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FRAÏSSÉ LIMITS FOR RELATIONAL METRIC STRUCTURES

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Abstract. The general theory developed by Ben Yaacov for metric structures provides Fraïssé limits which are approximately ultrahomogeneous. We show here that this result can be strengthened in the case of relational metric structures. We give an extra condition that guarantees exact ultrahomogenous limits. The condition is quite general. We apply it to stochastic processes, the class of diversities, and its subclass of L_1 diversities.

§1. Introduction. The concepts of homogeneity and universality pervade many areas of mathematics. Both concepts play a central role in the Fraïssé limit [8] of a class of finite structures with the amalgamation property. For instance, the Rado (or random) graph [19], which is the Fraïssé limit of the class of undirected finite graphs, is universal for the class of countable graphs, and ultrahomogeneous in the sense that its isomorphic finite subgraphs are automorphic in the graph. The conjunction of these two properties makes the Rado graph unique up to isomorphism. This behaviour is entirely typical for Fraïssé limits.

For structures in the classical sense, countability is essential to ensure this uniqueness. However, we are mainly interested in the setting of a complete metric space X with additional structure defined on it. In this context, algebraic embeddings turn into isometric embeddings preserving the structure; countability turns into separability, while the spaces themselves are usually uncountable. The Urysohn metric space $\mathbb{U}$ is analogous to the Rado graph; it was first described by Urysohn [19] in 1927, curiously, 26 years before the introduction of Fraïssé limits. The space $\mathbb{U}$ is the completion of the Fraïssé limit of finite metric spaces with rational distances. It is determined by being universal for the separable metric spaces, and ultrahomogeneous in the sense that its isometric finite subspaces are automorphic in the space.

Ben Yaacov [19] has developed a Fraïssé theory for metric structures that is analogous to the classical Fraïssé theory. Under his theory a class of finitely generated structures is confirmed to satisfy some conditions (analogues of the HP, JEP, and AP of the classical theory along with others) and then it is known that the class has a Fraïssé limit. Under this framework the Urysohn space is the Fraïssé limit of the finite metric spaces, the Urysohn sphere is the limit of finite metric spaces bounded by one, and ℓ_2 is the Fraïssé limit of the finite-dimensional Hilbert spaces. In each of these cases the limit is ultrahomogeneous, but a weaker result is

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actually established by Ben Yaacov's Fraïssé theory: approximate ultrahomogeneity, meaning that finite partial isomorphism can only be extended to all of the space up to some error. This is necessary for any theory that includes the Guararij space as the limit of the finite-dimensional Banach space, since this limit is only approximately ultrahomogeneous.

Here we will show that if we restrict ourselves to *relational* metric structures, we are able to add another condition to Ben Yaacov's theory to obtain limits which are exactly ultrahomogeneous, rather than just approximately ultrahomogeneous. The extra condition, which we call the bounded Amalgamation Property (bAP), requires that one-point amalgamations come with a bound on how far apart the two amalgamated points are in the amalgamation. For relational metric structures, Ben Yaacov's conditions along with bAP guarantee strictly ultrahomogeneous Fraïssé limits.

We provide three examples of our result. The first is to construct a universal ultrahomogeneous stochastic process taking values in any finite set. The second is to provide a short proof of the existence and uniqueness of the universal ultrahomogeneous diversity. Diversities were introduced in [5] as a generalization of metric spaces in which a non-negative value is assigned to all finite sets of points, and not just to pairs. We have established a construction of this diversity by independent means in previous work [4]; here it is derived easily from a more general theory. Finally, we investigate the existence of universal ultrahomogeneous L_1 metrics and diversities. L_1 diversities, in analogy with L_1 metric spaces, are diversities that can be embedded in the function space L_1. Both L_1 metrics and diversities are important in applications including combinatorial optimization [6, 7] and phylogenetics [1, 3]. We show that there is a universal ultrahomogeneous L_1 diversity, which supports the naturalness of the concept of L_1 diversities. In contrast, we establish that there cannot be a universal ultrahomogeneous L_1 metric space, since the class of finite L_1 metric spaces does satisfy the amalgamation property. These last results appears to be new, despite the fact that L_1 metrics have been studied for decades.

§2. Fraïssé limits for metric structures. Ben Yaacov's theory [19] concerns *metric structures*: these are metric spaces with collections of relations (which in continuous logic are real-valued functions of tuples of elements) and functions (which take tuples of elements to other elements). Here we will use Ben Yaacov's theory only for *relational metric structures*, which are metric structures having no functions, and consequently no constants. This restriction has the advantage that finite structures are precisely the same as finitely generated structures.

The following corresponds to [19, Definition 3.1].

Definition 1. Let $\mathcal{L}$ be a collection of symbols (all of which we think of as predicate symbols) each with an associated natural number which is its arity. An $\mathcal{L}$-*structure* $\mathfrak{A}$ consists of a complete metric space (A, d) together with, for each n-ary predicate symbol $R \in \mathcal{L}$, a uniformly continuous interpretation $R^{\mathfrak{A}} \colon A^n \to \mathbb{R}$. The symbol d is a distinguished binary predicate symbol. We will follow the convention that Latin letters correspond to the domain of a given structure, e.g., $\mathrm{dom}(\mathfrak{A}) = A$.

An *embedding* of $\mathcal{L}$-structures $\phi \colon \mathfrak{A} \to \mathfrak{B}$ is a map $\phi \colon A \to B$ such that for each n-ary predicate symbol $R \in \mathcal{L}$ and all $\bar{a} \in A^n$

$$R^{\mathfrak{B}}(\phi(\bar{a})) = R^{\mathfrak{A}}(\bar{a}).$$

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Note that since d is one of the predicate symbols, embeddings are always isometric. We define $\mathfrak{A}$ to be a *substructure* of $\mathfrak{B}$ if there is an embedding $\phi\colon \mathfrak{A} \to \mathfrak{B}$. An *isomorphism* is a surjective embedding. An *automorphism* is an isomorphism from a structure to itself.

A *partial isomorphism* $\phi\colon \mathfrak{A} \dashrightarrow \mathfrak{B}$ is an embedding $\phi\colon A_0 \to B$ where $A_0 \subseteq A$. We say that such a partial isomorphism is *finite* if its domain A_0 is finite.

Corresponding to [19, Definition 3.3], we say a separable structure $\mathfrak{M}$ is *approximately ultrahomogeneous* if every finite partial isomorphism $\phi\colon \mathfrak{M} \dashrightarrow \mathfrak{M}$ is arbitrarily close to the restriction of an automorphism of $\mathfrak{M}$: for every $\varepsilon > 0$, there is an automorphism f of $\mathfrak{M}$ such that $d(\phi a, fa) < \varepsilon$ for all $a \in \mathrm{dom}\phi$.

We say a separable structure $\mathfrak{M}$ is *ultrahomogeneous* if every finite partial isomorphism $\phi\colon \mathfrak{M} \dashrightarrow \mathfrak{M}$ extends to an automorphism.

Following [19, Definition 3.5], we have the following definitions.

DEFINITION 2. Let $\mathcal{K}$ be a class of finite structures.

1. By a $\mathcal{K}$-*structure* we mean an $\mathcal{L}$-structure $\mathfrak{A}$ such that all finite substructures of $\mathfrak{A}$ are in $\mathcal{K}$.
2. We say $\mathcal{K}$ has *HP* (*the Hereditary Property*) if all members of $\mathcal{K}$ are $\mathcal{K}$-structures.
3. Suppose $\mathcal{K}$ has HP. We say $\mathcal{K}$ has *AP* (*the Amalgamation Property*) if for every $\mathfrak{B}, \mathfrak{C} \in \mathcal{K}$ and embeddings $f_B\colon \mathfrak{A} \to \mathfrak{B}$ and $f_C\colon \mathfrak{A} \to \mathfrak{C}$ from a third finite structure $\mathfrak{A} \in \mathcal{K}$, there is a finite structure $\mathfrak{D} \in \mathcal{K}$ and embeddings $g_B\colon \mathfrak{B} \to \mathfrak{D}$, $g_C\colon \mathfrak{C} \to \mathfrak{D}$ with $g_B \circ f_B = g_C \circ f_C$ on A.
4. Suppose $\mathcal{K}$ has HP. The conditions on $\mathcal{K}$ for when *NAP* (*the Near Amalgamation Property*) holds are the same as for when AP holds, except we only require that for all $\varepsilon > 0$ the existence of $g_B, g_C, \mathfrak{D}$ such that
$$d(g_B(f_B(a)), g_C(f_C(a))) < \varepsilon$$
for all $a \in A$.
5. We say that $\mathcal{K}$ has *JEP* (*the Joint Embedding Property*) if every two structures in $\mathcal{K}$ embed into a third one.

Note that for the relational structures we consider here, AP implies NAP.

A key part of Ben Yaacov's framework is studying the space of tuples of elements from metric structures as a metric space itself; these are called the enumerated structures. The metric he defines on the enumerated structures is analogous to the Gromov–Hausdorff distance between compact metric spaces.

DEFINITION 3. Let $\mathcal{K}$ be a class of finite structures with NAP. Let $\mathcal{K}_n$ be the *enumerated structures of length n*, the set of all n-element tuples of members of some structures from $\mathcal{K}$.

So, $\bar{a} \in \mathcal{K}_n$ means that there is some structure $\mathfrak{A}$ with domain A such that $\bar{a} \in A^n$. The tuple $\bar{a}$ is not a structure in $\mathcal{K}$ itself, because there may be repeated entries, and order is relevant. However, in a natural way, for each k-ary predicate symbol R and each k-length sub-tuple of $\bar{a}$, we can apply R to that sub-tuple of $\bar{a}$ using the interpretation of R in $\mathcal{K}$. Similarly, we can talk of embedding the tuples in $\mathcal{K}_n$ in structures in $\mathcal{K}$. This is any map that takes the entries of $\bar{a} \in \mathcal{K}_n$ and maps them into a structure $\mathcal{K}$ while preserving the values of all the relations.

