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Scattering rigidity with trapped geodesics

CHRISTOPHER CROKE

Department of Mathematics, University of Pennsylvania, Philadelphia, PA 19104-6395, USA
(e-mail: ccroke@math.upenn.edu)

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Abstract. We prove that the flat product metric on $D^n \times S^1$ is scattering rigid where D^n is the unit ball in $\mathbb{R}^n$ and $n \geq 2$. The scattering data (loosely speaking) of a Riemannian manifold with boundary is the map $S : U^+\partial M \to U^-\partial M$ from unit vectors V at the boundary that point inward to unit vectors at the boundary that point outwards. The map (where defined) takes V to $\gamma_V'(T_0)$ where γ_V is the unit speed geodesic determined by V and T_0 is the first positive value of t (when it exists) such that $\gamma_V(t)$ again lies in the boundary. We show that any other Riemannian manifold $(M, \partial M, g)$ with boundary ∂M isometric to $\partial(D^n \times S^1)$ and with the same scattering data must be isometric to $D^n \times S^1$. This is the first scattering rigidity result for a manifold that has a trapped geodesic. The main issue is to show that the unit vectors tangent to trapped geodesics in $(M, \partial M, g)$ have measure zero in the unit tangent bundle.

1. *Introduction*

In this paper we prove scattering rigidity (see below) for a number of compact Riemannian manifolds with boundary that have trapped geodesics. A geodesic $\gamma(t)$ in a manifold with boundary is trapped if its domain of definition is unbounded.

Consider a compact Riemannian manifold $(M, \partial M, g)$ with boundary ∂M and metric g. We will let $U^+\partial M$ represent the space of inwardly pointing unit vectors at the boundary. That is, $V \in U^+\partial M$ means that V is a unit vector based at a boundary point and $\langle V, \eta^+ \rangle \geq 0$ where η^+ is the unit vector of M normal to ∂M and pointing inward. Similarly we let $U^-\partial M$ represent the outward vectors. Note that $U^+\partial M \cap U^-\partial M = U(\partial M)$ is the unit tangent bundle of ∂M.

For $V \in U^+\partial M$, let $\gamma_V(t)$ be the geodesic with $\gamma'(0) = V$. We let $TT(V) \in [0, \infty]$ (the travel time) be the first time $t > 0$ when $\gamma_V(t)$ hits the boundary again. If $\gamma_V(t)$ never hits the boundary again, then $TT(V) = \infty$ (i.e. γ_V is trapped) while if either $\gamma_V(t)$ does not exist for any $t > 0$ or there are arbitrarily small values of $t > 0$ such that $\gamma(t) \in \partial M$, then we let $TT(V) = 0$. Note that $TT(V) = 0$ implies that $V \in U(\partial M)$.

The scattering map $S : U^+\partial M \to U^-\partial M$ takes a vector $V \in U^+\partial M$ to the vector $\gamma'(TT(V)) \in U^-\partial M$. It will not be defined when $TT(V) = \infty$ and will be V itself

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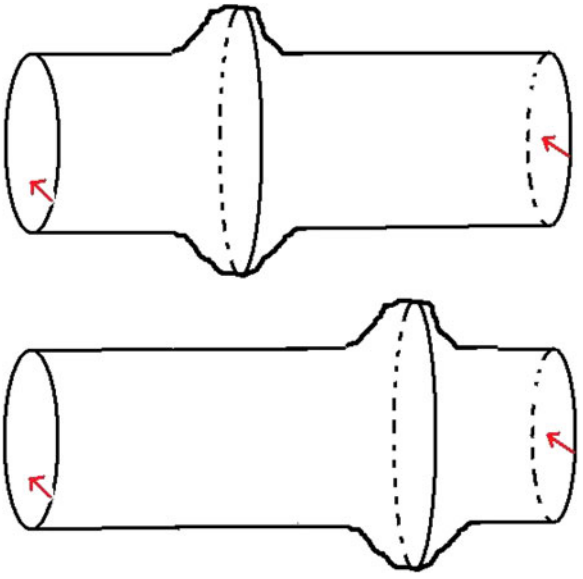


FIGURE 1. Not isometric, but the same scattering and lens data.

when $TT(V) = 0$. If another manifold $(M_1, \partial M_1, g_1)$ has isometric boundary to $(M, \partial M, g)$ in the sense that $(\partial M, g)$ (g restricted to ∂M) is isometric to $(\partial M_1, g_1)$, then we can identify $U^+\partial M_1$ with $U^+\partial M$ and $U^-\partial M_1$ with $U^-\partial M$. We say that $(M, \partial M, g)$ and $(M_1, \partial M_1, g_1)$ have the same scattering data if they have isometric boundaries and under the identifications given by the isometry they have the same scattering map. If in addition the travel times $TT(V)$ coincide then they are said to have the same lens data.

