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A symbolic approach to polyhedral scene analysis by parametric calotte propagation*

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SUMMARY
Polyhedral scene analysis studies whether a 2D line drawing of a 3D polyhedron is realizable in space, and if so, it gives the results of parameterizing the space of all possible realizations. For generic 2D data, symbolic computation with Grassmann–Cayley algebra is needed in the analysis. In this paper, we propose a method called *parametric calotte propagation* to solve the realization and parameterization problems for general polyhedral scenes at the same time. In algebraic manipulation, parametric propagation is more efficient than elimination. In applications, it can lead to linear construction sequences for nonspherical polyhedra whose resolvable sequences do not exist.

KEYWORDS: Structure geometry; Polyhedral scene analysis; Grassmann–Cayley algebra; Bracket algebra; Constraint satisfaction.

1. Introduction

Polyhedral scene analysis is a classical problem in artificial intelligence, robotics, and computer vision. The main topic is to determine whether a 2D line drawing of a 3D polyhedron is realizable in space, and if so, a parameterization of the space of all possible realizations is given.

In the computer vision community, the realization problem was first studied by Mackworth[14] and Huffman[9] using labeling schemes and reciprocal diagrams. Their approach can provide necessary but not sufficient conditions. Sugihara[21–24] established a necessary and sufficient condition on the general realizability, which is the existence of solutions in a linear programming problem. Sugihara[25] further proposed the concept "resolvable sequences" for polyhedra and proved that they always exist for spherical polyhedra, i.e., polyhedra which are homeomorphic to a sphere. This result is used by Ros and Thomas[18] to overcome the super-strictness in scene analysis.

In the combinatorial geometry community, the realization problem was studied in a different approach. Whiteley,[29–32] Crapo and Ryan,[4] Crapo,[5] and Crapo and Whiteley[6] studied the realization problem with structure geometry (Crapo and Whiteley,[6] White and Whiteley[27]), invariant theory (Doubilet *et al.*,[7] White,[26] Hestenes and Ziegler[8]), and synthetic geometry (Bokowski and Sturmfels,[2] Sturmfels,[20] Richter-Gebert[15]). Various necessary and sufficient conditions are established in terms of either the existence of compatible cross-sections, or strict self-stresses in frameworks, or Grassmann–Cayley algebraic, or bracket algebraic equalities.

According to Crapo (1991), the polyhedral scene analysis is an example of a system of linear equations whose rank controls the critical qualitative features. The combinatorial pattern determines a sparse matrix pattern that has both a generic rank, for general independent values of the nonzero entries, and a geometric rank, for special values of the coordinates of the points and planes in the scene. While considerable work has been carried out on the generic rank and the parameterization problem therein, only a few works deal with the geometric rank, and even fewer with the parameterization.

In our opinion, the major difference between Sugihara's linear programming approach and the invariant approach of Crapo *et al.* is that the former is numerical while the latter is symbolic. For symbolic computation, Grassmann–Cayley algebra and bracket algebra are a good help. Although the problem itself is linear, the realization conditions for a polyhedron whose 2D coordinates are generic can be very involved, bifurcated, and difficult to obtain. To make this clear, let us study an example.

Example 1. In space there is a torus composed of nine vertices and nine faces (see Fig. 1). Each vertex is in four faces and each face has four vertices. Assume that no three vertices in a face are collinear, reconstruct the 3D torus from its 2D image.

This example is taken from Sugihara (1986) as an example in which the realization conditions are difficult to find. Sugihara (1999) further showed that the torus does not have any resolvable sequence. His fundamental equations reflect the incidence relations of all pairs of incident vertices and faces. In this example, there are 36 fundamental equations in 36 unknowns. To reduce the size of the system, Crapo (1991) established a set of syzygy equations, which can be taken as

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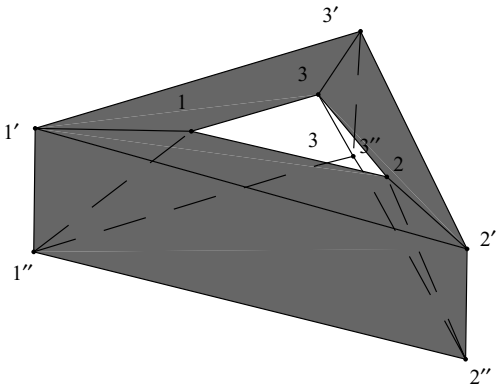


Fig. 1. Example 1.

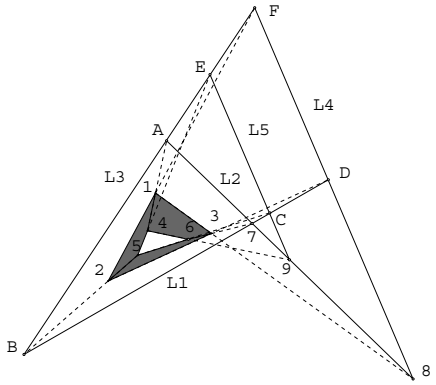


Fig. 2. A cross-section of the truncated pyramid.

the result of eliminating all the faces from the fundamental equations. For this example, there are nine syzygy equations in nine unknowns. Although the latter system is linear and sparse, it is still too difficult to solve symbolically. In this paper, following a different approach, we obtain the *complete rank classification* of the coefficient matrix M of the syzygy equations and the associated realization conditions as follows. It can be seen that the conditions are highly bifurcated.

1. $4 \leq \text{rank}(M) \leq 6$. The generic rank is 6.
   If $\text{rank}(M) = 6$, the line drawing can only be lifted to a spatial plane, the torus is said to be *trivial*, or *unrealizable*.
2. $\text{rank}(M) = 5$ if and only if either
   (a)

$$\mathbf{12}, \mathbf{1'2'}, \mathbf{1''2''};\ \ \mathbf{13}, \mathbf{1'3'}, \mathbf{1''3''};\ \ \mathbf{23}, \mathbf{2'3'}, \mathbf{2''3''} \quad (1.1)$$

   are all concurrent lines, one of

$$\mathbf{11'}, \mathbf{22'}, \mathbf{33'};\ \ \mathbf{11''}, \mathbf{22''}, \mathbf{33''};\ \ \mathbf{1'1''}, \mathbf{2'2''}, \mathbf{3'3''}, \quad (1.2)$$

   say A, is concurrent, and in the three triplets

$$\mathbf{123}, \mathbf{1'2'3'}, \mathbf{1''2''3''}, \quad (1.3)$$

   the one not involved in A is collinear; or
   (b) The triplets in (1.2) are all concurrent lines, one of (1.1), say B, is concurrent, and in

$$\mathbf{11'1''}, \mathbf{22'2''}, \mathbf{33'3''}, \quad (1.4)$$

   the one not involved in B is collinear.
   If $\text{rank}(M) = 5$, the lift can be lifted to three or five distinct spatial planes. It can only be lifted to three planes, i.e., two triangular tubes in Fig. 1 must be flat in 3D, if and only if in (a) all of (1.3) are collinear, or in (b) all of (1.4) are collinear.
3. $\text{rank}(M) = 4$ if and only if all the triplets of (1.1) and (1.2) are concurrent lines.
   If $\text{rank}(M) = 4$, the lift lies in nine distinct spatial planes, forming a 2D manifold.

In the literature, there are three strategies that can be used to find the realization conditions symbolically. The first is Crapo's syzygy equations. When the realization conditions contain only one equality, it can be obtained from the syzygy equations directly. Two important cases are covered by this strategy: (1) the truncated pyramid, (2) n-calotte. However, as in Example 1, this strategy may be difficult to employ for polyhedra whose realization conditions are bifurcated and contain multiple equalities.

The second strategy is the compatible cross-sections of Whiteley (1991). It is applicable only to spherical or disk polyhedra (Ros).[17] A typical example is the truncated pyramid (see fig. 2). Its cross-section can be constructed as follows. (1) Select a point 7 on line 36. (2) Draw two lines L_1 and L_2 through point 7. (3) Construct the intersections $8 = 13 \cap L_2$, $9 = 46 \cap L_2$, $A = 14 \cap L_2$, $B = 25 \cap L_1$, $C = 56 \cap L_1$, $D = 23 \cap L_1$, $E = 9C \cap AB$, $F = 8D \cap AB$. The line drawing has a compatible cross-section if and only if point E is on line 45 and point F is on line 12. The conditions can be represented by Grassmann–Cayley algebraic equalities, but are much more complicated than the well-known condition that the three lines 14, 25, 36 are concurrent. Even worse, they are not invariant due to the extraneous objects 7, L_1, L_2.

The third strategy is the star–delta reductions of Ros and Thomas (1998). It is applicable only to spherical polyhedra. The realization conditions can be represented by Grassmann–Cayley algebraic equalities, but are not invariant due to extraneous objects.

