Where is the centre of a polygon?

ANDREW JOBBINGS

1. *Introduction*

When a polygon has sufficient symmetry, there is usually only one realistic choice for the point at its ‘centre’. For example, for a rectangle this is the point of intersection of the diagonals (see Figure 1).

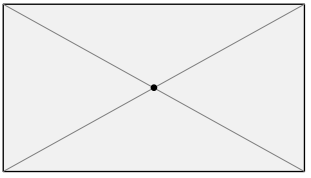


FIGURE 1

But even for a kite (a quadrilateral with two pairs of equal adjacent sides) there is more than one candidate, such as the point where the diagonals meet, or where the angle bisectors meet. Both these points lie on the line of symmetry (one of the diagonals), but in general they are different, as they are in Figures 2 and 3. Or should we choose some other point?

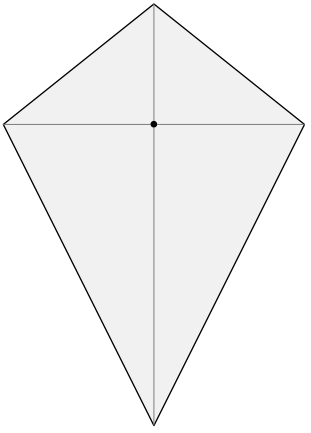


FIGURE 2

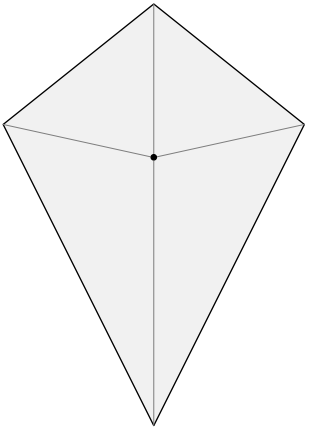


FIGURE 3

The situation gets less clear for a general quadrilateral, let alone a general polygon. This article attempts to provide an answer in any situation.

The original motivation was a seemingly simple practical matter: in a diagram, where should the label for a quadrilateral be placed? If the quadrilateral is considered to be a path, then the label can be placed somewhere along the perimeter. But suppose instead that the shape is to be labelled ‘in the middle’, then where should the label be placed? Of course, each case can be dealt with ‘by hand’, but is there an algorithm that always works?

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The reader may like to consider these questions before reading on.

In what follows we ignore the precise nature of the label—think of a label as being like a dot—so when we speak of adding a label, we are really just trying to find a suitable point.

2. *Triangles*

A triangle has many centres, and their investigation has a long history. Nowadays the website [1] is the definitive source, containing information on over 5500 triangle centres at the time of writing.

Which one of these centres is the best place for a label? We shall return to that question later—it is surprisingly tricky.

One triangle centre worth mentioning now is the *centroid*, the centre of mass of a triangle when it is made out of a uniform sheet of material, such as cardboard. A standard result is that the centroid is the point of intersection of the *medians*, each of which joins a vertex of the triangle to the midpoint of the opposite side (see Figure 4).

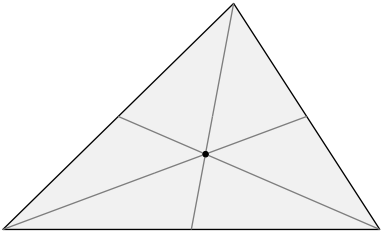


FIGURE 4

3. *Quadrilaterals*

There are several points that may be considered to be centres of a quadrilateral, such as the point of intersection of the diagonals or the centre of mass. Let us consider some of the possibilities.

3.1 *The intersection of the diagonals*

The diagonals of a quadrilateral meet at a point that does not look promising, even at first sight (see Figure 5).

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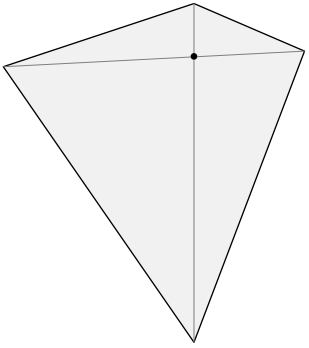


FIGURE 5

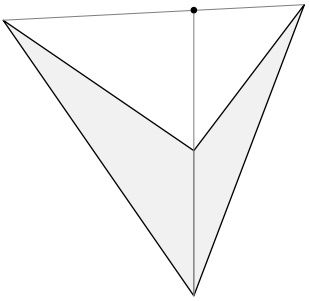


FIGURE 6

Furthermore, in some cases the diagonals meet outside the quadrilateral (see Figure 6). So the point of intersection of the diagonals has to be dismissed in our context.

