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RESEARCH ARTICLE

Quantum duality principle and quantum symmetric pairs

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Abstract

The *quantum duality principle* (QDP) by Drinfeld predicts a connection between the *quantized universal enveloping algebras* and the *quantized coordinate algebras*, where the underlying classical objects are related by the duality in Poisson geometry. The current paper gives an explicit formulization of the QDP for quantum symmetric pairs.

Let $\mathfrak{g}$ be a complex semi-simple Lie algebra, equipped with the standard Lie bialgebra structure. Let θ be a Lie algebra involution on $\mathfrak{g}$ and denote by $\mathfrak{k} = \mathfrak{g}^\theta$ the fixed point subalgebra. The quantum symmetric pair $(\mathrm{U}, \mathrm{U}^\imath)$ is originally defined to be a quantization of the symmetric pair of the universal enveloping algebras $(U(\mathfrak{g}), U(\mathfrak{k}))$. In this paper, we show that an explicit specialization of $(\mathrm{U}, \mathrm{U}^\imath)$ gives rise to the pair of the coordinate algebras $(\mathcal{O}(G^*), \mathcal{O}(K^\perp \backslash G^*))$, where G^* is the dual Poisson-Lie group with the Lie algebra $\mathfrak{g}^*$, and $K^\perp \backslash G^*$ is a G^*-Poisson homogeneous space. Here $K^\perp$ is the closed subgroup of G^* associated to the complementary dual of $\mathfrak{k}$. Therefore $(\mathrm{U}, \mathrm{U}^\imath)$ can be viewed as a pair of quantized coordinate algebras. This generalizes the result of De Concini–Procesi [14] that the quantum group U provides a quantization of the coordinate algebra of G^*.

Contents

1 Introduction **2**
2 Preliminaries **3**
2.1 Quantum groups . . . 3
2.2 Quantum symmetric pairs . . . 4
2.3 Poisson structures . . . 4
2.4 Specialization of quantum algebras . . . 5
2.5 Specialization of ${}_{\mathcal{A}}\mathrm{U}$ . . . 5
3 Proof of the main theorem **6**
3.1 Geometric preparations . . . 6
3.2 Quantization of the group P . . . 8
3.3 Quantization of the map ι^* . . . 8
3.4 Poisson generators . . . 9
3.5 The final proof . . . 11
4 Example: Dubrovin–Ugaglia Poisson structures **12**
References **12**

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1. Introduction

Let G be a complex semi-simple group, equipped with the standard multiplicative Poisson structure. Let $\mathrm{U} = U_q\mathfrak{g}$ be the Drinfeld–Jimbo quantum group, which is a Hopf algebra over the base field $\mathbb{C}(q^{1/2})$. When viewed as the quantized universal enveloping algebra of $\mathfrak{g}$, the algebra U is a central object in representation theory.

One can also view U as the quantized coordinate algebra on the dual Poisson group G^* as follows. Let $\mathcal{A} = \mathbb{C}[q^{1/2}, q^{-1/2}]$ be the subring of the base field $\mathbb{C}(q^{1/2})$. In comparison with Lusztig's integral form [21, 3.1.13], one can construct a non-standard $\mathcal{A}$-form ${}_{\mathcal{A}}\mathrm{U}$ of U (see §2.1), where the base change ${}_{\mathbb{C}}\mathrm{U} = \mathbb{C}\otimes_{\mathcal{A}}{}_{\mathcal{A}}\mathrm{U}$ becomes a commutative algebra. Moreover ${}_{\mathbb{C}}\mathrm{U}$ carries a Poisson algebra structure where the Poisson brackets are induced by taking the commutators in ${}_{\mathcal{A}}\mathrm{U}$ (see (2.6)).

The work of De Concini–Procesi [14, Theorem 12.1] gives an explicit Poisson algebra isomorphism

$$\varphi : {}_{\mathbb{C}}\mathrm{U} \xrightarrow{\sim} \mathcal{O}(G^*). \tag{1.1}$$

We recall the map φ in §2.5. Let us mention that although the QDP suggests that U shall provide a quantization of $\mathcal{O}(G^*)$, it takes much more effort to explicitly establish the isomorphism (1.1).

Let θ be a complex group involution on G, and $K = G^\theta$ be the θ-fixed point subgroup. Let $\mathfrak{g}$ and $\mathfrak{k}$ be the Lie algebras of G and K, respectively. Associated with the symmetric pair $(\mathfrak{g}, \mathfrak{k})$, the *quantum symmetric pair* $(\mathrm{U}, \mathrm{U}^\imath)$ is the pair which consists of the quantum group U, and a coideal subalgebra $\mathrm{U}^\imath$, called an *ıquantum group* [20, 8]. As a quantization of the universal enveloping algebra of $\mathfrak{k}$ inside U, the algebra $\mathrm{U}^\imath$ has found important applications in representation theory and algebraic geometry [6, 7, 9].

This paper studies the ıquantum group $\mathrm{U}^\imath$ from another point of view of quantized coordinate algebras. Let ${}_{\mathcal{A}}\mathrm{U}^\imath = \mathrm{U}^\imath \cap {}_{\mathcal{A}}\mathrm{U}$ be the $\mathcal{A}$-form of $\mathrm{U}^\imath$ induced from the non-standard integral form on U. This integral form is essentially different from the one studied by Bao and Wang [8]. Let ${}_{\mathbb{C}}\mathrm{U}^\imath = \mathbb{C}\otimes_{\mathcal{A}}{}_{\mathcal{A}}\mathrm{U}^\imath$ be the commutative Poisson algebra obtained by the specialization $q^{1/2} \mapsto 1$, and let ${}_{\mathbb{C}}\imath : {}_{\mathbb{C}}\mathrm{U}^\imath \to {}_{\mathbb{C}}\mathrm{U}$ be the Poisson algebra homomorphism obtained by the base change of the natural embedding $\imath : {}_{\mathcal{A}}\mathrm{U}^\imath \to {}_{\mathcal{A}}\mathrm{U}$.

Let $\mathfrak{k}^\perp$ be the space of linear forms on $\mathfrak{g}$ which vanish on $\mathfrak{k}$. We show (Proposition 3.1) that $\mathfrak{k}^\perp$ is moreover a Lie subalgebra of $\mathfrak{g}^*$. Let $K^\perp$ be the connected closed subgroup of G^* associated with $\mathfrak{k}^\perp$. The affine quotient $K^\perp\backslash G^*$ is automatically a G^*-Poisson homogeneous space, whose coordinate algebra is isomorphic to the Poisson subalgebra of $K^\perp$-invariant functions on G^*.

The main theorem of the paper is as follows.

Theorem 1. *There is a unique Poisson algebra isomorphism*

$$\varphi^\imath : {}_{\mathbb{C}}\mathrm{U}^\imath \xrightarrow{\sim} \mathcal{O}(K^\perp\backslash G^*), \tag{1.2}$$

such that the diagram

$$\begin{array}{ccc} \mathcal{O}(K^\perp\backslash G^*) & \hookrightarrow & \mathcal{O}(G^*) \\ {\scriptstyle\varphi^\imath}\uparrow & & \uparrow{\scriptstyle\varphi} \\ {}_{\mathbb{C}}\mathrm{U}^\imath & \xrightarrow[{}_{\mathbb{C}}\imath]{} & {}_{\mathbb{C}}\mathrm{U} \end{array} \tag{1.3}$$

commutes.

Our construction of $\varphi^\imath$ is explicit, and hence it is suitable for applications.

The Poisson structures $K^\perp\backslash G^*$ occur in many fields of mathematics and physics [1, 3, 5]. Therefore our result provides a bridge to utilize the theory of quantum symmetric pairs to study such Poisson structures. As an example, the Dubrovin–Ugaglia Poisson structures on the space of $(n \times n)$-upper triangular matrices with 1 on the diagonal appear in the study of Frobenius manifolds [13, 27], and are related to the connections on the Riemann surfaces as well as Poisson–Lie groups by the work of

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Boalch [1, 2]. In §4 we see that the $\imath$quantum group associated to the symmetric pair $(\mathfrak{sl}_n, \mathfrak{so}_n)$ is a quantization of the Dubrovin–Ugaglia Poisson structure, and the well-known braid group symmetries on such Poisson structures can be obtained from the braid group symmetries on the corresponding $\imath$quantum group via specialization.

