\Big[\frac{1}{H_1H_2}\frac{\partial}{\partial\rho}(H_1H_2u_\gamma)\Big]\Big\}\\
&+\frac{1}{\varepsilon}\Big\{\frac{2(1-\nu)}{1-2\nu}\frac{\partial}{\partial\rho}\Big[\frac{1}{H_1H_2}\Big(\frac{\partial}{\partial\alpha}(H_2u_\alpha)+\frac{\partial}{\partial\beta}(H_1u_\beta)\Big)\Big]\\
&-\frac{1}{H_1H_2}\frac{\partial}{\partial\alpha}\Big[\frac{H_2}{H_1}\frac{\partial(H_1u_\alpha)}{\partial\rho}\Big]-\frac{1}{H_1H_2}\frac{\partial}{\partial\beta}\Big[\frac{H_1}{H_2}\frac{\partial(H_2u_\beta)}{\partial\rho}\Big]\Big\}\\
&+\frac{1}{H_1H_2}\frac{\partial}{\partial\alpha}\Big[\frac{H_2}{H_1}\frac{\partial u_\gamma}{\partial\alpha}\Big]+\frac{1}{H_1H_2}\frac{\partial}{\partial\beta}\Big[\frac{H_1}{H_2}\frac{\partial u_\gamma}{\partial\beta}\Big]+\frac{F_\gamma}{\mu}=0.
\end{aligned} \tag{2.10}$$

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The traction boundary conditions (2.3) can be rewritten as follows:

$$\frac{1}{\varepsilon}\Big[\frac{\partial u_\alpha}{\partial \rho} - \frac{u_\alpha}{H_1}\frac{\partial H_1}{\partial \rho}\Big] + \frac{1}{H_1}\frac{\partial u_\gamma}{\partial \alpha} = \pm\varepsilon\frac{P_\alpha^\pm}{\mu}, \quad \rho = \pm 1, \tag{2.11}$$

$$\frac{1}{\varepsilon}\Big[\frac{\partial u_\beta}{\partial \rho} - \frac{u_\beta}{H_2}\frac{\partial H_2}{\partial \rho}\Big] + \frac{1}{H_2}\frac{\partial u_\gamma}{\partial \beta} = \pm\varepsilon\frac{P_\beta^\pm}{\mu}, \quad \rho = \pm 1, \tag{2.12}$$

and

$$\begin{aligned}&\frac{1}{\varepsilon}\Big\{\frac{v}{1-2v}\frac{1}{H_1H_2}\frac{\partial}{\partial \rho}(H_1H_2u_\gamma) + \frac{\partial u_\gamma}{\partial \rho}\Big\}\\ &+\frac{v}{1-2v}\Big\{\frac{1}{H_1H_2}\Big[\frac{\partial}{\partial \alpha}(H_2u_\alpha) + \frac{\partial}{\partial \beta}(H_1u_\beta)\Big]\Big\} = \pm\varepsilon\frac{P_\gamma^\pm}{2\mu},\ \rho = \pm 1.\end{aligned} \tag{2.13}$$

In terms of α, β and ρ, the metric coefficients H_1 and H_2 are given by

$$\begin{aligned}H_1 &= A(\alpha,\beta)\Big(1 + \frac{\varepsilon\rho}{R_1(\alpha,\beta)}\Big),\\ H_2 &= B(\alpha,\beta)\Big(1 + \frac{\varepsilon\rho}{R_2(\alpha,\beta)}\Big).\end{aligned} \tag{2.14}$$

The left-hand sides of equations (2.8)–(2.10) and boundary conditions (2.11)–(2.13) can be expanded in powers of ε with the use of (2.14).

As a result, one has

$$\frac{\partial^2 u_\alpha}{\partial \rho^2} + \varepsilon l_1^{(\alpha)}(\mathbf{u}) + \varepsilon^2 l_2^{(\alpha)}(\mathbf{u}) + \cdots = -\varepsilon^2\frac{F_\alpha}{\mu}, \tag{2.15}$$

$$\frac{\partial^2 u_\beta}{\partial \rho^2} + \varepsilon l_1^{(\beta)}(\mathbf{u}) + \varepsilon^2 l_2^{(\beta)}(\mathbf{u}) + \cdots = -\varepsilon^2\frac{F_\beta}{\mu}, \tag{2.16}$$

and

$$\frac{2(1-v)}{1-2v}\frac{\partial^2 u_\gamma}{\partial \rho^2} + \varepsilon l_1^{(\gamma)}(\mathbf{u}) + \varepsilon^2 l_2^{(\gamma)}(\mathbf{u}) + \cdots = -\varepsilon^2\frac{F_\gamma}{\mu}, \tag{2.17}$$

as $|\rho| < 1$. Further, for the boundary conditions

$$\frac{\partial u_\alpha}{\partial \rho} + \varepsilon m_1^{(\alpha)}(\mathbf{u}) + \varepsilon^2 m_2^{(\alpha)}(\mathbf{u}) + \cdots = \pm\varepsilon^2\frac{P_\alpha^\pm}{\mu}, \tag{2.18}$$

$$\frac{\partial u_\beta}{\partial \rho} + \varepsilon m_1^{(\beta)}(\mathbf{u}) + \varepsilon^2 m_2^{(\beta)}(\mathbf{u}) + \cdots = \pm\varepsilon^2\frac{P_\beta^\pm}{\mu}, \tag{2.19}$$

and

$$\frac{1-v}{1-2v}\frac{\partial u_\gamma}{\partial \rho} + \varepsilon m_1^{(\gamma)}(\mathbf{u}) + \varepsilon^2 m_2^{(\gamma)}(\mathbf{u}) + \cdots = \pm\varepsilon^2\frac{P_\gamma^\pm}{2\mu}, \tag{2.20}$$

as $\rho = \pm 1$.

Explicit definitions of the operators $l_j^{(q)}$ and $m_j^{(q)}$ are given in Appendix A.

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3 Formal asymptotic algorithm

3.1 The leading ansatz

Let the displacement vector be approximated by the asymptotic formulae

$$\mathbf{u} \sim \mathbf{u}^{(0)}(\alpha, \beta, \rho) + \varepsilon \mathbf{u}^{(1)}(\alpha, \beta, \rho) + \varepsilon^2 \mathbf{U}(\alpha, \beta, \rho). \tag{3.1}$$

Substituting (3.1) into (2.15)–(2.17), (2.18)–(2.20) we obtain the following recurrence sequence of problems.

First iteration

The vector function $\mathbf{u}^{(0)}$ satisfies the Neumann boundary value problem on the segment $\rho \in [-1, 1]$

$$\frac{\partial^2 \mathbf{u}^{(0)}}{\partial \rho^2} = 0 \quad |\rho| < 1, \tag{3.2}$$

$$\frac{\partial \mathbf{u}^{(0)}}{\partial \rho} = 0 \quad \rho = \pm 1. \tag{3.3}$$

Hence $\mathbf{u}^{(0)}$ is independent of ρ:

$$\mathbf{u}^{(0)} = \mathbf{u}^{(0)}(\alpha, \beta). \tag{3.4}$$

Second iteration

First, we note that

$$l_1^{(q)}(\mathbf{u}^{(0)}) = 0, \quad q = \alpha, \beta, \gamma. \tag{3.5}$$

Then, the vector function $\mathbf{u}^{(1)}$ satisfies the problem

$$\frac{\partial^2 \mathbf{u}^{(1)}}{\partial \rho^2} = 0, \quad |\rho| < 1, \tag{3.6}$$

and

$$\frac{\partial u_q^{(1)}}{\partial \rho} + m_1^{(q)}(\mathbf{u}^{(0)}) = 0, \quad \rho = \pm 1, \quad q = \alpha, \beta, \tag{3.7}$$

$$\frac{1 - \nu}{1 - 2\nu} \frac{\partial u_\gamma^{(1)}}{\partial \rho} + m_1^{(\gamma)}(\mathbf{u}^{(0)}) = 0, \quad \rho = \pm 1. \tag{3.8}$$

This problem is solvable, and its solution has the form

$$u_q^{(1)} = -m_1^{(q)}(\mathbf{u}^{(0)})\rho,$$

$$u_\gamma^{(1)} = -\frac{1 - 2\nu}{1 - \nu} m_1^{(\gamma)}(\mathbf{u}^{(0)})\rho, \quad q = \alpha, \beta. \tag{3.9}$$

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Third iteration

The components U_q, $q = \alpha, \beta, \gamma$, of the vector function $\mathbf{U}$ satisfy the equations

$$\frac{\partial^2 U_q}{\partial \rho^2} + l_1^{(q)}(\mathbf{u}^{(1)}) + l_2^{(q)}(\mathbf{u}^{(0)}) = -\frac{F_q}{\mu}, \quad q = \alpha, \beta, \quad |\rho| < 1, \tag{3.10}$$

$$\left[\frac{\partial U_q}{\partial \rho} + m_1^{(q)}(\mathbf{u}^{(1)}) + m_2^{(q)}(\mathbf{u}^{(0)})\right]_{\rho=\pm 1} = \pm\frac{P_q^{\pm}}{\mu}, \quad q = \alpha, \beta, \tag{3.11}$$

and

$$\frac{2(1-\nu)}{1-2\nu}\frac{\partial^2 U_\gamma}{\partial \rho^2} + l_1^{(\gamma)}(\mathbf{u}^{(1)}) + l_2^{(\gamma)}(\mathbf{u}^{(0)}) = -\frac{F_\gamma}{\mu}, \quad |\rho| < 1, \tag{3.12}$$

$$\left[\frac{1-\nu}{1-2\nu}\frac{\partial U_\gamma}{\partial \rho} + m_1^{(\gamma)}(\mathbf{u}^{(1)}) + m_2^{(\gamma)}(\mathbf{u}^{(0)})\right]_{\rho=\pm 1} = \pm\frac{P_\gamma^{\pm}}{2\mu}. \tag{3.13}$$

We note that $l_1^{(q)}(\mathbf{u}^{(1)}), l_2^{(q)}(\mathbf{u}^{(0)})$, $q = \alpha, \beta, \gamma$ are independent of ρ.

