J. Appl. Prob. **54,** 963–969 (2017)
*doi:*10.1017/jpr.2017.44
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ON THE THRESHOLD STRATEGIES IN OPTIMAL STOPPING PROBLEMS FOR DIFFUSION PROCESSES

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Abstract

We study a problem when the optimal stopping for a one-dimensional diffusion process is generated by a threshold strategy. Namely, we give necessary and sufficient conditions (on the diffusion process and the payoff function) under which a stopping set has a threshold structure.

Keywords: Diffusion process; optimal stopping; stopping set; threshold structure

2010 Mathematics Subject Classification: Primary 60G40
Secondary 60J60

1. Introduction

Let $(X_t,\ t \geq 0)$ be a diffusion process with values in the segment $D \subseteq \mathbb{R}^1$ with boundary points l and r defined on a stochastic basis $(\Omega, \mathcal{F}, \{\mathcal{F}_t,\ t \geq 0\}, \mathbb{P})$. Let us consider an optimal stopping problem for this process:

$$V(x) = \sup_{\tau} \mathbb{E}^x g(X_\tau) \mathrm{e}^{-\rho\tau}\, \mathbf{1}_{\{\tau<\infty\}}, \tag{1}$$

where $g\colon D \to \mathbb{R}^1$ is the payoff function, $\rho > 0$ is the discount rate, $\mathbf{1}_A$ is the indicator function of the set A, $\mathbb{E}^x$ means the expectation for the process X_t starting from the initial state x, and the supremum in (1) takes values over all stopping times τ (with respect to the natural filtration $\mathcal{F}_t = \sigma\{X_s,\ 0 \leq s \leq t\},\ t \geq 0$). We consider stopping times which can take infinite values (with positive probability).

Threshold strategies are among the most important and common strategies for one-dimensional optimal stopping problems, although there are simple examples for which the optimal stopping decision is not of threshold type; see, for example, [4].

In [7], Villeneuve was arguably the first to ask, can we find a set of easily testable conditions ensuring the existence of an optimal threshold strategy for a general state process? He stated the sufficient conditions under which the stopping set $S = \{x \in D\colon V(x) = g(x)\}$ for the geometric Itô diffusion process X_t has the threshold structure, i.e. $S = \{x \in D\colon x \geq x^*\}$ for some x^*; see [7, Theorem 4.2]. Later, Arkin [1] gave the necessary and sufficient conditions for the threshold structure of the stopping set in the case of Itô diffusion processes and the payoff functions from $C^2(D)$. Those conditions can be divided in three parts: to the left from threshold x^*, to the right from x^*, and locally at x^*. It is also necessary to mention that Crocce and Mordecki [3] studied the necessary and sufficient conditions under which the optimal

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https://doi.org/10.1017/jpr.2017.44 Published online by Cambridge University Press

stopping time in problem (1) is the first exit time when the underlying process exceeds a certain threshold. They considered more general diffusion processes also (not only Itô diffusions), however, their conditions were 'not local' at threshold x^*.

In this paper we state the necessary and sufficient conditions under which the stopping set in the optimal stopping problem for Itô diffusions has the threshold structure. We extend the results from [1] to the case of more general payoff functions and processes. The stated conditions can be viewed as a solution to the *inverse* optimal stopping problem when it is required to find the diffusion process and the payoff function such that the optimal decision (for example, the stopping set) will have the given threshold structure.

2. Assumptions

Let X_t, $t > 0$, be a diffusion process with values in the interval $D \subseteq \mathbb{R}^1$ with boundary points l and r, where $-\infty \le l < r \le +\infty$, open or closed (i.e. it may be (l, r), $[l, r)$, $(l, r]$, or $[l, r]$), which is the solution to the stochastic differential equation

$$\mathrm{d}X_t = a(X_t)\,\mathrm{d}t + \sigma(X_t)\,\mathrm{d}w_t, \qquad X_0 = x, \tag{2}$$

where w_t is a standard Wiener process, and $a\colon D \mapsto \mathbb{R}^1$ and $\sigma\colon D \mapsto \mathbb{R}^1_+$ are the drift and the diffusion coefficients, respectively. Following [6], we suppose that if the boundary point is inaccessible it is not included in D, and if the boundary point is not inaccessible then it is absorbing and the process remains forever at this point once reached. Denote $\mathcal{I} = \operatorname{int} D = (l, r)$.