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DEFINITION 4. Let $\bar{a}, \bar{b} \in \mathcal{K}_n$. We define $d_{\mathcal{K}}(\bar{a}, \bar{b}) = \inf_{\bar{a}', \bar{b}'} \max_i d(\bar{a}'_i, \bar{b}'_i)$ where $\bar{a}', \bar{b}'$ are images of $\bar{a}, \bar{b}$ under embeddings of structures into a third structure in $\mathcal{K}$.

Informally, for given enumerated structures $\bar{a}$ and $\bar{b}$ in $\mathcal{K}_n$ we consider embedding them simultaneously in a common structure in $\mathcal{K}$. We take the maximum distance in between corresponding a_i and b_i in the embedding for all i. Then we take the infimum of this quantity over all such embeddings to get $d_{\mathcal{K}}(\bar{a}, \bar{b})$.

Ben Yaacov makes the following observation as a comment, citing his Lemma 3.8.

LEMMA 5. *If $\mathcal{K}$ is a class of finite structures satisfying NAP and JEP then $d_{\mathcal{K}}$ is a pseudometric on $\mathcal{K}_n$ for each n.*

PROOF. Use JEP to show that $d_{\mathcal{K}}(\bar{a}, \bar{b})$ is well-defined and non-negative for all size-n structures $\bar{a}, \bar{b}$. Symmetry is straightforward, as is $d_{\mathcal{K}}(\bar{a}, \bar{a}) = 0$. For the triangle inequality, consider enumerated structures $\bar{a}, \bar{b}, \bar{c}$. Let $\varepsilon > 0$. Let $(\bar{a}', \bar{b}')$ be a joint embedding of $\bar{a}$ and $\bar{b}$ so that $\max_i d(\bar{a}'_i, \bar{b}'_i) \leq d_{\mathcal{K}}(\bar{a}, \bar{b}) + \varepsilon/3$, and likewise for $(\bar{b}'', \bar{c}'')$ so that $\max_i d(\bar{b}''_i, \bar{c}''_i) \leq d_{\mathcal{K}}(\bar{b}, \bar{c}) + \varepsilon/3$. Use NAP to get an amalgamation $(\bar{a}', \bar{b}', \bar{b}'', \bar{c}'')$ where $\max_i d(\bar{b}', \bar{b}'') \leq \varepsilon/3$, but distances between $\bar{a}'$ and $\bar{b}'$, and between $\bar{b}''$ and $\bar{c}''$ are preserved. Then for all i

$$d(a'_i, c''_i) \leq d(a'_i, b'_i) + d(b'_i, b''_i) + d(b''_i, c''_i) \leq d_{\mathcal{K}}(a_i, b_i) + d_{\mathcal{K}}(b_i, c_i) + \varepsilon$$

and so, taking the maximum over all i and letting ε go to zero gives $d_{\mathcal{K}}(a_i, c_i) \leq d_{\mathcal{K}}(a_i, b_i) + d_{\mathcal{K}}(b_i, c_i)$. ⊣

Our next two definitions follow [19, Definition 3.12]:

DEFINITION 6. Let $\mathcal{K}$ be a class of finite structures.

1. We say $\mathcal{K}$ satisfies *PP* (*the Polish Property*) if $\mathcal{K}_n$ is separable and complete under $d_{\mathcal{K}}$ for every n.
2. We say $\mathcal{K}$ satisfies *CP* (*the Continuity Property*) if for every n-ary predicate symbol P the map from $\mathcal{K}_n \to \mathbb{R}$ given by $\bar{a} \to P^{\bar{a}}(\bar{a})$ is continuous.

DEFINITION 7. Let $\mathcal{K}$ be a class of finite $\mathcal{L}$-structures. We say that $\mathcal{K}$ is a *Fraïssé class* if $\mathcal{K}$ satisfies HP, JEP, AP, PP, CP.

Our Def. 7 differs from Ben Yaacov's in that we use AP and not NAP, because we want to obtain ultrahomogeneous limits and not just approximately ultrahomogeneous limits (though we will require another, stronger condition).

Ben Yaacov does not define the extension properties explicitly, but they are implicit in his Corollary 3.20. We define them here and use them as an alternative definition of a limit of a Fraïssé class.

DEFINITION 8. Let $\mathcal{K}$ be a Fraïssé class and $\mathfrak{M}$ be a separable $\mathcal{K}$ structure.

1. $\mathfrak{M}$ has the *approximate extension property* if for all finite $\mathcal{K}$ structures $\mathfrak{B}$, finite enumerated structures $\bar{a}$ with elements in B, embedding $h: \bar{a} \to \mathfrak{M}$ and $\varepsilon > 0$, there is an embedding $f: \mathfrak{B} \to \mathfrak{M}$ such that $d(f\bar{a}, h\bar{a}) < \varepsilon$. Here d denotes the maximum distance between corresponding elements of the two enumerated structures.
2. $\mathfrak{M}$ has the *exact extension property* if for all finite $\mathcal{K}$ structures $\mathfrak{B}$, finite enumerated structures $\bar{a}$ with elements in B, and embedding $h: \bar{a} \to \mathfrak{M}$, there is an embedding $f: \mathfrak{B} \to \mathfrak{M}$ that extends h.

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In the following our approximate limits correspond to Ben Yaacov's limits (his Def. 3.15).

DEFINITION 9. Let $\mathcal{K}$ be a Fraïssé class and $\mathfrak{M}$ be a separable $\mathcal{K}$ structure.

1. $\mathfrak{M}$ is an *approximate limit* of $\mathcal{K}$ if $\mathfrak{M}$ has the approximate extension property.
2. $\mathfrak{M}$ is an *exact limit* of $\mathcal{K}$ if $\mathfrak{M}$ has the exact extension property.

Corresponding to Ben Yaacov's Lemma 3.17, Theorem 3.19, and Theorem 3.21 we have the following results.

LEMMA 10. *Every Fraïssé class has an approximate limit.*

THEOREM 11. *The approximate limit of a Fraïssé class is unique up to isomorphism.*

THEOREM 12. *Let $\mathcal{K}$ be a class of finite relational structures. Then the following are equivalent*:

(i) *$\mathcal{K}$ is a Fraïssé class*
(ii) *$\mathcal{K}$ is the class of all finite substructures of a separable approximately ultrahomogeneous structure $\mathfrak{M}$.*

Furthermore, $\mathfrak{M}$ is the approximate limit of $\mathcal{K}$, and is hence unique up to isomorphism and universal for separable $\mathcal{K}$-structures.

§3. Approximate versus exact Fraïssé limits. Here we show that if the $\mathcal{K}$-structures satisfy a property we call bAP (for "bounded Amalgamation Property") then approximate Fraïssé limits are in fact exact Fraïssé limits. Recall from Definition 3 that for a class $\mathcal{K}$ of finite $\mathcal{L}$-structures $\mathcal{K}_n$ is the class of enumerated $\mathcal{K}$-structures of length n.

There is another way to define the distance between two enumerated relational structures that does not appear to have an analogue in Ben Yaacov's paper. We define

$$d_\infty(\bar{a},\bar{b}) = \max_{m\le n} \max_{X\subseteq\{1,\dots,n\},|X|=m,R \text{ arity } m} |R(\bar{a}_X) - R(\bar{b}_X)|, \tag{1}$$

where we include the metric d as a binary predicate. For each m, $1 \le m \le n$, we look at all predicates of arity m. We then look at all subsets of $\bar{a}$ of size m and the corresponding subset of $\bar{b}$ and look at the difference between the predicates on those subsets. We take the max of the difference over all such m, predicates R, and index sets X. Unlike with $d_{\mathcal{K}}$, d_∞ does make use of any common embeddings of $\bar{a}$ and $\bar{b}$ into another metric structure. d_∞ can be thought of as a distance obtained by using the predicates to map enumerated structures into ℓ_∞^k for some k. We will establish the Lipschitz equivalence of $d_{\mathcal{K}}$ and d_∞ for given structures and then use this to prove statements about the space $(\mathcal{K}_n, d_{\mathcal{K}})$.

DEFINITION 13. We say that $\mathcal{K}$ satisfies *bAP* (*the bounded Amalgamation Property*) if there is a constant c depending only on n such that if $(\bar{a},w)$ and $(\bar{a},z)$ are two enumerated structures in $\mathcal{K}_n$ with common substructure $\bar{a}$, then there is an amalgamation $B = (\bar{a},w,z)$ in $\mathcal{K}_{n+1}$ such that

$$d_B(w,z) \le c\, d_\infty((\bar{a},w),(\bar{a},z)).$$

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The idea behind bAP is to strengthen AP so that we have some control on how far apart the non-common points are in the amalgamated space. So if every predicate R takes almost the same value on corresponding subsets of $(\bar{a}, w)$ and $(\bar{a}, z)$, then w and z are very close to each other in the amalgamation.

Note that bAP implies AP. To see this, observe that (forgetting the metric bound) this is a one-point amalgamation result for enumerated structures of the same length. By taking repeated elements if necessary, this is a one-point amalgamation result for finite structures. An induction argument gives the general amalgamation result.

The next result follows a result for metric spaces originally due to Urysohn; see [19, Theorem 3.4] or [9, Theorem 1.2.7]. (We proved a similar result, but in less generality, in [4, Lemma 17].) Recall Definition 8 where definitions of approximate and exact extension properties are given.