The solvability conditions for the boundary value problems (3.10)–(3.11) and (3.12)–(3.13) on the segment $\rho \in [-1, 1]$ can be written as follows:

$$l_1^{(q)}(\mathbf{u}^{(1)}) + l_2^{(q)}(\mathbf{u}^{(0)}) - \left[m_1^{(q)}(\mathbf{u}^{(1)}) + m_2^{(q)}(\mathbf{u}^{(0)})\right]_{\rho=1} = -\frac{1}{2\mu}\left\{\int_{-1}^{1} F_q d\rho + P_q^+ - P_q^-\right\}, \quad q = \alpha, \beta, \tag{3.14}$$

and

$$l_1^{(\gamma)}(\mathbf{u}^{(1)}) + l_2^{(\gamma)}(\mathbf{u}^{(0)}) - 2\left[m_1^{(\gamma)}(\mathbf{u}^{(1)}) + m_2^{(\gamma)}(\mathbf{u}^{(0)})\right]_{\rho=1} = -\frac{1}{2\mu}\left\{\int_{-1}^{1} F_\gamma d\rho + P_\gamma^+ - P_\gamma^-\right\}. \tag{3.15}$$

Using formulae (A 1)–(A 12) for the operators $l_k^{(q)}$ and $m_k^{(q)}$, and formulae (3.4), (3.9), we derive that the vector function $\mathbf{u}^{(0)}(\alpha, \beta)$ satisfies the equation

$$\mathscr{M}\left(\frac{\partial}{\partial \alpha}, \frac{\partial}{\partial \beta}\right)\mathbf{u}^{(0)}(\alpha, \beta) = \mathscr{F} \tag{3.16}$$

where $\mathscr{M}$ is the differential operator of the membrane shell theory

$$\mathscr{M}_{\alpha\alpha}u_\alpha^{(0)} = \frac{2}{1-\nu}\frac{1}{A}\frac{\partial}{\partial \alpha}\left[\frac{1}{AB}\frac{\partial(Bu_\alpha^{(0)})}{\partial \alpha}\right] + \frac{1}{B}\frac{\partial}{\partial \beta}\left[\frac{1}{AB}\frac{\partial(Au_\alpha^{(0)})}{\partial \beta}\right] + \frac{2}{R_1 R_2}u_\alpha^{(0)}, \tag{3.17}$$

$$\mathscr{M}_{\alpha\beta}u_\beta^{(0)} = \frac{2}{1-\nu}\frac{1}{A}\frac{\partial}{\partial \alpha}\left[\frac{1}{AB}\frac{\partial(Au_\beta^{(0)})}{\partial \beta}\right] - \frac{1}{B}\frac{\partial}{\partial \beta}\left[\frac{1}{AB}\frac{\partial(Bu_\beta^{(0)})}{\partial \alpha}\right], \tag{3.18}$$

$$\mathscr{M}_{\alpha\gamma}u_\gamma^{(0)} = \frac{2}{1-\nu}\frac{1}{A}\frac{\partial}{\partial \alpha}\left[\left(\frac{1}{R_1} + \frac{1}{R_2}\right)u_\gamma^{(0)}\right] - \frac{2}{AR_2}\frac{\partial u_\gamma^{(0)}}{\partial \alpha}, \tag{3.19}$$

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$$\mathscr{M}_{\beta\alpha}u_\alpha^{(0)} = \frac{2}{1-\nu}\frac{1}{B}\frac{\partial}{\partial\beta}\Big[\frac{1}{AB}\frac{\partial(Bu_\alpha^{(0)})}{\partial\alpha}\Big] - \frac{1}{A}\frac{\partial}{\partial\alpha}\Big[\frac{1}{AB}\frac{\partial(Au_\alpha^{(0)})}{\partial\beta}\Big], \tag{3.20}$$

$$\mathscr{M}_{\beta\beta}u_\beta^{(0)} = \frac{2}{1-\nu}\frac{1}{B}\frac{\partial}{\partial\beta}\Big[\frac{1}{AB}\frac{\partial(Au_\beta^{(0)})}{\partial\beta}\Big] + \frac{1}{A}\frac{\partial}{\partial\alpha}\Big[\frac{1}{AB}\frac{\partial(Bu_\beta^{(0)})}{\partial\alpha}\Big] + \frac{2}{R_1R_2}u_\beta^{(0)}, \tag{3.21}$$

$$\mathscr{M}_{\beta\gamma}u_\gamma^{(0)} = \frac{2}{1-\nu}\frac{1}{B}\frac{\partial}{\partial\beta}\Big[\Big(\frac{1}{R_1}+\frac{1}{R_2}\Big)u_\gamma^{(0)}\Big] - \frac{2}{BR_1}\frac{\partial u_\gamma^{(0)}}{\partial\beta}, \tag{3.22}$$

$$\mathscr{M}_{\gamma\alpha}u_\alpha^{(0)} = -\frac{2}{1-\nu}\frac{1}{AB}\Big(\frac{1}{R_1}+\frac{1}{R_2}\Big)\frac{\partial(Bu_\alpha^{(0)})}{\partial\alpha} + \frac{2}{AB}\frac{\partial}{\partial\alpha}\Big(\frac{B}{R_2}u_\alpha^{(0)}\Big), \tag{3.23}$$

$$\mathscr{M}_{\gamma\beta}u_\beta^{(0)} = -\frac{2}{1-\nu}\frac{1}{AB}\Big(\frac{1}{R_1}+\frac{1}{R_2}\Big)\frac{\partial(Au_\beta^{(0)})}{\partial\beta} + \frac{2}{AB}\frac{\partial}{\partial\beta}\Big(\frac{A}{R_1}u_\beta^{(0)}\Big), \tag{3.24}$$

$$\mathscr{M}_{\gamma\gamma}u_\gamma^{(0)} = -\frac{2}{1-\nu}\Big[\frac{1}{R_1^2}+2\nu\frac{1}{R_1R_2}+\frac{1}{R_2^2}\Big]u_\gamma^{(0)}. \tag{3.25}$$

The right-hand side $\mathscr{F}$ is given by

$$\mathscr{F}_q = -\frac{1}{2\mu}\Big\{\int_{-1}^{1} F_q d\rho + P_q^+ - P_q^-\Big\}, \quad q = \alpha,\beta,\gamma. \tag{3.26}$$

Theorem 1 *A solution of the static problems* (2.2), (2.3) *allows for a formal approximation* (3.1), *where the leading coefficient* $\mathbf{u}^{(0)}$ *satisfies the equation* (3.16) *of the membrane theory.*

Corollary 1 *If, in equations* (2.2), (2.3), *we replace* $\mathbf{P}^\pm$ *by zero and the term* $\mathbf{F}$ *by* $\rho_0\omega^2\mathbf{u}$, *then we shall have the eigenvalue problem*

$$\begin{aligned} \mathbf{Lu} + \rho_0\omega^2\mathbf{u} &= \mathbf{0} \quad \text{in} \quad \Omega_\varepsilon, \\ \boldsymbol{\sigma}^{(\mathbf{n})}(\mathbf{u}) &= \mathbf{0} \quad \text{on} \quad \Gamma_\pm. \end{aligned} \tag{3.27}$$

Here $\rho_0 > 0$ *can be interpreted as material density and* ω *as a radian frequency. In this case, the above asymptotic algorithm results in the equation*

$$\mathscr{M}\Big(\frac{\partial}{\partial\alpha},\frac{\partial}{\partial\beta}\Big)\mathbf{u}^{(0)}(\alpha,\beta) + \frac{\rho_0\omega^2}{\mu}\mathbf{u}^{(0)}(\alpha,\beta) = 0, \tag{3.28}$$

associated with the low-frequency oscillation of the thin elastic shell and solutions slowly varying in the space variable.

4 Moment theory of vibrating shells

In this section, we apply the asymptotic method to derive the equations for a vibrating elastic shell with large gradients of displacement components in space variables.