We suppose that at any $x \in \mathcal{I}$, the local integrability condition

$$\int_{x-\varepsilon}^{x+\varepsilon} \frac{1 + |a(y)|}{\sigma^2(y)}\,\mathrm{d}y < \infty \quad \text{for some } \varepsilon > 0, \tag{3}$$

holds. This condition guarantees the existence of a weak solution of (2) and its regularity, i.e. starting from an arbitrary point $x \in \mathcal{I}$, the process X_t reaches any point $y \in \mathcal{I}$ in finite time with positive probability; see, for example, [5, Theorem 5.15].

Under condition (3) there are (unique up to constant positive multipliers) increasing and decreasing functions $\psi(x)$ and $\varphi(x)$ with absolutely continuous derivatives, which are the fundamental solutions to the ordinary differential equation

$$a(x) f'(x) + \tfrac{1}{2}\sigma^2(x) f''(x) = \rho f(x)$$

almost sure (with respect to Lebesgue measure) on the interval $\mathcal{I}$; see, for example, [5, Lemma 5.26]. Moreover, $0 < \psi(p), \varphi(p) < \infty$ for $p \in \mathcal{I}$. If l is an absorbing boundary point, we can define $\psi(l)$ in continuity.

The payoff function $g(x)$ is bounded below, and $g(x_0) > 0$ for some $x_0 \in \mathcal{I}$.

We say the function $g(x)$ is *almost piecewise-smooth* (with the exception set A) on the interval $(a_0, b) \subset \mathcal{I}$ if

- $g(x)$ is continuous on (a_0, b);
- there is a (finite or countable) set $A = \{a_1, \ldots, a_n, \ldots\}$, where $a_0 < a_1 < a_2 < \cdots \le b$, such that, for any c, $a_0 < c < b$, the set $A \cap (a_0, c]$ consists of only a finite number of points;
- the derivative $g'(x)$ is absolutely continuous on the intervals (a_i, a_{i+1}), $i \ge 0$, and there are one-sided derivatives $g'(a_0 + 0)$ and $g'(a_i \pm 0)$, $i \ge 1$.

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Note that if the function $g(x)$ is almost piecewise-smooth on the interval (a_0, b) then there exists $g''(x)$ on $(a_0, b) \setminus A$, and we can define the differential operator $\mathbb{L}g(x) = a(x)g'(x) + \frac{1}{2}\sigma^2(x)g''(x)$ for $x \in [a_0, b) \setminus A$. Moreover, $g'(x-0)$ exists for all $x \in (a_0, b)$, and the second distributional derivative g'' is a measure which consists of an absolutely continuous component and jumps. Thus, $g'(x-0)$ is a function of bounded variation (at any segment $[c, d] \subset (a_0, b)$), i.e. it can be represented as the difference of two monotone functions. Therefore, $g(x)$ for $x \in (a_0, b)$ is the difference of two convex (or concave) functions.

3. Main results

Recall that the stopping set for problem (1) is $S = \{x \in D\colon V(x) = g(x)\}$, where $V(x)$ is specified in (1).

Theorem 1. *Let, for some $p^* \in \mathcal{I}$, the payoff function $g(x)$ be almost piecewise-smooth (with the exception set A) on the interval (p^*, r), and, moreover, $\sum_{a_i \in A}\sigma^2(a_i)|g'(a_i+0) - g'(a_i-0)| < \infty$.*

(i) *If the following conditions hold:*

$$g(x) < \frac{\psi(x)g(p^*)}{\psi(p^*)} \quad \text{for } x < p^*, \tag{4}$$

$$\psi'(p^*)g(p^*) \geq \psi(p^*)g'(p^*+0), \tag{5}$$

$$\mathbb{L}g(x) \leq \rho g(x) \quad \text{almost surely (in the Lebesgue measure) for } x > p^*, \tag{6}$$

$$g'(a_i+0) - g'(a_i-0) \leq 0 \quad \text{for all } a_i \in A, \tag{7}$$

$$g(r) \leq \liminf_{y \uparrow r} g(y) \quad \text{if } r \text{ is an absorbing boundary point}, \tag{8}$$

then the stopping set S is $\{x \in D\colon x \geq p^*\}$.

(ii) *Inversely, if l and r are natural boundary points of the process X_t, payoff $g(x)$ is a continuous function on D, and $\limsup_{x \downarrow l} g(x)/\varphi(x) = 0$, then the threshold structure of the stopping set $S = \{x \in D\colon x \geq p^*\}$ implies conditions (4)–(8).*

The proof follows from a straightforward application of the following criteria for ρ-excessive functions from [6, Theorem 5.1].