Viewing U as the quantized coordinate algebra is the fundamental philosophy to relate cluster algebras with quantum groups [26, 22]. Based on the same point of view, in the work [23] the author gave a cluster algebra realization of the $\imath$quantum group associated to $(\mathfrak{sl}_n, \mathfrak{so}_n)$. It is desirable to generalize the construction to other types. Our result provides the geometric foundation in this direction.

In [11], Ciccoli and Gavarini generalized Drinfeld's QDP to the setting of subgroups and homogeneous spaces. However, like the work of Drinfeld [16], the construction in [11] is based on the formal power series setting and hence cannot be applied to our situation. In their subsequent paper [12], by imposing various technical assumptions, the authors deal with the quantum algebras over the ordinary fractional field, which they call the global version of the QDP. Their results are general but abstract, and it is difficult to check those assumptions for explicit examples. Our result fits into the general philosophy of their QDP, but our proof is independent of theirs.

2. Preliminaries

In this section, we recall basic constructions on quantum groups, quantum symmetric pairs, (dual) Poisson groups and the specializations of quantum algebras.

2.1. Quantum groups

Fix a finite index set I, and a Cartan matrix $A = (a_{ij})_{i,j\in \mathrm{I}}$. Let $D = diag(\epsilon_i)_{i\in \mathrm{I}}$ be the diagonal matrix such that DA is symmetric positive definite, with $\epsilon_i \in \mathbb{Z}_{>0}$ and $\{\epsilon_i \mid i \in \mathrm{I}\}$ relatively prime.

Let $\mathfrak{g}$ be the complex semi-simple Lie algebra associated with the Cartan matrix A, and $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. Take Borel subalgebras $\mathfrak{b}^+$ and $\mathfrak{b}^-$ of $\mathfrak{g}$, such that $\mathfrak{b}^+ \cap \mathfrak{b}^- = \mathfrak{h}$. Denote by $\Phi \subset \mathfrak{h}^*$ the root system of $\mathfrak{g}$ relative to $\mathfrak{h}$. Let $\mathcal{R}^+ \subset \Phi$ be the set of positive roots associated with $\mathfrak{b}^+$, and $\Delta = \{\alpha_i \mid i \in \mathrm{I}\}$ be the subset of simple roots. Set $\mathcal{R}^- = -\mathcal{R}^+$. Let

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha$$

be the root space decomposition. Set $\mathfrak{u}^+ = \oplus_{\alpha\in\mathcal{R}^+}\mathfrak{g}_\alpha$ and $\mathfrak{u}^- = \oplus_{\alpha\in\mathcal{R}^-}\mathfrak{g}_\alpha$. Let Q be the root lattice, and W be the Weyl group of $\mathfrak{g}$ with generators $\{s_i \mid i \in \mathrm{I}\}$. The group W acts on Q in a natural way. For an element w in W, let $|w|$ be the length of w. Let w_0 be the longest element in W, and write $n = |w_0|$.

Let q be an indeterminate, and $q^{1/2} \in \overline{\mathbb{Q}(q)}$ be a fixed square root of q. Let $q_i = q^{\epsilon_i}$, for $i \in \mathrm{I}$. Let U be the Drinfeld–Jimbo quantum group associated with $\mathfrak{g}$ over the base field $\mathbb{C}(q^{1/2})$ (cf. [18, 4.3]). Let $\varepsilon : \mathrm{U} \to \mathbb{C}(q^{1/2})$ be the counit. For $i \in \mathrm{I}$, let $\mathbf{E}_i$, $\mathbf{F}_i$ and $\mathbf{K}_i$ be the generators as in *loc. cit.* For our purpose, we consider the rescaled generators

$$E_i = q_i^{1/2}(q_i^{-1} - q_i)\mathbf{E}_i, \quad F_i = q_i^{-1/2}(q_i - q_i^{-1})\mathbf{F}_\mathbf{i}, \quad K_i = \mathbf{K}_i, \quad \text{for } i \in \mathrm{I}.$$

Let U^- (resp., U^+) be the unital $\mathbb{C}(q^{1/2})$-subalgebra of U generated by F_i (resp., E_i), for various $i \in \mathrm{I}$. Let U^0 be the unital $\mathbb{C}(q^{1/2})$-subalgebra of U generated by $K_i^{\pm 1}$, for $i \in \mathrm{I}$. For any $\mu = \sum_{i\in\mathrm{I}} a_i\alpha_i \in Q$, we write $K_\mu = \prod_{i\in\mathrm{I}} K_i^{a_i}$.

For $i \in \mathrm{I}$, let T_i be the $\mathbb{C}(q^{1/2})$-algebra automorphism of U, denoted by $T''_{i,-1}$ in [21, 37.1.3]. It is known that automorphisms $\{\mathrm{T}_i \mid i \in \mathrm{I}\}$ satisfy the braid group relations. For $w \in W$, we write $\mathrm{T}_w = \mathrm{T}_{i_1}\mathrm{T}_{i_2}\cdots\mathrm{T}_{i_r}$, where $w = s_{i_1}s_{i_2}\cdots s_{i_r}$ and $r = |w|$.

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Let $\mathbf{i} = (i_1, i_2, \cdots, i_n) \in \mathrm{I}^n$ be a reduced expression of w_0. For $\mathbf{a} = (a_1, \cdots, a_n) \in \mathbb{N}^n$, set

$$F_{\mathbf{i}}(\mathbf{a}) = E_{i_1}^{a_1} \mathrm{T}_{i_1}(E_{i_2}^{a_2}) \cdots \mathrm{T}_{i_1} \cdots \mathrm{T}_{i_{n-1}}(E_{i_n}^{a_n}),$$
$$E_{\mathbf{i}}(\mathbf{a}) = F_{i_1}^{a_1} \mathrm{T}_{i_1}(F_{i_2}^{a_2}) \cdots \mathrm{T}_{i_1} \cdots \mathrm{T}_{i_{n-1}}(F_{i_n}^{a_n}).$$

For $1 \le k \le n$, we write $F_{\mathbf{i},k} = F_{\mathbf{i}}(\mathbf{e}_k)$ and $E_{\mathbf{i},k} = E_{\mathbf{i}}(\mathbf{e}_k)$, where $\mathbf{e}_k = (\delta_{kj})_{j=1}^n \in \mathbb{N}^n$.

Set $\mathcal{A} = \mathbb{C}[q^{1/2}, q^{-1/2}]$ to be the subring of $\mathbb{C}(q^{1/2})$. Let ${}_{\mathcal{A}}\mathrm{U}^-$ (resp., ${}_{\mathcal{A}}\mathrm{U}^+$) be the $\mathcal{A}$-submodule of U^- (resp., U^+), spanned by $F_{\mathbf{i}}(\mathbf{a})$ (resp., $E_{\mathbf{i}}(\mathbf{a})$), for various $\mathbf{a} \in \mathbb{N}^n$. It is known that ${}_{\mathcal{A}}\mathrm{U}^-$ and ${}_{\mathcal{A}}\mathrm{U}^+$ are independent of the choice of $\mathbf{i}$, and are closed under multiplications. Let ${}_{\mathcal{A}}\mathrm{U}^0$ be the unital $\mathcal{A}$-subalgebra of U^0, generated by $K_i^{\pm 1}$, for $i \in \mathrm{I}$. Let ${}_{\mathcal{A}}\mathrm{U}$ be the $\mathcal{A}$-subalgebra of U, generated by ${}_{\mathcal{A}}\mathrm{U}^-$, ${}_{\mathcal{A}}\mathrm{U}^0$ and ${}_{\mathcal{A}}\mathrm{U}^+$. It is different from the integral form defined by Lusztig [21, 3.1.13], and is the same as the one studied by Berenstein–Greenstein [4]. By [4, Corollary 3.13 & Theorem 3.11], the multiplication gives an isomorphism as $\mathcal{A}$-modules

$$ {}_{\mathcal{A}}\mathrm{U} \cong {}_{\mathcal{A}}\mathrm{U}^+ \otimes {}_{\mathcal{A}}\mathrm{U}^0 \otimes {}_{\mathcal{A}}\mathrm{U}^-. \tag{2.1}$$

Here the tensor products are over $\mathcal{A}$. It is easy to see that the $\mathcal{A}$-subalgebra ${}_{\mathcal{A}}\mathrm{U}$ is invariant under the braid group action T_i, for $i \in \mathrm{I}$.