It is assumed that the displacement $\mathbf{u}$ and the radian frequency ω satisfy the problem

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(3.27). In addition to the scaled variable γ/ε (see (2.7)), we introduce scaling in the tangential directions, i.e. we assume that

$$\mathbf{u} = diag\{\sqrt{\varepsilon}, \sqrt{\varepsilon}, 1\}\mathbf{v}(\xi, \eta, \rho; \varepsilon), \tag{4.1}$$

where

$$\xi = \frac{\alpha}{\sqrt{\varepsilon}}, \qquad \eta = \frac{\beta}{\sqrt{\varepsilon}}. \tag{4.2}$$

Expanding the operators of the problem in powers of ε, we obtain the equations similar to (2.15)–(2.17):

$$\frac{\partial^2 v_1}{\partial \rho^2} + \frac{1}{1-2\nu}\frac{1}{A}\frac{\partial^2 v_3}{\partial \xi \partial \rho} + \varepsilon l_1^{(1)}(\mathbf{v}) + \varepsilon^2 \rho_0 \omega^2 v_1 + \cdots = 0, \tag{4.3}$$

$$\frac{\partial^2 v_2}{\partial \rho^2} + \frac{1}{1-2\nu}\frac{1}{B}\frac{\partial^2 v_3}{\partial \eta \partial \rho} + \varepsilon l_1^{(2)}(\mathbf{v}) + \varepsilon^2 \rho_0 \omega^2 v_2 + \cdots = 0, \tag{4.4}$$

and

$$\frac{2(1-\nu)}{1-2\nu}\frac{\partial^2 v_2}{\partial \rho^2} + \varepsilon l_1^{(3)} + \varepsilon^2 l_2^{(3)}(\mathbf{v}) + \varepsilon^2 \rho_0 \omega^2 v_3 + \cdots = 0, \tag{4.5}$$

as $|\rho| < 1$. Further, for the boundary conditions we derive

$$\frac{\partial v_1}{\partial \rho} + \frac{1}{A}\frac{\partial v_3}{\partial \xi} + \varepsilon m_1^{(1)}(\mathbf{v}) + \cdots = 0, \tag{4.6}$$

$$\frac{\partial v_2}{\partial \rho} + \frac{1}{B}\frac{\partial v_3}{\partial \eta} + \varepsilon m_1^{(2)}(\mathbf{v}) + \cdots = 0, \tag{4.7}$$

and

$$\frac{1-\nu}{1-2\nu}\frac{\partial v_3}{\partial \rho} + \varepsilon m_1^{(3)}(\mathbf{v}) + \varepsilon^2 m_2^{(3)}(\mathbf{v}) + \cdots = 0 \tag{4.8}$$

as $\rho = \pm 1$.

Explicit definitions of the operators $l_j^{(k)}$ and $m_j^{(k)}$ are included in Appendix B.

4.1 Formal asymptotic algorithm

Let the components of the vector $\mathbf{v}$ from (4.1) be approximated by the asymptotic formulae

$$v_j \sim v_j^{(0)}(\xi, \eta, \rho) + \varepsilon V_j(\xi, \eta, \rho), \quad j = 1, 2, \tag{4.9}$$

$$v_3 \sim v_3^{(0)}(\xi, \eta, \rho) + \varepsilon v_3^{(1)}(\xi, \eta, \rho) + \varepsilon^2 V_3(\xi, \eta, \rho). \tag{4.10}$$

The recurrence sequence of problems can be obtained by simple substitution of (4.9)–(4.10) into (4.3)–(4.5), (4.6)–(4.8) which will be discussed in the text below.

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First iteration

The function $v_3^{(0)}$ satisfies the Neumann boundary value problem on the segment $\rho \in [-1, 1]$

$$\frac{\partial^2 v_3^{(0)}}{\partial \rho^2} = 0, \quad |\rho| < 1, \tag{4.11}$$

$$\frac{\partial v_3^{(0)}}{\partial \rho} = 0, \quad \rho = \pm 1. \tag{4.12}$$

Hence, $v_3^{(0)}$ is independent of ρ, i.e.

$$v_3^{(0)} = w(\xi, \eta). \tag{4.13}$$

The function $v_1^{(0)}$ satisfies the Neumann boundary problem on the same segment

$$\frac{\partial^2 v_1^{(0)}}{\partial \rho^2} = 0, \quad |\rho| < 1, \tag{4.14}$$

$$\frac{\partial v_1^{(0)}}{\partial \rho} = -\frac{1}{A}\frac{\partial w}{\partial \xi}, \quad \rho = \pm 1, \tag{4.15}$$

which is always solvable, and its solution has the form

$$v_1^{(0)} = -\frac{1}{A}\frac{\partial w}{\partial \xi}\rho + \mathscr{U}(\xi, \eta). \tag{4.16}$$

Similarly, one can find

$$v_2^{(0)} = -\frac{1}{B}\frac{\partial w}{\partial \eta}\rho + \mathscr{V}(\xi, \eta). \tag{4.17}$$

Second iteration

The solution for the problem for the function $v_3^{(1)}$

$$\frac{2(1-\nu)}{1-2\nu}\frac{\partial^2 v_3^{(1)}}{\partial \rho^2} + l_1^{(3)}(\mathbf{v}^{(0)}) = 0, \quad |\rho| < 1, \tag{4.18}$$

$$\frac{1-\nu}{1-2\nu}\frac{\partial v_3^{(1)}}{\partial \rho} + m_1^{(3)}(\mathbf{v}^{(0)}) = 0, \quad \rho = \pm 1, \tag{4.19}$$

always exists,

$$\begin{aligned} v_3^{(1)} = {} & \frac{\rho^2}{2}\frac{\nu}{1-\nu}\frac{1}{AB}\left[\frac{\partial}{\partial \xi}\left(\frac{B}{A}\frac{\partial w}{\partial \xi}\right) + \frac{\partial}{\partial \eta}\left(\frac{A}{B}\frac{\partial w}{\partial \eta}\right)\right] \\ & -\rho\frac{\nu}{1-\nu}\left(\frac{1}{R_1} + \frac{1}{R_2}\right)w - \rho\frac{\nu}{1-\nu}\frac{1}{AB}\left[\frac{\partial}{\partial \xi}(B\mathscr{U}) + \frac{\partial}{\partial \eta}(A\mathscr{V})\right]. \end{aligned} \tag{4.20}$$

The function V_1 satisfies the problem

$$\frac{\partial^2 V_1}{\partial \rho^2} + \frac{1}{1-2\nu}\frac{1}{A}\frac{\partial^2 v_3^{(1)}}{\partial \xi \partial \rho} + l_1^{(1)}(\mathbf{v}^{(0)}) = 0, \quad |\rho| < 1, \tag{4.21}$$

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$$\frac{\partial V_1}{\partial \rho} + \frac{1}{A}\frac{\partial v_3^{(1)}}{\partial \xi} + m_1^{(1)}(\mathbf{v}^{(0)}) = 0, \quad \rho = \pm 1, \tag{4.22}$$

or in compact form

$$\frac{\partial^2 V_1}{\partial \rho^2} = \mathscr{P}_1 + \mathscr{P}_2 \rho \quad |\rho| < 1, \tag{4.23}$$

$$\frac{\partial V_1}{\partial \rho} = \mathscr{Q}_1 + \mathscr{Q}_2 \rho \quad \rho = \pm 1, \tag{4.24}$$

where (we use formulae (B 3), (B 7), (4.13), (4.16) and (4.17)):

$$\begin{aligned}\mathscr{P}_1 = {} & \frac{\nu}{(1-2\nu)(1-\nu)}\frac{1}{A}\frac{\partial}{\partial \xi}\Big[\Big(\frac{1}{R_1}+\frac{1}{R_2}\Big)w\Big] \\ & -\frac{2-\nu}{1-\nu}\frac{1}{A}\frac{\partial}{\partial \xi}\Big[\frac{1}{AB}\frac{\partial}{\partial \xi}(B\mathscr{U}) + \frac{1}{AB}\frac{\partial}{\partial \eta}(A\mathscr{V})\Big] \\ & +\frac{1}{A}\frac{2}{R_2}\frac{\partial w}{\partial \xi} + \frac{1}{B}\frac{\partial}{\partial \eta}\Big[\frac{1}{AB}\frac{\partial}{\partial \xi}(B\mathscr{V}) - \frac{1}{AB}\frac{\partial}{\partial \eta}(A\mathscr{U})\Big],\end{aligned} \tag{4.25}$$

$$\mathscr{P}_2 = \frac{2-\nu}{1-\nu}\frac{1}{A}\frac{\partial}{\partial \xi}\Big[\frac{1}{AB}\frac{\partial}{\partial \xi}\Big(\frac{B}{A}\frac{\partial w}{\partial \xi}\Big) + \frac{1}{AB}\frac{\partial}{\partial \eta}\Big(\frac{A}{B}\frac{\partial w}{\partial \eta}\Big)\Big], \tag{4.26}$$

$$\mathscr{Q}_1 = -\frac{\nu}{2(1-\nu)}\frac{1}{A}\frac{\partial}{\partial \xi}\Big[\frac{1}{AB}\frac{\partial}{\partial \xi}\Big(\frac{B}{A}\frac{\partial w}{\partial \xi}\Big) + \frac{1}{AB}\frac{\partial}{\partial \eta}\Big(\frac{A}{B}\frac{\partial w}{\partial \eta}\Big)\Big] + \frac{1}{R_1}\mathscr{U}, \tag{4.27}$$

$$\begin{aligned}\mathscr{Q}_2 = {} & \frac{\nu}{1-\nu}\frac{1}{A}\frac{\partial}{\partial \xi}\Big[\Big(\frac{1}{R_1}+\frac{1}{R_2}\Big)w\Big] \\ & +\frac{\nu}{1-\nu}\frac{1}{A}\frac{\partial}{\partial \xi}\Big[\frac{1}{AB}\frac{\partial}{\partial \xi}(B\mathscr{U}) + \frac{1}{AB}\frac{\partial}{\partial \eta}(A\mathscr{V})\Big].\end{aligned} \tag{4.28}$$