A compact manifold $(M, \partial M, g)$ is said to be scattering (respectively, lens) rigid if for any other manifold $(M_1, \partial M_1, g_1)$ with the same scattering (respectively, lens) data there is an isometry from M_1 to M that agrees with the given isometry of the boundaries.

THEOREM 1.1. *For any $n \geq 2$ the flat product metric on $D^n \times S^1$ is scattering rigid where D^n is a ball in $\mathbb{R}^n$.*

The fact that not all manifolds are scattering rigid was pointed out in [**Cr91**]. For $\frac{1}{4} > \epsilon > 0$, let $h(t)$ be a small smooth bump function which is zero outside $(-\epsilon, \epsilon)$ and positive in $(-\epsilon, \epsilon)$. For $s \in (-1 + 2\epsilon, 1 - 2\epsilon)$ consider surfaces of revolution g_s with smooth generating functions $F_s(t) = 1 + h(s + t)$ for $t \in [-1, 1]$. These surfaces of revolution look like flat cylinders with bumps on them that are shifted depending on s but otherwise look the same (see Figure 1). The Clairaut relations show that, independent of s, geodesics entering one side with a given initial condition exit out the other side after the same distance at the same point with the same angle. Hence, all metrics have the same scattering data (and in fact lens data) but are not isometric. A much larger class of examples was given in [**CK94**, §6]. All of the examples have in common that there are trapped geodesics.

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The scattering and lens rigidity problems are closely related to other inverse problems. In particular the boundary rigidity problem is equivalent to the lens rigidity question in the simple and SGM (strong geodesic minimizing) cases. See [**Cr91, Cr04**] for definitions and relations to some other problems. There is a vast literature on these problems (see for example [**Be83, BI06, Cr91, Cr90, Gr83, Mi81, Mu77, Ot90, PS88, PU05, SU09**]). Most of the results in these papers concern manifolds with no trapped geodesics. An exception is [**SU09**] where they prove local scattering rigidity (i.e. if the two metrics are in a particular C^k neighborhood they must be isometric) for a class of Riemannian manifolds that includes those discussed in this paper. However, to date all of the global rigidity results concern manifolds without trapped geodesics. The results in this paper constitute the first examples of (global) scattering rigid manifolds that have trapped geodesics.

The key difficulty in our case is to show that the set of unit vectors tangent to trapped geodesic rays in the metric g_1 has measure zero in the unit tangent bundle. This allows us (with an application of Santaló's formula) to conclude that g and g_1 have the same volumes. Since the metric g has a real factor (i.e. $D^n \subset \mathbb{R} \times \mathbb{R}^{n-1}$) we can use a result from [**CK98**] to complete the argument. In fact, the argument in Theorem 1.1 extends (see §3) to the case where D^n above is replaced by a ball in $\mathbb{R} \times N^{n-1}$ where N is a complete simply connected Riemannian manifold with non-positive curvature. (In fact, with more work one could extend this to the case of no conjugate points but we chose not to give the slightly different arguments here.)

One case that was not dealt with in Theorem 1.1 is the two-dimensional case, namely the flat cylinder $[-1, 1] \times S^1$ and the Möbius strip. There are ways in which this case is easier and ways in which it is harder. The major differences are that the scattering data does not determine the lens data and we cannot conclude that the C^∞ jets of the metrics agree at the boundary. The problem of lens rigidity in the two-dimensional case will be taken up in a future paper with Pilar Herreros. In particular, it turns out that the Möbius strip is not scattering rigid if $(M_1, \partial M_1, g_1)$ is allowed to be C^1.

2. *The $D^n \times S^1$ for $n \geq 2$ case*

In this section we prove Theorem 1.1. We consider generalizations in §3. Throughout the section $n \geq 2$ and g will be the standard flat product metric on $M = D^n \times S^1$. For concreteness we will take D^n to be the unit ball and S^1 to have length 2π. Here $(M_1, S^{n-1} \times S^1, g_1)$ will be another Riemannian metric on a manifold M_1 whose boundary is isometric to that of M. We use this isometry to identify the two boundaries. We assume that g_1 has the same scattering data as g.

We do not *a priori* assume that M_1 is diffeomorphic to M. For each $p \in \partial M = \partial M_1$ we let $\tau_p \subset \partial M = \partial M_1$ be the closed curve in the vertical (i.e. S^1) direction.