3.2 *The intersection of the bimedians*

A *bimedian* of a quadrilateral is the line segment joining the midpoints of opposite sides (see Figure 7). A quadrilateral has two bimedians, so they determine a point, their point of intersection, as shown in Figure 8.

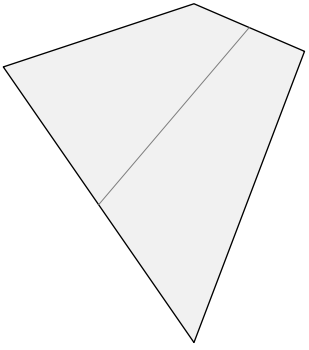


FIGURE 7

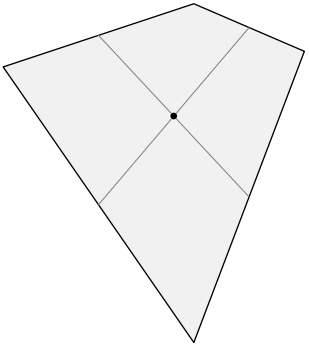


FIGURE 8

In fact the bimedians bisect one another. This is a consequence of *Varignon's theorem*, which states that the figure obtained by joining the midpoints of the sides of a quadrilateral is a parallelogram. We encourage the reader to find a proof. It follows that the bimedians of the original quadrilateral are just the diagonals of Varignon's parallelogram.

The point of intersection of the bimedians is also the centre of mass of the edges of the quadrilateral. As a consequence, for some quadrilaterals the bimedians intersect outside the quadrilateral (see Figure 9). This rules out the point of intersection of the bimedians for our purposes.

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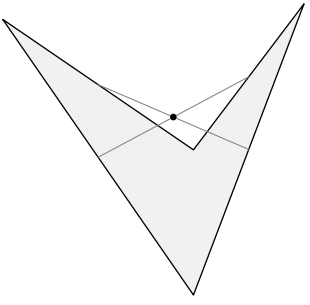


FIGURE 9

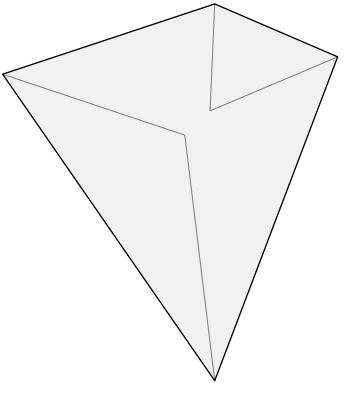


FIGURE 10

3.3 *The intersection of the angle bisectors*

In general, the four angle bisectors of a quadrilateral are not concurrent, that is, they do not meet (see Figure 10).

For which quadrilaterals are the angle bisectors concurrent? It turns out that having concurrent angle bisectors is equivalent to a quadrilateral being *inscribable*, in other words, a circle may be drawn touching all four sides. The interested reader may like to find a proof of this result.

Not all quadrilaterals are inscribable, so not all of them have concurrent angle bisectors. This is a pity, since the points on an angle bisector are equidistant from the corresponding edges of the quadrilateral, which is exactly the sort of fact that is useful when trying to place a label. However, we shall not give up on angle bisectors entirely, since the method we describe later makes use of them.

3.4 *The centre of mass*

A quadrilateral is divided into two triangles by a diagonal. The centre of mass of each triangle is at its centroid, so the centre of mass of the quadrilateral lies on the line joining these centroids (see Figure 11). Hence the centre of mass of the quadrilateral is the point of intersection of two such lines, obtained by considering the two diagonals in turn (see Figure 12).

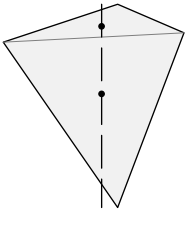


FIGURE 11

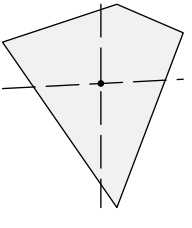


FIGURE 12

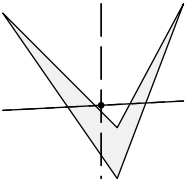


FIGURE 13

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Unfortunately, for some quadrilaterals the centre of mass lies outside the quadrilateral, as shown in Figure 13, so this point is also not suitable for our purposes.