Now, the solvability condition for the boundary value problem (4.23)–(4.24) on the segment $\rho \in [-1, 1]$ can be written as follows:

$$\mathscr{P}_1 - \mathscr{Q}_2 = 0. \tag{4.29}$$

Using formulae (4.25), (4.28) for the functions $\mathscr{P}_1$ and $\mathscr{Q}_2$, one can rewrite the solvability condition (4.29) as

$$\begin{aligned}& -\frac{1}{A}\frac{\partial}{\partial \xi}\Big[\frac{1}{AB}\frac{\partial}{\partial \xi}(B\mathscr{U})\Big] - \frac{1-\nu}{2}\frac{1}{B}\frac{\partial}{\partial \eta}\Big[\frac{1}{AB}\frac{\partial}{\partial \eta}(A\mathscr{U})\Big] \\ & -\frac{1}{A}\frac{\partial}{\partial \xi}\Big[\frac{1}{AB}\frac{\partial}{\partial \eta}(A\mathscr{V})\Big] + \frac{1-\nu}{2}\frac{1}{B}\frac{\partial}{\partial \eta}\Big[\frac{1}{AB}\frac{\partial}{\partial \xi}(B\mathscr{V})\Big] \\ & -\frac{1}{A}\Big(\frac{1}{R_1}+\frac{\nu}{R_2}\Big)\frac{\partial w}{\partial \xi} - \frac{1}{A}\frac{\partial}{\partial \xi}\Big(\frac{1}{R_1}+\frac{1}{R_2}\Big)w = 0.\end{aligned} \tag{4.30}$$

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The other solvability condition for the boundary value problem for the component V_2 has the same form as (4.30), and is obtained by a circular permutation of indices (α, β):

$$
\begin{aligned}
&-\frac{1}{B}\frac{\partial}{\partial\eta}\Big[\frac{1}{AB}\frac{\partial}{\partial\eta}(A\mathscr{V})\Big]-\frac{1-\nu}{2}\frac{1}{A}\frac{\partial}{\partial\xi}\Big[\frac{1}{AB}\frac{\partial}{\partial\xi}(B\mathscr{V})\Big]\\
&-\frac{1}{B}\frac{\partial}{\partial\eta}\Big[\frac{1}{AB}\frac{\partial}{\partial\xi}(B\mathscr{U})\Big]+\frac{1-\nu}{2}\frac{1}{A}\frac{\partial}{\partial\xi}\Big[\frac{1}{AB}\frac{\partial}{\partial\eta}(A\mathscr{U})\Big]\\
&-\frac{1}{B}\Big(\frac{\nu}{R_1}+\frac{1}{R_2}\Big)\frac{\partial w}{\partial\eta}-\frac{1}{B}\frac{\partial}{\partial\eta}\Big(\frac{1}{R_1}+\frac{1}{R_2}\Big)w=0.
\end{aligned}
\tag{4.31}
$$

The function V_1, which solves the boundary value problem (4.23)–(4.24), can be represented in the following form:

$$
\begin{aligned}
V_1=&\frac{\rho^3}{6}\frac{2-\nu}{1-\nu}\frac{1}{A}\frac{\partial}{\partial\xi}\Big[\frac{1}{AB}\frac{\partial}{\partial\xi}\Big(\frac{B}{A}\frac{\partial w}{\partial\xi}\Big)+\frac{1}{AB}\frac{\partial}{\partial\eta}\Big(\frac{A}{B}\frac{\partial w}{\partial\eta}\Big)\Big]\\
&+\frac{\rho^2}{2}\frac{\nu}{1-\nu}\Big\{\frac{1}{A}\frac{\partial}{\partial\xi}\Big[\Big(\frac{1}{R_1}+\frac{1}{R_2}\Big)w\Big]+\frac{1}{A}\frac{\partial}{\partial\xi}\Big[\frac{1}{AB}\frac{\partial}{\partial\xi}(B\mathscr{U})+\frac{1}{AB}\frac{\partial}{\partial\eta}(A\mathscr{V})\Big]\Big\}\\
&+\rho\Big\{-\frac{1}{1-\nu}\frac{1}{A}\frac{\partial}{\partial\xi}\Big[\frac{1}{AB}\frac{\partial}{\partial\xi}\Big(\frac{B}{A}\frac{\partial w}{\partial\xi}\Big)+\frac{1}{AB}\frac{\partial}{\partial\eta}\Big(\frac{A}{B}\frac{\partial w}{\partial\eta}\Big)\Big]+\frac{1}{R_1}\mathscr{U}\Big\}.
\end{aligned}
\tag{4.32}
$$

Similarly,

$$
\begin{aligned}
V_2=&\frac{\rho^3}{6}\frac{2-\nu}{1-\nu}\frac{1}{B}\frac{\partial}{\partial\eta}\Big[\frac{1}{AB}\frac{\partial}{\partial\xi}\Big(\frac{B}{A}\frac{\partial w}{\partial\xi}\Big)+\frac{1}{AB}\frac{\partial}{\partial\eta}\Big(\frac{A}{B}\frac{\partial w}{\partial\eta}\Big)\Big]\\
&+\frac{\rho^2}{2}\frac{\nu}{1-\nu}\Big\{\frac{1}{B}\frac{\partial}{\partial\eta}\Big[\Big(\frac{1}{R_1}+\frac{1}{R_2}\Big)w\Big]+\frac{1}{B}\frac{\partial}{\partial\eta}\Big[\frac{1}{AB}\frac{\partial}{\partial\eta}(B\mathscr{U})+\frac{1}{AB}\frac{\partial}{\partial\xi}(A\mathscr{V})\Big]\Big\}\\
&+\rho\Big\{-\frac{1}{1-\nu}\frac{1}{B}\frac{\partial}{\partial\eta}\Big[\frac{1}{AB}\frac{\partial}{\partial\xi}\Big(\frac{B}{A}\frac{\partial w}{\partial\xi}\Big)+\frac{1}{AB}\frac{\partial}{\partial\eta}\Big(\frac{A}{B}\frac{\partial w}{\partial\eta}\Big)\Big]+\frac{1}{R_1}\mathscr{U}\Big\}.
\end{aligned}
\tag{4.33}
$$

Third iteration

The function V_3 satisfies the equations

$$
\frac{2(1-\nu)}{1-2\nu}\frac{\partial^2 V_3}{\partial\rho^2}+l_1^{(3)}(V_1,V_2,v_3^{(1)})+l_2^{(3)}(\mathbf{v}^{(0)})=-\Lambda v_3^{(0)},\quad |\rho|<1,
\tag{4.34}
$$

$$
\frac{1-\nu}{1-2\nu}\frac{\partial V_3}{\partial\rho}+m_1^{(3)}(V_1,V_2,v_3^{(1)})+m_2^{(3)}(\mathbf{v}^{(0)})=0,\quad \rho=\pm1.
\tag{4.35}
$$

The structure of (4.34), (4.35) is characterised by

$$
\frac{2(1-\nu)}{1-2\nu}\frac{\partial^2 V_3}{\partial\rho^2}+\mathscr{S}_1+\mathscr{S}_2\rho+\mathscr{S}_3\rho^2+\mathscr{S}_4\rho^3=0,\quad |\rho|<1,
\tag{4.36}
$$

$$
\frac{1-\nu}{1-2\nu}\frac{\partial V_3}{\partial\rho}+\mathscr{T}_1\rho+\mathscr{T}_2=0,\quad \rho=\pm1,
\tag{4.37}
$$

where $\mathscr{S}_j$ and $\mathscr{T}_j$ are known smooth functions of ξ and η. The problem (4.36), (4.37) is solvable, provided