THEOREM 14. *Let $\mathcal{K}$ be a Fraïssé class and $\mathfrak{M}$ be a $\mathcal{K}$-structure. If $\mathcal{K}$ satisfies bAP and $\mathfrak{M}$ satisfies the approximate extension property, then $\mathfrak{M}$ satisfies the exact extension property.*

PROOF. Let $\bar{a}$ be an enumerated structure of length n with elements taken from $\mathcal{M}$. Let z be a point such that $(\bar{a}, z) \in \mathcal{K}$ with $\bar{a}$ embedded in $(\bar{a}, z)$. It suffices to show that there is a sequence $w_0, w_1, \ldots$ in $\mathfrak{M}$ such that for all p,

$$d(w_p, w_{p+1}) \leq 3 \cdot 2^{-(p+1)} \text{ and } d_\infty((\bar{a}, z), (\bar{a}, w_p)) \leq 2^{-p}. \tag{2}$$

Because $\mathfrak{M}$ is complete, the first part of (2) shows that $\{w_p\}$ has a limit w. The second part of (2) shows that $(\bar{a}, w)$ is isomorphic to $(\bar{a}, z)$, as required.

Using induction, we will construct the sequence $\{w_p\}$ satisfying conditions (2), along with structures $M_p = (\bar{a}, z, w_0, \ldots, w_p)$ which are extensions of both $(\bar{a}, z)$ and $(\bar{a}, w_0, \ldots, w_p)$. In particular, the points $\{w_p\}$ and the structures M_p will satisfy for all $p \geq 0$:

I. $d_\infty((\bar{a}, z), (\bar{a}, w_p)) \leq 2^{-p}$,
II. (a) M_p is an extension of $(\bar{a}, w_0, \ldots, w_p)$,
    (b) M_p is an extension of $(\bar{a}, z)$, and
    (c) $d(w_p, z) \leq 2^{-p}$ in M_p.

First, we use the approximate extension property to get a $w_0 \in M$ such that

$$d_\infty((\bar{a}, w_0), (\bar{a}, z)) \leq \min(c^{-1}, 1),$$

where c is the constant (depending on n) in bAP for $\mathcal{K}$. So w_0 satisfies condition (I), for $p = 0$. Then using bAP, there is an amalgamation $M_0 = (\bar{a}, z, w_0)$ in which $d(z, w_0) < 1$, thereby satisfying condition (II) for $p = 0$.

For the inductive step, suppose, for $p \geq 0$, we have $w_0, \ldots, w_p$ and a structure M_p satisfying the conditions (I) and (II) above. We show there exist w_{p+1} and M_p, satisfying the corresponding conditions for $p + 1$.

Condition (II.a) allows us to use the approximate extension property to get $w_{p+1} \in M$ so that

$$d_\infty((\bar{a}, w_0, \ldots w_{p+1}), (\bar{a}, w_0, \ldots, w_p, z)) \leq \min(c^{-1}, 1)2^{-(p+1)}.$$

This immediately gives us $d_\infty((\bar{a}, w_{p+1}), (\bar{a}, z)) \leq 2^{-(p+1)}$ which is condition (I) for $p + 1$. bAP allows us to amalgamate $(\bar{a}, w_0, \ldots w_{p+1})$ and $(\bar{a}, w_0, \ldots w_p, z)$ to get

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$(\bar{a}, w_0, \ldots w_{p+1}, z)$. bAP also gives us $d(w_{p+1}, z) \leq 2^{-(p+1)}$ in the amalgamation, and so we have all of condition (II) for $p+1$. This concludes the inductive argument.

Now it remains to show that $d(\{w_p, w_{p+1}\}) \leq 3 \cdot 2^{-(p+1)}$ for each $p \geq 1$. In the structure $(\bar{a}, w_0, \ldots, w_p, w_{p+1}, z)$ we have both $d(w_p, z) \leq 2^{-p}$ and $d(w_{p+1}, z) \leq 2^{-(p+1)}$, so the triangle inequality gives the result. ⊣

THEOREM 15. *Let $\mathcal{K}$ be a class of finite relational structures satisfying HP, JEP, bAP, PP, CP. Then $\mathcal{K}$ is a Fraïssé class, and its limit $\mathfrak{M}$ is ultrahomogeneous (therefore an exact limit in our terminology), in addition to being universal for separable $\mathcal{K}$-structures. It is the unique such structure.*

PROOF. bAP implies AP, so $\mathcal{K}$ is a Fraïssé class. Hence $\mathcal{K}$ has a unique approximate limit. By the previous lemma, this approximate limit is in fact an exact limit. ⊣

§4. A universal ultrahomogeneous stochastic process. In this section we apply our extension of Ben Yaacov's metric Fraïssé theory to stochastic processes. We show the existence of a unique universal ultrahomogeneous stochastic process with a separable index set taking values in a finite set. A key step will be showing how to view stochastic processes as metric structures.

We recall the definition and basic theory of stochastic processes. Let the state space S be a finite set and index set T be arbitrary. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A *stochastic process* is an S-valued family of random variables $(X_t)_{t \in T}$ on Ω. It is customary to denote the random variable X_t by $X(t)$.

Every stochastic process has an associated family of finite-dimensional distributions. For every tuple $(t_1, \ldots, t_k)$ of distinct elements, $t_i \in T$ and tuple $(s_1, \ldots, s_k)$, $s_i \in S$ we define

$$F_{t_1,\ldots,t_k}(s_1, \ldots, s_k) = \mathbb{P}(X(t_1) = s_1, \ldots, X(t_k) = s_k).$$

In order to simplify notation, in what follows we will denote tuples of values by using an overbar, so that $\bar{t} = (t_1, \ldots, t_k)$ and $\bar{s} = (s_1, \ldots, s_k)$. For example, we would write the above definition as $F_{\bar{t}}(\bar{s}) = \mathbb{P}(X(\bar{t}) = \bar{s})$. For any stochastic process on T taking values in S, the finite-dimensional distributions satisfy, for any k-tuple of distinct values $\bar{t}$ and any k-tuple of states $\bar{s}$:

1. $F_{\bar{t}}(\bar{s}) \in [0, 1]$.
2. $\sum_{\bar{s} \in S^k} F_{\bar{t}}(\bar{s}) = 1$.
3. $\sum_{z \in S} F_{\bar{t}, t_{k+1}}(\bar{s}, z) = F_{\bar{t}}(\bar{s})$.
4. For any permutation σ of k-tuples, $F_{\sigma(\bar{t})}(\sigma(\bar{s})) = F_{\bar{t}}(\bar{s})$.

The Kolmogorov existence theorem guarantees that any such family of functions $F_{\bar{t}}$ satisfying properties (1), (2), (3), (4) are the finite-dimensional distributions of some stochastic process. Of course, there is not a one-to-one relationship between stochastic processes and families of finite-dimensional distributions: any set of finite-dimensional distributions identifies a whole class of stochastic processes. In what follows, our theory will work at the level of families of finite-dimensional distributions, or in other words, stochastic processes in the weak sense. However, in order to make arguments more transparent we will often refer to a given stochastic process $(X_t)_{t \in T}$ whose distribution is $F_{\bar{t}}$.

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We can identify each such family of finite-dimensional distributions with a metric structure, once we add a single non-degeneracy condition: We assume that

5. $\sum_{s\in S} F_{t_1,t_2}(s,s) < 1$ for $t_1 \neq t_2$.

This is equivalent to requiring $\mathbb{P}(X(t_1) = X(t_2)) < 1$ for $t_1 \neq t_2$. We can view this either as a restriction on the stochastic processes or as identifying the point t_1 and t_2 in T when $X(t_1) = X(t_2)$ with probability 1. In either case, it ensures that the index set T is endowed with a metric, as the following lemma shows.

LEMMA 16. *Let T be a set and F a family of finite-dimensional distributions on T for a process taking values in a finite set S that satisfies condition* (5). *For $t_1, t_2 \in T$, define the function*

$$d(t_1,t_2) = 1 - \sum_{s\in S} F_{t_1,t_2}(s,s).$$

Then (T,d) is a metric space.

Note that if F consists of the finite-dimensional distributions for a stochastic process X, then

$$d(t_1,t_2) = \mathbb{P}(X(t_1) \neq X(t_2)).$$

PROOF. First note that $d(t,t) = 1 - \sum_{s\in S} F_{t,t}(s,s) = 0$ by property (2). Also, property (5) implies that $d(t_1,t_2) > 0$ if $t_1 \neq t_2$.

Next, property (4) implies $d(t_1,t_2) = d(t_2,t_1)$.

To check the triangle inequality, we switch to the stochastic process viewpoint. Note that $X(t_1) \neq X(t_3)$ implies at least one of $X(t_1) \neq X(t_2)$ and $X(t_2) \neq X(t_3)$. So

$$\begin{aligned} d(t_1,t_3) &= \mathbb{P}(X(t_1) \neq X(t_3)) \\ &\leq \mathbb{P}(X(t_1) \neq X(t_2)) + \mathbb{P}(X(t_2) \neq X(t_3)) \\ &= d(t_1,t_2) + d(t_2,t_3). \end{aligned}$$

⊣

To view stochastic processes satisfying property (5) as relational metric structures, with domain being the index set T, we only need to define for every k-tuple of values from S the predicates

$$R_{\bar{s}}(\bar{t}) = F_{\bar{t}}(\bar{s}) = \mathbb{P}(X(\bar{t}) = \bar{s}).$$

So the language $\mathcal{L}$ consists of the set of predicate symbols $R_{\bar{s}}$ for any finite tuple of values $\bar{s}$, along with metric d, which we always interpret by $d(t_1,t_2) = 1 - \sum_{s\in S} R_{s,s}(t_1,t_2)$.

We need to show that each of these predicates is uniformly continuous.