2.2. *Quantum symmetric pairs*

Let $(\mathrm{I} = \mathrm{I}_\circ \sqcup \mathrm{I}_\bullet, \tau)$ be a Satake diagram (cf. [8]). Recall that τ is a graph involution on I, which leaves $\mathrm{I}_\circ$ and $\mathrm{I}_\bullet$ invariant. We fix once for all a subset $\mathrm{I}'_\circ \subset \mathrm{I}_\circ$, consisting of exactly one element in each τ-orbits of $\mathrm{I}_\circ$. Let $\Phi_\bullet \subset \Phi$ be the sub-root system with simple roots $\{\alpha_i \mid i \in \mathrm{I}_\bullet\}$, and $W_\bullet = \langle s_i \mid i \in \mathrm{I}_\bullet \rangle$ be the parabolic subgroup. Let $w_\bullet$ be the longest element in $W_\bullet$. Let $\mathcal{R}_\bullet^+ = \mathcal{R}^+ \cap \Phi_\bullet$ and $\mathcal{R}_\bullet^- = \mathcal{R}^- \cap \Phi_\bullet$. Let $\theta = -w_\bullet \tau : Q \to Q$ be the group involution on Q, and $Q^\theta = \{\mu \in Q \mid \theta(\mu) = \mu\}$.

We fix parameters $\varsigma_i \in \pm q^{\mathbb{Z}}$, for $i \in \mathrm{I}_\circ$, which satisfy the conditions in [8, Definition 3.5]. The associated *ıquantum group* $\mathrm{U}^\imath$ is the unital $\mathbb{C}(q^{1/2})$-subalgebra of U, generated by the following elements,

$$B_i = F_i - \varsigma_i \mathrm{T}_{w_\bullet}(E_{\tau i}) K_i^{-1}, \ k_i = K_i K_{\tau i}^{-1}, \qquad \text{for } i \in \mathrm{I}_\circ;$$
$$E_i, \ F_i, \ K_i^{\pm 1} \qquad \text{for } i \in \mathrm{I}_\bullet.$$

We write $B_i = F_i$, for $i \in \mathrm{I}_\bullet$. The pair $(\mathrm{U}, \mathrm{U}^\imath)$ is called a *quantum symmetric pair*.

2.3. *Poisson structures*

For any complex affine algebraic variety V, we write $\mathcal{O}(V)$ to denote the $\mathbb{C}$-algebra of regular functions on V.

Let G be the complex semi-simple adjoint group with the Lie algebra $\mathfrak{g}$. Let B^+, B^- and H be the connected closed subgroups of G, whose Lie algebras are $\mathfrak{b}^+$, $\mathfrak{b}^-$ and $\mathfrak{h}$, respectively. Let U^+ (resp., U^-) be the unipotent radical of B^+ (resp., B^-). For $\alpha \in \Phi$, let U_α be the root subgroup of G associated with α.

Let us briefly recall the standard Poisson structure associated with G, and the dual Poisson group G^*. We refer to [10, §1] for a detailed exposition. Let $\langle \, , \, \rangle$ be the Killing form of $\mathfrak{g}$, and $\mathfrak{g}^*$ be the space of $\mathbb{C}$-linear forms on $\mathfrak{g}$. The *Drinfeld double* $\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{g}$ admits a non-degenerate invariant bilinear form $\langle\langle \, , \, \rangle\rangle$, given by

$$\langle\langle (X_1, Y_1), (X_2, Y_2) \rangle\rangle = \langle X_1, X_2 \rangle - \langle Y_1, Y_2 \rangle, \quad \forall \, X_1, X_2, Y_1, Y_2 \in \mathfrak{g}.$$

Let us identify $\mathfrak{g}$ with the diagonal Lie subalgebra of $\mathfrak{d}$, and identify $\mathfrak{g}^*$ with another Lie subalgebra,

$$\mathfrak{g}^* \cong \{(X, Y) \in \mathfrak{b}^+ \times \mathfrak{b}^- \mid \text{the } \mathfrak{h}\text{-component of } X + Y \text{ is } 0\}, \tag{2.2}$$

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via the bilinear form $\langle\langle\ ,\ \rangle\rangle$. Under these identifications the triple $(\mathfrak{d}, \mathfrak{g}, \mathfrak{g}^*)$ forms a *Manin triple*, which determines a *Lie bialgebra* structure on $\mathfrak{g}$, and determines the *standard Poisson structure* on G.

By a dual construction, one gets a Lie bialgebra structure on $\mathfrak{g}^*$, which determines a Poisson structure on the *dual Poisson group* G^*, where

$$G^* = \{(u_+t, t^{-1}u_-) \mid u_\pm \in U^\pm, t \in H\}$$

is the connected closed subgroup of $B^+ \times B^-$ with the Lie algebra $\mathfrak{g}^*$.

We have an isomorphism as varieties

$$U^+ \times H \times U^- \xrightarrow{\sim} G^*, \qquad (u_+, t, u_-) \mapsto (u_+t, t^{-1}u_-), \tag{2.3}$$

which determines an isomorphism as $\mathbb{C}$-algebras

$$\mathcal{O}(G^*) \cong \mathcal{O}(U^+) \otimes \mathcal{O}(H) \otimes \mathcal{O}(U^-). \tag{2.4}$$

2.4. Specialization of quantum algebras

Let R_q be a non-commutative unital $\mathcal{A}$-algebra, which satisfies the condition

$$[R_q, R_q] \subseteq (q^{1/2} - 1)R_q. \tag{2.5}$$

Let $R = \mathbb{C} \otimes_{\mathcal{A}} R_q$ where the tensor product is given by $q^{1/2} \mapsto 1$. The condition (2.5) is equivalent to saying that the $\mathbb{C}$-algebra R is commutative. Denote by $f \mapsto \overline{f}$ the canonical map $R_q \to R$. Then R carries a Poisson bracket $\{\ ,\ \} : R \times R \to R$, defined by

$$\{\overline{f}, \overline{g}\} = \overline{\frac{1}{2(q^{1/2} - 1)}(fg - gf)}, \quad \forall f, g \in R_q. \tag{2.6}$$

2.5. Specialization of ${}_{\mathcal{A}}\mathrm{U}$

Let us fix a pinning $\{x_i, y_i \mid i \in \mathrm{I}\}$ of the group G, where $x_i : \mathbb{C} \to U_{\alpha_i}$ and $y_i : \mathbb{C} \to U_{-\alpha_i}$ are one-parameter subgroups.

For $i \in \mathrm{I}$, set

$$\dot{s}_i = x_i(1)y_i(-1)x_i(1) \in G.$$

It is well-known that elements $\{\dot{s}_i \mid i \in \mathrm{I}\}$ satisfy the braid group relations. For $w \in W$, denote by $\dot{w} = \dot{s}_{i_1} \cdot \dot{s}_{i_2} \cdots \dot{s}_{i_k}$, for one (and hence for all) reduced expression $w = s_{i_1}s_{i_2}\cdots s_{i_k}$.

Take a reduced expression $\mathbf{i} = (i_1, \cdots, i_n)$ of the longest w_0 of W. For $1 \le k \le n$, set $\beta_{\mathbf{i},k} = s_{i_1}\cdots s_{i_{k-1}}(\alpha_{i_k}) \in \mathcal{R}^+$. Set $x_{\mathbf{i},k} = \mathrm{Ad}_{\dot{s}_{i_1}\cdots\dot{s}_{i_{k-1}}} \circ x_{i_k} : \mathbb{C} \to U_{\beta_{\mathbf{i},k}}$, and $y_{\mathbf{i},k} = \mathrm{Ad}_{\dot{s}_{i_1}\cdots\dot{s}_{i_{k-1}}} \circ y_{i_k} : \mathbb{C} \to U_{-\beta_{\mathbf{i},k}}$. We have $x_{\mathbf{i},k} = x_j$ and $y_{\mathbf{i},k} = y_j$, if $\beta_{\mathbf{i},k} = \alpha_j$, for some $j \in \mathrm{I}$.