$$
\mathscr{S}_1+\tfrac{1}{3}\mathscr{S}_3-2\mathscr{T}_2=0.
\tag{4.38}
$$

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Here,

$$\begin{aligned}\mathscr{S}_1 = &-\frac{2(1-\nu)}{1-2\nu}\Big(\frac{1}{R_1^2}+\frac{1}{R_2^2}\Big)w\\ &+\frac{2(1-\nu)}{1-2\nu}\frac{1}{AB}\Big\{\frac{\partial}{\partial\xi}\Big(\frac{B}{R_2}\mathscr{U}\Big)+\frac{\partial}{\partial\eta}\Big(\frac{A}{R_1}\mathscr{V}\Big)\\ &-\Big(\frac{1}{R_1}+\frac{1}{R_2}\Big)\Big[\frac{\partial}{\partial\xi}(B\mathscr{U})+\frac{\partial}{\partial\eta}(A\mathscr{V})\Big]\Big\}\\ &-\frac{1}{AB}\Big\{\frac{\partial}{\partial\xi}\Big(\frac{B}{R_1}\mathscr{U}\Big)-\frac{\partial}{\partial\eta}\Big(\frac{A}{R_2}\mathscr{V}\Big)\Big\}\\ &-\frac{2\nu}{1-2\nu}\Big(\frac{1}{R_1}+\frac{1}{R_2}\Big)\Big\{\Big(\frac{1}{R_1}+\frac{1}{R_2}\Big)w+\frac{1}{AB}\Big[\frac{\partial}{\partial\xi}(B\mathscr{U})+\frac{\partial}{\partial\eta}(A\mathscr{V})\Big]\Big\}\\ &-\frac{1}{(1-\nu)(1-2\nu)}\Delta_{\xi\eta}^2 w+\frac{1}{1-2\nu}\frac{1}{AB}\Big[\frac{\partial}{\partial\xi}\Big(\frac{B}{R_1}\mathscr{U}\Big)+\frac{\partial}{\partial\eta}\Big(\frac{A}{R_2}\mathscr{V}\Big)\Big],\end{aligned}\tag{4.39}$$

$$\mathscr{S}_3 = \frac{1+\nu}{1-2\nu}\Delta_{\xi\eta}^2 w,\tag{4.40}$$

$$\begin{aligned}\mathscr{T}_2 = &-\frac{\nu}{1-2\nu}\Big(\frac{1}{R_1^2}+\frac{1}{R_2^2}\Big)w\\ &+\frac{\nu}{1-2\nu}\Big\{-\frac{1}{AR_1}\frac{\partial\mathscr{U}}{\partial\xi}+\frac{1}{AB}\frac{\partial}{\partial\xi}\Big(B\frac{1}{R_2}\Big)\mathscr{U}-\frac{1}{AB}\Big(\frac{1}{R_1}+\frac{1}{R_2}\Big)\frac{\partial B}{\partial\xi}\mathscr{U}\\ &-\frac{1}{BR_2}\frac{\partial\mathscr{V}}{\partial\eta}+\frac{1}{AB}\frac{\partial}{\partial\eta}\Big(\frac{A}{R_1}\Big)\mathscr{V}-\frac{1}{AB}\Big(\frac{1}{R_1}+\frac{1}{R_2}\Big)\frac{\partial A}{\partial\eta}\mathscr{V}\Big\}\\ &+\frac{\nu}{1-2\nu}\Big(\frac{1}{R_1}+\frac{1}{R_2}\Big)\Big\{-\frac{\nu}{1-\nu}\Big(\frac{1}{R_1}+\frac{1}{R_2}\Big)w\\ &-\frac{\nu}{1-\nu}\frac{1}{AB}\Big[\frac{\partial}{\partial\xi}(B\mathscr{U})+\frac{\partial}{\partial\eta}(A\mathscr{V})\Big]\Big\}\\ &+\frac{\nu}{1-2\nu}\frac{1}{AB}\Big\{\frac{\partial}{\partial\xi}\Big(\frac{B}{R_1}\mathscr{U}\Big)+\frac{\partial}{\partial\eta}\Big(\frac{A}{R_2}\mathscr{V}\Big)\Big\}\\ &-\frac{\nu(4+\nu)}{6(1-\nu)(1-2\nu)}\Delta_{\xi\eta}^2 w,\end{aligned}\tag{4.41}$$

where

$$\Delta_{\xi\eta}w = \frac{1}{AB}\Big[\frac{\partial}{\partial\xi}\Big(\frac{B}{A}\frac{\partial w}{\partial\xi}\Big)+\frac{\partial}{\partial\eta}\Big(\frac{A}{B}\frac{\partial w}{\partial\eta}\Big)\Big]\tag{4.42}$$

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denotes the Laplacian in curvilinear coordinates (ξ, η). Direct substitution of (4.39)–(4.41) into (4.38) gives

$$\frac{1}{3}\Delta^2_{\xi\eta} w + \Big(\frac{1}{R_1^2} + \frac{2\nu}{R_1 R_2} + \frac{1}{R_2^2}\Big) w + \frac{1}{A}\Big(\frac{1}{R_1} + \frac{\nu}{R_2}\Big)\frac{\partial \mathscr{U}}{\partial \xi} + \frac{1}{AB}\Big(\frac{\nu}{R_1} + \frac{1}{R_2}\Big)\frac{\partial B}{\partial \xi}\mathscr{U} + \frac{1}{B}\Big(\frac{\nu}{R_1} + \frac{1}{R_2}\Big)\frac{\partial \mathscr{V}}{\partial \eta} + \frac{1}{AB}\Big(\frac{1}{R_1} + \frac{\nu}{R_2}\Big)\frac{\partial A}{\partial \eta}\mathscr{V} = \Lambda w, \tag{4.43}$$

where

$$\Lambda = \frac{\rho_0 \omega^2 (1 - \nu^2)}{E}.$$

Remark 1 *Derivation of the above formulae* (4.30), (4.31) *and* (4.43) *also required the following differential relations:*

$$A\frac{\partial}{\partial \eta}\Big(\frac{1}{R_1}\Big) = \Big(\frac{1}{R_2} - \frac{1}{R_1}\Big)\frac{\partial A}{\partial \eta},$$

$$B\frac{\partial}{\partial \xi}\Big(\frac{1}{R_2}\Big) = \Big(\frac{1}{R_1} - \frac{1}{R_2}\Big)\frac{\partial B}{\partial \xi}. \tag{4.44}$$

Equations (4.30), (4.31) and (4.43) are known in the literature (Vlasov, 1958; Aslanyan & Lidskii, 1974) as the equations of moment theory due to Vlasov.

We conclude with the following.

Theorem 2 *A solution of the spectral problem* (3.27) *with the scaling* (4.1), (4.2) *allows for a formal approximation* (4.9), (4.10)*, where the leading coefficients satisfy the equations* (4.30), (4.31), (4.43) *of Vlasov's moment theory of elastic shells.*

5 Examples

In this section we consider simpler versions of equations that describe circular cylindrical shells. In this case

$$A = 1, \quad B = R, \quad R_1^{-1} = 0, \quad R_2 = R.$$

5.1 Moment theory

We use the system of cylindrical coordinates (r, φ, z), so that

$$r = R, \quad \varphi = \sqrt{\varepsilon}\,\eta, \quad z = \sqrt{\varepsilon}\,\xi,$$

and introduce the normalised displacement components in accordance with (4.1), so that on the limit (neutral) surface we have

$$u_r = w, \quad u_\varphi = \sqrt{\varepsilon}\,\mathscr{V}, \quad u_z = \sqrt{\varepsilon}\,\mathscr{U}.$$

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Then, in the conventional engineering dimensional notations the system of equations (4.30), (4.31), (4.43) takes the form

$$\frac{\partial^2 u_z}{\partial z^2} + \frac{1-\nu}{2R^2}\frac{\partial^2 u_z}{\partial \varphi^2} + \frac{1+\nu}{2R}\frac{\partial^2 u_\varphi}{\partial \varphi \partial z} + \frac{\nu}{R}\frac{\partial u_r}{\partial z} = 0, \tag{5.1}$$

$$\frac{1-\nu}{2}\frac{\partial^2 u_\varphi}{\partial z^2} + \frac{1}{R^2}\frac{\partial^2 u_\varphi}{\partial \varphi^2} + \frac{1+\nu}{2R}\frac{\partial^2 u_z}{\partial \varphi \partial z} + \frac{1}{R^2}\frac{\partial u_r}{\partial \varphi} = 0, \tag{5.2}$$

$$\frac{(\varepsilon h)^2}{3}\left[\frac{\partial^4 u_r}{\partial z^4} + \frac{2}{R^2}\frac{\partial^4 u_r}{\partial \varphi^2 \partial z^2} + \frac{1}{R^4}\frac{\partial^4 u_r}{\partial \varphi^4}\right] + \frac{1}{R^2}u_r + \frac{\nu}{R}\frac{\partial u_z}{\partial z} + \frac{1}{R^2}\frac{\partial u_\varphi}{\partial \varphi} = \Lambda u_r, \tag{5.3}$$

where $2\varepsilon h$ is the small thickness of the shell, which is consistent with the equations of Vlasov's theory (see Vlasov (1958) and Aslanyan & Lidskii (1974)).

5.2 Membrane theory

We refer to the expansion (3.1) and let

$$u_r = u_r^{(0)}, \quad u_\varphi = u_\varphi^{(0)}, \quad u_z = u_z^{(0)}.$$

Then, in accordance with (3.28), the following equations hold for a cylindrical shell vibrating with a low frequency:

$$\frac{\partial^2 u_z}{\partial z^2} + \frac{1-\nu}{2R^2}\frac{\partial^2 u_z}{\partial \varphi^2} + \frac{1+\nu}{2R}\frac{\partial^2 u_\varphi}{\partial \varphi \partial z} + \frac{\nu}{R}\frac{\partial u_r}{\partial z} = -\Lambda u_z, \tag{5.4}$$

$$\frac{1-\nu}{2}\frac{\partial^2 u_\varphi}{\partial z^2} + \frac{1}{R^2}\frac{\partial^2 u_\varphi}{\partial \varphi^2} + \frac{1+\nu}{2R}\frac{\partial^2 u_z}{\partial \varphi \partial z} + \frac{1}{R^2}\frac{\partial u_r}{\partial \varphi} = -\Lambda u_\varphi, \tag{5.5}$$

$$\frac{1}{R^2}u_r + \frac{\nu}{R}\frac{\partial u_z}{\partial z} + \frac{1}{R^2}\frac{\partial u_\varphi}{\partial \varphi} = \Lambda u_r. \tag{5.6}$$

The simple equations given in this section are, of course, known in the engineering literature, and they are appropriate for the cylindrical foundation of the multi-structure mentioned in the introduction.