LEMMA 17. *For each k-tuple $\bar{s}$, the relation $R_{\bar{s}}$ is 1-Lipschitz in each of its arguments.*

PROOF. Without loss of generality, consider the first argument. For $t_1, t_1^* \in T$ we have

$$\begin{aligned} R_{s_1,\ldots,s_k}(t_1,\ldots,t_k) &= \mathbb{P}(X(t_1) = s_1, \ldots, X(t_k) = s_k) \\ &\leq \mathbb{P}(X(t_1^*) = s_1, \ldots, X(t_k) = s_k) + \mathbb{P}(X(t_1^*) \neq X(t_1)) \\ &= R_{s_1,\ldots,s_k}(t_1^*,\ldots,t_k) + d(t_1,t_1^*), \end{aligned}$$

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where we have used the fact that $X(t_1) = s_1$ can only happen if at least one of $X(t_1^*) = s_1$ or $X(t_1^*) \neq X(t_1)$ occurs. Switching t_1 and t_1^* gives

$$R_{s_1,\dots,s_k}(t_1^*, \dots, t_k) \leq R_{s_1,\dots,s_k}(t_1, \dots, t_k) + d(t_1, t_1^*),$$

and hence

$$|R_{s_1,\dots,s_k}(t_1^*, \dots, t_k) - R_{s_1,\dots,s_k}(t_1, \dots, t_k)| \leq d(t_1, t_1^*). \qquad \dashv$$

Now we formally define the class of metric structures $\mathcal{SP}$.

DEFINITION 18. A *separable non-degenerate S-valued stochastic process* is a separable metric space (T, d) with k-ary predicates $R_{\bar{s}}$ for $k \geq 2$, $\bar{s} \in S^k$ such that, if we let $F_{\bar{t}}(\bar{s}) = R_{\bar{s}}(\bar{t})$ for all $k, \bar{s} \in S^k, \bar{t} \in T^k$, then conditions (1) through (4) as well as

$$(5') \qquad 1 - \sum_{s \in S} F_{t_1,t_2}(s, s) = d(t_1, t_2)$$

hold. We denote the set of all such metric structures by $\mathcal{SP}$.

We will not actually use the notation $R_{\bar{s}}$ in what follows. Rather, we will stay with either $F_{\bar{t}}(\bar{s})$ or $X(\bar{t})$ to keep in line with probabilistic notation.

To summarize, any metric structure satisfying Definition 18 gives a family of finite-dimensional distributions corresponding to a non-degenerate stochastic process on T taking values in S. On the other hand, any such non-degenerate stochastic process corresponds to a metric structure of the type in Definition 18. This metric structure view of non-degenerate stochastic processes immediately gives a definition of completion and separability for stochastic processes, namely, whether the space (T, d) with the induced metric d is complete and separable, respectively.

Let $\mathcal{K}$ be the class of all finite metric structures in $\mathcal{SP}$. We will show that $\mathcal{K}$ is a Fraïssé class satisfying bAP. Note that for any two n-tuples in $\mathcal{K}_n$, we have

$$\begin{aligned} d_\infty(\bar{a}, \bar{b}) &= \max_{|A| \leq n, \bar{s} \in S^{|A|}} |R_{\bar{s}}(\bar{a}_A) - R_{\bar{s}}(\bar{b}_A)| \\ &= \max_{|A| \leq n, \bar{s} \in S^{|A|}} |\mathbb{P}(X(\bar{a}_A) = \bar{s}) - \mathbb{P}(X(\bar{b}_A) = \bar{s})| \end{aligned}$$

where A runs over all subsets of $\{1, \dots, n\}$.

Here is our main result for this section.

THEOREM 19. *The class $\mathcal{K}$ of all finite non-degenerate stochastic processes satisfies HP, JEP, bAP, PP, and CP, and therefore has a unique Fraïssé limit.*

A key component of the proof of this theorem is the Coupling Lemma of probability theory (see, for example [19, p. 19]). Between any two random variables X and Y taking values in some finite state space U, we can define the total variation distance between their distributions as

$$d_{TV}(X, Y) = \sup_{A \subseteq U} |\mathbb{P}(X \in A) - \mathbb{P}(Y \in A)| = \frac{1}{2} \sum_{u \in U} |\mathbb{P}(X = u) - \mathbb{P}(Y = u)|.$$

The Coupling Lemma asserts the following: given any two U-valued random variables X and Y, there are U-valued random variables $\tilde{X}$ and $\tilde{Y}$ defined on

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the same probability space so that $\tilde{X}$ has the same distribution as X, $\tilde{Y}$ has the same distribution as Y, and

$$\mathbb{P}(\tilde{X} \neq \tilde{Y}) = d_{TV}(X, Y).$$

In what follows, we will use the Coupling Lemma with X and Y being tuples of random variables of length n, and U being S^n.

PROOF OF THEOREM 19. HP. This follows immediately.

JEP. This follows from bAP if we identify a single point in each of the two index sets of the stochastic processes we want to jointly embed.

bAP. Let $X = (\bar{a}, w)$ and $Y = (\bar{a}, z)$ be two enumerated structures in $\mathcal{K}_{n+1}$, where X and Y agree on $\bar{a}$. We wish to amalgamate X and Y to obtain $Z = (\bar{a}, w, z)$.

To do this, we shift to random variable notation. X and Y correspond to two random pairs $(\mathbf{X}_a, \mathbf{X}_w)$ and $(\mathbf{Y}_a, \mathbf{Y}_z)$ such that $\mathbf{X}_a$ and $\mathbf{Y}_a$ (taking values in S^n) have the same distribution and $\mathbf{X}_w$ and $\mathbf{Y}_z$ take values in S. We will create a random variable $(\mathbf{Z}_a, \mathbf{Z}_w, \mathbf{Z}_z)$ such that $(\mathbf{Z}_a, \mathbf{Z}_w)$ has the same distribution as $(\mathbf{X}_a, \mathbf{X}_w)$, $(\mathbf{Z}_a, \mathbf{Z}_z)$ has the same distribution as $(\mathbf{Y}_a, \mathbf{Y}_z)$, and we can bound the probability that $\mathbf{Z}_w$ and $\mathbf{Z}_z$ are different.

Fix $\overline{s} \in S^n$. We let $\mathbf{X}_{a|\overline{s}}$ denote the random variable $\mathbf{X}_w$ conditioned on $\mathbf{X}_a = \overline{s}$; this is the random variable defined by the distribution

$$\mathbb{P}[\mathbf{X}_{a|\overline{s}} = s] = \frac{\mathbb{P}[(\mathbf{X}_a, \mathbf{X}_w) = (\overline{s}, s)]}{\mathbb{P}[\mathbf{X}_a = \overline{s}]}.$$

Let $\mathbf{Y}_{a|\overline{s}}$ be defined similarly.

For each value of $\overline{s}$ we apply the Coupling Lemma to $\mathbf{X}_{a|\overline{s}}$ and $\mathbf{Y}_{a|\overline{s}}$ to obtain the jointly defined random variables $\mathbf{Z}^X_{a|\overline{s}}$ and $\mathbf{Z}^Y_{a|\overline{s}}$. $\mathbf{Z}^X_{a|\overline{s}}$ has the same distribution as $\mathbf{X}_{a|\overline{s}}$, $\mathbf{Z}^Y_{a|\overline{s}}$ has the same distribution as $\mathbf{Y}_{a|\overline{s}}$, and

$$\mathbb{P}\left(\mathbf{Z}^X_{a|\overline{s}} \neq \mathbf{Z}^Y_{a|\overline{s}}\right) = d_{TV}(\mathbf{X}_{a|\overline{s}}, \mathbf{Y}_{a|\overline{s}}).$$

We can assemble $(\mathbf{Z}_a, \mathbf{Z}_w, \mathbf{Z}_z)$ as follows. For each $\overline{s} \in S^n$ define

$$\mathbb{P}[(\mathbf{Z}_a, \mathbf{Z}_w, \mathbf{Z}_z) = (\overline{s}, s_1, s_2)] = \mathbb{P}[\mathbf{X}_a = \overline{s}]\mathbb{P}[\mathbf{Z}^X_{a|\overline{s}} = s_1, \mathbf{Z}^Y_{a|\overline{s}} = s_2].$$

We let the tuple (a, w, z) correspond to the random variable $(\mathbf{Z}_a, \mathbf{Z}_w, \mathbf{Z}_z)$. It remains to show that we can bound $d(w, z)$. First we re-express the quantity in terms of $\mathbf{Z}^X_{a|\overline{s}}$ and $\mathbf{Z}^Y_{a|\overline{s}}$ by conditioning on $\mathbf{Z}_a$.

$$\begin{aligned} d(w, z) &= \mathbb{P}(\mathbf{Z}_w \neq \mathbf{Z}_z) \\ &= \sum_{\overline{s}} \mathbb{P}[\mathbf{Z}_a = \overline{s}]\mathbb{P}[\mathbf{Z}_w \neq \mathbf{Z}_z | \mathbf{Z}_a = \overline{s}] \\ &= \sum_{\overline{s}} \mathbb{P}[\mathbf{Z}_a = \overline{s}]\mathbb{P}[\mathbf{Z}^X_{a|\overline{s}} \neq \mathbf{Z}^Y_{a|\overline{s}}]. \end{aligned}$$

By our use of the Coupling Lemma we then have

$$d(w, z) = \sum_{\overline{s}} \mathbb{P}[\mathbf{X}_a = \overline{s}] d_{TV}(\mathbf{X}_{a|\overline{s}}, \mathbf{Y}_{a|\overline{s}})$$

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$$= \sum_{\overline{s}} \mathbb{P}[\mathbf{X}_a = \overline{s}] \frac{1}{2} \sum_{s \in S} \left| \mathbb{P}(\mathbf{X}_{a|\overline{s}} = s) - \mathbb{P}(\mathbf{Y}_{a|\overline{s}} = s) \right|$$

$$\leq \frac{1}{2} \sum_{\overline{s}} \sum_{s \in S} |\mathbb{P}[(\mathbf{X}_a, \mathbf{X}_w) = (\overline{s}, s)] - \mathbb{P}[(\mathbf{Y}_a, \mathbf{Y}_z) = (\overline{s}, s)]|$$

$$\leq \frac{1}{2} |S|^{n+1} d_\infty((a, w), (a, z)),$$

as required.