LEMMA 2.1. *The metric g_1 has the same lens data as g.*

Proof. This is an application of the first variation formula. For $V \in U^+\partial M$ let $G(V) = L(\gamma_{1V}) - L(\gamma_V)$. We need to show that $G(V) = 0$ for all V. A smooth curve of initial conditions $s \mapsto V(s)$ in the interior of $U^+\partial M$ gives rise to smooth variations $\gamma_{V(s)}$ through unit speed geodesics in M and $\gamma_{1V(s)}$ through unit speed geodesics in M_1 whose initial and final tangents agree. (Note that this uses the convexity of the

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boundary since for more general manifolds with boundary there may be a discontinuous jump in the endpoints of geodesics.) The first variation formula (along with the fact that the metrics agree at the boundary) tells us that $\frac{d}{ds}L(\gamma_{V(s)}) = \frac{d}{ds}L(\gamma_{1V(s)})$. Hence, $G(V(s)) = L(\gamma_{1V(s)}) - L(\gamma_{V(s)})$ is independent of s. Since $U^+\partial M$ is connected, G is a constant C. Further when V approaches a non-vertical vector (i.e. one not tangent to the S^1 factor) in $\partial(U^+\partial M) = U\partial M$, then $L(\gamma_V)$ approaches zero and hence $C > 0$. If we knew $L(\gamma_{1V})$ approached zero for this or any sequence, then $C = 0$ and the lemma would follow. We now show that this must happen.

If this is not the case (i.e. if $C > 0$), then the boundary must be *concave* everywhere and further when V_i approaches a non-vertical vector V in $\partial(U^+\partial M) = U\partial M$ as above, then $L(\gamma_{1V})$ approaches C and the limiting geodesic γ_{1V} must be a closed geodesic of length C with initial (and final) tangent vector V. By taking limits this is also true for the vertical vectors. (This certainly looks unlikely to happen. However, there is an example [**HHL06**], an Eaton lens, of a manifold with a singularity having the same scattering data as the flat 2-disc but different lens data. In that case the boundary is a closed geodesic.)

Note that in our case (since the dimension of the boundary is at least two) all of these closed geodesics γ_{1V} for V based at a boundary point p are homotopic to each other since there is a curve of initial tangent vectors V_t between any two initial tangents (i.e. the homotopy is through the curves γ_{1V_t}). In particular, they are all homotopic to their negatives (i.e. running around in the opposite direction). Thus, going twice around such a geodesic is a contractible curve.

In fact, the closed geodesic γ_V of length C tangent to vertical direction V at $p \in \partial M$ is a multiple of τ_p. To see this let $x = \tau_p(t)$ be a point on τ_p close to p (say $t < \pi/2$) and x_i a sequence of points on ∂M approaching x but not on τ_p. Let γ_{V_i} be the minimal g geodesics from p to x_i with $x_i = \gamma_{V_i}(t_i)$. We see that V_i approaches V and t_i approaches t. Looking at the other metric we see $x_i = \gamma_{1V_i}(t_i + C)$ and, hence, by taking limits $x = \gamma_{1V}(t + C)$. Since this is true for all p and $x = \tau_p(t)$ for $t < \pi/2$, we see that $\tau_p(t) = \gamma_{1V_i}(t + C)$ and, hence, τ_p is the g_1 geodesic with initial tangent V. Thus, γ_{1V_i} simply goes around τ_p a number of times (i.e. C is an integer multiple of 2π). Thus, we know that going twice as many times around τ_p yields a contractible curve. Thus, the following sublemma gives the desired contradiction. □

SUBLEMMA 2.2. *No multiple of τ_p is contractible in M_1.*

Proof. This argument is an oriented intersection number argument. We begin by seeing that when $C > 0$, then M_1 is orientable. Fix x on the boundary. Then for every element of π_1 the shortest loop representing this class is either a geodesic loop in M_1 or partly runs along the boundary. By the scattering data assumption the only geodesic loops at p are those that start tangent to the boundary (i.e. the closed geodesics we are discussing). All of these are homotopic to a multiple of τ_p. If the minimizing path runs along the boundary some of the time, then (since when it leaves the boundary it must be tangent) the only parts not on the boundary are the closed geodesic loops again. Hence, every element of π_1 has a representative that lies in the boundary. Since running along such curves does not change the orientation, M_1 must be orientable.

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Choose an orientation reversing diffeomorphism $F : S^{n-1} \to S^{n-1}$. This induces a map $H : S^{n-1} \times [0, 1] \to M_1$ by

$$H(x, 0) = (x, 0) \in \partial M_1 = S^{n-1} \times S^1, \quad H(x, 1) = \left(F(x), \frac{\pi}{10}\right).$$

For fixed x, $H(x, t)$ is the g_1 geodesic (parameterized proportional to arclength) with the same initial tangent as the minimizing g geodesic from $(x, 0)$ to $(F(x), \pi/10)$ (and, hence, has length C longer). Note that if $F(x) = x$, then the geodesic $H(x, t)$ may wrap around $\tau_{(x,0)}$ many times before ending at $(x, \pi/10)$.