6 Remarks on geometrically nonlinear theories

We note that the previous sections describe the asymptotic algorithms applied to a linear problem. It can be extended to the case of geometrically nonlinear deformations. Derivation of the resulting lower-dimensional equations for shells can be a subject of a separate paper. However, we can give a simple illustration for the case of a thin plate $\{\mathbf{x} : (x_1, x_2) \in G \subset \mathbf{R}^2,\ |x_3| < \varepsilon\}$ whose neutral surface has zero curvature; here $x_i, i = 1, 2, 3$, denote Cartesian coordinates and, as before, ε stands for a small non-dimensional parameter. For the sake of simplicity, we assume that the stress components satisfy homogeneous equilibrium equations within the plate and homogeneous traction boundary conditions on the upper and lower surfaces of the plate. For geometrically

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nonlinear deformations, Green's strain tensor components are given by

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j}\right),$$

where k is the index of summation, and the indices i, j, k may take values $1, 2, 3$; here, u_j represent Cartesian components of the displacement.

One can consider the following scaling for the displacement field:

$$u_j = \varepsilon^2 U_j\left(x_1, x_2, \frac{x_3}{\varepsilon}; \varepsilon\right), \quad j = 1, 2,$$
$$u_3 = \varepsilon U_3\left(x_1, x_2, \frac{x_3}{\varepsilon}; \varepsilon\right). \tag{6.1}$$

Analysis of equilibrium equations and homogeneous traction boundary conditions yield that the leading-order terms in the representation of the displacement do not depend on x_3

$$U_j = U_j^{(0)}(x_1, x_2) + \text{small terms},$$

and, to leading order, the stress components are constant across the plate thickness, and $\sigma_{j3}^{(0)} = 0, j = 1, 2, 3$. It implies that we deal with the case of plane stress, and that one can neglect the components $\sigma_{i3}, i = 1, 2, 3$, and the x_3-derivatives in the equilibrium equations. The resulting equilibrium equations take the form

$$\frac{\partial \sigma_{11}^{(0)}}{\partial x_1} + \frac{\partial \sigma_{12}^{(0)}}{\partial x_2} = 0, \quad \frac{\partial \sigma_{12}^{(0)}}{\partial x_1} + \frac{\partial \sigma_{22}^{(0)}}{\partial x_2} = 0,$$

where the superscript 0 denotes the leading-order approximation. Assuming that the stress components are small (the external load can be defined via the boundary conditions at the edges of the plate) we can introduce a stress function

$$\Phi = \varepsilon^2 \Phi^{(0)}(x_1, x_2) + \text{smaller terms}$$

in such a way that

$$\sigma_{11}^{(0)} = \varepsilon^2 \frac{\partial^2 \Phi^{(0)}}{\partial x_2^2}, \quad \sigma_{22}^{(0)} = \varepsilon^2 \frac{\partial^2 \Phi^{(0)}}{\partial x_1^2}, \quad \sigma_{12}^{(0)} = -\varepsilon^2 \frac{\partial^2 \Phi^{(0)}}{\partial x_1 \partial x_2}. \tag{6.2}$$

Hooke's law and equations (6.2) yield, to the leading order,

$$\frac{1}{E}\left(\frac{\partial^2 \Phi^{(0)}}{\partial x_2^2} - \nu \frac{\partial^2 \Phi^{(0)}}{\partial x_1^2}\right) = \frac{\partial U_1^{(0)}}{\partial x_1} + \frac{1}{2}\left(\frac{\partial U_3^{(0)}}{\partial x_1}\right)^2, \tag{6.3}$$

$$\frac{1}{E}\left(\frac{\partial^2 \Phi^{(0)}}{\partial x_1^2} - \nu \frac{\partial^2 \Phi^{(0)}}{\partial x_2^2}\right) = \frac{\partial U_2^{(0)}}{\partial x_2} + \frac{1}{2}\left(\frac{\partial U_3^{(0)}}{\partial x_2}\right)^2, \tag{6.4}$$

$$-\frac{2(1+\nu)}{E}\frac{\partial^2 \Phi^{(0)}}{\partial x_1 \partial x_2} = \frac{\partial U_1^{(0)}}{\partial x_2} + \frac{\partial U_2^{(0)}}{\partial x_1} + \frac{\partial U_3^{(0)}}{\partial x_1}\frac{\partial U_3^{(0)}}{\partial x_2}. \tag{6.5}$$

It follows immediately that the stress function $\Phi^{(0)}$ satisfies the differential equation

$$\nabla^4 \Phi^{(0)} = E\left[\left(\frac{\partial^2 U_3^{(0)}}{\partial x_1 \partial x_2}\right)^2 - \frac{\partial^2 U_3^{(0)}}{\partial x_1^2}\frac{\partial^2 U_3^{(0)}}{\partial x_2^2}\right] \tag{6.6}$$

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known in the literature (see, for example, Berdichevski (1983)) as von Kármán's equation. Rigorous derivation, based on a variational method, and justification of equations of von Kármán's type were published by Ciarlet (1980).

We note that the scaling could also be chosen in such a way that

$$u_j = \varepsilon^{2\delta} U_j\Big(x_1, x_2, \frac{x_3}{\varepsilon}; \varepsilon\Big), \quad j = 1, 2,$$
$$u_3 = \varepsilon^{\delta} U_3\Big(x_1, x_2, \frac{x_3}{\sqrt{\varepsilon}}; \varepsilon\Big), \tag{6.7}$$

where $1/2 < \delta < 1$; in this case the corresponding stress function would be

$$\Phi = \varepsilon^{2\delta} \Phi^{(0)}(x_1, x_2) + \text{smaller terms}.$$

For this particular choice of scaling, the first-order derivatives of u_j are small, and the second-order derivatives are large, as $\varepsilon \to 0$. The formal asymptotic analysis would also lead to the von Kármán equation (6.6).

7 Concluding remarks

We have deduced that a static cylindrical foundation associated with a 2D–3D multi-structure, described in the first section of the paper, can be treated in accordance with the membrane theory of shells. On the other hand, for problems of vibration, involving the comparatively large (see (4.1)) derivatives of the displacement in the tangential directions, elastic fields are described by the moment theory equations due to Vlasov (1958). Also, the asymptotic algorithm presented in this work enables one to derive lower-dimensional models for thin elastic shells; it is suitable for shells of variable thickness, layered shells, shells with imperfect interfaces. The obvious advantage of this algorithm is that it does not require any conjectures based on the physical intuition. The final section of the paper has shown that the change of scaling may result in different limit theories. In particular, for geometrically nonlinear deformations the scaling (6.1) corresponds to equations of von Kármán's type.

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Appendix A

The operators $l_j^{(q)}$ and $m_j^{(q)}$, used in (2.15)–(2.17) and (2.18)–(2.20), are defined by

$$l_1^{(\alpha)} = \Big(\frac{1}{R_1} + \frac{1}{R_2}\Big)\frac{\partial u_\alpha}{\partial \rho} + \frac{1}{1-2\nu}\frac{1}{A}\frac{\partial^2 u_\gamma}{\partial\alpha\partial\rho}, \tag{A 1}$$

$$\begin{aligned} l_2^{(\alpha)} = &-\Big(\frac{1}{R_1^2} + \frac{1}{R_2^2}\Big)\rho\frac{\partial u_\alpha}{\partial\rho} + \frac{1}{R_1}\Big(\frac{1}{R_2} - \frac{1}{R_1}\Big)u_\alpha - \frac{1}{1-2\nu}\frac{\rho}{AR_1}\frac{\partial^2 u_\gamma}{\partial\alpha\partial\rho} \\ &+\frac{2(1-\nu)}{1-2\nu}\frac{1}{A}\frac{\partial}{\partial\alpha}\Big[\Big(\frac{1}{R_1} + \frac{1}{R_2}\Big)u_\gamma\Big] - \frac{1}{A}\Big(\frac{1}{R_2} - \frac{1}{R_1}\Big)\frac{\partial u_\gamma}{\partial\alpha} \\ &+\frac{2(1-\nu)}{1-2\nu}\frac{1}{A}\frac{\partial}{\partial\alpha}\Big[\frac{1}{AB}\frac{\partial(Bu_\alpha)}{\partial\alpha} + \frac{1}{AB}\frac{\partial(Au_\beta)}{\partial\beta}\Big] \\ &-\frac{1}{B}\frac{\partial}{\partial\beta}\Big[\frac{1}{AB}\frac{\partial(Bu_\beta)}{\partial\alpha} - \frac{1}{AB}\frac{\partial(Au_\alpha)}{\partial\beta}\Big], \end{aligned} \tag{A 2}$$