It remains an open problem to find a general and efficient strategy for the classification of the geometric ranks and the parameterization of the corresponding solution spaces. Our first task is to use elimination methods in polynomial systems to manipulate the fundamental equations. We write the equations in terms of vectors, bivectors, and their brackets for easy manipulation by vectorial equation solving (Li, 2000). By a suitable choice of an order for elimination, we can arrive at an algebraic classification. However, bifurcation in the procedure of elimination will occur inevitably. For complicated problems, elimination methods often fail to be efficient.

The main contribution of this paper includes (1) a *parametric propagation* method for vectorial equation solving and (2) a *parametric calotte propagation* method for polyhedral scene analysis. In some sense, parametric propagation is the reverse of elimination. While in elimination, the leading unknown is to be eliminated first, in

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parametric propagation, the least leading unknown is labeled as the "origin," and new parameters are introduced to solve the unknown next to the origin. The origin is gradually enlarged by encompassing more and more solved unknowns, and the propagation of the solving goes from the ever-refreshing origin to its surroundings. The system is ultimately transformed into such a system that the unknowns become the parameters introduced in every round of propagation. The new system is generally much easier to deal with than the original one. In scene analysis, the system is the fundamental equations, and parametric calotte propagation is an improved version of the general method in this special setting.

Compared with elimination, parametric propagation has the following advantages: (1) the order of propagation is easy to determine, (2) the propagation procedure generally does not bifurcate, and (3) there are more factored results. As a consequence, parametric propagation is more efficient in manipulation and better in result. In scene analysis, it is a general method for finding both realization conditions and parameterizations in the realizable cases.

We have implemented parametric calotte propagation with Maple 8, and have used it to test 20 examples in polyhedral scene analysis. Surprising results include the discovery of *linear construction sequences* for some nonspherical polyhedra whose resolvable sequences do not exist. The linear construction sequences have all the properties of resolvable sequences needed in applications.

2. Grassmann–Cayley Algebra

2.1. Definition

Grassmann–Cayley algebra is the most general structure in which projective properties are expressed in a coordinate-free way. The following is a coordinate-free definition of this algebra.

Let $\mathcal{V}$ be an n-dimensional vector space over a field $\mathcal{F}$ whose characteristic is not 2. Then $\mathcal{V}$ generates the *Grassmann algebra* $(\Lambda(\mathcal{V}), \vee)$ which is composed of the *Grassmann space* $\Lambda(\mathcal{V})$ and the *outer product* "$\vee$." The Grassmann space is a linear space spanned by the outer products of vectors in $\mathcal{V}$. The outer product of k vectors is called a *k-flat*, or *k-extensor*, and k is called the *grade*, or *rank*, of the flat. The linear subspace spanned by all k-flats is called the *k-vector space*. Its elements are called *k-vectors*. The whole space $\Lambda(\mathcal{V})$ is the direct sum of the k-vector spaces with k ranging from 0 to n. An element in $\Lambda(\mathcal{V})$ is called a *multivector*. The k-graded part of a multivector x is denoted by $\langle x\rangle_k$. Any k-vector is called a *homogeneous multivector*. In applications, the outer product is usually denoted by the juxtaposition of elements.

Let I_n be a fixed nonzero n-vector in $\Lambda(\mathcal{V})$. The *bracket* of a multivector x with respect to I_n is defined by

$$[x] = \langle x\rangle_n / I_n. \tag{2.1}$$

The bilinear form $(x, y) \mapsto [xy]$ for $x, y \in \Lambda(\mathcal{V})$ is nonsingular, and induces a linear invertible mapping i from $\Lambda(\mathcal{V})$ to its dual space $\Lambda(\mathcal{V}^*)$, where $\mathcal{V}^*$ is the dual space of $\mathcal{V}$. The pullback of the outer product from $\Lambda(\mathcal{V}^*)$ to $\Lambda(\mathcal{V})$ through i is called the *wedge product*, or *meet product*, or *shuffle product*, in $\Lambda(\mathcal{V})$, and is denoted by "$\wedge$." The Grassmann space $\Lambda(\mathcal{V})$ equipped with the two products is called the *Grassmann–Cayley algebra* over $\mathcal{V}$.

The outer product and the wedge product are both associative, and satisfy the following graded-anticommutative rules: let A_r, B_s be, respectively, r-vector and s-vector of $\Lambda(\mathcal{V})$, then

$$\begin{aligned} A_r B_s &= (-1)^{rs} B_s A_r, \\ A_r \wedge B_s &= (-1)^{(n-r)(n-s)} B_s \wedge A_r. \end{aligned} \tag{2.2}$$

The wedge product has the following famous *shuffle formula*: for any vectors $\mathbf{V}_1, \ldots, \mathbf{V}_r$ and $\mathbf{V}'_1, \ldots, \mathbf{V}'_s$, where $r + s \geq n$,

$$\begin{aligned} &(\mathbf{V}_1 \cdots \mathbf{V}_r) \wedge (\mathbf{V}'_1 \cdots \mathbf{V}'_s) \\ &\quad = \sum_{\sigma} \operatorname{sign}(\sigma) \big[\mathbf{V}_{\sigma(1)} \cdots \mathbf{V}_{\sigma(n-s)} \mathbf{V}'_1 \cdots \mathbf{V}'_s\big] \\ &\qquad \times \mathbf{V}_{\sigma(n-s+1)} \cdots \mathbf{V}_{\sigma(r)} \\ &\quad = \sum_{\tau} \operatorname{sign}(\tau) \big[\mathbf{V}_1 \cdots \mathbf{V}_r \mathbf{V}'_{\tau(r+s+1-n)} \cdots \mathbf{V}'_{\tau(s)}\big] \\ &\qquad \times \mathbf{V}'_{\tau(1)} \cdots \mathbf{V}'_{\tau(r+s-n)}. \end{aligned} \tag{2.3}$$

Here, σ is a *bi-partition* of $1, \ldots, r$ into lengths $n - s$ and $r + s - n$, i.e., it is a permutation of $1, \ldots, r$ such that $\sigma(1) < \cdots < \sigma(n - s)$ and $\sigma(n - s + 1) < \cdots < \sigma(r)$. Similarly, τ is a bi-partition of $1, \ldots, s$ into lengths $r + s - n$ and $n - r$.

As a special case, we have the following *Cramer's rule* for any k-vectors A_k and any n vectors $\mathbf{V}_1, \ldots, \mathbf{V}_n$:

$$\begin{aligned} [\mathbf{V}_1 \cdots \mathbf{V}_n] A_k &= (\mathbf{V}_1 \cdots \mathbf{V}_n) \wedge A_k \\ &= \sum_{\sigma} \operatorname{sign}(\sigma) \big[\mathbf{V}_{\sigma(1)} \cdots \mathbf{V}_{\sigma(n-k)} A_k\big] \\ &\qquad \times \mathbf{V}_{\sigma(n-k+1)} \cdots \mathbf{V}_{\sigma(n)}. \end{aligned} \tag{2.4}$$

Here, σ is a bi-partition of $1, \ldots, n$ into lengths $n - k$ and k.

The following are representations of some geometric objects and incidence relations in 2D projective geometry by Grassmann–Cayley algebra.

(1) A point is represented by a nonzero vector, which is unique up to scale. In this paper, a point is always denoted by a boldfaced integer or character.
(2) A line passing through points $\mathbf{1}, \mathbf{2}$ is represented by $\mathbf{12}$. Three points $\mathbf{1}, \mathbf{2}, \mathbf{3}$ are collinear if and only if their outer product is zero, or equivalently, $[\mathbf{123}] = 0$.
(3) The intersection of two lines $\mathbf{12}, \mathbf{1'2'}$ is represented by $\mathbf{12} \wedge \mathbf{1'2'}$.
(4) Three lines $\mathbf{12}, \mathbf{1'2}, \mathbf{1''2''}$ are collinear if and only if their wedge product is zero, i.e., $\mathbf{12} \wedge \mathbf{1'2'} \wedge \mathbf{1''2''} = 0$.

2.2. Bracket algebra

A bracket is a determinant whose columns are the homogeneous coordinates of vectors. For a finite set of vectors, all the brackets generated by them will generate a ring, called the *bracket ring*, or *bracket algebra*.