Although H may not be transverse to the boundary that is easy to fix. For some small $\epsilon > 0$ we can parameterize the ϵ neighborhood N_ϵ in M_1 of ∂M_1 as $[0, \epsilon) \times S^{n-1} \times S^1$ (where $\partial M_1 = \{0\} \times S^{n-1} \times S^1$). Further we have a diffeomorphism $D_\epsilon : M_1 \to M_1 - N_\epsilon$ such that $D_\epsilon(0, x, \theta) = (\epsilon, x, \theta)$. Thus, we can take $\tilde{H} : S^{n-1} \times [-\epsilon, 1 + \epsilon] \to M_1$ by letting $\tilde{H}(x, t) = (t, x, 0)$ for $t \le 0$, $\tilde{H}(x, t) = D_\epsilon(H(x, t))$ for $0 \le t \le 1$, and let $\tilde{H}(x, t) = (1 + \epsilon - t, F(x), \pi/10)$ for $t \ge 1$. This is now transverse to ∂M_1 and we can tweak this to make it smooth. We can double this picture in the manifold double $M_1 \times M_1$ to get a smooth map $\bar{H} : S^{n-1} \times S^1 \to M_1 \cup M_1$ which is transverse to the curve τ_p. Here τ_p passes through each of $S^{n-1} \times \{0\} \in \partial M_1$ and $S^{n-1} \times \{\pi/10\} \in \partial M_1$ once each time around and our choice of orientations guarantees that the two contributions to the intersection number of $\bar{H}$ with τ_p have the same sign. Thus, $\bar{H}$ has a non-zero intersection with any multiple of τ_p. Thus, the homology class of any multiple of τ_p is non-zero in $M_1 \cup M_1$ and, hence, in M_1. □

Since the lens data and, hence, distances between nearby boundary points agree, the C^∞ jets of g and g_1 also must agree at the boundary. This follows from [**LSU03**], [**UW03**] or [**Zh11**] since, for the flat metric g, the second fundamental form of the boundary has a positive eigenvalue at every point. (Note that this argument will not work in the two-dimensional case $n = 1$.) This in particular means that we can glue $(\mathbb{R}^n - D^n) \times S^1$ along the boundary of M_1 to yield a smooth metric M_1^{ext} which is isometric to $\mathbb{R}^n \times S^1$ outside of M_1.

LEMMA 2.3. *We have $\pi_1(M_1) = \mathbb{Z}$ and the generator is represented by the S^1 factor of the boundary.*

Proof. To see this fix a base point $p \in \partial M_1$. The lens data being the same tells us that the only geodesic loops at a p are the multiples of τ_p. Further the convexity of the boundary guarantees that there is at least one geodesic loop in each homotopy class (the shortest curve in that class). In particular, we see as before that M_1 is orientable. The lemma now follows from the proof of Sublemma 2.2. □

Thus, the Riemannian universal cover $\widetilde{M}_1$ of M_1 sits naturally in $\widetilde{M_1^{ext}}$, the universal cover of M_1^{ext}, and further $\widetilde{M_1^{ext}} - \widetilde{M}_1$ is isometric to $(R^n - D^n) \times \mathbb{R}$. Also $\widetilde{M} = D^n \times \mathbb{R}$ has the same scattering data as $\widetilde{M}_1$. We will slightly abuse notation and call the metrics on the universal covers g and g_1 as well.

LEMMA 2.4. *An M_1 geodesic γ between boundary points is the shortest path in its homotopy class (relative to boundary points).*

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Proof. This is the same as saying that such geodesics in the universal cover are the minimizing paths between the endpoints. This is true for $\widetilde{M}$ where there is a unique geodesic between any two boundary points. Thus, there is also a unique geodesic between boundary points in $\widetilde{M}_1$ which must thus be the minimizing geodesic. □

In fact, this implies that all geodesic segments in $\widetilde{M_1^{\text{ext}}}$ are minimizing except possibly in the case that they are segments of geodesics trapped in $\widetilde{M}_1$. If p and q are points in $\widetilde{M_1^{\text{ext}}} - \widetilde{M}_1 = \mathbb{R}^{n+1} - D^n \times \mathbb{R}$, then this implies that $d_1(p, q) = d(p, q)$. In particular, for $p \in \widetilde{M_1^{\text{ext}}} - \widetilde{M}_1$ all geodesics from p minimize. Hence, the exponential map is a diffeomorphism which allows us to conclude that not only are the fundamental groups the same but M_1 is diffeomorphic to M.