PP. Note that, for each n, the conditions for a non-degenerate stochastic process define a closed subspace of $\mathbb{R}^k$ for a sufficiently large k. It suffices to show that $d_{\mathcal{K}}$ and d_∞ are Lipschitz equivalent, since d_∞ is Lipschitz equivalent to the Euclidean metric. We will show, for any pair of n-tuples $\bar{a}, \bar{b}$ that

$$\frac{1}{n} d_\infty(\bar{a}, \bar{b}) \leq d_{\mathcal{K}}(\bar{a}, \bar{b}) \leq |S|^n d_\infty(\bar{a}, \bar{b}).$$

To prove the first inequality, suppose that we have $\bar{a}$ and $\bar{b}$ embedded together in $(\bar{a}, \bar{b})$ so that $\max_i \mathbb{P}(X(a_i) \neq X(b_i)) \leq d_{\mathcal{K}}(\bar{a}, \bar{b}) + \varepsilon$ for some $\varepsilon > 0$. For any subset X of $\{1, \ldots, n\}$ and any element α of $S^{|X|}$ we have

$$|\mathbb{P}(X(a_X) = \alpha) - \mathbb{P}(X(b_X) = \alpha)| \leq \mathbb{P}(X(a_X) \neq X(b_X)) \leq \mathbb{P}(X(\bar{a}) \neq X(\bar{b}))$$
$$\leq n \max_i \mathbb{P}(X(a_i) \neq X(b_i)) \leq n d_{\mathcal{K}}(\bar{a}, \bar{b}) + n\varepsilon.$$

Dividing by n and letting ε go to zero gives the first inequality.

To prove the second inequality, note that for any $A \subseteq S^n$, we have

$$|\mathbb{P}(X(\bar{a}) \in A) - \mathbb{P}(X(\bar{b}) \in A)| \leq \sum_{\alpha \in A} |\mathbb{P}(X(\bar{a}) = \alpha) - \mathbb{P}(X(\bar{b}) = \alpha)| \leq |S|^n d_\infty(\bar{a}, \bar{b}).$$

So we obtain the following from the definition of total variation distance:

$$d_{TV}(X(\bar{a}), X(\bar{b})) \leq |S|^n d_\infty(\bar{a}, \bar{b}).$$

Using the Coupling Lemma on the whole random vectors $X(\bar{a})$ and $X(\bar{b})$, we can find a joint embedding so that $\mathbb{P}(X(\bar{a}) \neq X(\bar{b})) \leq d_{TV}(X(\bar{a}), X(\bar{b})) \leq |S|^n d_\infty(\bar{a}, \bar{b})$. Now we have that

$$\max_i \mathbb{P}(X(a_i) \neq X(b_i)) \leq \mathbb{P}(X(\bar{a}) \neq X(\bar{b})) \leq |S|^n d_\infty(\bar{a}, \bar{b})$$

and the inequality is established.

CP. The n-ary predicates on $\mathcal{K}$ are, for tuples $\alpha \in S^n$,

$$R_\alpha(t_1, \ldots, t_n) = \mathbb{P}(X(t_1) = \alpha_1, \ldots, X(t_n) = \alpha_n).$$

Let $\bar{a}$ and $\bar{b}$ be two structures of length n such that $d_{\mathcal{K}}(\bar{a}, \bar{b}) = \varepsilon$. Let them be jointly embedded in $(\bar{a}, \bar{b})$ such that $\max_i d(a_i, b_i) \leq 2\varepsilon$. Then Lemma 17 implies that

$$|R_\alpha(\bar{a}) - R_\alpha(\bar{b})| \leq 2n\varepsilon.$$

as required. ⊣

Now we will translate the existence of a Fraïssé limit for the class $\mathcal{K}$ into the language of stochastic processes. We will need to introduce some terminology that is non-standard in the context of stochastic processes. Let X_t for $t \in T$ and Y_u for

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$u \in U$ be two S-valued stochastic processes. Let $\phi\colon T \to U$ be given. The map ϕ *embeds* (T, X) in (U, Y) if the finite-dimensional distributions of $Y_{\phi(t)}, t \in T$ are identical to those of $X_t, t \in T$. Furthermore, ϕ is an *isomorphism* if ϕ is onto. An *automorphism* is an isomorphism from a stochastic process to itself. A stochastic process is *non-degenerate* if $d(t_1, t_2) := \mathbb{P}(X(t_1) \neq X(t_2)) > 0$ is non-zero for all $t_1 \neq t_2$. A stochastic process is *separable* if (T, d) is a separable metric space. (This conflicts with the notion of separability in the stochastic processes literature.) A stochastic process $X_t, t \in T$ is *finite* if T is finite.

A stochastic process is *universal* if any finite S-valued stochastic process can be embedded in it. A map ϕ is a *partial isomorphism* from $X_t, t \in T$ to $Y_u, u \in U$ if $\phi\colon T_0 \to U$ is an embedding for some $A_0 \subseteq T$. We say ϕ is a finite partial isomorphism if A_0 is finite. A stochastic processes is *ultrahomogeneous* if any finite partial isomorphism from the process to itself can be extended to an automorphism.

COROLLARY 20. *There is a separable universal ultrahomogeneous process* $\mathbb{X}_t, t \in \mathbb{T}$ *that is unique up to isomorphism*.

§5. Diversities. A *diversity* [5] is a pair (X, δ) where X is a set and δ is a function from the finite subsets of X to $\mathbb{R}$ satisfying

(D1) $\delta(A) \geq 0$, and $\delta(A) = 0$ if and only if $|A| \leq 1$.
(D2) If $B \neq \emptyset$ then $\delta(A \cup B) + \delta(B \cup C) \geq \delta(A \cup C)$

for all finite $A, B, C \subseteq X$. Diversities form an extension of the concept of a metric space. Property (D2) is the *triangle inequality* and is the analogue of the triangle inequality for metric spaces. Every diversity has an *induced metric*, given by $d(a, b) = \delta(\{a, b\})$ for all $a, b \in X$. Note also that δ is *monotonic*: $A \subseteq B$ implies $\delta(A) \leq \delta(B)$. Also δ is *subadditive on sets with nonempty intersection*: $\delta(A \cup B) \leq \delta(A) + \delta(B)$ when $A \cap B \neq \emptyset$ [5, Proposition 2.1]. Monotonicity and subadditivity on overlapping sets are also sufficient to establish the triangle inequality: If $B \neq \emptyset$,

$$\delta(A \cup C) \leq \delta(A \cup B \cup C) \leq \delta(A \cup B) + \delta(B \cup C). \tag{3}$$

Just as a semimetric generalizes a metric space by allowing $d(x, y) = 0$ for $x \neq y$, we define a *semidiversity* to be a pair (X, δ) that satisfy (D1) and (D2) above except that we may have $\delta(A) = 0$ for $|A| > 1$.

In [4] we constructed the diversity analog $(\mathbb{U}, \delta_{\mathbb{U}})$ of the Urysohn metric space. It is determined uniquely by being universal for separable diversities, and ultrahomogeneous in the sense that isomorphic finite subdiversities are automorphic. We also showed that the induced metric space of $(\mathbb{U}, \delta_{\mathbb{U}})$ is the Urysohn metric space. Our method of constructing this Urysohn diversity in [4] was analogous to Katětov's construction of the Urysohn metric space [19]. Andreas Hallbäck [10] has published results connecting our research on the universal ultrahomogeneous diversity to descriptive set theory. For instance, he shows that the automorphism group is a universal Polish group.

Here we demonstrate the existence of $(\mathbb{U}, \delta_{\mathbb{U}})$ via Ben Yaacov's theory of metric Fraïssé limits [19], as we described in Section 2, along with our results in Section 3 is order to prove that the Fraïssé limit is ultrahomogeneous and not just approximately ultrahomogeneous.

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In order to study diversities using Ben Yaacov's theory, it is necessary to describe them as metric structures. Every diversity immediately has a metric space associated with it, the induced metric. The diversity function δ takes a variable number of distinct arguments, so this does not immediately fit into model theory. Hence we define δ_k as a predicate for $k \geq 1$. We define δ_k of a tuple in X^k to be δ of the set of values that the tuple takes. For example,

$$\delta_3(x, y, x) = \delta(\{x, y\})$$

for all $x, y \in X$. In this way, diversities are relational metric structures with a countable number of predicates, one with arity m for every $m \geq 2$.

What axioms do diversities satisfy as metric structures? In the following all tuples are assumed to be of non-zero length. The set function δ being a diversity is equivalent to

1. $\delta_1(x) = 0$ for all points $x \in X$.
2. δ_k is permutation invariant.
3. If $\delta_k(a_1, \ldots, a_k) = 0$ then $a_1 = \cdots = a_k$.
4. For all $a_1, \ldots, a_k \in X$, $\delta_{k+1}(a_1, \ldots, a_{k-1}, a_k, a_k) = \delta_k(a_1, \ldots, a_{k-1}, a_k)$.
5. $\delta_{k+1}(\bar{a}, b) \geq \delta_k(\bar{a})$ for all $\bar{a} \in X^k$, $b \in X$.
6. For all tuples $\bar{a}, \bar{b}, \bar{c}$ with lengths j, k, ℓ, if $k \geq 1$

$$\delta_{j+\ell}(\bar{a}, \bar{c}) \leq \delta_{j+k}(\bar{a}, \bar{b}) + \delta_{k+\ell}(\bar{b}, \bar{c}).$$

Note that these conditions imply that $d \equiv \delta_2$ is a metric on X. To be a metric structure, we need to confirm that each predicate is uniformly continuous, as shown in the following lemma.