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Bracket algebra can be defined in a purely symbolic manner. The following definition is from White (1991). Let $\mathbf{V}_1, \ldots, \mathbf{V}_m$ be symbols, and let $[\mathbf{V}_{i_1} \cdots \mathbf{V}_{i_n}]$ be indeterminates over a field $\mathcal{F}$ of characteristic $\neq 2$ for each n-tuple $1 \leq i_1, \ldots, i_n \leq m$, such that they are algebraically independent over $\mathcal{F}$. The n-dimensional *bracket algebra* generated by the m symbols over $\mathcal{F}$ is the quotient of the polynomial ring $\mathcal{F}[\{[\mathbf{V}_{i_1} \cdots \mathbf{V}_{i_n}] \mid 1 \leq i_1, \ldots, i_n \leq m\}]$ by the ideal generated by elements of the following three types:

B1. $[\mathbf{V}_{i_1} \cdots \mathbf{V}_{i_n}]$, if $i_j = i_k$ for some $j \neq k$.
B2. $[\mathbf{V}_{i_1} \cdots \mathbf{V}_{i_n}] - \text{sign}(\sigma)[\mathbf{V}_{i_{\sigma(1)}} \cdots \mathbf{V}_{i_{\sigma(n)}}]$ for any permutation σ of $1, \ldots, n$.
GP. $\sum_{k=1}^{n+1} (-1)^{k+1} [\mathbf{V}_{i_1} \cdots \mathbf{V}_{i_{n-1}} \mathbf{V}_{j_k}][\mathbf{V}_{j_1} \cdots \check{\mathbf{V}}_{j_k} \ldots \mathbf{V}_{j_{n+1}}]$.
Here, $\check{\mathbf{V}}_{j_k}$ means $\mathbf{V}_{j_k}$ is absent.

The equalities by assigning each **GP**-typed polynomial to zero are called *Grassmann–Plücker* (GP) *syzygies*.

2.3. Cayley factorization

Cayley factorization is the transformation from a bracket polynomial to an expression in Grassmann–Cayley algebra—*Cayley expression*, which involves only the outer product and the meet product. It is a crucial step from algebraic characterization to geometric description. White (1991) developed a general Cayley factorization algorithm for multilinear bracket polynomials. Li and Wu (2003) developed a set of Cayley factorization formulas for nonmultilinear bracket polynomials based on *Cayley expansion*, the inverse of Cayley factorization.

In this paper, three formulas for $n = 3$ from Li and Wu (2003) are needed:

1. The following equality is the reverse of a shuffle expansion:

$$[\mathbf{V}_1\mathbf{V}_3\mathbf{V}_4][\mathbf{V}_2\mathbf{V}_5\mathbf{V}_6] - [\mathbf{V}_2\mathbf{V}_3\mathbf{V}_4][\mathbf{V}_1\mathbf{V}_5\mathbf{V}_6] = \mathbf{V}_1\mathbf{V}_2 \wedge \mathbf{V}_3\mathbf{V}_4 \wedge \mathbf{V}_5\mathbf{V}_6. \tag{2.5}$$

2. The following equality is a GP syzygy:

$$-[\mathbf{V}_1\mathbf{V}_2\mathbf{V}_1'][\mathbf{V}_1\mathbf{V}_2'\mathbf{V}_3'] + [\mathbf{V}_1\mathbf{V}_2\mathbf{V}_2'][\mathbf{V}_1\mathbf{V}_1'\mathbf{V}_3'] = \mathbf{V}_1\mathbf{V}_2\mathbf{V}_3'][\mathbf{V}_1\mathbf{V}_1'\mathbf{V}_2']. \tag{2.6}$$

3. The following equality combines (2.5) and (2.6):

$$[\mathbf{V}_1\mathbf{V}_2\mathbf{V}_1'][\mathbf{V}_2\mathbf{V}_3\mathbf{V}_2'][\mathbf{V}_1\mathbf{V}_3\mathbf{V}_3'] - [\mathbf{V}_1\mathbf{V}_2\mathbf{V}_2'][\mathbf{V}_2\mathbf{V}_3\mathbf{V}_3'][\mathbf{V}_1\mathbf{V}_3\mathbf{V}_1'] = [\mathbf{V}_1\mathbf{V}_2\mathbf{V}_3]\mathbf{V}_1\mathbf{V}_1' \wedge \mathbf{V}_2\mathbf{V}_2' \wedge \mathbf{V}_3\mathbf{V}_3'. \tag{2.7}$$

3. 3D Scene Analysis with Grassmann–Cayley Algebra

A polyhedral line drawing is either a perspective or a parallel projection of a polyhedron. A polyhedron can be described by its vertices $\{V_1, \ldots, V_m\}$, faces $\{F_1, \ldots, F_n\}$, and the incidence structure composed of all the incidence pairs (V_i, F_j). Let $\mathbf{n}$ be either a point or a direction in space. Then by either the perspective projection centered at the point $\mathbf{n}$, or the parallel projection along the direction $\mathbf{n}$, the vertices of a polyhedron are projected into an image plane $\mathcal{I}$, with 2D coordinates (x_i, y_i) for the image $\mathbf{i}$ of vertex V_i. Both $\mathbf{n}$ and the image plane are already given in space.

Conventional Assumption:

(1) Any face of the polyhedron has at least three noncollinear vertices.
(2) $\mathbf{n}$ is not incident to any plane supporting a nontriangular face of the polyhedron.

The 3D *projective reconstruction*, or *scene analysis* of the polyhedron, is to recover under the Conventional Assumption the 3D positions of the vertices and faces from the 2D coordinates so that the incidence structure is preserved.

In 3D projective geometry, the vector space is 4D in which the image plane is a 3D vector subspace $\mathcal{I}$, and $\mathbf{n}$ is a 1D subspace outside $\mathcal{I}$. Let $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_4$ be a basis of $\mathcal{I}$. When $\mathbf{n} = (n_1, n_2, n_3)$ is a space point, let $\mathbf{e}_3 = (n_1, n_2, n_3, 1)$; when $\mathbf{n}$ is a spatial direction, let $\mathbf{e}_3 = (n_1, n_2, n_3, 0)$. Then $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4$ form a basis of the 4D vector space. The homogeneous coordinates of image $\mathbf{i}$ with respect to the basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_4$ are $(x_i, y_i, 1)$, so that the homogeneous coordinates of point V_i with respect to the basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4$ are $(x_i, y_i, h_i, 1)$, where h_i is the unknown "height." The homogeneous coordinates of face F_j is $(a_j, b_j, -1, c_j)$. Now point V_i is in face F_j if and only if

$$a_j x_i + b_j y_i + c_j = h_i. \tag{3.8}$$

When (i, j) ranges over all incidence pairs, we obtain a set of linear homogeneous equations in the unknowns a, b, c, h's, called *Sugihara's fundamental equations*. Let

$$\mathbf{B}_j = a_j \mathbf{e}_2\mathbf{e}_4 - b_j \mathbf{e}_1\mathbf{e}_4 + c_j \mathbf{e}_1\mathbf{e}_2. \tag{3.9}$$

Then $\mathbf{B}_j$ is a bivector in the Grassmann space generated by $\mathcal{I}$, called the *inhomogeneous coordinates* of face F_j. Now (3.8) can be written as an equality in the Grassmann–Cayley algebra generated by $\mathcal{I}$:

$$h_i = [\mathbf{i}\mathbf{B}_j]. \tag{3.10}$$

For any four-tuple of vertices V_1, V_2, V_3, V_4 in face F_j, their 2D coordinates in plane $\mathcal{I}$ satisfy the GP syzygy:

$$[\mathbf{1}\mathbf{B}_j][\mathbf{234}] - [\mathbf{2}\mathbf{B}_j][\mathbf{134}] + [\mathbf{3}\mathbf{B}_j][\mathbf{124}] - [\mathbf{4}\mathbf{B}_j][\mathbf{123}] = 0. \tag{3.11}$$

When j ranges over all faces with more than three vertices, and the four-tuples of vertices range over all vertices in the same face, we obtain a set of linear equations of the h's by substituting (3.10) into (3.11):

$$h_1[\mathbf{234}] - h_2[\mathbf{134}] + h_3[\mathbf{124}] - h_4[\mathbf{123}] = 0. \tag{3.12}$$

They are called *Crapo's syzygy equations*.

Theorem 3.1. Under the Conventional Assumption, the fundamental equations and the syzygy equations have the same solutions for the h's.

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Proof. We only need to prove that any solution of the syzygy equations is also a solution of the fundamental equations.