3.5 *What is left*?

We have ruled out four possibilities, but we have not found a suitable centre. In fact, finding an appropriate centre for a general quadrilateral is not straightforward, and it turns out to be easier to deal with polygons with more than four sides! So we shall leave the discussion of the quadrilateral and return to it later.

4. *Polygons in general*

We shall now see how to define the centre of a polygon with n vertices (where $n > 3$) as the centre of a related polygon with $n - 1$ vertices.

4.1 *Parallel polygons*

Starting from a given polygon $\mathcal{P}_0$, it is possible to construct a family of 'parallel polygons'. Each parallel polygon has vertices on the angle bisectors of the original, and its edges are parallel to the edges of the original (see Figure 14).

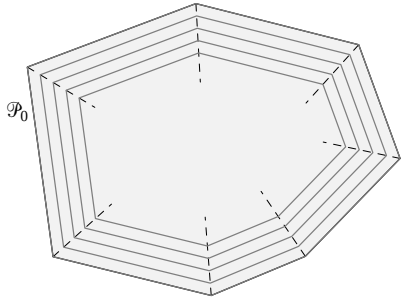


FIGURE 14

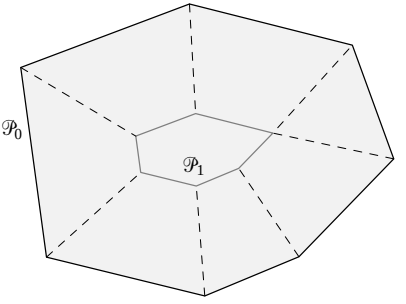


FIGURE 15

The edges of a parallel polygon always 'close up'. In other words, if we start at a point on a bisector and work around the polygon, then we will always return to the starting point. We leave the proof of this fact as an exercise for the reader.

If smaller and smaller parallel polygons are constructed, eventually two vertices will coincide. This happens when one vertex of the parallel polygon $\mathcal{P}_1$ is at the point of intersection of two adjacent angle bisectors, as shown in Figure 15.

What are the properties of $\mathcal{P}_1$? Of course, each side of $\mathcal{P}_1$ is parallel to a side of the original polygon $\mathcal{P}_0$ since that is how the polygon was constructed. But also, in going from $\mathcal{P}_0$ to $\mathcal{P}_1$ we have *reduced the number of vertices by one*. In other words, if $\mathcal{P}_0$ has n vertices, then $\mathcal{P}_1$ has $n - 1$.

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Now let us define the centre of $\mathcal{P}_0$ to be the centre of $\mathcal{P}_1$. Since we do not know where the centre of $\mathcal{P}_1$ is, at first sight this does not look particularly helpful. But we can continue the process: from $\mathcal{P}_1$ we can find a corresponding parallel polygon $\mathcal{P}_2$ with $n - 2$ vertices, and so on, forming a sequence of polygons $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3, \ldots$, where the number of vertices is reduced by one each time, and the edges are parallel to edges of $\mathcal{P}_0$.

We need to be careful. For example, we may be unlucky and reduce the number of vertices by two in one go, such as happens with a rectangle. For a general polygon, however, this will not be the case, and we shall ignore the possibility from now on. At least, that is so if there are more than three vertices! In the case of a triangle, the three angle bisectors are *always* concurrent—they meet at the *incentre*, the centre of the inscribed circle, shown in Figure 16—so that the number of vertices jumps from 3 to 1. As we shall see, the incentre is not a good choice when placing a label, so we shall stop the process when we arrive at a triangle.

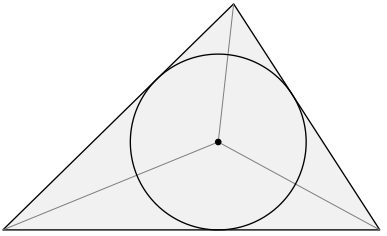


FIGURE 16

Figure 17 shows the complete process, starting with a six-sided polygon $\mathcal{P}_0$ and ending with a triangle $\mathcal{P}_3$. The shaded triangles indicate the relevant intersecting angle bisectors.