Lemma 1. *A function $F\colon D \to \mathbb{R}_+$ is ρ-excessive, i.e. satisfies the inequality*

$$\mathbb{E}^x F(X_\tau)\mathrm{e}^{-\rho\tau}\,\mathbf{1}_{\{\tau<\infty\}} \leq F(x)$$

for all stopping times τ and all initial states $x \in D$, if and only if the following statements hold:

(i) *$F(x)$ is the difference of two convex functions for $x \in \mathcal{I}$;*

(ii) *the measure $(\mathcal{L}F)(\mathrm{d}x) = \frac{1}{2}\sigma^2(x)F''(\mathrm{d}x) + a(x)F'(x-0)\,\mathrm{d}x - \rho F(x)\,\mathrm{d}x$, where $F''(\mathrm{d}x)$ means the second distributional derivative, is nonpositive;*

(iii) *F is lower semicontinuous at absorbing boundary points.*

Proof of Theorem 1. Let us define the functions $h(p) = g(p)/\psi(p)$ and

$$\Phi(x) = \begin{cases} h(p^*)\psi(x) & \text{for } x < p^*, \\ g(x) & \text{for } x \geq p^*. \end{cases} \tag{9}$$

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Note that $\Phi(x) = \mathbb{E}^x g(X_{\tau^*})\mathrm{e}^{-\rho\tau^*}\mathbf{1}_{\{\tau^*<\infty\}}$, where $\tau^* = \inf\{t \geq 0\colon X_t \geq p^*\}$; see, for example, [2, Chapter II].

(i) Let conditions (4)–(8) hold. Apply Lemma 1 to the function $\Phi(x)$.

First, $\Phi(x)$ can be expressed as the difference of two convex functions (see the arguments at the end of the previous section).

Second, for any Borel set $B \subset D$, we have

$$\begin{aligned}(\mathcal{L}\Phi)(B) &= h(p^*)\int_B (\mathbb{L}\psi - \rho\psi)(x)\,\mathbf{1}_{\{x<p^*\}}\,\mathrm{d}x + \int_{B\setminus A}(\mathbb{L}g - \rho g)(x)\,\mathbf{1}_{\{x>p^*\}}\,\mathrm{d}x\\ &\quad + \frac{1}{2}\sigma^2(p^*)[g'(p^*+0) - h(p^*)\psi'(p^*)]\\ &\quad + \frac{1}{2}\sum_{a_i\in A\cap B}\sigma^2(a_i)[g'(a_i+0) - g'(a_i-0)]\\ &\leq 0.\end{aligned}$$

Thus, by Lemma 1, for all states $x \in D$ and stopping times τ, we have

$$\mathbb{E}^x g(X_\tau)\mathrm{e}^{-\rho\tau}\,\mathbf{1}_{\{\tau<\infty\}} \leq \mathbb{E}^x\Phi(X_\tau)\mathrm{e}^{-\rho\tau}\,\mathbf{1}_{\{\tau<\infty\}} \leq \Phi(x)$$

since $g(x) \leq \Phi(x)$ for all x due to (4). Hence, $V(x) \leq \Phi(x)$. On the other hand, obviously, $V(x) \geq \Phi(x)$.

Thus, $V(x) = \Phi(x)$ and for the stopping set, we have

$$S = \{x \in D\colon V(x) = g(x)\} = \{x \in D\colon \Phi(x) = g(x)\} = \{x \in D\colon x \geq p^*\}$$

since $\Phi(x) = h(p^*)\psi(x) > h(x)\psi(x) = g(x)$ for $x < p^*$ due to (4).

(ii) Let $\{x \in D\colon x \geq p^*\}$ be the stopping set. Then $V(x) < \infty$ for all $x \in \mathcal{I}$. Propositions 5.13 and 5.14 in [4] imply that under the requirements in Theorem 1(ii), the stopping time $\tau^* = \inf\{t \geq 0\colon X_t \geq p^*\}$ is optimal for problem (1). Therefore, $V(x) = \Phi(x)$, where $\Phi(x)$ is defined in (9).