The syzygy equations are on four-tuples of coplanar vertices. For the face containing the four-tuple, its inhomogeneous coordinates $\mathbf{B}$ exist. The face has at least three noncollinear vertices V_1, V_2, V_3, and their images $\mathbf{1}$, $\mathbf{2}$, $\mathbf{3}$ are not collinear. $\mathbf{B}$ can be computed by the following formula, which is obtained from (2.4) and (3.10):

$$\mathbf{B} = \frac{h_3\mathbf{12} - h_2\mathbf{13} + h_1\mathbf{23}}{[\mathbf{123}]}. \tag{3.13}$$

If a triangular face $V_1V_2V_3$ is incident to $\mathbf{n}$, then it has no inhomogeneous coordinates, and does not appear in the fundamental equations. If it is not incident to $\mathbf{n}$, its inhomogeneous coordinates $\mathbf{B}$ satisfy (3.10) for $i = 1, 2, 3$. Since V_1, V_2, V_3 are not collinear, neither are $\mathbf{1}$, $\mathbf{2}$, $\mathbf{3}$. Then $\mathbf{B}$ can be computed by the same formula (3.13).

Thus, any inhomogeneous coordinates can be computed from the h's satisfying the syzygy equations. □

In practice, the following stronger assumption covers all the interesting cases in scene analysis. Henceforth, it will be adopted throughout this paper.

Practical Assumption:

(1) No three neighboring vertices in a face are collinear.
(2) $\mathbf{n}$ is not incident to any plane supporting a nontriangular face.

Corollary 3.2. The images of any three neighboring vertices in a nontriangular face are not collinear.

Although the syzygy equations have much fewer unknowns than the fundamental equations, they are often difficult to solve. In this paper, we split the fundamental equations into two systems: the first being that the height of a vertex V_i computed from any incident face $F_1, \ldots, F_k$ is the same:

$$[\mathbf{iB}_1] = [\mathbf{iB}_2] = \cdots = [\mathbf{iB}_k], \quad \text{for any } 1 \le j \le k; \tag{3.14}$$

the second being that one face F_1 is sufficient to characterize the height of vertex V_i:

$$h_i = [\mathbf{iB}_1]. \tag{3.15}$$

The latter system is obviously trivial. The first system is on the $\mathbf{B}$'s. It is much easier to solve than both the fundamental equations and the syzygy equations.

4. Vectorial Equation Solving

4.1. Vectorial equations

In this paper, by a *vectorial equation* we mean a Grassmann–Cayley algebraic or bracket algebraic equality in which the leading variable is either a vector or a bivector (two-vector). To solve a set of vectorial equations with the same leading variable is to find all solutions and their existence conditions.

Below we list some linear bracket equations in 3D vector space and their solutions represented by Cayley expressions. In the list, the $\mathbf{V}$'s are vectors, the $\mathbf{B}$'s are bivectors, the μ's are scalars, and the ω's are new parameters. The proofs of the solutions are easy and are omitted.

Equations with vector unknowns

Type V.1:

$$\begin{cases} [\mathbf{VB}_1] = 0, \\ [\mathbf{VB}_2] = 0. \end{cases} \tag{4.1}$$

Nonzero homogeneous solution:
either (1) $\mathbf{B}_1 \wedge \mathbf{B}_2 = 0$ and $[\mathbf{VB}_1] = 0$ or (2) $\mathbf{B}_1 \wedge \mathbf{B}_2 \neq 0$ and $\mathbf{V} = \mathbf{B}_1 \wedge \mathbf{B}_2$.

Type V.2: For $m \ge 3$,

$$\begin{cases} [\mathbf{VB}_1] = 0, \\ [\mathbf{VB}_2] = 0, \\ \ldots \\ [\mathbf{VB}_m] = 0, \\ \mathbf{B}_1 \wedge \mathbf{B}_2 \neq 0. \end{cases} \tag{4.2}$$

Nonzero homogeneous solution: $\mathbf{V} = \mathbf{B}_1 \wedge \mathbf{B}_2$ and $\mathbf{B}_1 \wedge \mathbf{B}_2 \wedge \mathbf{B}_k = 0$ for any $3 \le k \le m$.

Equations with bivector unknowns:

Type B.1:

$$\begin{cases} [\mathbf{V}_1\mathbf{B}] = \mu_1, \\ [\mathbf{V}_2\mathbf{B}] = \mu_2, \\ [\mathbf{V}_3\mathbf{B}] = \mu_3, \\ \mathbf{V}_1\mathbf{V}_2 \neq 0. \end{cases} \tag{4.3}$$

Solution: either

$$(1) \begin{cases} [\mathbf{V}_1\mathbf{V}_2\mathbf{V}_3] \neq 0, \\ [\mathbf{V}_1\mathbf{V}_2\mathbf{V}_3]\,\mathbf{B} = \mu_1\mathbf{V}_2\mathbf{V}_3 - \mu_2\mathbf{V}_1\mathbf{V}_3 + \mu_3\mathbf{V}_1\mathbf{V}_2; \end{cases}$$

or

$$(2) \begin{cases} [\mathbf{V}_1\mathbf{V}_2\mathbf{V}_3] = 0, \\ \mu_1\mathbf{V}_2\mathbf{V}_3 - \mu_2\mathbf{V}_1\mathbf{V}_3 + \mu_3\mathbf{V}_1\mathbf{V}_2 = 0, \\ [\mathbf{V}_1\mathbf{B}] = \mu_1, \\ [\mathbf{V}_2\mathbf{B}] = \mu_2. \end{cases}$$

In particular, if $\mu_i = [\mathbf{V}_i\mathbf{B}']$ for $i = 1, 2, 3$, then either $[\mathbf{V}_1\mathbf{V}_2\mathbf{V}_3] \neq 0$ and $\mathbf{B} = \mathbf{B}'$, or

$$\begin{cases} [\mathbf{V}_1\mathbf{V}_2\mathbf{V}_3] = 0, \\ \mathbf{B} = \mathbf{B}' + \omega\mathbf{V}_1\mathbf{V}_2. \end{cases}$$

Type B.2: For $m \ge 4$,

$$\begin{cases} [\mathbf{V}_1\mathbf{B}] = \mu_1, \\ [\mathbf{V}_2\mathbf{B}] = \mu_2, \\ \ldots \\ [\mathbf{V}_k\mathbf{B}] = \mu_m, \\ \mathbf{V}_1\mathbf{V}_2 \neq 0. \end{cases} \tag{4.4}$$

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Solution: either

$$(1)\quad \begin{cases} [\mathbf{V}_1\mathbf{V}_2\mathbf{V}_k] \neq 0 \text{ for some } 3 \leq k \leq m, \\ [\mathbf{V}_1\mathbf{V}_2\mathbf{V}_k]\mathbf{B} = \mu_1\mathbf{V}_2\mathbf{V}_k - \mu_2\mathbf{V}_1\mathbf{V}_k + \mu_k\mathbf{V}_1\mathbf{V}_2, \\ \mu_1[\mathbf{V}_2\mathbf{V}_k\mathbf{V}_l] - \mu_2[\mathbf{V}_1\mathbf{V}_k\mathbf{V}_l] + \mu_k[\mathbf{V}_1\mathbf{V}_2\mathbf{V}_l] \\ \quad -\mu_l[\mathbf{V}_1\mathbf{V}_2\mathbf{V}_k] = 0, \qquad \text{for any } l \neq 1, 2, k; \end{cases}$$

or

$$(2)\quad \begin{cases} [\mathbf{V}_1\mathbf{V}_2\mathbf{V}_k] = 0, \quad \text{for any } 3 \leq k \leq m, \\ \mu_1\mathbf{V}_2\mathbf{V}_k - \mu_2\mathbf{V}_1\mathbf{V}_k \\ \quad + \mu_k\mathbf{V}_1\mathbf{V}_2 = 0, \quad \text{for any } 3 \leq k \leq m, \\ [\mathbf{V}_1\mathbf{B}] = \mu_1, \\ [\mathbf{V}_2\mathbf{B}] = \mu_2. \end{cases}$$

Type B.3:

$$\begin{cases} [\mathbf{V}_1\mathbf{B}] = [\mathbf{V}_1\mathbf{B}'], \\ [\mathbf{V}_2\mathbf{B}] = [\mathbf{V}_2\mathbf{B}'], \\ \mathbf{V}_1\mathbf{V}_2 \neq 0. \end{cases} \tag{4.5}$$

Solution: $\mathbf{B} = \mathbf{B}' + \omega\mathbf{V}_1\mathbf{V}_2$.

This type includes the special case $\mathbf{B}' = 0$.

Type B.4:

$$\begin{cases} [\mathbf{V}_1\mathbf{B}] = \mu_1, \\ [\mathbf{V}_2\mathbf{B}] = \mu_2, \\ [\mathbf{V}\mathbf{V}_1\mathbf{V}_2] \neq 0. \end{cases} \tag{4.6}$$

Solution: $[\mathbf{V}\mathbf{V}_1\mathbf{V}_2]\mathbf{B} = \omega\mathbf{V}_1\mathbf{V}_2 - \mu_1\mathbf{V}\mathbf{V}_2 + \mu_2\mathbf{V}\mathbf{V}_1$.