By [25, Lemma 8.3.5], elements $a \in U^+$, $b \in U^-$ can be factored as

$$a = x_{\mathbf{i},n}(a_n)x_{\mathbf{i},n-1}(a_{n-1})\cdots x_{\mathbf{i},1}(a_1) \quad \text{and} \quad b = y_{\mathbf{i},1}(b_1)y_{\mathbf{i},2}(b_2)\cdots y_{\mathbf{i},n}(b_n), \tag{2.7}$$

for unique tuples $(a_1, \cdots, a_n) \in \mathbb{C}^n$ and $(b_1, \cdots, b_n) \in \mathbb{C}^n$. We define $\chi^+_{\mathbf{i},k} \in \mathcal{O}(U^+)$ by $\chi^+_{\mathbf{i},k}(a) = a_k$, and define $\chi^-_{\mathbf{i},k} \in \mathcal{O}(U^-)$ by $\chi_{\mathbf{i},k}(b) = b_k$. If $\beta_{\mathbf{i},k} = \alpha_j$ is a simple root, elements $\chi^+_{\mathbf{i},k}$ and $\chi^-_{\mathbf{i},k}$ are independent of the reduced expression, in which case we denote by $\chi^+_j = \chi^+_{\mathbf{i},k}$ and $\chi^-_j = \chi^-_{\mathbf{i},k}$.

Recall the $\mathcal{A}$-algebras ${}_{\mathcal{A}}\mathrm{U}$, ${}_{\mathcal{A}}\mathrm{U}^+$, ${}_{\mathcal{A}}\mathrm{U}^-$ and ${}_{\mathcal{A}}\mathrm{U}^0$ in §2.1. Let ${}_{\mathbb{C}}\mathrm{U}$, ${}_{\mathbb{C}}\mathrm{U}^+$, ${}_{\mathbb{C}}\mathrm{U}^-$ and ${}_{\mathbb{C}}\mathrm{U}^0$ be the $\mathbb{C}$-algebras obtained by the base change $q^{1/2} \mapsto 1$ of the corresponding $\mathcal{A}$-algebras, which are known

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to be commutative [4, Corollary 3.15]. Moreover the isomorphism (2.1) induces the isomorphism ${}_{\mathbb{C}}\mathrm{U} \cong {}_{\mathbb{C}}\mathrm{U}^+ \otimes {}_{\mathbb{C}}\mathrm{U}^0 \otimes {}_{\mathbb{C}}\mathrm{U}^-$ as $\mathbb{C}$-algebras. Here tensor products are over $\mathbb{C}$.

Take a reduced expression $\mathbf{i}$ of w_0. One has $\mathbb{C}$-algebra isomorphisms

$$\varphi^+ : {}_{\mathbb{C}}\mathrm{U}^+ \xrightarrow{\sim} \mathcal{O}(U^+) \qquad \text{and} \qquad \varphi^- : {}_{\mathbb{C}}\mathrm{U}^- \xrightarrow{\sim} \mathcal{O}(U^-), \tag{2.8}$$

given by $\varphi^+(\overline{E_{\mathbf{i},k}}) = \chi^+_{\mathbf{i},k}$ and $\varphi^-(\overline{F_{\mathbf{i},k}}) = \chi^-_{\mathbf{i},k}$, for $1 \le k \le n$. In particular $\varphi^+(\overline{E_i}) = \chi^+_i$ and $\varphi^-(\overline{F_i}) = \chi^-_i$, for $i \in \mathrm{I}$. The maps $\varphi^\pm$ are independent of the choice of $\mathbf{i}$.

Since G has trivial center, one has the canonical $\mathbb{C}$-algebra isomorphism $\mathcal{O}(H) \cong \mathbb{C}[Q]$, which induces the $\mathbb{C}$-algebra isomorphism $\varphi^0 : {}_{\mathbb{C}}\mathrm{U}^0 \xrightarrow{\sim} \mathcal{O}(H)$, given by $\varphi^0(\overline{K_\mu}) = \mu$, for $\mu \in Q$.

The following theorem asserts that ${}_{\mathcal{A}}\mathrm{U}$ specializes to the coordinate algebra $\mathcal{O}(G^*)$. The proof essentially follows from the work of De Concini–Kac–Procesi.

Theorem 2.1 [15, Theorem 7.6]. *The map*

$$\varphi = \varphi^+ \otimes \varphi^0 \otimes \varphi^- : {}_{\mathbb{C}}\mathrm{U} \xrightarrow{\sim} \mathcal{O}(G^*) \tag{2.9}$$

is an isomorphism as Poisson algebras.

Since ${}_{\mathcal{A}}\mathrm{U}$ is invariant under the action T_i, for $i \in \mathrm{I}$, each T_i descends to a Poisson algebra automorphism on ${}_{\mathbb{C}}\mathrm{U} \cong \mathcal{O}(G^*)$. We use the same notation T_i to denote the induced action.

3. Proof of the main theorem

We give the proof of Theorem 1 in this section. We need some preparation before the final proof. Retain the notations in the previous section.

3.1. Geometric preparations

For $i \in \mathrm{I}_\circ$, recall the parameter ς_i in §2.2. Set $c_i = \overline{\varsigma_i} \in \{\pm 1\}$. By [24], there is a unique complex group involution $\theta : G \to G$, such that,

$$\begin{aligned} \theta(x_i(\xi)) = x_i(\xi), \quad \theta(y_i(\xi)) = y_i(\xi), \qquad & \text{for } i \in \mathrm{I}_\bullet, \text{ and } \xi \in \mathbb{C}; \\ \theta(x_i(\xi)) = \mathrm{Ad}_{\dot{w}_\bullet}(y_{\tau i}(c_i\xi)), \quad \theta(y_i(\xi)) = \mathrm{Ad}_{\dot{w}_\bullet}(x_{\tau i}(c_i\xi)), \qquad & \text{for } i \in \mathrm{I}_\circ, \text{ and } \xi \in \mathbb{C}. \end{aligned}$$

Let $K = G^\theta$ be the fixed-point subgroup, and $\mathfrak{k}$ be the Lie algebra of K. We use the same notation θ to denote the induced Lie algebra involution on $\mathfrak{g}$.

Proposition 3.1. *The closed subgroup K is coisotropic, that is, the complementary dual*

$$\mathfrak{k}^\perp = \{f \in \mathfrak{g}^* \mid f(X) = 0, \ \forall X \in \mathfrak{k}\}$$

is a Lie subalgebra of $\mathfrak{g}^*$.

Proof. We identify $\mathfrak{g}^*$ with the subalgebra of the double $\mathfrak{d}$ as in (2.2). We *claim* that under the identification one has

$$\mathfrak{k}^\perp = \{(X_1, X_2) \in \mathfrak{g}^* \mid X_1 = \theta(X_2)\}. \tag{3.1}$$

Let $\mathfrak{p} = \{X \in \mathfrak{g} \mid \theta(X) = -X\}$. Then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ as vector spaces. Since the involution $\theta : \mathfrak{g} \to \mathfrak{g}$ preserves the Killing form, we have $\mathfrak{p} = \{X \in \mathfrak{g} \mid \langle X, Y\rangle = 0, \ \forall\, Y \in \mathfrak{k}\}$.

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Take any $x = (X_1, X_2) \in \mathfrak{g}^*$. We have

$$\begin{aligned} x \in \mathfrak{k} &\iff 0 = \langle\langle x, y\rangle\rangle = \langle X_1 - X_2, Y\rangle, \quad \text{for any } y = (Y, Y) \text{ with } Y \in \mathfrak{k}, \\ &\iff X_1 - X_2 \in \mathfrak{g}_1, \\ &\iff \theta(X_1) - X_2 = \theta(X_2) - X_1. \end{aligned} \tag{3.2}$$

Let us write $X_1 = X_1' + h$ and $X_2 = X_2' - h$, where $X_1' \in \mathfrak{u}^+$, $X_2' \in \mathfrak{u}^-$ and $h \in \mathfrak{h}$. Then the condition (3.2) holds if and only if $\theta(X_1') - X_2' = \theta(X_2') - X_1'$ and $\theta(h) = -h$. Therefore $\theta(X_1') - X_2' \in \mathfrak{p}$. On the other hand, by the construction of θ, we have

$$\theta(X_1') - X_2' \in \bigoplus_{\alpha \in \mathcal{R}^- - \Phi_\bullet} \mathfrak{g}_\alpha.$$

Note that $\mathfrak{g}_\alpha \subset \mathfrak{k}$, for $\alpha \in \Phi_\bullet$, and $\theta(\mathcal{R}^- - \Phi_\bullet) \cap \mathcal{R}^- = \emptyset$. We conclude that $\theta(X_1') - X_2' = 0$. Therefore we have $\theta(X_1) = X_2$. This proves the claim.