LEMMA 21. *Let (X, δ) be a diversity. For each n, the function $\delta^{(k)}$ is 1-Lipschitz in each argument.*

PROOF. Consider varying the ith argument of $\delta^{(k)}$ from x_i to x_i'. We know from the triangle inequality that

$$\begin{aligned}\delta^{(k)}(x_1, \ldots, x_i, \ldots, x_k) &= \delta(\{x_1, \ldots, x_i, \ldots, x_k\}) \\ &\leq \delta(\{x_1, \ldots, x_i', \ldots, x_k\}) + \delta(\{x_i, x_i'\}) \\ &= \delta^{(k)}(x_1, \ldots, x_i', \ldots, x_k) + d(x_i, x_i').\end{aligned}$$

Similarly, $\delta^{(k)}(x_1, \ldots, x_i', \ldots, x_k) \leq \delta^{(k)}(x_1, \ldots, x_i, \ldots, x_k) + d(x_i, x_i')$. So

$$|\delta^{(k)}(x_1, \ldots, x_i, \ldots, x_k) - \delta^{(k)}(x_1, \ldots, x_i', \ldots, x_k)| \leq d(x_i, x_i')$$

as required. ⊣

Since these are closed conditions, for any finite set X with $|X| = n$, the set of diversities can be viewed as a closed subset of $\mathbb{R}^k$ for $k = n + n^2 + \cdots + n^n$.

Diversities have already been given a definition of completeness and separability in [19]: a diversity is complete if its induced metric space is complete and it is separable if its induced metric space is separable. Fortunately, these correspond precisely with the definitions one would get with viewing diversities as metric structures, so the Fraïssé limit will give us exactly what we want.

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We define a special one-point amalgamation of diversities that yields the bounded Amalgamation Property (bAP).

DEFINITION 22. Let $(X \cup \{z_1\}, \delta)$ and $(X \cup \{z_2\}, \delta)$ be two diversities that agree on X. The amalgamation of the diversities is defined on $X \cup \{z_1, z_2\}$ by

$$\delta(A \cup \{z_1, z_2\}) \tag{4}$$

$$= \max\left[\sup_{B \subseteq X} \delta(A \cup B \cup \{z_1\}) - \delta(B \cup \{z_2\}),\ \sup_{C \subseteq X} \delta(A \cup C \cup \{z_2\}) - \delta(C \cup \{z_1\})\right].$$

The idea of this definition of the amalgamation is to obtain the minimal diversity that extends both the diversities on $X \cup \{z_1\}$ and $X \cup \{z_2\}$. To see this, note that however we define the amalgamation, the triangle inequality requires

$$\delta(A \cup \{z_1, z_2\}) \geq \delta(A \cup B \cup \{z_1\}) - \delta(B \cup \{z_2\})$$

for all $B \subseteq X$ and

$$\delta(A \cup \{z_1, z_2\}) \geq \delta(A \cup C \cup \{z_2\}) - \delta(C \cup \{z_1\})$$

for all $C \subseteq X$. So the value of $\delta(A \cup \{z_1, z_2\})$ can not be less than the one we have defined.

LEMMA 23. *The amalgamation given by Definition 22 is a diversity on* $X \cup \{z_1, z_2\}$.

PROOF. To establish that δ is a diversity, it suffices to prove

(a) monotonicity: $\delta(A) \leq \delta(B)$ when $A \subseteq B$ and
(b) subadditivity on overlapping sets: $A \cap B \neq \emptyset$ implies $\delta(A \cup B) \leq \delta(A) + \delta(B)$,

see (3).

Monotonicity with respect to A follows easily from where A appears in the definition. To see $\delta(A \cup \{z_1, z_2\}) \geq \delta(A \cup \{z_1\})$, just let $B = \emptyset$, and likewise for $\delta(A \cup \{z_1, z_2\}) \geq \delta(A \cup \{z_2\})$ with $C = \emptyset$.

For subadditivity on overlapping sets, start with the sets $A_1 \cup \{z_1\}$ and $A_2 \cup \{z_2\}$ where $A_1 \cap A_2 \neq \emptyset$. Let $\varepsilon > 0$ be given. Without loss of generality, suppose we have a set B such that

$$\delta(A_1 \cup A_2 \cup \{z_1, z_2\}) - \varepsilon \leq \delta(A_1 \cup A_2 \cup B \cup \{z_1\}) - \delta(B \cup \{z_2\}).$$

The triangle inequality gives $\delta(A_1 \cup A_2 \cup B \cup \{z_1\}) \leq \delta(A_1 \cup \{z_1\}) + \delta(A_2 \cup B)$, and so

$$\delta(A_1 \cup A_2 \cup \{z_1, z_2\}) - \varepsilon \leq \delta(A_1 \cup \{z_1\}) + \delta(A_2 \cup B) - \delta(B \cup \{z_2\}).$$

The triangle inequality again gives $\delta(A_2 \cup B) \leq \delta(A_2 \cup \{z_2\}) + \delta(B \cup \{z_2\})$ and so

$$\delta(A_1 \cup A_2 \cup \{z_1, z_2\}) - \varepsilon \leq \delta(A_1 \cup \{z_1\}) + \delta(A_2 \cup \{z_2\}).$$

Since $\varepsilon > 0$ was arbitrary, we have $\delta(A_1 \cup A_2 \cup \{z_1, z_2\}) \leq \delta(A_1 \cup \{z_1\}) + \delta(A_2 \cup \{z_2\})$.

Next, consider the sets $A_1 \cup \{z_1\}$ and $A_2 \cup \{z_1, z_2\}$, where A_1 and A_2 do not necessarily intersect. Let $\varepsilon > 0$ be given. Suppose there exists $B \subseteq X$ such that

$$\delta(A_1 \cup A_2 \cup \{z_1, z_2\}) - \varepsilon \leq \delta(A_1 \cup A_2 \cup B \cup \{z_1\}) - \delta(B \cup \{z_2\}).$$

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The triangle inequality gives $\delta(A_1 \cup A_2 \cup B \cup \{z_1\}) \leq \delta(A_1 \cup \{z_1\}) + \delta(A_2 \cup B \cup \{z_1\})$ and so

$$\begin{aligned}\delta(A_1 \cup A_2 \cup \{z_1, z_2\}) - \varepsilon &\leq \delta(A_1 \cup \{z_1\}) + \delta(A_2 \cup B \cup \{z_1\}) - \delta(B \cup \{z_2\}) \\ &\leq \delta(A_1 \cup \{z_1\}) + \delta(A_2 \cup \{z_1, z_2\}),\end{aligned}$$

where we have used the definition of $\delta(A_2 \cup \{z_1, z_2\})$.

On the other hand, suppose there exists a $C \subseteq X$ such that

$$\delta(A_1 \cup A_2 \cup \{z_1, z_2\}) - \varepsilon \leq \delta(A_1 \cup A_2 \cup C \cup \{z_2\}) - \delta(C \cup \{z_1\}).$$

Add and subtract $\delta(A_1 \cup C \cup \{z_1\})$ from the left-hand side to get

$$\begin{aligned}\delta(A_1 \cup A_2 \cup \{z_1, z_2\}) - \varepsilon &\leq \delta(A_1 \cup A_2 \cup C \cup \{z_2\}) - \delta(A_1 \cup C \cup \{z_1\}) + \\ &\quad \delta(A_1 \cup C \cup \{z_1\}) - \delta(C \cup \{z_1\}) \\ &\leq \delta(A_2 \cup \{z_1, z_2\}) + \delta(A_1 \cup \{z_1\}),\end{aligned}$$

where we have used the definition of the amalgamation and the fact that

$$\delta(A_1 \cup C \cup \{z_1\}) - \delta(C \cup \{z_1\} \leq \delta(A_1 \cup \{z_1\})$$

by the triangle inequality. ⊣

THEOREM 24. *The class of finite diversities viewed as relational metric structures satisfies HP, JEP, bAP, PP, CP, and is therefore a Fraïssé class with an ultrahomogeneous limit.*

PROOF. To obtain HP, observe that the diversity axioms are just equalities and inequalities that hold for all points in the diversity, so taking a subset of the diversity cannot violate any of the axioms.

To obtain JEP, for any two finite diversities (X, δ_X) and (Y, δ_Y), let Z be the disjoint union of X and Y. Let k be the largest value taken by either of the diversities δ_X and δ_Y. Let δ_Z be defined as the extension of δ_X and δ_Y such that $\delta_Z(A) = k$ for any set A with nonzero intersection with both X and Y. It is straightforward to check that (Z, δ_Z) is a diversity.

To show bAP, we apply the bounded amalgamation in Def. 22 to the two tuples $(\bar{a}, z_1)$ and $(\bar{a}, z_2)$ to get $(\bar{a}, z_1, z_2)$. Within this tuple, we have that

$$d(z_1, z_2) = \delta(\{z_1, z_2\}) = \sup_{A \subseteq \bar{a}} |\delta(A \cup \{z_1\}) - \delta(A \cup \{z_2\})|$$

giving $d(z_1, z_2) \leq d_\infty((\bar{a}, z_1), (\bar{a}, z_2))$, as required.

To show PP: The set of all enumerated diversities on n points can be viewed as a subset of $\mathbb{R}^k$ for some finite k. This subset is closed and separable under the Euclidean metric, since it is the set of points that satisfies a family of non-strict inequalities. So this subset of $\mathbb{R}^k$ is a Polish space. We just need to show that for each n, $d_{\mathcal{K}}$ is Lipschitz equivalent to the Euclidean metric. Since the Euclidean metric is Lipschitz equivalent to d_∞, it suffices to show that $d_{\mathcal{K}}$ is Lipschitz equivalent to d_∞ which is the content of Lemma 25.