Type B.5:

$$\begin{cases} [\mathbf{V}\mathbf{B}] = \mu, \\ [\mathbf{V}\mathbf{V}_1\mathbf{V}_2] \neq 0. \end{cases} \tag{4.7}$$

Solution: $[\mathbf{V}\mathbf{V}_1\mathbf{V}_2]\mathbf{B} = \mu\mathbf{V}_1\mathbf{V}_2 + \omega_1\mathbf{V}\mathbf{V}_1 + \omega_2\mathbf{V}\mathbf{V}_2$.

In particular, if $\mu = [\mathbf{V}\mathbf{B}']$, then $\mathbf{B} = \mathbf{B}' + \omega_1\mathbf{V}\mathbf{V}_1 + \omega_2\mathbf{V}\mathbf{V}_2$.

4.2. Elimination for vectorial equation solving

To solve a set of vectorial equations with multiple unknowns, a common technique is to triangulate the equations following a prescribed order of the unknowns. The first round is to solve the highest ordered unknown by taking the other unknowns as "known," and substitute the solution back to the equations to "eliminate" the highest ordered unknown. Then a new round of solving and eliminating takes place among the new equations. Ultimately, if all the unknowns are solved and eliminated, the system is split into two systems in which the first is the solutions and the second is the parametric equations. To solve the second system, some standard techniques in computer algebra can be applied (Wu, 1984; Buchberger, 1988).

As an example, we solve the system of the $\mathbf{B}$'s in Example 1 (see Fig. 3). Let i, j, k be an even permutation of 1, 2, 3. The face with vertices V_i, V_j, $V_{i'}$, $V_{j'}$ is denoted by $F_{k''}$. Similarly, the face with vertices V_i, V_j, $V_{i''}$, $V_{j''}$ is denoted by $F_{k'}$, and the face with vertices $V_{i'}$, $V_{j'}$, $V_{i''}$, $V_{j''}$ is denoted by F_k. The inhomogeneous coordinates of faces $F_{k''}$, $F_{k'}$, F_k are denoted, respectively, by $\mathbf{B}_{k''}$, $\mathbf{B}_{k'}$, $\mathbf{B}_k$.

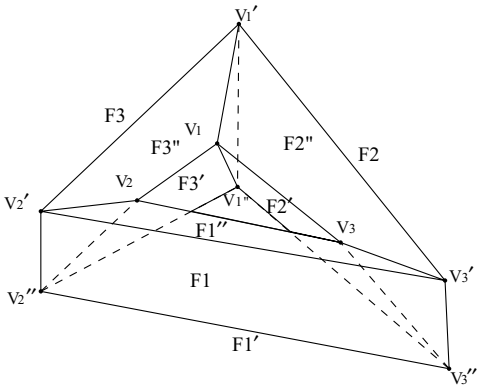


Fig. 3. Example 1 continued.

The equations are

$$\begin{cases} [\mathbf{1}\mathbf{B}_{2'}] = [\mathbf{1}\mathbf{B}_{2''}] = [\mathbf{1}\mathbf{B}_{3'}] = [\mathbf{1}\mathbf{B}_{3''}], \\ [\mathbf{2}\mathbf{B}_{1'}] = [\mathbf{2}\mathbf{B}_{1''}] = [\mathbf{2}\mathbf{B}_{3'}] = [\mathbf{2}\mathbf{B}_{3''}], \\ [\mathbf{3}\mathbf{B}_{2'}] = [\mathbf{3}\mathbf{B}_{2''}] = [\mathbf{3}\mathbf{B}_{1'}] = [\mathbf{3}\mathbf{B}_{1''}], \\ [\mathbf{1}'\mathbf{B}_2] = [\mathbf{1}'\mathbf{B}_{2''}] = [\mathbf{1}'\mathbf{B}_3] = [\mathbf{1}'\mathbf{B}_{3''}], \\ [\mathbf{2}'\mathbf{B}_1] = [\mathbf{2}'\mathbf{B}_{1''}] = [\mathbf{2}'\mathbf{B}_3] = [\mathbf{2}'\mathbf{B}_{3''}], \\ [\mathbf{3}'\mathbf{B}_1] = [\mathbf{3}'\mathbf{B}_{1''}] = [\mathbf{3}'\mathbf{B}_2] = [\mathbf{3}'\mathbf{B}_{2''}], \\ [\mathbf{1}''\mathbf{B}_2] = [\mathbf{1}''\mathbf{B}_{2'}] = [\mathbf{1}''\mathbf{B}_3] = [\mathbf{1}''\mathbf{B}_{3'}], \\ [\mathbf{2}''\mathbf{B}_1] = [\mathbf{2}''\mathbf{B}_{1'}] = [\mathbf{2}''\mathbf{B}_3] = [\mathbf{2}''\mathbf{B}_{3'}], \\ [\mathbf{3}''\mathbf{B}_1] = [\mathbf{3}''\mathbf{B}_{1'}] = [\mathbf{3}''\mathbf{B}_2] = [\mathbf{3}''\mathbf{B}_{2'}]. \end{cases} \tag{4.8}$$

By replacing $\mathbf{B}_i$ with $\mathbf{B}_i - \mathbf{B}$, where $\mathbf{B}$ is any fixed bivector, the system is invariant. So if we set one of the $\mathbf{B}$'s to be zero, for example, set $\mathbf{B}_{3''} = 0$, and obtain a solution $\mathbf{B}_i = \mathbf{C}_i$ for $i \neq 3''$, then the original system has the solution $\mathbf{B}_i = \mathbf{C}_i + \mathbf{B}_{3''}$ for $i \neq 3''$, in which $\mathbf{B}_{3''}$ is free.

We set $\mathbf{B}_{3''} = 0$. Under the following order of the other $\mathbf{B}$'s,

$$\mathbf{B}_3 \prec \mathbf{B}_{1''} \prec \mathbf{B}_{3'} \prec \mathbf{B}_{2''} \prec \mathbf{B}_1 \prec \mathbf{B}_2 \prec \mathbf{B}_{1'} \prec \mathbf{B}_{2'}, \tag{4.9}$$

the elimination proceeds as follows.

1. Elimination of $\mathbf{B}_{2'}$:

$$[\mathbf{1}\mathbf{B}_{2'}] = 0,\ [\mathbf{1}''\mathbf{B}_{2'}] = [\mathbf{1}''\mathbf{B}_3],\ [\mathbf{3}\mathbf{B}_{2'}] = [\mathbf{3}\mathbf{B}_{1''}],$$
$$[\mathbf{3}''\mathbf{B}_{2'}] = [\mathbf{3}''\mathbf{B}_1]$$
$$\xrightarrow{B.2} \begin{cases} [\mathbf{131}'']\mathbf{B}_{2'} = [\mathbf{1}''\mathbf{B}_3]\mathbf{13} - [\mathbf{3}\mathbf{B}_{1''}]\mathbf{11}'', \\ d(\mathbf{B}_1) : [\mathbf{3}''\mathbf{B}_1][\mathbf{131}''] = [\mathbf{1}''\mathbf{B}_3][\mathbf{133}''] \\ \qquad - [\mathbf{3}\mathbf{B}_{1''}][\mathbf{11}''\mathbf{3}'']. \end{cases}$$

There are four equations involving $\mathbf{B}_{2'}$, each equation has three different forms. For solving these equations, the form which is the lowest under the lexicographic order induced by (4.9) is used. The equations belong to type B.2, and have only one branch because of the Practical Assumption. $d(\mathbf{B}_1)$ is an equation with leading unknown $\mathbf{B}_1$.