The proposition follows immediately. □

Let $K^\perp$ be the connected closed subgroup of G^* with the Lie algebra $\mathfrak{k}^\perp$. It is also a coisotropic subgroup. Hence the quotient $K^\perp\backslash G^*$ is automatically a Poisson homogeneous space of G^* [17].

In order to study the geometry of $K^\perp\backslash G^*$, let us introduce another closed subgroup

$$P = \{(u_+ t, t^{-1} u_-) \mid u_+ \in U^+_{w_\bullet}, u_- \in U^-, t \in H'\} \subset G^*,$$

where $U^+_{w_\bullet}$ is the closed subgroup of U^+ with the Lie algebra $\oplus_{\alpha \in \mathcal{R}^+_\bullet} \mathfrak{g}_\alpha$, and

$$H' = \{t \in H \mid \alpha_i(t) = 1, \forall i \in \mathrm{I}'_\circ\}.$$

It follows immediately from the definition that the map

$$U^+_{w_\bullet} \times H' \times U^- \xrightarrow{\sim} P, \qquad (u_+, t, u_-) \mapsto (u_+ t, t^{-1} u_-), \tag{3.3}$$

defines an isomorphism as varieties.

Proposition 3.2. *The map*

$$K^\perp \times P \xrightarrow{\sim} G^*, \qquad (g_1, g_2) \mapsto g_1 g_2, \tag{3.4}$$

is an isomorphism as varieties. Therefore we have $K^\perp\backslash G^* \cong P$ as varieties.

Proof. Let $U^+_{w_0 w_\bullet}$ be the closed subgroups of U^+, with the Lie algebra $\oplus_{\alpha \in \mathcal{R}^+ - \mathcal{R}^+_\bullet} \mathfrak{g}_\alpha$. Let H_1 be the closed connected subgroup of H, with the Lie algebra $\{X \in \mathfrak{h} \mid \theta(X) = -X\}$. By the proof of Proposition 3.1, the subgroup $K^\perp$ is the identity component of the subgroup

$$\widetilde{K} = \{(b_+, b_-) \in G^* \mid b_+ = \theta(b_-)\}.$$

Also note that $U^+_{w_\circ w_\bullet} = U^+ \cap \theta(U^-)$. Therefore the map

$$U^+_{w_0 w_\bullet} \times H_1 \xrightarrow{\sim} K^\perp, \qquad (u_+, t) \mapsto (u_+ t, \theta(u_+ t)), \tag{3.5}$$

defines an isomorphism as varieties.

By [25, Lemma 8.3.5] the group multiplication $U^+_{w_0 w_\bullet} \times U^+_{w_\bullet} \to U^+$ is an isomorphism as varieties. Since G is of adjoint type, the multiplication $H_1 \times H' \to H$ also defines an isomorphism as varieties.

Since the adjoint action of H preserves the subgroups $U^+_{w_0 w_\bullet}$ and $U^+_{w_\bullet}$, the proposition then follows from the isomorphism (2.3). □

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By Proposition 3.2, the quotient variety $K^\perp\backslash G^*$ is actually affine. Therefore, by [25, Exercise 5.5.9 (8)], the coordinate ring $\mathcal{O}(K^\perp\backslash G^*)$ is isomorphic to the subalgebra

$$\{f \in \mathcal{O}(G^*) \mid f(kg) = f(g),\ \forall\, k \in K^\perp, g \in G^*\}$$

of $K^\perp$-invariant functions in $\mathcal{O}(G^*)$. We shall make this identification throughout the paper.

Let $\iota^* : \mathcal{O}(G^*) \to \mathcal{O}(P)$ be the comorphism of the embedding $\iota : P \hookrightarrow G^*$. Thanks to Proposition 3.2, we conclude that the restriction map

$$\iota^*\,|_{\mathcal{O}(K^\perp\backslash G^*)} \colon \mathcal{O}(K^\perp\backslash G^*) \xrightarrow{\sim} \mathcal{O}(P) \tag{3.6}$$

defines an isomorphism as $\mathbb{C}$-algebras.

In light of the isomorphism (3.6), the map ι^* provides a tool to control the size of the algebra $\mathcal{O}(K^\perp\backslash G^*)$. We also need a similar map on the quantum level.

3.2. *Quantization of the group P*

We firstly describe a quantization of the algebra $\mathcal{O}(P)$. Following [19, 2.3], define the *partial parabolic subalgebra* U_P to be the unital $\mathbb{C}(q^{1/2})$-subalgebra of U, generated by F_i ($i \in \mathrm{I}$), E_i ($i \in \mathrm{I}_\bullet$) and K_μ ($\mu \in Q^\theta$). Let $\mathrm{U}^+_{w_\bullet}$ be the unital subalgebra of U^+ generated by E_i, for various $i \in \mathrm{I}_\bullet$. Let $\mathrm{U}^{0\theta}$ be the unital subalgebra of U^0 generated by K_μ, for $\mu \in Q^\theta$. Then the multiplication gives an isomorphism as $\mathbb{C}(q^{1/2})$-vector spaces

$$\mathrm{U}_P \cong \mathrm{U}^+_{w_\bullet} \otimes \mathrm{U}^{0\theta} \otimes \mathrm{U}^-. \tag{3.7}$$

Recall the $\mathcal{A}$-subalgebra ${}_{\mathcal{A}}\mathrm{U}$ in §2.1. Set ${}_{\mathcal{A}}\mathrm{U}_P = \mathrm{U}_P \cap {}_{\mathcal{A}}\mathrm{U}$. Similarly define ${}_{\mathcal{A}}\mathrm{U}^{0\theta}$ and ${}_{\mathcal{A}}\mathrm{U}^+_{w_\bullet}$. By the PBW-bases, the multiplication gives an isomorphism as $\mathcal{A}$-modules

$${}_{\mathcal{A}}\mathrm{U}_P \cong {}_{\mathcal{A}}\mathrm{U}^+_{w_\bullet} \otimes {}_{\mathcal{A}}\mathrm{U}^{0\theta} \otimes {}_{\mathcal{A}}\mathrm{U}^-, \tag{3.8}$$

where the tensor products are over $\mathcal{A}$.

Let ${}_{\mathbb{C}}\mathrm{U}_P = \mathbb{C} \otimes {}_{\mathcal{A}}\mathrm{U}_P$ be the commutative $\mathbb{C}$-algebra obtained by the base change $q^{1/2} \mapsto 1$. Similarly define the $\mathbb{C}$-algebras ${}_{\mathbb{C}}\mathrm{U}^+_{w_\bullet}$, ${}_{\mathbb{C}}\mathrm{U}^{0\theta}$ and ${}_{\mathbb{C}}\mathrm{U}^-$. Thanks to (3.8), we have the isomorphism as $\mathbb{C}$-algebras

$${}_{\mathbb{C}}\mathrm{U}_P \cong {}_{\mathbb{C}}\mathrm{U}^+_{w_\bullet} \otimes {}_{\mathbb{C}}\mathrm{U}^{0\theta} \otimes {}_{\mathbb{C}}\mathrm{U}^-. \tag{3.9}$$

Recall the $\mathbb{C}$-algebra isomorphisms $\varphi^\pm : {}_{\mathbb{C}}\mathrm{U}^\pm \xrightarrow{\sim} \mathcal{O}(U^\pm)$ in (2.8). The map φ^+ induces an isomorphism $\varphi^+_{w_\bullet} : {}_{\mathbb{C}}\mathrm{U}^+_{w_\bullet} \xrightarrow{\sim} \mathcal{O}(U^+_{w_\bullet})$. Let $\varphi^0 : {}_{\mathbb{C}}\mathrm{U}^{0\theta} \xrightarrow{\sim} \mathcal{O}(H')$ be the $\mathbb{C}$-algebra isomorphism given by $\overline{k_i} \mapsto \alpha_i\,|_{H'}$, for $i \in \mathrm{I}_\circ - \mathrm{I}'_\circ$, and $\overline{K_i} \mapsto \alpha_i\,|_{H'}$, for $i \in \mathrm{I}_\bullet$.