To show CP: The only n-ary predicate on $\mathcal{K}$ is the nth diversity predicate $\delta^{(n)}(x_1, \ldots, x_n) = \delta(\{x_1, \ldots, x_n\})$, deleting repeated elements in the set listing. Let $\bar{a}$,

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$\bar{b}$ be two diversities in $\mathcal{K}_n$. Suppose $d_{\mathcal{K}}(\bar{a}, \bar{b}) = \varepsilon$. Let $(\bar{a}', \bar{b}')$ be an embedding of $\bar{a}$, $\bar{b}$ so that $\max_i(a_i', b_i') \leq 2\varepsilon$. By Lemma 21,

$$|\delta^{(n)}(\bar{a}) - \delta^{(n)}(\bar{b})| = |\delta'(\bar{a}') - \delta'(\bar{b}')| \leq 2n\varepsilon,$$

showing that the map $\delta^{(n)} \colon \mathcal{K}_n \to \mathbb{R}$ is Lipschitz continuous. ⊣

LEMMA 25. *For all n and* $\bar{a}, \bar{b}$

$$\frac{1}{n} d_\infty(\bar{a}, \bar{b}) \leq d_{\mathcal{K}}(\bar{a}, \bar{b}) \leq d_\infty(\bar{a}, \bar{b}).$$

PROOF. The first inequality follows from the predicates being one-Lipschitz. In particular, suppose we have an embedding of $\bar{a}$ and $\bar{b}$ into a third diversity $((\bar{a}', \bar{b}'), \delta)$ so that $\max_i d(a_i', b_i') < d_{\mathcal{K}}(\bar{a}, \bar{b}) + \varepsilon$. For any set $X \subseteq \{1, \ldots, n\}$

$$\begin{aligned} |\delta_a(\bar{a}_X) - \delta_b(\bar{b}_X)| = |\delta(\bar{a}'_X) - \delta(\bar{b}'_X)| &\leq \sum_{i=1}^{n} d(\bar{a}_i', \bar{b}_i') \leq n \max_i d(\bar{a}_i', \bar{b}_i') \\ &< n d_{\mathcal{K}}(\bar{a}, \bar{b}) + n\varepsilon, \end{aligned}$$

where we have used Lemma 21. Taking the limit as $\varepsilon \to 0$ and dividing by n gives the first inequality.

To prove the second inequality we will repeatedly use the bAP for diversities. Suppose $d_\infty(\bar{a}, \bar{b}) = \varepsilon$. We will construct a joint embedding of $\bar{a}$ and $\bar{b}$ into a new diversity where corresponding points in the tuples are never further than ε away from each other.

First we identify a_1 and b_1. Consider the two tuples (a_1, b_1, a_2) and (a_1, b_1, b_2). Since $d_\infty(\bar{a}, \bar{b}) = \varepsilon$, we know that $d_\infty((a_1, b_1, a_2), (a_1, b_1, b_2)) \leq \varepsilon$, and so we can use bAP for diversities to find a joint embedding (a_1, b_1, a_2, b_2) where $d(a_2, b_2) \leq \varepsilon$. We repeat this step for the tuples $(a_1, b_1, a_2, b_2, a_3)$ and $(a_1, b_1, a_2, b_2, b_3)$, and so forth. In the end we have a joint embedding of all the points in $\bar{a}$ and $\bar{b}$ such that $d(a_i, b_i) \leq \varepsilon$ for all $i = 1, \ldots, k$, as required. ⊣

§6. L_1 metrics and L_1 diversities. Recall that $L_1(\Omega, \mathcal{A}, \mu)$ is the set of $\mathcal{A}$-measurable functions f defined on Ω with $\int_\Omega |f(\omega)| d\mu(\omega) < \infty$ equipped with the metric

$$d(f, g) = \int_\Omega |f(\omega) - g(\omega)| d\mu(\omega).$$

An L_1 *metric space* is a metric space that can be embedded in $L_1(\Omega, \mathcal{A}, \mu)$ for some measure space $(\Omega, \mathcal{A}, \mu)$. (Here an embedding is a injective map that preserves the metric.) There is a well-developed theory of L_1 metric spaces, including many alternative characterizations of them in the finite case [7, Chapters 3 and 4]. In particular, a metric on a finite set X is L_1 if and only if it can be written as a non-negative linear combination of *cut semimetrics*, $d_{U|\bar{U}}$ [7, Theorem 4.2.6]. To explain, letting $\bar{U} = X \setminus U$, the cut semimetric $d_{U|\bar{U}}$ is defined by

$$d_{U|\bar{U}}(x, y) = \begin{cases} 1, & \text{if } (x \in U \text{ and } y \in \bar{U}) \text{ or } (y \in U \text{ and } x \in \bar{U}), \\ 0, & \text{otherwise.} \end{cases}$$

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Then d is an L_1 metric if and only if

$$d(x,y) = \sum_{U \subseteq X} \lambda_U d_{U|\bar{U}}(x,y)$$

for some $\lambda_U \geq 0$.

L_1 diversities were introduced in [6]. To define this class of diversities, we first define a particular diversity function δ on $L_1(\Omega, \mathcal{A}, \mu)$. For any finite set of functions F in $L_1(\Omega, \mathcal{A}, \mu)$ we define the diversity of F to be

$$\delta(F) = \int_\Omega \max_{f \in F} f(\omega) - \min_{g \in F} g(\omega)\, d\omega. \tag{5}$$

In [6, p. 4] we showed that this is a diversity, and if we restrict F to only having two points this gives the L_1 metric as its induced metric. Now we define an L_1 *diversity* to be a diversity that can be embedded in $L_1(\Omega, \mathcal{A}, \mu)$ with the diversity function δ given by (5), for some measure space $(\Omega, \mathcal{A}, \mu)$.

Analogous to cut semimetrics, for any partition $U|\bar{U}$ of a set X there is a cut semidiversity given by [6, p. 9]

$$\delta_{U|\bar{U}}(A) = \begin{cases} 1, & \text{if } (A \cap U \neq \emptyset) \text{ and } (A \cap \bar{U} \neq \emptyset), \\ 0, & \text{otherwise.} \end{cases}$$

In [6, Proposition 9] we showed that a finite diversity (X, δ) is L_1 if and only if δ can be written as a non-negative linear combination of *cut semidiversities*:

$$\delta(A) = \sum_{U \subseteq X} \lambda_U \delta_{U|\bar{U}}(A),$$

where all $\lambda_U \geq 0$. The induced metric of an L_1 diversity is L_1, and conversely, every L_1 metric is the induced metric of some L_1 diversity. Both of these facts follow from the definition of both L_1 metrics and L_1 diversities in terms of embedding into $L_1(\Omega, \mathcal{A}, \mu)$.

In general, finite L_1 metrics do not have a unique decomposition in cut semimetrics [7, Section 4.3]; see [2, Equation 9] for an example. An attractive feature of L_1 diversities is that each L_1 diversity has a unique decomposition into cut semidiversities; see [6, Proposition 10] which is derived from [3, Theorem 3 and 4]. Later we show another nice feature of finite L_1 diversities: they satisfy the bounded Amalgamation Property (bAP) and in turn have an exact Fraïssé limit. Conversely, we show that the class of L_1 metric spaces does not even satisfy NAP, and hence does not have even an approximate Fraïssé limit.

6.1. There is no approximate Fraïssé limit for L_1 metrics. We will prove that there is no Fraïssé limit (neither approximate nor exact) for finite L_1 metrics by showing that the class does not satisfy NAP. This can be done with a simple example.

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We start by considering the simplest finite metric space that cannot be embedded in L_1. All metric spaces on four or fewer points can be embedded in L_1 [19, p. 190]. The following five-point metric space cannot:

	a	b	c	z_1	z_2
a	0	2	2	1	1
b		0	2	1	1
c			0	1	1
z_1				0	2
z_2					0

(This is the shortest path metric in the complete bipartite graph $K_{2,3}$.) This metric is proved to not be L_1 by showing that it does not satisfy the following pentagonal inequality [7, Section 6.1]

$$\sum_{x,y\in\{a,b,c\}} d(x,y) + \sum_{x,y\in\{z_1,z_2\}} d(x,y) - 2 \sum_{x\in\{a,b,c\},y\in\{z_1,z_2\}} d(x,y) \leq 0. \quad (6)$$

A natural way to approach finding a counter-example to the NAP is to choose two metric spaces with a common substructure that when amalgamated must yield this metric or another one violating the pentagonal inequality. Some experimentation gave the following pair of metric spaces

	a	b	c	e	z_1
a	0	2	2	2	1
b		0	2	2	1
c			0	1	1
e				0	1
z_1					0

	a	b	c	e	z_2
a	0	2	2	2	1
b		0	2	2	1
c			0	1	1
e				0	2
z_2					0

Amalgamating these two metric spaces while identifying the common substructure on $\{a, b, c, e\}$ gives

	a	b	c	e	z_1	z_2
a	0	2	2	2	1	1
b		0	2	2	1	1
c			0	1	1	1
e				0	1	2
z_1					0	γ
z_2						0

where $\gamma > 0$ needs to be determined. For the amalgamation to be a valid metric space, we need $1 \leq \gamma \leq 2$. However, the pentagonal inequality is not satisfied for any value of γ in this range. Evaluating (6) for this metric space gives $2\gamma \leq 0$, which cannot hold for any γ in the range. So the class of all finite L_1 metrics does not satisfy AP. Hence, there is not a exact Fraïssé limit for finite L_1 metrics.

Is it still possible for finite L_1 metrics to have an approximate Fraïssé limit? To rule out this possibility we show that the set of finite L_1 metrics does not even satisfy NAP. The approximate amalgamation property requires us to embed the two metric spaces above in a common metric space, where instead of the images of a, b, c being

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common, they just have to be arbitrarily close to each other. But the pentagonal inequality is a closed condition, and so any metric space arbitrarily close to our counter-example must also fail to satisfy it. So NAP does not hold and there is no approximate Fraïssé limit for finite L_1 metric spaces.