2. Elimination of $\mathbf{B}_{1'}$:

$$[\mathbf{2}\mathbf{B}_{1'}] = 0,\ [\mathbf{3}\mathbf{B}_{1'}] = [\mathbf{3}\mathbf{B}_{1''}],\ [\mathbf{2}''\mathbf{B}_{1'}] = [\mathbf{2}''\mathbf{B}_3],$$

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$$[\mathbf{3''B}_{1'}] = [\mathbf{3''B}_1]$$

$$\xrightarrow{B.2} \begin{cases} [\mathbf{232''}]\mathbf{B}_{1'} = [\mathbf{2''B}_3]\mathbf{23} - [\mathbf{3B}_{1''}]\mathbf{22''}, \\ e(\mathbf{B}_1) : [\mathbf{3''B}_1][\mathbf{232''}] = [\mathbf{2''B}_3][\mathbf{233''}] \\ \qquad - [\mathbf{3B}_{1''}][\mathbf{22''3''}]. \end{cases}$$

3. Elimination of $\mathbf{B}_2$:

$$[\mathbf{1'B}_2] = 0,\ [\mathbf{3'B}_2] = [\mathbf{3'B}_{1''}],\ [\mathbf{1''B}_2] = [\mathbf{1''B}_3],$$
$$[\mathbf{3''B}_2] = [\mathbf{3''B}_1]$$

$$\xrightarrow{B.2} \begin{cases} [\mathbf{1'3'1''}]\mathbf{B}_2 = [\mathbf{1''B}_3]\mathbf{1'3'} - [\mathbf{3'B}_{1''}]\mathbf{1'1''}, \\ f(\mathbf{B}_1) : [\mathbf{3''B}_1][\mathbf{1'3'1''}] = [\mathbf{1''B}_3][\mathbf{1'3'3''}] \\ \qquad - [\mathbf{3'B}_{1''}][\mathbf{1'1''3''}]. \end{cases}$$

4. Elimination of $\mathbf{B}_1$:

$$[\mathbf{2''B}_1] = [\mathbf{2''B}_3],\ [\mathbf{2'B}_1] = 0,\ [\mathbf{3'B}_1] = [\mathbf{3'B}_{1''}],$$
$$d(\mathbf{B}_1),\ e(\mathbf{B}_1),\ f(\mathbf{B}_1)$$

$$\xrightarrow{B.2} \begin{cases} [\mathbf{2'3'2''}]\mathbf{B}_1 = [\mathbf{2''B}_3]\mathbf{2'3'} - [\mathbf{3'B}_{1''}]\mathbf{2'2''}, \\ d(\mathbf{B}_{1''}) : [\mathbf{2'3'2''}]([\mathbf{1''B}_3][\mathbf{133''}] - [\mathbf{3B}_{1''}][\mathbf{11''3''}]) \\ \qquad = [\mathbf{131''}]([\mathbf{2''B}_3][\mathbf{2'3'3''}] - [\mathbf{3'B}_{1''}][\mathbf{2'2''3''}]), \\ e(\mathbf{B}_{1''}) : [\mathbf{2'3'2''}]([\mathbf{2''B}_3][\mathbf{233''}] - [\mathbf{3B}_{1''}][\mathbf{22''3''}]) \\ \qquad = [\mathbf{232''}]([\mathbf{2''B}_3][\mathbf{2'3'3''}] - [\mathbf{3'B}_{1''}][\mathbf{2'2''3''}]), \\ f(\mathbf{B}_{1''}) : [\mathbf{2'3'2''}]([\mathbf{1''B}_3][\mathbf{1'3'3''}] - [\mathbf{3'B}_{1''}][\mathbf{1'1''3''}]) \\ \qquad = [\mathbf{1'3'1''}]([\mathbf{2''B}_3][\mathbf{2'3'3''}] - [\mathbf{3'B}_{1''}][\mathbf{2'2''3''}]). \end{cases}$$

By (2.5), $e(\mathbf{B}_{1''})$ can be written as

$$[\mathbf{2''B}_3](\mathbf{23} \wedge \mathbf{2'3'} \wedge \mathbf{2''3''}) = [\mathbf{3'B}_{1''}][\mathbf{232''}][\mathbf{2'2''3''}] - [\mathbf{3B}_{1''}][\mathbf{22''3''}][\mathbf{2'3'2''}],$$

and $f(\mathbf{B}_{1''})$ can be written as

$$[\mathbf{3'B}_{1''}](\mathbf{1'1''} \wedge \mathbf{2'2''} \wedge \mathbf{3'3''}) = [\mathbf{1''B}_3][\mathbf{1'3'3''}][\mathbf{2'3'2''}] - [\mathbf{2''B}_3][\mathbf{1'3'1''}][\mathbf{2'3'3''}].$$

5. Elimination of $\mathbf{B}_{2''}$:

$$[\mathbf{1B}_{2''}] = 0,\ [\mathbf{1'B}_{2''}] = 0,\ [\mathbf{3B}_{2''}] = [\mathbf{3B}_{1''}],$$
$$[\mathbf{3'B}_{2''}] = [\mathbf{3'B}_{1''}]$$

$$\xrightarrow{B.2} \begin{cases} [\mathbf{131'}]\mathbf{B}_{2''} = -[\mathbf{3B}_{1''}]\mathbf{11'}, \\ g(\mathbf{B}_{1''}) : [\mathbf{3B}_{1''}][\mathbf{11'3'}] + [\mathbf{3'B}_{1''}][\mathbf{131'}] = 0. \end{cases}$$

By (2.5), $g(\mathbf{B}_{1''})$ can be written as $\mathbf{B}_{1''} \wedge \mathbf{11'} \wedge \mathbf{33'} = 0$.

6. Elimination of $\mathbf{B}_{3'}$:

$$[\mathbf{1B}_{3'}] = 0,\ [\mathbf{2B}_{3'}] = 0,\ [\mathbf{1''B}_{3'}] = [\mathbf{1''B}_3],$$
$$[\mathbf{2''B}_{3'}] = [\mathbf{2''B}_3]$$

$$\xrightarrow{B.2} \begin{cases} [\mathbf{121''}]\mathbf{B}_{3'} = [\mathbf{1''B}_3]\mathbf{12}, \\ h(\mathbf{B}_3) : [\mathbf{1''B}_3][\mathbf{122''}] - [\mathbf{2''B}_3][\mathbf{121''}] = 0. \end{cases}$$

By (2.5), $h(\mathbf{B}_3)$ can be written as $\mathbf{B}_3 \wedge \mathbf{12} \wedge \mathbf{1''2''} = 0$.

7. Elimination of $\mathbf{B}_{1''}$:

$$[\mathbf{2B}_{1''}] = 0,$$
$$[\mathbf{2'B}_{1''}] = 0,\ d(\mathbf{B}_{1''}),\ e(\mathbf{B}_{1''}),\ f(\mathbf{B}_{1''}),\ g(\mathbf{B}_{1''}) \xrightarrow{B.2,\ B.3}$$

either

$$\mathbf{B}_{1''} = 0,\ d(\mathbf{B}_{1''}),\ e(\mathbf{B}_{1''}),\ f(\mathbf{B}_{1''}), \tag{4.10}$$

or

$$\begin{cases} \mathbf{11'} \wedge \mathbf{22'} \wedge \mathbf{33'} = 0, \\ \mathbf{1'1''} \wedge \mathbf{2'2''} \wedge \mathbf{3'3''} \neq 0, \\ [\mathbf{22'3'}](\mathbf{1'1''} \wedge \mathbf{2'2''} \wedge \mathbf{3'3''})\mathbf{B}_{1''} \\ \quad = ([\mathbf{1''B}_3][\mathbf{1'3'3''}][\mathbf{2'3'2''}] \\ \qquad -[\mathbf{2''B}_3][\mathbf{1'3'1''}][\mathbf{2'3'3''}])\mathbf{22'}, \\ d(\mathbf{B}_{1''}), \\ e(\mathbf{B}_{1''}), \end{cases} \tag{4.11}$$

or

$$\begin{cases} \mathbf{11'} \wedge \mathbf{22'} \wedge \mathbf{33'} = 0, \\ \mathbf{1'1''} \wedge \mathbf{2'2''} \wedge \mathbf{3'3''} = 0, \\ [\mathbf{1''B}_3][\mathbf{1'3'3''}][\mathbf{2'3'2''}] = [\mathbf{2''B}_3][\mathbf{1'3'1''}][\mathbf{2'3'3''}], \\ [\mathbf{22'3'}][\mathbf{232''}][\mathbf{2'2''3''}] + [\mathbf{232'}][\mathbf{22''3''}][\mathbf{2'3'2''}] \\ \qquad = -[\mathbf{22'2''}](\mathbf{23} \wedge \mathbf{2'3'} \wedge \mathbf{2''3''}) \neq 0, \\ (23 \wedge \mathbf{2'3'} \wedge \mathbf{2''3''})([\mathbf{22'2''}]\mathbf{B}_{1''} + [\mathbf{2''B}_3]\mathbf{22'}) = 0, \\ d(\mathbf{B}_{1''}), \end{cases} \tag{4.12}$$