By (3.3), we have the natural isomorphism $\mathcal{O}(P) \cong \mathcal{O}(U^+_{w_\bullet}) \otimes \mathcal{O}(H') \otimes \mathcal{O}(U^-)$ as $\mathbb{C}$-algebras. Under this isomorphism, one gets the $\mathbb{C}$-algebra isomorphism

$$\varphi_P = \varphi^+_{w_\bullet} \otimes \varphi^0 \otimes \varphi^- : {}_{\mathbb{C}}\mathrm{U}_P \xrightarrow{\sim} \mathcal{O}(P).$$

Therefore the algebra U_P provides a quantization of $\mathcal{O}(P)$.

3.3. *Quantization of the map ι^**

We next describe a quantization of the map $\iota^* : \mathcal{O}(G^*) \to \mathcal{O}(P)$. Note that $\mathrm{U}^+_{w_\bullet} = \mathrm{U}^+ \cap \mathrm{T}_{w_\bullet}(\mathrm{U}^-)$. Set $\mathrm{U}^{+\prime}_{w_\bullet} = \mathrm{U}^+ \cap \mathrm{T}_{w_\bullet}(\mathrm{U}^+)$ to be another subalgebra of U^+. By the PBW-bases we have the tensor product decomposition $\mathrm{U}^+ \cong \mathrm{U}^+_{w_\bullet} \otimes \mathrm{U}^{+\prime}_{w_\bullet}$. Under this isomorphism, set

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$$\pi^+_{w_\bullet} = id\,|_{\mathrm{U}^+_{w_\bullet}} \otimes \varepsilon : \mathrm{U}^+ \longrightarrow \mathrm{U}^+_{w_\bullet}.$$

Set $Q' = \mathbb{Z}[\alpha_i \mid i \in \mathrm{I}'_\circ]$ to be the sublattice of Q. Let $\mathrm{U}^{0\prime}$ be the unital subalgebra of U^0, generated by $K_i^{\pm 1}$, for $i \in \mathrm{I}'_\circ$. Then we have the natural isomorphism $\mathrm{U}^0 \cong \mathrm{U}^{0\theta} \otimes \mathrm{U}^{0\prime}$. Under this isomorphism, set

$$\pi^0 = id\,|_{\mathrm{U}^{0\theta}} \otimes \varepsilon : \mathrm{U}^0 \longrightarrow \mathrm{U}^{0\theta}.$$

Finally, under the isomorphisms (3.7), set

$$\pi = \pi^+_{w_\bullet} \otimes \pi^0 \otimes id\,|_{\mathrm{U}^-} : \mathrm{U} \cong \mathrm{U}^+ \otimes \mathrm{U}^0 \times \mathrm{U}^- \longrightarrow \mathrm{U}_P. \tag{3.10}$$

The map π restricts to integral forms ${}_{\mathcal{A}}\pi : {}_{\mathcal{A}}\mathrm{U} \to {}_{\mathcal{A}}\mathrm{U}_P$.

The following proposition asserts that the map ${}_{\mathcal{A}}\pi$ quantize the map ι^*.

Proposition 3.3. *Let ${}_{\mathbb{C}}\pi : {}_{\mathbb{C}}\mathrm{U} \to {}_{\mathbb{C}}\mathrm{U}_P$ be the base change of the map ${}_{\mathcal{A}}\pi$. Then the diagram*

$$\begin{array}{ccc} \mathcal{O}(G^*) & \xrightarrow{\iota^*} & \mathcal{O}(P) \\ {\scriptstyle\varphi}\uparrow & & \uparrow{\scriptstyle\varphi_P} \\ {}_{\mathbb{C}}\mathrm{U} & \xrightarrow[{}_{\mathbb{C}}\pi]{} & {}_{\mathbb{C}}\mathrm{U}_P \end{array} \tag{3.11}$$

commutes.

Proof. The maps φ, ι^* and φ_P are clearly algebra homomorphisms. Since ${}_{\mathbb{C}}\mathrm{U}$ is commutative, the map ${}_{\mathbb{C}}\pi$ is also an algebra homomorphism. Therefore it will suffice to check the diagram for generators of ${}_{\mathbb{C}}\mathrm{U}$.

Take a reduced expression $\mathbf{i} = (i_1, i_2, \cdots, i_n)$ of w_0, such that $(i_1, i_2, \cdots, i_{n'})$ is a reduced expression of $w_\bullet$. The elements $\overline{E_{\mathbf{i},k}}$, $\overline{F_{\mathbf{i},k}}$, and $\overline{K_i}$, for $1 \le k \le n$ and $i \in \mathrm{I}$, generate the $\mathbb{C}$-algebra ${}_{\mathbb{C}}\mathrm{U}$. It will suffice to check the diagram when acting on these generators, which is direct and will be omitted. □

3.4. Poisson generators

The last ingredient that is needed for the proof is a set of Poisson generators for the algebra $\mathcal{O}(K^\perp \backslash G^*)$.

Let us consider the following elements in $\mathcal{O}(G^*)$:

(i) χ_i^+, χ_i^-, for $i \in \mathrm{I}_\bullet$;
(ii) $\chi_i^- - c_i \mathrm{T}_{w_\bullet}(\chi_{\tau i}^+)\alpha_i^{-1}$, for $i \in \mathrm{I}_\circ$;
(iii) $\alpha_i^{\pm 1}$, for $i \in \mathrm{I}_\bullet$;
(iv) $(\alpha_i \alpha_{\tau i}^{-1})^{\pm 1}$, for $i \in \mathrm{I}_\circ - \mathrm{I}'_\circ$.

Lemma 3.1. *The elements in (i)–(iv) belong to $\mathcal{O}(K^\perp \backslash G^*)$. Moreover they form a set of Poisson generators of $\mathcal{O}(K^\perp \backslash G^*)$.*

Proof. Let $\pi_2 : G^* \to P$ be the projection map onto P under the isomorphism (3.4). By Proposition 3.2, the image of the comorphism $\pi_2^* : \mathcal{O}(P) \to \mathcal{O}(G^*)$ is exactly $\mathcal{O}(K^\perp \backslash G^*)$.

In order to compute images of various functions under π_2^*, we give a precise description on the map π_2. Take any element $(u_+ t, t^{-1} u_-)$ in G^*. Decompose $u_+ = u'_+ u''_+$, with $u'_+ \in U^+_{w_0 w_\bullet}$ and $u''_+ \in U^+_{w_\bullet}$. Decompose $t = t' t''$, with $t' \in H_0$ and $t'' \in H'$. It is direct to verify that

$$\pi_2\big((u_+ t, t^{-1} u_-)\big) = \big(\mathrm{Ad}_{t'^{-1}}(u''_+) t'', t''^{-1} \mathrm{Ad}_t(\theta(u'_+))^{-1} u_-\big). \tag{3.12}$$

Let us identify regular functions on U^+, $U^-_{w_\bullet}$ and H' as regular functions on P, via the isomorphism $P \cong U^+_{w_\bullet} \times H' \times U^-$ (3.3).

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(a) *For $i \in \mathrm{I}_\bullet$, we have $\pi_2^*(\chi_i^+ \mid_{U^+_{w_\bullet}}) = \chi_i^+$ and $\pi_2^*(\chi_i^-) = \chi_i^-$.*
Take $i \in \mathrm{I}_\bullet$, and $(u_+t, t^{-1}u_-) \in G^*$. By (3.12), we have

$$\pi_2^*(\chi_i^+ \mid_{U^+_{w_\bullet}})\big((u_+t, t^{-1}u_-)\big) = \chi_i^+\big(\mathrm{Ad}_{t'^{-1}}(u''_+)\big) = \alpha_i(t'^{-1})\chi_i^+(u_+).$$

Note that $\alpha_i(t') = 1$, because $\theta(\alpha_i) = \alpha_i$ and $\theta(t') = t'^{-1}$. Therefore we have $\pi_2^*(\chi^+_{P,i} \mid_{U^+_{w_\bullet}}) = \chi_i^+$. Since $\theta(U^+_{w_0 w_\bullet}) \cap U^-_{w_\bullet} = \{e\}$, we deduce that

$$\pi_2^*(\chi_i^-)\big((u_+t, t^{-1}u_-)\big) = \chi_i^-\big(\mathrm{Ad}_t(\theta(u'_+))^{-1}u_-\big) = \chi_i^-(u_-).$$

This implies $\pi_2^*(\chi_i^-) = \chi_i^-$.