Since the value function is ρ-excessive (see [6, Lemma 6.1]), we can apply Lemma 1 to the function $\Phi(x)$. Then, nonpositivity of the measure $(\mathcal{L}\Phi)(\mathrm{d}x)$ and almost piecewise-smoothness of $g(x)$ for $x \geq p^*$ imply conditions (6)–(7). Condition (5) follows from the relations

$$\begin{aligned}0 &\geq (\mathcal{L}\Phi)(\{p^*\})\\ &= \frac{1}{2}\sigma^2(p^*)[\Phi'(p^*+0) - \Phi'(p^*-0)]\\ &= \frac{1}{2}\sigma^2(p^*)\left[g'(p^*+0) - \frac{g(p^*)}{\psi(p^*)}\psi'(p^*)\right].\end{aligned}$$

Finally, in order to prove (4), note that in the continuation set $\{x \in D\colon x < p^*\}$ we have $g(x) < V(x) = \Phi(x) = (g(p^*)/\psi(p^*))\psi(x)$, which implies (4). □

Remark 1. The requirement that $\limsup_{x\downarrow l} g(x)/\varphi(x) = 0$ (Theorem 1(ii)) is essential for the necessity of condition (4). We give the appropriate example in the next section.

Remark 2. The above theorem extends the corresponding results in [1] to the case of more general payoff functions (not only twice-differentiable) and Itô diffusions (not only with natural boundary points). Crocce and Mordecki [3] studied the optimal stopping time of the threshold

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form for more general diffusion processes, and payoff functions satisfying so-called 'right regularity conditions'. Their conditions are close to ours, but in contrast to (5), which is 'the local' condition (at the threshold point p^* only), the corresponding condition in [3, Theorem 3.1] is nonlocal and depends also on values of the underlying functions for $x > p^*$.

The requirement that the payoff is almost a piecewise-smooth function is introduced for deriving the conditions of Theorem 1 in the explicit form. Without this requirement the above theorem can be slightly extended in more general terms. We say the process X_t^B is a restriction of the process X_t on the set B if $X_0^B \in \operatorname{int} B$ and $X_t^B = X_{t\wedge\tau_B}$, where $\tau_B = \inf\{t \geq 0\colon X_t \notin B\}$. In other words, the restricted (on B) process is started at a point in the interior of B and is absorbed when it reaches the boundary of B.

Theorem 2. *Let, for some $p^* \in \mathbb{1}$, there exist $g'(p^* + 0)$.*

(i) *If conditions (4), (5), and*

$$g(x) \text{ is } \rho\text{-excessive for the restriction of the process } X_t \text{ on } (p^*, r), \tag{10}$$

hold, then the stopping set S is $\{x \in D\colon x \geq p^*\}$.

(ii) *Inversely, if l and r are natural boundary points of the process X_t, payoff $g(x)$ is a continuous function on D, and $\limsup_{x\downarrow l} g(x)/\varphi(x) = 0$, then the threshold structure of the stopping set $S = \{x \in D\colon x \geq p^*\}$ implies conditions (4), (5), and (10).*

This result may be proved by similar arguments using Lemma 1 also.

4. Examples

Example 1. The requirement that $\limsup_{x\downarrow l} g(x)/\varphi(x) = 0$ in Theorem 1(ii) is essential, i.e. if it fails then assertion Theorem 1(ii) may not be valid.

Let us consider the process of geometric Brownian motion $\mathrm{d}X_t = \sigma X_t\,\mathrm{d}w_t$, the payoff function $g(x) = \max(1/x, x - 1)$, and the discount $\rho = \sigma^2$. For this case, we have $\psi(x)=x^2$, $\varphi(x)=1/x$, and $g(x)/\varphi(x)=1$ when $x \leq (1 + \sqrt{5})/2$. It is not difficult to prove that the value function for problem (1) in this case is $V(x) = 1/x + \frac{5}{27}x^2$ if $0 < x < p^* = 3$ and $V(x) = x - 1$ if $x \geq p^*$; see Figure 1. Indeed, in the continuation set, the value function is the linear combination $V(x) = c_1x^{-1} + c_2x^2$ with some $c_1, c_2 \geq 0$. But, if $c_1 > 1$ then the corresponding $V(x)$ is not a minimal majorant of $g(x)$. And if $c_1 < 1$ then $G(x) = \max(g(x), V(x))$ is not a ρ-excessive function since at the point p_0, where $g(p_0) = V(p_0)$, we have $\mathcal{L}G(\{p_0\}) = \frac{1}{2}\sigma^2(p^*)[G'(p_0+0) - G'(p_0-0)] = \frac{1}{2}\sigma^2(p^*)[(1-c_1)p_0^{-2} + 2c_2p_0] > 0$, which contradicts the ρ-excessiveness; see Lemma 1. So, in this example, the stopping set is $S = \{x \geq 3\}$, but condition (4) has obviously failed.