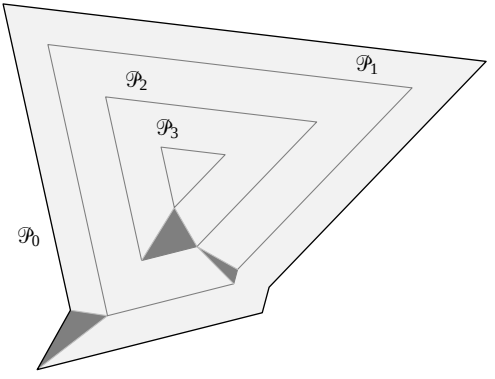


FIGURE 17

4.2 *Self-intersections*

There is another way in which we need to be careful. In some cases one of the resulting parallel polygons may be self-intersecting. The original

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polygon $\mathcal{P}_0$ does not even need to have a large number of sides for this to happen: Figure 18 shows a self-intersecting quadrilateral obtained from a pentagon.

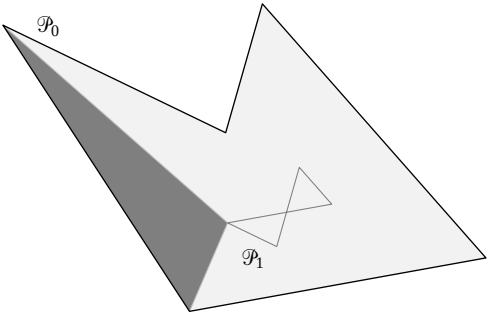


FIGURE 18

Why does this cause a problem? Consider the self-intersecting quadrilateral $\mathcal{P}_0$ shown in Figures 19 and 20. The two figures show two different ways to construct a corresponding parallel polygon $\mathcal{P}_1$. We cannot say that either of them is inside (or outside) $\mathcal{P}_0$ because a self-intersecting polygon does not have an inside or outside.

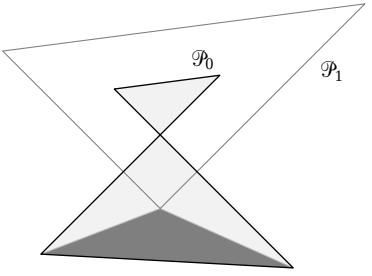


FIGURE 19

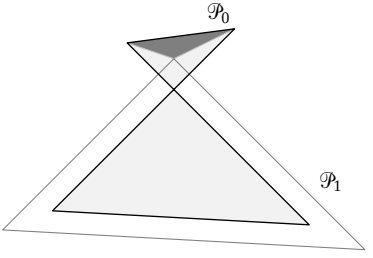


FIGURE 20

In view of this, when we used the phrase ‘smaller and smaller’ in describing the family of parallel polygons, we should have been more precise. In any case, we need to have a means of choosing which of the two possibilities to use whenever the situation arises.

In the example shown in Figures 19 and 20, the triangle $\mathcal{P}_1$ in Figure 20 would be the better one to use when positioning a label, because the resulting label would be placed in the larger component of $\mathcal{P}_0$. What is it that distinguishes this triangle from that in Figure 19?

The key difference is that the sides of the polygon $\mathcal{P}_1$ in Figure 20 are closer to the corresponding parallel sides of $\mathcal{P}_0$. So the criterion we need to use is the ‘distance’ between the two polygons $\mathcal{P}_0$ and $\mathcal{P}_1$, measured as the distance between corresponding parallel sides. This gives a means of making a choice when we have to do so.

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4.3 *Left and right*

There is one more situation that may arise where a choice becomes necessary.

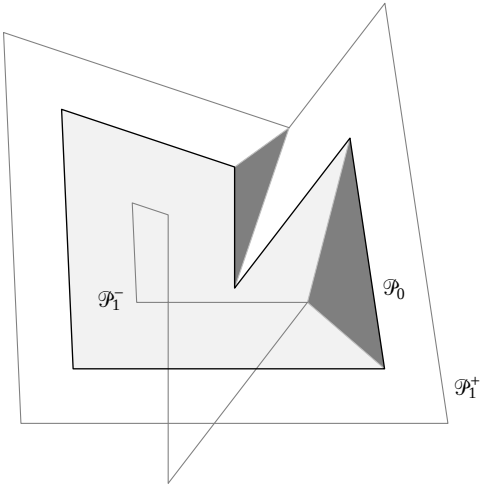


FIGURE 21

Consider Figure 21. The angles of the polygon $\mathcal{P}_0$ in this example are such that, as one proceeds around the perimeter, adjacent angle bisectors sometimes meet on the left, sometimes on the right (indeed, the self-intersecting case may also be thought of in this way). Once again this means that there are two possible parallel polygons to choose from, labelled $\mathcal{P}_1^+$ and $\mathcal{P}_1^-$ in the figure.