(b) *For $i \in \mathrm{I}_\circ$, we have $\pi_2^*(\chi^-_{P,i}) = \chi_i^- - c_i \mathrm{T}_{w_\bullet}(\chi^+_{\tau i})\alpha_i^{-1}$.*
Take $i \in \mathrm{I}_\circ$, and $(u_+t, t^{-1}u_-) \in G^*$. Then by (3.12) one has

$$\pi_2^*(\chi_i^-)\big((u_+t, t^{-1}u_-)\big) = \chi_i^-\big(\mathrm{Ad}_t(\theta(u'_+))^{-1}u_-\big) = \chi_i^-(u_-) + \chi_i^-(\mathrm{Ad}_t(\theta(u_+'^{-1}))).$$

Take a reduced expression $\mathbf{i} = (i_1, i_2, \cdots, i_n)$ of w_0, such that $(i_1, i_2, \cdots, i_{n'})$ is a reduced expression of $w_\bullet$, and $i_{n'+1} = \tau i$. Write $u'_+ = x_{\mathbf{i},n}(a_n)x_{\mathbf{i},n-1}(a_{n-1})\cdots x_{\mathbf{i},1}(a_1)$, for $a_k \in \mathbb{C}$. Then $u'_+ \in U^+_{w_0 w_\bullet}$. Hence we have $a_k = 0$ for $1 \le k \le n'$. Therefore

$$\begin{aligned}\theta(u_+'^{-1}) &= \theta(x_{\mathbf{i},n'+1}(a_{n'+1}))\theta(x_{\mathbf{i},n'+2}(a_{n'+2}))\cdots\theta(x_{\mathbf{i},n}(a_n))\\ &= y_i(c_i a_{n'+1})\theta(x_{\mathbf{i},n'+2}(a_{n'+2}))\cdots\theta(x_{\mathbf{i},n}(a_n))\end{aligned}$$

is a factorisation as in (2.7). Hence

$$\chi_i^-(\mathrm{Ad}_t(\theta(u_+'^{-1}))) = \chi_i^-(y_i(\alpha_i^{-1}(t)c_i a_{n'+1})) = c_i\alpha_i^{-1}(t)a_{n'+1} = c_i\alpha_i^{-1}(t)\mathrm{T}_{w_\bullet}(\chi^+_{\tau i})(u_+).$$

This implies that $\pi_2^*(\chi^-_{P,i}) = \chi_i^- - c_i \mathrm{T}_{w_\bullet}(\chi^+_{\tau i})\alpha_i^{-1}$.

(c) *For $i \in \mathrm{I}_\bullet$, we have $\pi_2^*(\alpha_i \mid_{H'}) = \alpha_i$. For $i \in \mathrm{I}_\circ - \mathrm{I}'_\circ$, we have $\pi_2^*(\alpha_i \mid_{H'}) = \alpha_i\alpha_{\tau i}^{-1}A_i$, where A_i is a monomial of the form $\prod_{i\in\mathrm{I}_\bullet}\alpha_i^{n_i}$ with $n_i \in \mathbb{Z}$.*

It follows from the similar argument. Statement (c) is actually easier to prove since we only need to consider elements in the torus. We skip the details.

Thanks to the statements (a)–(c), we conclude that elements in (i)–(iv) belong to the image of π_2^* and hence belong to $\mathcal{O}(K^\perp\backslash G^*)$.

Next we show that these elements form a set of Poisson generators.

Let $\mathcal{O}'$ be the Poisson subalgebra of $\mathcal{O}(G^*)$ generated by elements in (i)–(iv). By the previous argument we have $\mathcal{O}' \subset \mathcal{O}(K^\perp\backslash G^*)$. Recall that the restriction of the map $\iota^* : \mathcal{O}(G^*) \to \mathcal{O}(P)$ to $\mathcal{O}(K^\perp\backslash G^*)$ is an isomorphism. In order to show $\mathcal{O}' = \mathcal{O}(K^\perp\backslash G^*)$, it suffices to show that $\iota^*(\mathcal{O}') = \mathcal{O}(P)$. Still let us identify $\mathcal{O}(U^+_{w_\bullet})$, $\mathcal{O}(H')$ and $\mathcal{O}(U^-)$ as the subalgebras of $\mathcal{O}(P)$. It suffices to show that $\iota^*(\mathcal{O}')$ contains these subalgebras.

Firstly, by the proof of [15, Theorem 7.6], the algebra $\mathcal{O}(U^+_{w_\bullet})$ is generated by functions $\chi_i^+ \mid_{U^+_{w_\bullet}}$, for $i \in \mathrm{I}_\bullet$, as a Poisson algebra. Therefore as subalgebras of $\mathcal{O}(G^*)$, we have $\mathcal{O}' \supset \mathcal{O}(U^+_{w_\bullet})$, thanks to (a). This implies $\iota^*(\mathcal{O}') \supset \mathcal{O}(U_{w_\bullet})$, where $\mathcal{O}(U_{w_\bullet})$ on the right hand side is viewed as a subalgebra of $\mathcal{O}(P)$.

Next, the algebra $\mathcal{O}(H')$ is generated by $(\alpha_i \mid_{H'})^{\pm 1}$, for $i \in \mathrm{I} - \mathrm{I}'_\circ$. Thanks to (c) we conclude that $\iota^*(\mathcal{O}')$ contains $\mathcal{O}(H')$.

Lastly, still by the proof of [15, Theorem 7.6], as a Poisson algebra $\mathcal{O}(U^-)$ is generated by χ_i^-, for $i \in \mathrm{I}$. Write $b_i = \chi_i^-$, for $i \in \mathrm{I}_\bullet$, and $b_i = \chi_i^- - c_i\mathrm{T}_{w_\bullet}(\chi^+_{\tau i})\alpha_i^{-1}$, for $i \in \mathrm{I}_\circ$, to be elements in $\mathcal{O}(G^*)$. Let $f(x_1, x_2, \cdots, x_k)$ be a Poisson polynomial, that is, a polynomial possibly with Poisson brackets. Note that there is a $\mathbb{Z}[\mathrm{I}]$-grading on the algebra $\mathcal{O}(G^*)$ respecting the Poisson brackets, where $\deg(\chi_i^-) = i$,

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$\deg(\chi_i^+) = -i$ and $\deg(\mu) = 0$, for $i \in \mathrm{I}$ and $\mu \in Q$. We define a partial order on $\mathbb{Z}[\mathrm{I}]$ by setting $\mu \leq \mu'$ if $\mu' - \mu$ is a non-negative combination of various $i \in \mathrm{I}$. Then for any $i_1, i_2, \cdots, i_k$ in I, we have

$$f(b_{i_1}, b_{i_2}, \cdots, b_{i_k}) = f(\chi_{i_1}^-, \chi_{i_2}^-, \cdots, \chi_{i_k}^-) + \text{lower terms}$$

in $\mathcal{O}(G^*)$. Therefore we deduce that $\iota^*(\widetilde{\mathcal{O}})$ contains $\mathcal{O}(U^-) \subset \mathcal{O}(P)$ by induction on the degree.

We complete the proof of the lemma. □

3.5. The final proof

We are now ready to prove the main theorem of the paper.

Proof of Theorem 1. The uniqueness of $\varphi^\imath$ is clear. We show the existence. Consider the following diagram

$$\begin{array}{ccccc} \mathcal{O}(K^\perp\backslash G^*) & \hookrightarrow & \mathcal{O}(G^*) & \xrightarrow{\iota^*} & \mathcal{O}(P) \\ \uparrow{\scriptstyle \varphi^\imath} & & \uparrow{\scriptstyle \varphi} & \overset{\pi_2^*}{\nwarrow} & \uparrow{\scriptstyle \varphi_P} \\ {}_{\mathbb{C}}\mathrm{U}^\imath & \xrightarrow[{}_{\mathbb{C}}\iota]{} & {}_{\mathbb{C}}\mathrm{U} & \xrightarrow[{}_{\mathbb{C}}\pi]{} & {}_{\mathbb{C}}\mathrm{U}_P \end{array} \tag{3.13}$$

Let $\widetilde{\mathcal{O}} \subset \mathcal{O}(G^*)$ be the image of the map $\varphi \circ {}_{\mathbb{C}}\iota : {}_{\mathbb{C}}\mathrm{U}^\imath \to \mathcal{O}(G^*)$.