Example 2. Let the boundary points $l = 0$ and $r = \infty$ be natural, and the process X_t satisfy the following condition:

$$F(x) = x - \frac{\psi(x)}{\psi'(x)} \quad \text{is an increasing function.}$$

This is valid, for example, for the geometric Brownian motion $\mathrm{d}X_t = X_t(\alpha\,\mathrm{d}t + \sigma\,\mathrm{d}w_t)$, where $\alpha < \rho$.

Consider the payoff function $g(x) = (kx - I_1) \wedge I_2$, where $k > 0$. This case (with the process of the geometric Brownian motion) appears in the perpetual American capped call

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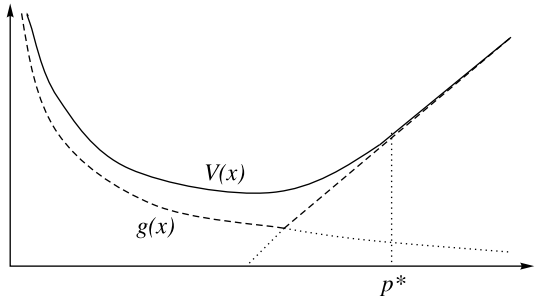


FIGURE 1: The payoff and the value function for Example 1.

option; see, for example, [4, Section 6.3]. To avoid the 'degenerative' case, we suppose also that $I_1 + I_2 > 0$.

For the natural boundary l, we have (see, for example, [2, Chapter II])

$$\lim_{x \downarrow l} \psi(x) = 0, \qquad \lim_{x \downarrow l} \psi'(x) = 0, \qquad \lim_{x \downarrow l} \varphi(x) = +\infty. \tag{11}$$

Denote $\bar{x} = (I_1 + I_2)/k$. An application of Theorem 1 to the above piecewise linear case yields the following result.

Corollary 1. *Suppose that* $\lim_{x \downarrow 0} F(x) < I_1/k < \lim_{x \uparrow \infty} F(x)$. *The stopping set for problem (1) has the threshold structure* $S = \{x \in D\colon x \geq p^*\}$ *if and only if*

$$I_1 > 0, \qquad p^* = \bar{x} \wedge F^{-1}\left(\frac{I_1}{k}\right), \tag{12}$$

$$a(x) \leq \rho x - \rho \frac{I_1}{k} \qquad \text{for } F^{-1}\left(\frac{I_1}{k}\right) < x < \bar{x}, \tag{13}$$

where F^{-1} *is the inverse function of* F.

Proof. Let $p_1 = F^{-1}(I_1/k)$ and $h(x) = g(x)/\psi(x)$. Note that, for $0 < x < \bar{x}$,

$$h'(x) = -\frac{k\psi'(x)}{\psi^2(x)}\left[F(x) - \frac{I_1}{k}\right].$$

First, let the stopping set have a threshold structure. Condition (4) implies that $I_1 \geq 0$, but $I_1 = 0$ contradicts (4) for small x (due to (11)). Therefore, $I_1 > 0$.

If $p^* > \bar{x}$ then, for $\bar{x} \leq x < p^*$, we have $g(x) = g(p^*) = h(p^*)\psi(p^*) > h(p^*)\psi(x)$, which contradicts (4). So, $p^* \leq \bar{x}$.

Let $p_1 < \bar{x}$. It is easy to see that inequalities $p^* < p_1$ or $p^* > p_1$ contradicts (5) or (4), respectively. The $p_1 = \bar{x}$ and $p^* < p_1$ cases are analogous.

Therefore, (12) holds. Now, (13) follows directly from (6).

Second, inversely, let (12) and (13) hold. Then $h'(x) > 0$ for $0 < x < p^*$, which proves (4). Condition (5) follows from $h'(p_1) = 0$ (if $p_1 < \bar{x}$), or $g'(\bar{x} + 0) = 0$ (if $p_1 \geq \bar{x}$). Then

$$Lg(x) - \rho g(x) = \rho I_1 - k[\rho x - a(x)] \leq 0 \quad \text{for } p_1 \leq x < \bar{x},$$
$$Lg(x) - \rho g(x) = -\rho I_2 < 0 \quad \text{for } x > \bar{x},$$

therefore, (6) is valid. Finally, (7) follows from $k > 0$.

Thus, by Theorem 1, the stopping set is $S = \{x \in D\colon x \geq p^*\}$. □

https://doi.org/10.1017/jpr.2017.44 Published online by Cambridge University Press

Acknowledgements

This work was supported by the Russian Foundation for Basic Research (grant number 15-06-03723). The authors would like to thank the anonymous reviewers for their helpful comments and suggestions which helped to improve the paper.

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