We can use the same criterion as in the self-intersecting case to decide which parallel polygon to select, namely the one closer to $\mathcal{P}_0$ (in the sense of having the smaller distance between parallel edges). In the example shown in Figure 21 we would therefore choose $\mathcal{P}_1^+$.

5. *Quadrilaterals and triangles revisited*

The process described in Section 4 reduces the problem of finding the centre of a general polygon to generating a sequence of parallel polygons, until eventually a triangle is obtained. We now turn our attention to finding an appropriate centre of this final triangle. In fact, as we shall see, the preceding quadrilateral also has a part to play in determining the location of the centre. Before describing the process, we provide some motivation for our choice of centre.

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5.1 *Inscribing a circle*

When trying to find the centre of a quadrilateral or a triangle, one approach that may be considered is to inscribe a circle. The required centre would then be the centre of the circle.

There are two drawbacks to this approach. Firstly, it is not possible to inscribe a circle in a general quadrilateral. We met this result in Section 3.3.

Secondly, even though all triangles do have a unique inscribed circle (see Figure 16), the incentre is not in general a good place for a label. To see why, consider a quadrilateral which is very close to being a long thin rectangle, as shown in Figure 22. The parallel polygon is a long thin triangle, whose incentre is very near one end. However, this would place the label near that end of the quadrilateral, which is clearly not an appropriate place,

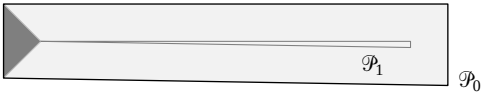


FIGURE 22

5.2 *Inscribing an ellipse*

Ellipses give more scope than circles: a triangle has many inscribed ellipses, and it is always possible to inscribe an ellipse in a quadrilateral. Remember, *inscribed* means that all the sides are tangents of the ellipse. Perhaps we can inscribe an ellipse, then take the centre of the ellipse to be the centre of the polygon.

Suppose that an ellipse is inscribed in a given quadrilateral. What can be said about the centre of the ellipse? This problem was considered by Newton, who gave the solution: the centre of the ellipse lies on the line segment joining the midpoints of the diagonals of the quadrilateral.

Unfortunately, Newton's result means that the idea of inscribing an ellipse in a quadrilateral is doomed as a method of finding an appropriate centre. To see why, consider the quadrilateral shown in Figure 23. Using the result, we know that the centre of any inscribed ellipse lies on the segment MN, where M and N are the midpoints of the diagonals. But no point on this segment is really suitable as the location of a label, which should probably be placed in the vicinity of the point labelled L.

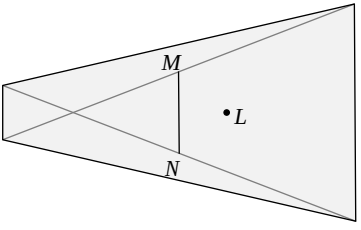


FIGURE 23

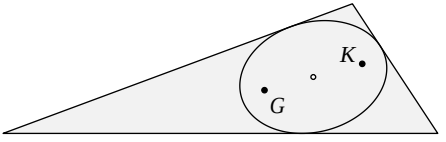


FIGURE 24

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What about inscribing an ellipse in a triangle? As mentioned above, there are many possible candidates. Indeed, it is known that any point in the *medial triangle*—the triangle joining the midpoints of the sides of the triangle—is the centre of some inscribed ellipse.

One possibility is the *Lemoine ellipse*, shown in Figure 24. The centre of the ellipse is at the midpoint of GK, where G is the *centroid* of the triangle, and K is the *symmedian point*. Indeed, the points G and K are the *foci* of the ellipse. We mentioned the centroid in Section 2; more detail about the symmedian point can be found on the website [1], or in [2], for example.