(a) *We have* $\widetilde{\mathcal{O}} \supset \mathcal{O}(K^\perp\backslash G^*)$.
Recall the generators B_i ($i \in \mathrm{I}$), E_i ($i \in \mathrm{I}_\bullet$), $K_i^{\pm 1}$ ($i \in \mathrm{I}_\bullet$) and k_i ($i \in \mathrm{I}_\bullet$) of the subalgebra $\mathrm{U}^\imath$ in §2.2. It is clear that these elements belong to the integral form ${}_{\mathcal{A}}\mathrm{U}^\imath$. Moreover these elements specialize exactly to the elements (i)–(iv) in §3.4. Since the maps ${}_{\mathbb{C}}\iota$ and φ are Poisson, $\widetilde{\mathcal{O}}$ is a Poisson subalgebra of $\mathcal{O}(G^*)$. Then (a) follows from Lemma 3.1.

(b) *The map* ${}_{\mathbb{C}}\pi \circ {}_{\mathbb{C}}\iota : {}_{\mathbb{C}}\mathrm{U} \to {}_{\mathbb{C}}\mathrm{U}_P$ *is surjective.*
Recall that φ_P is an isomorphism. Therefore it suffices to show that $\varphi_P \circ {}_{\mathbb{C}}\pi \circ {}_{\mathbb{C}}\iota$ is surjective. By Proposition 3.3, we have $\varphi_P \circ {}_{\mathbb{C}}\pi \circ {}_{\mathbb{C}}\iota = \iota^* \circ \varphi \circ {}_{\mathbb{C}}\iota$. Recall that $\iota^*(\mathcal{O}(K^\perp\backslash G^*)) = \mathcal{O}(P)$. Then (b) follows from (a).

(c) *The map* ${}_{\mathbb{C}}\pi \circ {}_{\mathbb{C}}\iota : {}_{\mathbb{C}}\mathrm{U} \to {}_{\mathbb{C}}\mathrm{U}_P$ *is an isomorphism.*
Since ${}_{\mathcal{A}}\mathrm{U}_P$ is a free $\mathcal{A}$-module, we take an $\mathcal{A}$-basis B_P of ${}_{\mathcal{A}}\mathrm{U}_P$. Then B_P is a $\mathbb{C}(q^{1/2})$-basis of U_P. Thanks to [19, Lemma 2.10], the Letzter's map $\pi^\imath = \pi \mid_{\mathrm{U}^\imath} : \mathrm{U}^\imath \to \mathrm{U}_P$ is an isomorphism as vector spaces. Therefore, the set $(\pi^\imath)^{-1}(\mathrm{B}_P) \subset \mathrm{U}^\imath$ gives a $\mathbb{C}(q^{1/2})$-basis of $\mathrm{U}^\imath$. For $b \in \mathrm{B}_P$, let $d_b \in \mathbb{C}[q^{1/2}]$ be the monic polynomial with the lowest degree, such that $d_b(\pi^\imath)^{-1}(b) \in {}_{\mathcal{A}}\mathrm{U}^\imath$. Then $\{d_b(\pi^\imath)^{-1}(b) \mid b \in \mathrm{B}\}$ is an $\mathcal{A}$-basis of ${}_{\mathcal{A}}\mathrm{U}^\imath$. Hence the map ${}_{\mathbb{C}}\pi^\imath = {}_{\mathbb{C}}\pi \circ {}_{\mathbb{C}}\iota : {}_{\mathbb{C}}\mathrm{U}^\imath \to {}_{\mathbb{C}}\mathrm{U}_P$, obtained by the base change of the map $\pi^\imath \mid_{{}_{\mathcal{A}}\mathrm{U}^\imath} : {}_{\mathcal{A}}\mathrm{U}^\imath \to {}_{\mathcal{A}}\mathrm{U}_P$, is an isomorphism if $d_b(1) \neq 0$, for any $b \in \mathrm{B}$. Suppose otherwise $d_b(1) = 0$ for some $b \in \mathrm{B}$. Then $\overline{b} \in {}_{\mathbb{C}}\mathrm{U}_P$ does not belong to the image of ${}_{\mathbb{C}}\pi^\imath$, which contradicts with the surjectivity established in (b). This completes the proof of (c).

(d) *We have* $\widetilde{\mathcal{O}} = \mathcal{O}(K^\perp\backslash G^*)$. *Hence there exists* $\varphi^\imath$ *which makes the diagram* (3.13) *commute.*
By (c) and the Proposition 3.3, we deduce that $\iota^* \circ \varphi \circ {}_{\mathbb{C}}\iota = \varphi_P \circ {}_{\mathbb{C}}\pi \circ {}_{\mathbb{C}}\iota$ is an isomorphism. In particular $\iota^* \mid_{\widetilde{\mathcal{O}}}$ is injective. Since $\iota^* \mid_{\mathcal{K}\mathcal{O}(G^*)}$ is an isomorphism, (d) follows from (a).

(e) *The map* $\varphi^\imath$ *is an isomorphism as Poisson algebras.*

By the commuting diagram (3.13), we have $\varphi^\imath = \pi_2^* \circ \varphi_P \circ {}_{\mathbb{C}}\pi \circ {}_{\mathbb{C}}\iota$. Since the maps π_2^*, φ_P and ${}_{\mathbb{C}}\pi \circ {}_{\mathbb{C}}\iota$ are isomorphisms, we deduce $\varphi^\imath$ is an isomorphism. It is moreover Poisson since ${}_{\mathbb{C}}\iota$ and φ are Poisson.

We complete the proof of the theorem. □

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4. Example: Dubrovin–Ugaglia Poisson structures

Let $G = \mathrm{PSL}_n(\mathbb{C})$ and $\theta : G \to G$ be the map given by $g \mapsto {}^T g^{-1}$. Then $K = \mathrm{PSO}_n(\mathbb{C})$, and

$$K^\perp = \{(ut, {}^T u^{-1} t^{-1}) \mid u \in U^+, t \in H\}.$$

One has an isomorphism as varieties

$$K^\perp \backslash G^* \xrightarrow{\sim} U^+; \quad K^\perp (tu_+, t^{-1} u_-) \mapsto {}^T u_- u_+. \tag{4.1}$$

The unipotent subgroup U^+ consists of $(n \times n)$-upper triangular matrices with 1 on the diagonal. When viewing U^+ as the space of Stokes matrices in the study of Frobenius manifolds, it is naturally equipped with a Poisson structure, called the Dubrovin–Ugaglia Poisson structure [13, 27]. Boalch [1] relates such Poisson structure with the stable locus Poisson structure of G^*, and then Xu [29] interprets the Poisson structure as a Poisson homogeneous space of G^*. By [29, Theorem 5.12], up to rescaling of Poisson brackets, (4.1) is an isomorphism as Poisson manifolds .

Let $\mathrm{U}^\imath$ be the $\imath$quantum group associated to the symmetric pair $(\mathfrak{sl}_n, \mathfrak{so}_n)$ and let ${}_{\mathbb{C}}\mathrm{U}^\imath$ be the $\mathbb{C}$-algebra obtained by the base change as before. By Theorem 1 and the isomorphism (4.1), we get the Poisson algebra isomorphism

$${}_{\mathbb{C}}\mathrm{U}^\imath \xrightarrow{\sim} \mathcal{O}(U^+). \tag{4.2}$$

It is also known that the braid group acts naturally on the space U^+ via Poisson automorphisms [13, 27]. On the other hand, by the work [28] the $\imath$quantum groups also admit braid group actions, generalising Lusztig's braid group actions on the quantum groups. Thanks to [23, Proposition 3.6], for the $\imath$quantum group associated to $(\mathfrak{sl}_n, \mathfrak{so}_n)$, the braid group action actually preserves the integral form ${}_{\mathcal{A}}\mathrm{U}^\imath$. Therefore it induces the action on ${}_{\mathbb{C}}\mathrm{U}$. It is then direct to verify that the isomorphism (4.2) is moreover equivariant with respect to the braid group actions. Therefore the braid group symmetries on the Dubrovin–Ugaglia Poisson structure can be understood from the theory of quantum symmetric pairs.

Competing interest. The author has no competing interest to declare.

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