A geodesic γ_{1V} will either be trapped or coincide with an oriented Euclidean line L_V outside $\widetilde{M}_1$. By the direction of L_V we mean the oriented line through the origin parallel to L_V. Here L_V will be called 'positive' if it makes a (strictly) positive inner product with the upward vertical direction.

There are two cases that are exceptional. These are trapped geodesics and vertical geodesics (i.e. $\{x\} \times \mathbb{R}$ for $x \in R^n - D^n$). We will exclude both these cases by the phrase 'L_v is not vertical'. For $p \in \widetilde{M_1^{\text{ext}}}$ we let $\mathbf{A}(p) = \{V \in U_p \widetilde{M_1^{\text{ext}}} | L_V \text{ is not vertical}\}$. Note that for $p \in \widetilde{M_1^{\text{ext}}} - \widetilde{M}_1$ we have that $\mathbf{A}(p)$ is just the unit sphere with the north and south pole removed. Here $\mathbf{A}$ will represent the union of the $\mathbf{A}(p)$.

LEMMA 2.5. *If $V_i \in \mathbf{A}(p)$ converges to a vector $V \in U_p - \mathbf{A}(p)$, then the directions of the lines L_{V_i} become vertical.*

Proof. Assume that this is not the case. Then there is a subsequence of the V_i (which we will still call V_i) such that the directions of the lines L_{V_i} converge to a non-vertical direction L. We claim that a subsequence of these L_{V_i} converge to a line L_W. To see this we only need to note that the lines L_{V_i} intersect a common compact set. Now the length of $\gamma_{V_i} \cap \widetilde{M}_1$ is the same as the length of $L_{V_i} \cap D^n \times \mathbb{R} \subset \mathbb{R}^{n+1}$ which is uniformly bounded above (say by B) since the directions of the L_{V_i} converge to L which is not vertical. Thus, all of the L_{V_i} intersect the boundary of $\widetilde{M}_1$ inside the compact ball about p of radius B. This means that the geodesics γ_{V_i} (for the subsequence) converge to L_V outside $\widetilde{M}_1$ but they converge to γ_V which is supposed to be trapped. This yields the desired contradiction. □

We next see that even though $\widetilde{M_1^{\text{ext}}}$ might *a priori* have conjugate points (along geodesics trapped in $\widetilde{M}_1$), Busemann functions along rays where L_V is not vertical behave like those in manifolds without conjugate points. In particular, they are $C^{1,1}$ smooth, $|\nabla b_{1V}| = 1$, and the Lipschitz constant for ∇b_{1V} is uniformly bounded.

For $V \in U\widetilde{M_1^{\text{ext}}}$ such that $\gamma_V(t)$ minimizes for all positive t, let $b_{1V} : \widetilde{M_1^{\text{ext}}} \to \mathbb{R}$ be the Busemann function defined by V, i.e.

$$b_{1V}(p) = \lim_{t \to \infty} d_1(p, \gamma_v(t)) - t.$$

Since $d_1(p, q) = d(p, q)$ when p and q are points in $\widetilde{M_1^{\text{ext}}} - \widetilde{M}_1$, b_{1V} coincides with the Euclidean b_V outside $\widetilde{M}_1$ as long as γ_{1V} is not trapped. That is, b_{1V} will coincide with the height function (up to a constant) in the direction L_V.

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For all reals s we will let $H_V(s) = \{p \in \widetilde{M_1^{\text{ext}}} | b_{1V}(p) = s\}$ be the s level set of b_{1V}. Of course, outside of $\widetilde{M}_1$, $H_V(s)$ is just a hyperplane perpendicular to L_V.

LEMMA 2.6. *For all $V \in \mathbf{A}$, b_{1V} is $C^{1,1}$ and the Lipschitz constant of ∇b_{1V} is bounded by a constant independent of V.*

Proof. The proof is the usual proof that such a statement holds on manifolds without conjugate points. This is done by showing that the approximating functions, $f_t(p) = d_1(p, \gamma_{1V}(t)) - t$, are C^∞, have $|\nabla f_t| = 1$ and have uniformly bounded Hessian. If $\gamma_{1V}(t) \in \widetilde{M_1^{\text{ext}}} - \widetilde{M}_1$, then all geodesics from $\gamma_{1V}(t)$ minimize so the distance function from $\gamma_{1V}(t)$ is C^∞ for large t. The fact that $|\nabla f_t| = 1$ is clear. The uniform control on the Hessian is also the same as we will see. Fix a number r less than the convexity radius of M_1^{ext}, which exists since $M_1^{\text{ext}} - M_1$ is flat. Since there is a compact set $K \subset M_1^{\text{ext}}$ of base points such that for $q \notin K$ the ball $B(q, r)$ is flat we conclude that the eigenvalues of the second fundamental forms of the boundaries of $B(q, r)$ are uniformly bounded above and below independent of q. This same bound applies to balls in the universal cover $\widetilde{M_1^{\text{ext}}}$. Now to bound the Hessian of f_t at $q \in \widetilde{M_1^{\text{ext}}}$ let $\tau(s)$ be the (unique) geodesic from $\gamma_{1V}(t)$ to $q = \tau(s_0)$ (we can assume that $s_0 \gg r$ since we will be taking the limit as $t \to \infty$). Then by the triangle inequality the level set of f_t at q (i.e. $\partial B(\gamma_{1V}(t), s_0)$) lies outside both $B(\tau(s_0 - r), r)$ and $B(\tau(s_0 + r), r)$ which are tangent to the level set at q. Hence, the second fundamental forms of the level sets are uniformly bounded and, hence, so is the Hessian. Thus, the lemma follows. □