As can be seen from Figure 25, for a long thin triangle the centre of the Lemoine ellipse is a better choice, in terms of labelling, than the centre of the incircle. So the centre of the Lemoine ellipse is a good candidate for an appropriate centre of a triangle.



FIGURE 25

However, when the triangle is the parallel polygon of a quadrilateral we need to consider a different point. Referring again to Figure 22, it is clear that the appropriate centre in that example should be about 'halfway along' the quadrilateral, and thus about halfway along the triangle. The centre of the Lemoine ellipse is too far to the right, towards the shortest edge of the triangle.

Fortunately, there is a triangle centre with the desired properties, which we describe next.

5.3 *The centre of mass of the edges of a triangle*

Instead of considering a triangle as a uniform sheet, suppose we think of it as made from uniform wire, so that its centre of mass is the centre of mass of the perimeter. The centre of mass of the edges of a triangle is a point known as the *Spieker centre* of the triangle (see [2]).

Figure 26 shows the Spieker centre of a long thin triangle. As we would expect from the definition, and as we require when the triangle is the parallel polygon of a quadrilateral, the centre is roughly halfway along. However, as Figure 27 reveals, the Spieker centre S is not always appropriate. In that particular example, the centre X of the Lemoine ellipse of $\mathcal{P}_1$ might be better.

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FIGURE 26

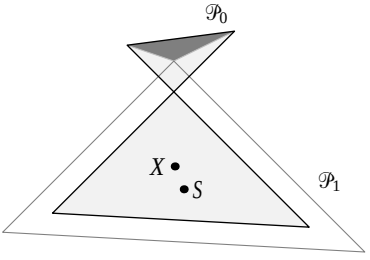


FIGURE 27

6. *Putting everything together*

What have we found? For a triangle, the centre X of the Lemoine ellipse is a suitable point. But if the triangle is the parallel polygon of a quadrilateral, then the Spieker centre S may be a better choice. Neither centre is appropriate in all circumstances.

In the case when the triangle is the parallel polygon of a quadrilateral, therefore, we choose a point lying on the line segment SX joining the two centres. Precisely which point we choose depends on the triangle and the quadrilateral; we shall spare the reader the details.

Putting this all together, the procedure for finding the centre of a polygon with more than three sides is as follows. By finding parallel polygons, reduce the number of sides until a triangle is reached. Finally, choose an appropriate point on the segment SX, where S is the Spieker centre of the triangle and X is the centre of the Lemoine ellipse.

Figure 28 shows the complete process applied to a seven-sided polygon, so that the parallel polygons have 6, 5, 4 and 3 sides.

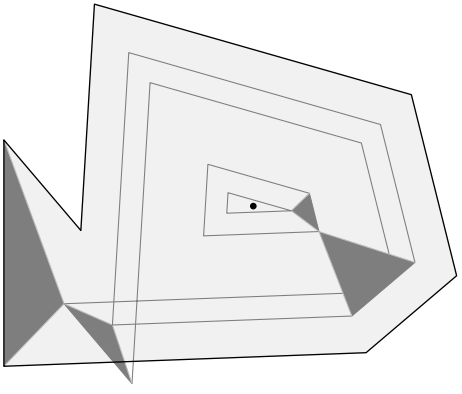


FIGURE 28

The procedure even copes with fairly pathological initial polygons, as indicated by Figure 29, where the original polygon has more than two self-intersections.

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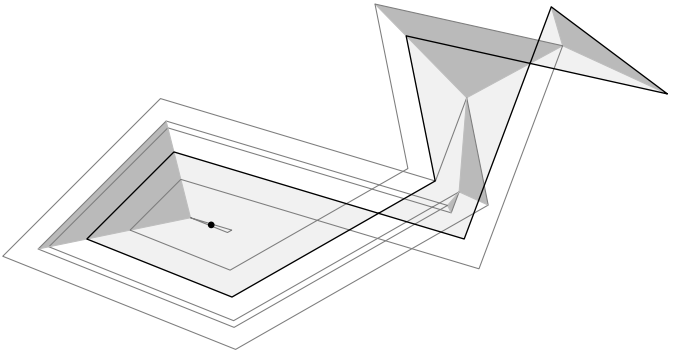


FIGURE 29

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