$$l_1^{(\beta)} = \Big(\frac{1}{R_1} + \frac{1}{R_2}\Big)\frac{\partial u_\beta}{\partial\rho} + \frac{1}{1-2\nu}\frac{1}{B}\frac{\partial^2 u_\gamma}{\partial\beta\partial\rho}, \tag{A 3}$$

$$\begin{aligned} l_2^{(\beta)} = &-\Big(\frac{1}{R_1^2} + \frac{1}{R_2^2}\Big)\rho\frac{\partial u_\beta}{\partial\rho} - \frac{1}{R_2}\Big(\frac{1}{R_2} - \frac{1}{R_1}\Big)u_\beta - \frac{1}{1-2\nu}\frac{\rho}{BR_2}\frac{\partial^2 u_\gamma}{\partial\beta\partial\rho} \\ &+\frac{2(1-\nu)}{1-2\nu}\frac{1}{B}\frac{\partial}{\partial\beta}\Big[\Big(\frac{1}{R_1} + \frac{1}{R_2}\Big)u_\gamma\Big] + \frac{1}{B}\Big(\frac{1}{R_2} - \frac{1}{R_1}\Big)\frac{\partial u_\gamma}{\partial\beta} \\ &+\frac{2(1-\nu)}{1-2\nu}\frac{1}{B}\frac{\partial}{\partial\beta}\Big[\frac{1}{AB}\frac{\partial(Au_\beta)}{\partial\beta} + \frac{1}{AB}\frac{\partial(Bu_\alpha)}{\partial\alpha}\Big] \\ &-\frac{1}{A}\frac{\partial}{\partial\alpha}\Big[\frac{1}{AB}\frac{\partial(Au_\alpha)}{\partial\beta} - \frac{1}{AB}\frac{\partial(Bu_\beta)}{\partial\alpha}\Big], \end{aligned} \tag{A 4}$$

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$$l_1^{(\gamma)} = \frac{2(1-\nu)}{1-2\nu}\Big[\Big(\frac{1}{R_1}+\frac{1}{R_2}\Big)\frac{\partial u_\gamma}{\partial\rho} + \frac{1}{A}\frac{\partial^2 u_\alpha}{\partial\alpha\partial\rho} + \frac{1}{AB}\frac{\partial B}{\partial\alpha}\frac{\partial u_\alpha}{\partial\rho} + \frac{1}{B}\frac{\partial^2 u_\beta}{\partial\beta\partial\rho} + \frac{1}{AB}\frac{\partial A}{\partial\beta}\frac{\partial u_\beta}{\partial\rho}\Big] - \frac{1}{A}\frac{\partial^2 u_\alpha}{\partial\alpha\partial\rho} - \frac{1}{AB}\frac{\partial B}{\partial\alpha}\frac{\partial u_\alpha}{\partial\rho} - \frac{1}{B}\frac{\partial^2 u_\beta}{\partial\beta\partial\rho} - \frac{1}{AB}\frac{\partial A}{\partial\beta}\frac{\partial u_\beta}{\partial\rho}, \tag{A 5}$$

$$\begin{aligned} l_2^{(\gamma)} = {} & \frac{2(1-\nu)}{1-2\nu}\Big[-\Big(\frac{1}{R_1^2}+\frac{1}{R_2^2}\Big)\rho\frac{\partial u_\gamma}{\partial\rho} - \Big(\frac{1}{R_1^2}+\frac{1}{R_2^2}\Big)u_\gamma\Big] \\ & + \frac{2(1-\nu)}{1-2\nu}\Big[-\frac{\rho}{AR_1}\frac{\partial^2 u_\alpha}{\partial\alpha\partial\rho} - \frac{1}{AR_1}\frac{\partial u_\alpha}{\partial\alpha} + \frac{\rho}{AB}\frac{\partial(B/R_2)}{\partial\alpha}\frac{\partial u_\alpha}{\partial\rho} \\ & - \frac{1}{AB}\Big(\frac{1}{R_1}+\frac{1}{R_2}\Big)\rho\frac{\partial B}{\partial\alpha}\frac{\partial u_\alpha}{\partial\rho} + \frac{1}{AB}\frac{\partial(B/R_2)}{\partial\alpha}u_\alpha - \frac{1}{AB}\Big(\frac{1}{R_1}+\frac{1}{R_2}\Big)\frac{\partial B}{\partial\alpha}u_\alpha \\ & - \frac{\rho}{BR_2}\frac{\partial^2 u_\beta}{\partial\beta\partial\rho} - \frac{1}{BR_2}\frac{\partial u_\beta}{\partial\beta} + \frac{1}{AB}\rho\frac{\partial(A/R_1)}{\partial\beta}\frac{\partial u_\beta}{\partial\rho} \\ & - \frac{1}{AB}\Big(\frac{1}{R_1}+\frac{1}{R_2}\Big)\rho\frac{\partial A}{\partial\beta}\frac{\partial u_\beta}{\partial\rho} + \frac{1}{AB}\frac{\partial(A/R_1)}{\partial\beta}u_\beta - \frac{1}{AB}\Big(\frac{1}{R_1}+\frac{1}{R_2}\Big)\frac{\partial A}{\partial\beta}u_\beta\Big] \\ & + \frac{\rho}{AR_1}\frac{\partial^2 u_\alpha}{\partial\alpha\partial\rho} - \frac{\rho}{AB}\frac{\partial(B/R_2)}{\partial\alpha}\frac{\partial u_\alpha}{\partial\rho} + \frac{1}{AB}\Big(\frac{1}{R_1}+\frac{1}{R_2}\Big)\rho\frac{\partial B}{\partial\alpha}\frac{\partial u_\alpha}{\partial\rho} \\ & - \frac{1}{AB}\frac{\partial}{\partial\alpha}\Big(\frac{B}{R_1}u_\alpha\Big) + \frac{\rho}{BR_2}\frac{\partial^2 u_\beta}{\partial\beta\partial\rho} - \frac{1}{AB}\frac{\partial(A/R_1)}{\partial\beta}\rho\frac{\partial u_\beta}{\partial\rho} \\ & + \frac{1}{AB}\Big(\frac{1}{R_1}+\frac{1}{R_2}\Big)\rho\frac{\partial A}{\partial\beta}\frac{\partial u_\beta}{\partial\rho} - \frac{1}{AB}\frac{\partial}{\partial\beta}\Big(\frac{A}{R_2}u_\beta\Big) \\ & + \frac{1}{AB}\frac{\partial}{\partial\alpha}\Big(\frac{B}{A}\frac{\partial u_\gamma}{\partial\alpha}\Big) + \frac{1}{AB}\frac{\partial}{\partial\beta}\Big(\frac{A}{B}\frac{\partial u_\gamma}{\partial\beta}\Big), \end{aligned} \tag{A 6}$$

$$m_1^{(\alpha)} = -\frac{1}{R_1}u_\alpha + \frac{1}{A}\frac{\partial u_\gamma}{\partial\alpha}, \tag{A 7}$$

$$m_2^{(\alpha)} = \frac{\rho}{R_1^2}u_\alpha - \frac{\rho}{AR_1}\frac{\partial u_\gamma}{\partial\alpha}, \tag{A 8}$$

$$m_1^{(\beta)} = -\frac{1}{R_2}u_\beta + \frac{1}{B}\frac{\partial u_\gamma}{\partial\beta}, \tag{A 9}$$

$$m_2^{(\beta)} = \frac{\rho}{R_2^2}u_\beta - \frac{\rho}{BR_2}\frac{\partial u_\gamma}{\partial\beta}, \tag{A 10}$$

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$$m_1^{(\gamma)} = \frac{v}{1-2v}\Big[\Big(\frac{1}{R_1}+\frac{1}{R_2}\Big)u_\gamma + \frac{1}{A}\frac{\partial u_\alpha}{\partial \alpha} + \frac{1}{AB}\frac{\partial B}{\partial \alpha}u_\alpha + \frac{1}{B}\frac{\partial u_\beta}{\partial \beta} + \frac{1}{AB}\frac{\partial A}{\partial \beta}u_\beta\Big], \tag{A 11}$$

$$\begin{aligned} m_2^{(\gamma)} = {} & \frac{v}{1-2v}\Big[-\Big(\frac{1}{R_1^2}+\frac{1}{R_2^2}\Big)\rho u_\gamma\Big] + \frac{v}{1-2v}\Big[-\frac{\rho}{AR_1}\frac{\partial u_\alpha}{\partial \alpha} \\ & + \frac{1}{AB}\frac{\partial (B/R_2)}{\partial \alpha}\rho u_\alpha - \frac{1}{AB}\Big(\frac{1}{R_1}+\frac{1}{R_2}\Big)\rho\frac{\partial B}{\partial \alpha}u_\alpha \\ & - \frac{\rho}{BR_2}\frac{\partial u_\beta}{\partial \beta} + \frac{1}{AB}\frac{\partial (A/R_1)}{\partial \beta}\rho u_\beta - \frac{1}{AB}\Big(\frac{1}{R_1}+\frac{1}{R_2}\Big)\rho\frac{\partial A}{\partial \beta}u_\beta. \end{aligned} \tag{A 12}$$