The usual properties of Busemann functions (see [**Es77**] for the basic properties of Busemann functions) tell us that if $W(p) = \nabla b_{1V}(p)$, then $\gamma'_{1W}(t) = \nabla b_{1V}(\gamma_{1W}(t))$ for all t. Hence, if γ_{1W} is not trapped, then L_W will be parallel to L_V. A straightforward open and closed argument shows that for all p, $\gamma_{1W(p)}$ is not trapped.

LEMMA 2.7. *Let V and W in $\mathbf{A}$ be such that L_V and L_W are positive and not parallel to each other. Then for any given s the maximum and minimum values of b_{1V} on the compact $H_W(s) \cap \widetilde{M}_1$ are achieved on the boundary of $H_W(s) \cap \widetilde{M}_1$.*

Proof. We first note that $H_W(s) \cap \widetilde{M}_1$ is compact. Indeed, if D is the diameter of M_1, then for every $p \in H_W(s) \cap \widetilde{M}_1$ there is a point $q \in \partial(\widetilde{M}_1)$ with $d(p, q) \leq D$. Since b_{1W} has Lipschitz constant one, we know that $s - D \leq b_{1W}(q) \leq s + D$ and, hence, p lies in the (compact) set of points that are at distance $\leq D$ from the compact (as W is not horizontal as it is positive) set of boundary points $\{q \in \partial \widetilde{M}_1 | s - D \leq b_{1W}(q) \leq s + D\} = \{q \in \partial(B^{n-1} \times \mathbb{R}) | s - D \leq b_W(q) \leq s + D\}$.

If the maximum (or minimum) value of b_{1V} occurs in the interior, then ∇b_{1V} must be perpendicular to $H_W(s)$ there and, hence, coincides with $\pm \nabla b_{1W}$ at that point. However, this contradicts the condition that L_V and L_W are positive and not parallel. □

We now see that if V and W in $\mathbf{A}$ are such that L_V and L_W are parallel, then $b_{1V} - b_{1W}$ is constant. Since they agree with the height functions outside $\widetilde{M}_1$, $b_{1V} - b_{1W} = C$ outside $\widetilde{M}_1$. However, then they must also differ by C along any geodesic whose corresponding line is parallel to L_V and L_W. Since such a geodesic passes though every point $p \in \widetilde{M}_1$ (i.e. take the geodesic in the direction of $\nabla b_{1V}(p)$), this says $b_{1V} - b_{1W} = C$ everywhere.

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In particular, for every p there is a unique geodesic passing through p and parallel to a given line. This gives a natural identification of $\mathbf{A}(p)$ with the space of non-vertical directions.

PROPOSITION 2.8. *Through every $p \in \widetilde{M}_1$ there is exactly one trapped geodesic.*

Proof. This is equivalent to showing that for every $p \in \widetilde{M_1^{\text{ext}}}$, $\mathbf{A}(p)$ consists of the unit sphere $U_p\widetilde{M_1^{\text{ext}}}$ minus a pair of antipodal points. Assume that p is a point with more trapped geodesics. Note that if there is a trapped half geodesic at p then the other half must also be trapped by the assumption that the scattering data coincides with the flat case. Thus, the tangent directions to trapped geodesics come in antipodal pairs. Of course there is at least one trapped geodesic through p since one could take the limit of a subsequence of geodesics from p to a boundary points q_i where q_i runs off to infinity. We only need to consider p in the interior of $\widetilde{M}_1$.