or

$$\begin{cases} \mathbf{11'} \wedge \mathbf{22'} \wedge \mathbf{33'} = 0, \\ \mathbf{1'1''} \wedge \mathbf{2'2''} \wedge \mathbf{3'3''} = 0, \\ [\mathbf{1''B}_3][\mathbf{1'3'3''}][\mathbf{2'3'2''}] = [\mathbf{2''B}_3][\mathbf{1'3'1''}][\mathbf{2'3'3''}], \\ \mathbf{23} \wedge \mathbf{2'3'} \wedge \mathbf{2''3''} \neq 0, \\ [\mathbf{22'2''}] = 0, \\ [\mathbf{2''B}_3] = 0, \\ [\mathbf{232'}][\mathbf{11''3''}][\mathbf{2'3'2''}] + [\mathbf{22'3'}][\mathbf{131''}][\mathbf{2'2''3''}] \neq 0, \\ ([\mathbf{232'}][\mathbf{11''3''}][\mathbf{2'3'2''}] + [\mathbf{22'3'}][\mathbf{131''}][\mathbf{2'2''3''}])\mathbf{B}_{1''} \\ \quad = ([\mathbf{2''B}_3][\mathbf{131''}][\mathbf{2'3'3''}] - [\mathbf{1''B}_3][\mathbf{133''}][\mathbf{2'3'2''}])\mathbf{22'}, \end{cases} \tag{4.13}$$

or

$$\begin{cases} \mathbf{11'} \wedge \mathbf{22'} \wedge \mathbf{33'} = 0, \\ \mathbf{1'1''} \wedge \mathbf{2'2''} \wedge \mathbf{3'3''} = 0, \\ [\mathbf{1''B}_3][\mathbf{1'3'3''}][\mathbf{2'3'2''}] = [\mathbf{2''B}_3][\mathbf{1'3'1''}][\mathbf{2'3'3''}], \\ \mathbf{23} \wedge \mathbf{2'3'} \wedge \mathbf{2''3''} \neq 0, \\ [\mathbf{22'2''}] = 0, \\ [\mathbf{2''B}_3] = 0, \\ [\mathbf{232'}][\mathbf{11''3''}][\mathbf{2'3'2''}] + [\mathbf{22'3'}][\mathbf{131''}][\mathbf{2'2''3''}] = 0, \\ [\mathbf{2''B}_3][\mathbf{131''}][\mathbf{2'3'3''}] - [\mathbf{1''B}_3][\mathbf{133''}][\mathbf{2'3'2''}] = 0, \\ \mathbf{B}_{1''} = \omega_{22'}\mathbf{22'}, \end{cases} \tag{4.14}$$

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or

$$\begin{cases} \mathbf{11}' \wedge \mathbf{22}' \wedge \mathbf{33}' = 0, \\ \mathbf{1}'\mathbf{1}'' \wedge \mathbf{2}'\mathbf{2}'' \wedge \mathbf{3}'\mathbf{3}'' = 0, \\ [\mathbf{1}''\mathbf{B}_3][\mathbf{1}'\mathbf{3}'\mathbf{3}''][\mathbf{2}'\mathbf{3}'\mathbf{2}''] = [\mathbf{2}''\mathbf{B}_3][\mathbf{1}'\mathbf{3}'\mathbf{1}''][\mathbf{2}'\mathbf{3}'\mathbf{3}''], \\ \mathbf{23} \wedge \mathbf{2}'\mathbf{3}' \wedge \mathbf{2}''\mathbf{3}'' = 0, \\ [\mathbf{232}'][\mathbf{11}''\mathbf{3}''][\mathbf{2}'\mathbf{3}'\mathbf{2}''] + [\mathbf{22}'\mathbf{3}'][\mathbf{131}''][\mathbf{2}'\mathbf{2}''\mathbf{3}''] \neq 0, \\ ([\mathbf{232}'][\mathbf{11}''\mathbf{3}''][\mathbf{2}'\mathbf{3}'\mathbf{2}''] + [\mathbf{22}'\mathbf{3}'][\mathbf{131}''][\mathbf{2}'\mathbf{2}''\mathbf{3}''])\mathbf{B}_{1''} \\ \quad = ([\mathbf{2}''\mathbf{B}_3][\mathbf{131}''][\mathbf{2}'\mathbf{3}'\mathbf{3}''] - [\mathbf{1}''\mathbf{B}_3][\mathbf{133}''][\mathbf{2}'\mathbf{3}'\mathbf{2}''])\mathbf{22}', \end{cases} \tag{4.15}$$

or

$$\begin{cases} \mathbf{11}' \wedge \mathbf{22}' \wedge \mathbf{33}' = 0, \\ \mathbf{1}'\mathbf{1}'' \wedge \mathbf{2}'\mathbf{2}'' \wedge \mathbf{3}'\mathbf{3}'' = 0, \\ [\mathbf{1}''\mathbf{B}_3][\mathbf{1}'\mathbf{3}'\mathbf{3}''][\mathbf{2}'\mathbf{3}'\mathbf{2}''] = [\mathbf{2}''\mathbf{B}_3][\mathbf{1}'\mathbf{3}'\mathbf{1}''][\mathbf{2}'\mathbf{3}'\mathbf{3}''], \\ \mathbf{23} \wedge \mathbf{2}'\mathbf{3}' \wedge \mathbf{2}''\mathbf{3}'' = 0, \\ [\mathbf{232}'][\mathbf{11}''\mathbf{3}''][\mathbf{2}'\mathbf{3}'\mathbf{2}''] + [\mathbf{22}'\mathbf{3}'][\mathbf{131}''][\mathbf{2}'\mathbf{2}''\mathbf{3}''] = 0, \\ [\mathbf{2}''\mathbf{B}_3][\mathbf{131}''][\mathbf{2}'\mathbf{3}'\mathbf{3}''] = [\mathbf{1}''\mathbf{B}_3][\mathbf{133}''][\mathbf{2}'\mathbf{3}'\mathbf{2}''], \\ \mathbf{B}_{1''} = \omega_{22'}\mathbf{22}'. \end{cases} \tag{4.16}$$

8. Elimination of $\mathbf{B}_3$:
The solution of $\mathbf{B}_{1''}$ has seven branches. $\mathbf{B}_3$ is to be solved in each branch separately. As an example, we solve $\mathbf{B}_3$ in the last branch:

$$[\mathbf{1}'\mathbf{B}_3] = 0,$$

$$[\mathbf{2}'\mathbf{B}_3] = 0,\ h(\mathbf{B}_3),\ \text{two equations of } \mathbf{B}_3 \text{ in (4.16)} \xrightarrow{B.2,\ B.3}$$

either

$$\begin{cases} \mathbf{12} \wedge \mathbf{1}'\mathbf{2}' \wedge \mathbf{1}''\mathbf{2}'' \neq 0, \\ \mathbf{B}_3 = 0, \end{cases} \tag{4.17}$$

or

$$\begin{cases} \mathbf{12} \wedge \mathbf{1}'\mathbf{2}' \wedge \mathbf{1}''\mathbf{2}'' = 0, \\ [\mathbf{133}''][\mathbf{1}'\mathbf{2}'\mathbf{1}''][\mathbf{2}'\mathbf{3}'\mathbf{2}''] \neq [\mathbf{131}''][\mathbf{1}'\mathbf{2}'\mathbf{2}''][\mathbf{2}'\mathbf{3}'\mathbf{3}''], \\ \mathbf{B}_3 = 0, \end{cases} \tag{4.18}$$

or

$$\begin{cases} \mathbf{12} \wedge \mathbf{1}'\mathbf{2}' \wedge \mathbf{1}''\mathbf{2}'' = 0, \\ [\mathbf{133}''][\mathbf{1}'\mathbf{2}'\mathbf{1}''][\mathbf{2}'\mathbf{3}'\mathbf{2}''] = [\mathbf{131}''][\mathbf{1}'\mathbf{2}'\mathbf{2}''][\mathbf{2}'\mathbf{3}'\mathbf{3}''], \\ \mathbf{B}_3 = \omega_{1'2'}\mathbf{1}'\mathbf{2}'. \end{cases} \tag{4.19}$$

5. Parametric Propagation in Vectorial Equation Solving

As shown in the previous section, a typical problem with elimination is that bifurcation occurs inevitably, and the number of branches grows exponentially. If the order of elimination is not appropriate, the bifurcation may become much more complicated. On the contrary, if the unknowns can be sequentially parameterized and constructed, the elimination following the reverse order of the construction should have less branches. The idea of *parametric propagation* is to construct the solutions parametrically. New parameters are introduced every time the number of equations is not sufficient in the construction. The parameters may be either free or constrained, and the constraints may not be obtained until all the unknowns are constructed. The goal is to transform the system into one in which the unknowns are the new parameters, and which is usually much easier to solve.

Algorithm: Parametric propagation in vectorial equation solving

Input: A set of vectorial equations eqn, a set of unknowns var which are either vectors or bivectors. Assume that the unknowns in $var_0 \subseteq var$ can be set to be free.

Output: A set of parameters $para$, a set of vectorial and parametric equations $soln$.

Initialization: Set $para$ and $soln$ to be the empty set.