6.2. An exact Fraïssé limit for L_1 diversities. Finally, if we consider L_1 diversities, there is an exact Fraïssé limit, as we show here. Its induced metric is a universal L_1 metric, but is not ultrahomogeneous, by the results of the previous subsection.

THEOREM 26. *The class of finite L_1 diversities viewed as relational metric structures satisfies HP, JEP, bAP, PP, CP, and is therefore a Fraïssé class with a separable ultrahomogeneous limit. Furthermore, the limit is universal with respect to separable L_1 diversities, and is the unique such L_1 diversity.*

PROOF. HP. As in the general diversity case, taking a subset of points cannot violate any diversity axioms. Furthermore, if a diversity can be embedded in $L_1(\Omega, \mathcal{A}, \mu)$ so can any subdiversity of it.

JEP. This follows from bAP if we identify a single point in each of the finite L_1 diversities.

bAP. Suppose that (A, δ_A) embeds into (B, δ_B) and into (C, δ_C). Suppose that all three diversities are L_1-embeddable. We need to show that (B, δ_B) and (C, δ_C) can be simultaneously embedded into a L_1 diversity (D, δ_D).

We assume that $B \setminus A$ is disjoint from $C \setminus A$. To help index the splits, we fix $a \in A$. Then the three diversities can be written as

$$\delta_A = \sum_{U \subseteq A \setminus \{a\}, U \neq \emptyset} \alpha_U \delta_{U|(A \setminus U)},$$

$$\delta_B = \sum_{V \subseteq B \setminus \{a\}, V \neq \emptyset} \beta_V \delta_{V|(B \setminus V)},$$

$$\delta_C = \sum_{W \subseteq C \setminus \{a\}, W \neq \emptyset} \gamma_W \delta_{W|(C \setminus W)},$$

where $\alpha_U, \beta_V, \gamma_W$ are all non-negative.

We know that the three diversities all agree on subsets of A:

$$\delta_A = \delta_B|_A = \delta_C|_A.$$

L_1 diversities on A are uniquely expressed as a weighted sum of splits of A. We need to figure out how to write $\delta_B|_A$ and $\delta_C|_A$ as a weighted sum of splits of A. But for each split of A, there are many corresponding splits of B (or C) that have the same effect on subsets of A. So we can break down $\delta_B|_A$ as

$$\delta_B = \sum_{U \subseteq A \setminus \{a\}, U \neq \emptyset} \left[\sum_{V \subseteq B:\ V \cap A = U} \beta_V \delta_{V|(B \setminus V)} \right],$$

and a similar expression holds for δ_C. For each split $U \subseteq A$, $a \notin U$, $U \neq \emptyset$, we then have

$$\sum_{V \subseteq B : V \cap A = U} \beta_V = \sum_{W \subseteq C : W \cap A = U} \gamma_W = \alpha_U.$$

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We will now define the amalgamated L_1 diversity. Let $M^{(U)}$ be a matrix with rows indexed by elements of $\{V \subseteq B : V \cap A = U\}$; columns indexed by elements of $\{W \subseteq C : W \cap A = U\}$; such that for all V

$$\sum_{W \subseteq C : W \cap A = U} M^{(U)}_{VW} = \beta_V$$

and for all W,

$$\sum_{V \subseteq B : V \cap A = U} M^{(U)}_{VW} = \gamma_W.$$

Any such matrix provides an amalgamation δ_D given by

$$\delta_D = \sum_{U \subseteq A \setminus \{a\}, U \neq \emptyset} \sum_{V \subseteq B : V \cap A = U} \sum_{W \subseteq C : W \cap A = U} M^{(U)}_{VW} \delta_{V \cup W | ((B \cup C) \setminus (V \cup W))}.$$

We need to find a choice of $M^{(U)}$ such that bAP holds. We just consider two-point amalgamation; the general case follows by induction. We let $B = A \cup \{z_1\}$ and $C = A \cup \{z_2\}$. Now U runs over all nonempty subsets of A not containing $\{a\}$. But for each U, V just takes the values U and $U \cup \{z_1\}$ and W just takes the values U and $U \cup \{z_2\}$.

Now our amalgamated diversity simplifies to (only writing the one half of the splits in the split notation, i.e., U for $U|((A \cup \{z_1, z_2\}) \setminus U)$

$$\begin{aligned}\delta_D = \sum_{U \subseteq A \setminus \{a\}} & M^{(U)}_{U,U} \delta_U + M^{(U)}_{U \cup \{z_1\}, U} \delta_{U \cup \{z_1\}} + M^{(U)}_{U, U \cup \{z_2\}} \delta_{U \cup \{z_2\}} \\ & + M^{(U)}_{U \cup \{z_1\}, U \cup \{z_2\}} \delta_{U \cup \{z_1, z_2\}}.\end{aligned}$$

If we only want to know the value of the diversity on $\{z_1, z_2\}$ then it simplifies to

$$\delta_D(\{z_1, z_2\}) = \sum_{U \subseteq A \setminus \{a\}} M^{(U)}_{U \cup \{z_1\}, U} + M^{(U)}_{U, U \cup \{z_2\}}. \tag{7}$$

We choose the entries of $M^{(U)}$ analogously to how we chose to amalgamate stochastic processes in Section 4. We let

$$\begin{aligned} M^{(U)}_{U,U} &= \min(\beta_U, \gamma_U), \\ M^{(U)}_{U \cup \{z_1\}, U \cup \{z_2\}} &= \min(\beta_{U \cup \{z_1\}}, \gamma_{U \cup \{z_2\}}). \\ M^{(U)}_{U \cup \{z_1\}, U} &= \gamma_U - \min(\beta_U, \gamma_U). \\ M^{(U)}_{U, U \cup \{z_2\}} &= \beta_U - \min(\beta_U, \gamma_U) \end{aligned}$$

for each $U \subseteq A, a \notin U$.

Plugging these choices into (7) gives

$$\delta_D(\{z_1, z_2\}) = \sum_{U \subseteq A \setminus \{a\}} \beta_U + \gamma_U - 2\min(\beta_U, \gamma_U) = \sum_{U \subseteq A \setminus \{a\}} |\beta_U - \gamma_U|.$$

We now have to bound this expression in terms of differences of $\delta_B(U \cup \{z_1\})$ and $\delta_C(U \cup \{z_2\})$. To do this we need to express β_U and γ_U in terms of the diversities

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evaluated on set. Equation (7) in [6, p. 12] gives the expression for the weights when $\lambda = \bar{\lambda}$. We are only taking splits $U|\bar{U}$ where U doesn't contain A. So we have to double the weight in that paper. We get, for all $U \subseteq A$, $a \notin U$:

$$\beta_U = \sum_{V: U \subseteq V, V \subseteq A} (-1)^{|U|-|V|+1}(\delta(V) - \delta(V \cup \{z_1\})).$$
$$\gamma_U = \sum_{V: U \subseteq V, V \subseteq A} (-1)^{|U|-|V|+1}(\delta(V) - \delta(V \cup \{z_2\})).$$

This gives us

$$\begin{aligned}\delta_D(\{z_1, z_2\}) &= \sum_{U \subseteq A \setminus \{a\}} \left| \sum_{V: U \subseteq V, V \subseteq A} (-1)^{|U|-|V|+1}(\delta(V \cup \{z_1\}) - \delta(V \cup \{z_2\})) \right| \\ &\leq \sum_{U \subseteq A \setminus \{a\}} \sum_{V: U \subseteq V, V \subseteq A} |\delta(V \cup \{z_1\}) - \delta(V \cup \{z_2\})| \\ &\leq 2^{2n} \max_{V \subseteq A} |\delta(V \cup \{z_1\}) - \delta(V \cup \{z_2\})|\end{aligned}$$

as required.

PP. We follow the exact same argument as in Theorem 24, noting that L_1 diversities also can be viewed as a closed subset of $\mathbb{R}^k$ for large enough k. We only need to establish the analogue of Lemma 25 for L_1 diversities. The inequality $d_\infty(\bar{a}, \bar{b}) \leq n d_{\mathcal{K}}(\bar{a}, \bar{b})$ follows exactly as in Lemma 25. To get a bound on $d_{\mathcal{K}}(\bar{a}, \bar{b})$ in terms of $d_\infty(\bar{a}, \bar{b})$ we use bAP for L_1 diversities repeatedly.

First, suppose that $d_\infty(\bar{a}, \bar{b}) = \varepsilon$. We will construct a joint embedding of $\bar{a}$ and $\bar{b}$ into a new L_1 diversity where corresponding points in the tuples are never further than $2^{2n}\varepsilon$ from each other.

First we identify a_1 and b_1. Consider the two tuples (a_1, b_1, a_2), (a_1, b_1, b_2). Since $d_\infty(\bar{a}, \bar{b}) = \varepsilon$, we know that $d_\infty((a_1, b_1, a_2), (a_1, b_1, b_2)) \leq \varepsilon$, and so be can use bAP for L_1 diversities to obtain a joint embedding (a_1, b_1, a_2, b_2) where $d(a_2, b_2) \leq 2^{2n}\varepsilon$. We repeat this step for the tuples $(a_1, b_1, a_2, b_2, a_3)$ and $(a_1, b_1, a_2, b_2, b_3)$, and so forth. In the end we have a joint embedding of all the points in $\bar{a}$ and $\bar{b}$ such that $d(a_i, b_i) \leq 2^{2n}\varepsilon$ for all $i = 1, \ldots, k$. Thus we have obtained

$$\frac{1}{n} d_\infty(\bar{a}, \bar{b}) \leq d_{\mathcal{K}}(\bar{a}, \bar{b}) \leq 2^{2n} d_\infty(\bar{a}, \bar{b}),$$

as required.

CP. This follows exactly like the proof of the property CP in Theorem 24, the restriction to L_1 diversities not making any difference. ⊣

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