A limiting half geodesic of a sequence of trapped half geodesics starting at p will be a half geodesic starting at p that stays in $\widetilde{M}_1$. (In fact, it stays in the interior since the only half geodesics in $\widetilde{M}_1$ that are tangent to the boundary are the vertical ones, hence stay in the boundary.) Thus, the set of tangent directions to trapped geodesics (i.e. $U_p - \mathbf{A}(p)$) is closed in U_p and, thus, $\mathbf{A}(p)$ is open and non-empty (by the correspondence with non-vertical directions). The set of boundary points of $U_p - \mathbf{A}(p)$ is thus non-empty and if it consisted of a single antipodal pair, then $U_p - \mathbf{A}(p)$ would also be a single antipodal pair. Thus, there is a pair of distinct unit vectors V and W in $U_p - \mathbf{A}(p)$ such that $\langle V, W\rangle = C$ with $-1 < C < 1$ (one could take $C \geq 0$) and such that there exists sequences $V_i \in \mathbf{A}(p)$ and $W_i \in \mathbf{A}(p)$ such that V_i converges to V and W_i converges to W. We extend V_i and W_i to vector fields by letting $V_i(q) = \nabla b_{1V_i}(q)$ and $W_i(q) = \nabla b_{1W_i}(q)$. By the uniform bound on Lipschitz constants on Busemann functions, i.e. Lemma 2.6, there is an $\epsilon > 0$ (depending only on C and the Lipschitz constant of the Busemann functions but not on i) such that for all $q \in B_p(\epsilon)$ (the ϵ ball about p) and all sufficiently large i we have

$$-\frac{1+C}{2} < \langle V_i(q), W_i(q)\rangle < \frac{1+C}{2}.$$

This holds since for large i we have $\langle V_i(p), W_i(p)\rangle$ is approximately C and then, with respect to a parallel frame along geodesics of length ϵ, the change of V_i and W_i is uniformly bounded by the Lipschitz constant and ϵ. We can further take ϵ less than the distance from p to $\partial\widetilde{M}_1$.

Now consider the Busemann function b_{1V_i} on the zero level set $H_{W_i}(0)$ of b_{1W_i} (i.e. the level set through p). By the inner product condition above we can find unit speed differentiable curves τ_1 and τ_2 in $H_{W_i}(0)$ starting at p of length ϵ (and, hence, in $H_{W_i}(0) \cap B_p(\epsilon)$) such that

$$\langle \tau_1'(s), V_i(\tau_1(s))\rangle > \bar{C} = \sqrt{1 - \left(\frac{1+C}{2}\right)^2}$$

and

$$\langle \tau_2'(s), V_i(\tau_2(s))\rangle < -\bar{C} = -\sqrt{1 - \left(\frac{1+C}{2}\right)^2}.$$

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Thus, for every sufficiently large i there are points $z_i^1, z_i^2 \in H_{W_i}(0) \cap B_p(\epsilon)$ such that $b_{1V_i}(z_i^1) > \bar{C}\epsilon$ and $b_{1V_i}(z_i^2) < -\bar{C}\epsilon$. Thus, by Lemma 2.7 for every sufficiently large i there are points y_i^1 and y_i^2 on the boundary of $H_{W_i}(0) \cap \widetilde{M}_1$ with $b_{1V_i}(y_i^1) > \bar{C}\epsilon$ and $b_{1V_i}(y_i^2) < -\bar{C}\epsilon$.

Since V_i and W_i converge to trapped geodesics, Lemma 2.5 says that as $i \to \infty$ the lines L_{V_i} and L_{W_i} converge to vertical. However, this means that the Busemann functions b_{1V_i} and b_{1W_i} (which are height functions outside $\widetilde{M}_1$) approximate the vertical height function. In particular, for i large enough the values of b_{1V_i} on the boundary of $H_{W_i}(0) \cap \widetilde{M}_1$ vary by less than $\bar{C}\epsilon$. This contradicts the simultaneous existence of y_i^1 and y_i^2 for large i. □

The first consequence of this proposition is that the trapped geodesics are also minimizing (as limits of minimizing geodesics) and, hence, g has no conjugate points. Start with a large enough flat $n+1$ torus $T^n \times S^1$ so that D^n sits isometrically in T^n. Now if we replace $D^n \times S^1$ with $(M_1, \partial M_1, g)$ we get an $n+1$ torus since M_1 is diffeomorphic to M (as was pointed out after the proof of Lemma 2.4). Further it has no conjugate points. Then the theorem of Burago–Ivanov [**BI94**] proving the E. Hopf conjecture says that the metric is flat. This gives a proof of Theorem 1.1.

However, the above proof does not generalize very far. In the next section we give an alternative proof that does generalize.

3. *Generalizations*

In the previous section we considered only flat metrics so as to make the proof more transparent. However, the arguments extend almost without change to give the following result.