Appendix B

The operators $l_j^{(k)}$ and $m_j^{(k)}$, used in (4.3)–(4.5) and (4.6)–(4.8), are defined by

$$\begin{aligned} l_1^{(1)}(v) = {} & \Big(\frac{1}{R_1}+\frac{1}{R_2}\Big)\frac{\partial v_1}{\partial \rho} - \frac{1}{1-2v}\frac{1}{AR_1}\rho\frac{\partial^2 v_3}{\partial \xi \partial \rho} \\ & + \frac{2(1-v)}{1-2v}\frac{1}{A}\frac{\partial}{\partial \xi}\Big[\Big(\frac{1}{R_1}+\frac{1}{R_2}\Big)v_3\Big] - \frac{1}{A}\Big(\frac{1}{R_2}-\frac{1}{R_1}\Big)\frac{\partial v_3}{\partial \xi} \\ & + \frac{2(1-v)}{1-2v}\frac{1}{A}\frac{\partial}{\partial \xi}\Big[\frac{1}{AB}\frac{\partial}{\partial \xi}(Bv_1) + \frac{1}{AB}\frac{\partial}{\partial \eta}(Av_2)\Big] \\ & - \frac{1}{B}\frac{\partial}{\partial \eta}\Big[\frac{1}{AB}\frac{\partial}{\partial \xi}(Bv_2) - \frac{1}{AB}\frac{\partial}{\partial \eta}(Av_1)\Big], \end{aligned} \tag{B 1}$$

$$\begin{aligned} l_1^{(2)}(v) = {} & \Big(\frac{1}{R_1}+\frac{1}{R_2}\Big)\frac{\partial v_2}{\partial \rho} - \frac{1}{1-2v}\frac{1}{BR_2}\rho\frac{\partial^2 v_3}{\partial \eta \partial \rho} \\ & + \frac{2(1-v)}{1-2v}\frac{1}{B}\frac{\partial}{\partial \eta}\Big[\Big(\frac{1}{R_1}+\frac{1}{R_2}\Big)v_3\Big] - \frac{1}{B}\Big(\frac{1}{R_1}-\frac{1}{R_2}\Big)\frac{\partial v_3}{\partial \eta} \\ & + \frac{2(1-v)}{1-2v}\frac{1}{B}\frac{\partial}{\partial \eta}\Big[\frac{1}{AB}\frac{\partial}{\partial \eta}(Av_2) + \frac{1}{AB}\frac{\partial}{\partial \xi}(Bv_1)\Big] \\ & - \frac{1}{A}\frac{\partial}{\partial \xi}\Big[\frac{1}{AB}\frac{\partial}{\partial \eta}(Av_1) - \frac{1}{AB}\frac{\partial}{\partial \xi}(Bv_2)\Big], \end{aligned} \tag{B 2}$$

$$\begin{aligned} l_1^{(3)}(v) = {} & \frac{2(1-v)}{1-2v}\Big(\frac{1}{R_1}+\frac{1}{R_2}\Big)\frac{\partial v_3}{\partial \rho} + \frac{1}{1-2v}\frac{1}{AB}\Big[\frac{\partial}{\partial \xi}\Big(B\frac{\partial v_1}{\partial \rho}\Big) + \frac{\partial}{\partial \eta}\Big(A\frac{\partial v_2}{\partial \rho}\Big)\Big] \\ & + \frac{1}{AB}\Big[\frac{\partial}{\partial \xi}\Big(\frac{B}{A}\frac{\partial v_3}{\partial \xi}\Big) + \frac{\partial}{\partial \eta}\Big(\frac{A}{B}\frac{\partial v_3}{\partial \eta}\Big)\Big], \end{aligned} \tag{B 3}$$

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$$
\begin{aligned}
l_2^{(3)}(v) = &-\rho\frac{2(1-\nu)}{1-2\nu}\Big(\frac{1}{R_1^2}+\frac{1}{R_2^2}\Big)\frac{\partial v_3}{\partial\rho} - \frac{2(1-\nu)}{1-2\nu}\Big(\frac{1}{R_1^2}+\frac{1}{R_2^2}\Big)v_3\\
&+\frac{2(1-\nu)}{1-2\nu}\frac{1}{AB}\Big\{\Big[\frac{\partial}{\partial\xi}\Big(\frac{B}{R_2}v_1\Big)+\frac{\partial}{\partial\eta}\Big(\frac{A}{R_1}v_2\Big)\Big]\\
&+\rho\frac{\partial}{\partial\rho}\Big[\frac{\partial}{\partial\xi}\Big(\frac{B}{R_2}v_1\Big)+\frac{\partial}{\partial\eta}\Big(\frac{A}{R_1}v_2\Big)\Big] - \Big(\frac{1}{R_1}+\frac{1}{R_2}\Big)\Big[\frac{\partial}{\partial\xi}(Bv_1)+\frac{\partial}{\partial\eta}(Av_2)\Big]\\
&-\rho\Big(\frac{1}{R_1}+\frac{1}{R_2}\Big)\frac{\partial}{\partial\rho}\Big[\frac{\partial}{\partial\xi}(Bv_1)+\frac{\partial}{\partial\eta}(Av_2)\Big]\Big\}\\
&-\frac{1}{AB}\Big\{\rho\frac{\partial}{\partial\xi}\Big(\frac{B}{R_2}\frac{\partial v_1}{\partial\rho}\Big)+\frac{\partial}{\partial\xi}\Big(\frac{B}{R_1}v_1\Big)-\rho\Big(\frac{1}{R_1}+\frac{1}{R_2}\Big)\frac{\partial}{\partial\xi}\Big(B\frac{\partial v_1}{\partial\rho}\Big)\Big\}\\
&-\frac{1}{AB}\Big\{\rho\frac{\partial}{\partial\eta}\Big(\frac{A}{R_1}\frac{\partial v_2}{\partial\rho}\Big)+\frac{\partial}{\partial\eta}\Big(\frac{A}{R_2}v_2\Big)-\rho\Big(\frac{1}{R_1}+\frac{1}{R_2}\Big)\frac{\partial}{\partial\eta}\Big(A\frac{\partial v_2}{\partial\rho}\Big)\Big\}\\
&+\frac{1}{AB}\rho\Big\{\frac{\partial}{\partial\xi}\Big[\frac{B}{A}\Big(\frac{1}{R_2}-\frac{1}{R_1}\Big)\frac{\partial v_3}{\partial\xi}\Big]-\Big(\frac{1}{R_1}+\frac{1}{R_2}\Big)\frac{\partial}{\partial\xi}\Big(\frac{B}{A}\frac{\partial v_3}{\partial\xi}\Big)\Big\}\\
&+\frac{1}{AB}\rho\Big\{\frac{\partial}{\partial\eta}\Big[\frac{A}{B}\Big(\frac{1}{R_1}-\frac{1}{R_2}\Big)\frac{\partial v_3}{\partial\eta}\Big]-\Big(\frac{1}{R_1}+\frac{1}{R_2}\Big)\frac{\partial}{\partial\eta}\Big(\frac{A}{B}\frac{\partial v_3}{\partial\eta}\Big)\Big\},
\end{aligned}
\tag{B 4}
$$

$$
m_1^{(1)}(v) = -\frac{1}{R_1}v_1 - \frac{1}{AR_1}\rho\frac{\partial v_3}{\partial\xi}
\tag{B 5}
$$

$$
m_1^{(2)}(v) = -\frac{1}{R_2}v_2 - \frac{1}{BR_2}\rho\frac{\partial v_3}{\partial\eta}
\tag{B 6}
$$

$$
m_1^{(3)}(v) = \frac{\nu}{1-2\nu}\Big\{\Big(\frac{1}{R_1}+\frac{1}{R_2}\Big)v_3+\frac{1}{AB}\Big[\frac{\partial}{\partial\xi}(Bv_1)+\frac{\partial}{\partial\eta}(Av_2)\Big]\Big\}
\tag{B 7}
$$

$$
\begin{aligned}
m_2^{(3)}(v) = &-\rho\frac{\nu}{1-2\nu}\Big(\frac{1}{R_1^2}+\frac{1}{R_2^2}\Big)v_3+\rho\frac{\nu}{1-2\nu}\Big\{-\frac{1}{AR_1}\frac{\partial v_1}{\partial\xi}\\
&+\frac{1}{AB}\frac{\partial}{\partial\xi}\Big(\frac{B}{R_2}\Big)v_1-\frac{1}{AB}\Big(\frac{1}{R_1}+\frac{1}{R_2}\Big)\frac{\partial B}{\partial\xi}v_1-\frac{1}{BR_2}\frac{\partial v_2}{\partial\eta}\\
&+\frac{1}{AB}\frac{\partial}{\partial\eta}\Big(\frac{A}{R_1}\Big)v_2-\frac{1}{AB}\Big(\frac{1}{R_1}+\frac{1}{R_2}\Big)\frac{\partial A}{\partial\eta}v_2\Big\}
\end{aligned}
\tag{B 8}
$$

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