PROPOSITION 3.1. *Let D^n be a ball in a complete simply connected manifold N^n with non-positive curvature, and $(M_1, \partial M_1, g_1)$ a Riemannian manifold with boundary that has the same scattering data as $(D \times S^1, \partial D \times S^1, g)$, where g is the product metric. Then through every point of M_1 there is exactly one trapped geodesic.*

The main change that affects the proof is that geodesics in $\widetilde{M_1^{\text{ext}}}$ are not lines outside $\widetilde{M}_1$ but geodesics in N. Oriented geodesic rays in N are thought of as parallel if they have the same limit point at infinity (which means that they stay a bounded distance from each other). The notion of 'positive' geodesics also makes sense. This allows us to relate $\mathbf{A}(p)$ to $\mathbf{A}(q)$. Most of the arguments go through exactly as before. In particular Lemmas 2.3–2.4 go through as is. Lemma 2.5 also goes through as before where convergence of directions needs to be interpreted as the endpoints at infinity converging (where the topology of infinity is the cone topology, hence homeomorphic to the standard $n-1$ sphere). The argument in Lemma 2.7 needs to be viewed carefully since there may be many oriented geodesics corresponding to $-\nabla b_{1W}$ as the point on $H_W(s)$ varies. However, none of these will be positive so again the proof goes through. The only part of Proposition 2.8 that needs to be noted is that in non-positive curvature the level sets of Busemann functions vary continuously with the initial vector.

Remark 3.2. The above arguments likely can be modified to cover manifolds without conjugate points. One first needs to deal with the fact that balls may not be convex. This

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would seem to give us problems with the differentiability of the metric on $\widetilde{M_1^{\text{ext}}}$ at the boundary of $\widetilde{M}_1$. However (except possibly in the case where the boundary of D contains a region that is totally geodesic), since there are no conjugate points, Theorem 1 of [**SU09**] will still tell us that the metric will be smooth. In fact, very little of the argument really needs the boundary to be convex (or the metric to be smooth for that mater) but extending the arguments would look somewhat different from the above. Also one has to worry about relating $\mathbf{A}(p)$ and $\mathbf{A}(q)$. This can be done by fixing a base point $x_0 \in \widetilde{M_1^{\text{ext}}} - \widetilde{M}_1$ and looking only at Busemann functions defined by vectors in U_{x_0}. Also Busemann functions are not as well behaved. In any event, the arguments would look very different and we do not pursue them here.

We now want to generalize Theorem 1.1 to the non-flat case. The first point to note is that for g and g_1 as in Proposition 3.1 we have

$$\text{Vol}(g_1) = \text{Vol}(g).$$

Let $\mathbf{T_1} \subset UM_1$ be the set of unit vectors tangent to trapped geodesic rays. Similarly define $\mathbf{T} \subset UM$ (here $M = D \times S^1$). Using the fact that the metrics are lens equivalent we consider the standard measure preserving map $F : UM_1 - \mathbf{T} \to UM - \mathbf{T}$ which assigns to each vector $V \in UM_1 - \mathbf{T}$ the unique vector $W \in UM_1 - \mathbf{T}$ such that $V = \gamma_1'(t)$ and $W = \gamma'(t)$, where γ and γ_1 are geodesics with corresponding initial conditions on the boundary. That is, $\gamma(0) \in \partial M$ and $\gamma_1(0) \in \partial M_1$ are corresponding points while $\gamma'(0)$ and $\gamma_1'(0)$ are corresponding inwardly pointing unit vectors. That F (which conjugates the geodesic flows) is measure preserving is a standard fact which follows for example from Santaló's formula (see, for example, [**Cr91**]). The fact that $\mathbf{T}$ and $\mathbf{T_1}$ have measure zero tells us that the unit tangent bundles (and, hence, the manifolds) have the same volume.

The generalization of Theorem 1.1 is as follows.

THEOREM 3.3. *Let D^n be a ball in $N^{n-1} \times \mathbb{R}$, where N is a complete simply connected manifold with non-positive curvature. Then $D \times S^1$ is scattering rigid.*

Proof. One proves this via [**CK98**, Proposition 2.2]. That result compares two metrics g and g_1 on manifolds without conjugate points with an additional condition on g that it contain a real factor (which is satisfied by our assumption on D). Under a weak lens equivalency condition (which is satisfied since the metrics are lens equivalent) one concludes that $\text{Vol}(g_1) \geq \text{Vol}(g)$ with equality holding if and only if g_1 is isometric to g. This (along with the above fact that $\text{Vol}(g_1) = \text{Vol}(g)$) proves the theorem. □

Note that we do not claim the result for D^n a ball in a complete simply connected manifold with non-positive curvature. Here D^n is a ball in $N^{n-1} \times \mathbb{R}$. This means that in the universal cover of D (and, hence, of $M = D \times S^1$) there is a bounded $\mathbb{R}$ factor (as opposed to the unbounded $\mathbb{R}$ factor coming from the S^1 factor in M). This is because [**CK98**, Proposition 2.2] requires a set whose $\mathbb{R}$ component in the universal cover is bounded.

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