Step 1. Choose an element $\mathbf{X}$ in var_0. Set

$$para = para \cup \{\mathbf{X}\}, \qquad var = var - \{\mathbf{X}\}.$$

Step 2. For i from 0 to 1, do the following.
Find an element $\mathbf{Y}$ in var which can be solved from a subset $eqn(\mathbf{Y})$ of eqn, such that the solution $s(\mathbf{Y})$ contains i new parameters $\omega(\mathbf{Y})$ but no variables in $var - \{\mathbf{Y}\}$. If there are more than one such $\mathbf{Y}$, then choose one whose $eqn(\mathbf{Y})$, has least terms and lowest degree. If $\mathbf{Y}$ is found, then (1) set

$$para = para \cup \{\omega(\mathbf{Y})\}, \qquad var = var - \{\mathbf{Y}\},$$
$$soln = soln \cup \{s(\mathbf{Y})\}, \qquad eqn = eqn - eqn(\mathbf{Y});$$

(2) substitute $s(\mathbf{Y})$ into eqn; (3) if eqn is empty then exit, else go to the beginning of Step 2.

As an example, we use parametric propagation to solve the system (4.8). Choose $\mathbf{B}_{3''}$ as $\mathbf{X}$ by setting $\mathbf{B}_{3''} = 0$. The following are different rounds of Step 2.

Round 1:
When $i = 0$, $\mathbf{Y}$ is not found. When $i = 1$, $\mathbf{Y}$ can be any of $\mathbf{B}_3, \mathbf{B}_{3'}, \mathbf{B}_{1''}, \mathbf{B}_{2''}$. Choose $\mathbf{B}_3$. Then

$$[\mathbf{1}'\mathbf{B}_3] = 0,\ [\mathbf{2}'\mathbf{B}_3] = 0 \xrightarrow{B.3} \mathbf{B}_3 = \omega_{1'2'}\mathbf{1}'\mathbf{2}'. \tag{5.1}$$

Substitute (5.1) into the system.

Round 2:
When $i = 0$,

$$\begin{cases} [\mathbf{1}\mathbf{B}_{3'}] = 0 \\ [\mathbf{2}\mathbf{B}_{3'}] = 0 \\ [\mathbf{1}''\mathbf{B}_{3'}] = \omega_{1'2'}[\mathbf{1}'\mathbf{2}'\mathbf{1}''] \\ [\mathbf{2}''\mathbf{B}_{3'}] = \omega_{1'2'}[\mathbf{1}'\mathbf{2}'\mathbf{2}''] \end{cases} \xrightarrow{B.2} \begin{cases} [\mathbf{121}'']\mathbf{B}_{3'} = \omega_{1'2'}[\mathbf{1}'\mathbf{2}'\mathbf{1}'']\mathbf{12}, \\ \omega_{1'2'}([\mathbf{1}'\mathbf{2}'\mathbf{1}''][\mathbf{122}''] \\ \quad - [\mathbf{1}'\mathbf{2}'\mathbf{2}''][\mathbf{121}'']) = 0. \end{cases}$$

The second equality in the solution has Cayley factorization $\omega_{1'2'}\mathbf{12} \wedge \mathbf{1}'\mathbf{2}' \wedge \mathbf{1}''\mathbf{2}'' = 0$.

Round 3:
When $i = 0$, $\mathbf{Y}$ is not found. When $i = 1$, $\mathbf{Y}$ can be any of $\mathbf{B}_1, \mathbf{B}_2, \mathbf{B}_{1'}, \mathbf{B}_{2'}, \mathbf{B}_{1''}, \mathbf{B}_{2''}$. Choose $\mathbf{B}_{1''}$ because its equations are simpler. Then

$$[\mathbf{2}\mathbf{B}_{1''}] = 0,\ [\mathbf{2}'\mathbf{B}_{1''}] = 0 \xrightarrow{B.3} \mathbf{B}_{1''} = \omega_{22'}\mathbf{22}'. \tag{5.2}$$

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Substitute (5.2) into the system.

Round 4:

$$\begin{cases}[\mathbf{1}\mathbf{B}_{2''}]=0\\ [\mathbf{1}'\mathbf{B}_{2''}]=0\\ [\mathbf{3}\mathbf{B}_{2''}]=-\omega_{22'}[\mathbf{232}']\\ [\mathbf{3}'\mathbf{B}_{2''}]=\omega_{22'}[\mathbf{22}'\mathbf{3}']\end{cases}\xrightarrow{B.2}\begin{cases}[\mathbf{131}']\mathbf{B}_{2''}=\omega_{22'}[\mathbf{232}']\mathbf{11}',\\ \omega_{22'}([\mathbf{11}'\mathbf{3}'][\mathbf{232}']\\ \quad-[\mathbf{131}'][\mathbf{22}'\mathbf{3}'])=0.\end{cases}$$

The second equality in the solution has Cayley factorization $\omega_{22'}\mathbf{11}'\wedge\mathbf{22}'\wedge\mathbf{33}'=0$.

Round 5:

$$\begin{cases}[\mathbf{2}'\mathbf{B}_1]=0\\ [\mathbf{3}'\mathbf{B}_1]=\omega_{22'}[\mathbf{22}'\mathbf{3}']\\ [\mathbf{2}''\mathbf{B}_1]=\omega_{1'2'}[\mathbf{1}'\mathbf{2}'\mathbf{2}'']\end{cases}\xrightarrow{B.1}\begin{array}{r}[\mathbf{2}'\mathbf{3}'\mathbf{2}'']\mathbf{B}_1=\omega_{1'2'}[\mathbf{1}'\mathbf{2}'\mathbf{2}'']\mathbf{2}'\mathbf{3}'\\ -\omega_{22'}[\mathbf{22}'\mathbf{3}']\mathbf{2}'\mathbf{2}''.\end{array}\tag{5.3}$$

Substitute (5.3) into the system.

Round 6:

$$\begin{cases}[\mathbf{1}'\mathbf{B}_2]=0\\ [\mathbf{3}'\mathbf{B}_2]=\omega_{22'}[\mathbf{22}'\mathbf{3}']\\ [\mathbf{1}''\mathbf{B}_2]=\omega_{1'2'}[\mathbf{1}'\mathbf{2}'\mathbf{1}'']\\ [\mathbf{2}'\mathbf{3}'\mathbf{2}''][\mathbf{3}''\mathbf{B}_2]=\omega_{1'2'}[\mathbf{1}'\mathbf{2}'\mathbf{2}''][\mathbf{2}'\mathbf{3}'\mathbf{3}'']\\ \quad-\omega_{22'}[\mathbf{22}'\mathbf{3}'][\mathbf{2}'\mathbf{2}''\mathbf{3}'']\end{cases}$$

$$\xrightarrow{B.2}\begin{cases}[\mathbf{1}'\mathbf{3}'\mathbf{1}'']\mathbf{B}_2=\omega_{1'2'}[\mathbf{1}'\mathbf{2}'\mathbf{1}'']\mathbf{1}'\mathbf{3}'-\omega_{22'}[\mathbf{22}'\mathbf{3}']\mathbf{1}'\mathbf{1}'',\\ \omega_{1'2'}([\mathbf{1}'\mathbf{2}'\mathbf{1}''][\mathbf{1}'\mathbf{3}'\mathbf{3}''][\mathbf{2}'\mathbf{3}'\mathbf{2}'']\\ \qquad-[\mathbf{1}'\mathbf{2}'\mathbf{2}''][\mathbf{1}'\mathbf{3}'\mathbf{1}''][\mathbf{2}'\mathbf{3}'\mathbf{3}''])\\ \quad=\omega_{22'}[\mathbf{22}'\mathbf{3}'])([\mathbf{2}'\mathbf{3}'\mathbf{2}''][\mathbf{1}'\mathbf{1}''\mathbf{3}'']-[\mathbf{1}'\mathbf{3}'\mathbf{1}''][\mathbf{2}'\mathbf{2}''\mathbf{3}'']).\end{cases}$$

The second equality in the solution has Cayley factorization

$$(\mathbf{1}'\mathbf{1}''\wedge\mathbf{2}'\mathbf{2}''\wedge\mathbf{3}'\mathbf{3}'')(\omega_{1'2'}[\mathbf{1}'\mathbf{2}'\mathbf{3}']-\omega_{22'}[\mathbf{22}'\mathbf{3}'])=0.$$

Round 7:

$$\begin{cases}[\mathbf{2}\mathbf{B}_{1'}]=0\\ [\mathbf{3}\mathbf{B}_{1'}]=-\omega_{22'}[\mathbf{232}']\\ [\mathbf{2}''\mathbf{B}_{1'}]=\omega_{1'2'}[\mathbf{1}'\mathbf{2}'\mathbf{2}'']\\ [\mathbf{2}'\mathbf{3}'\mathbf{2}''][\mathbf{3}''\mathbf{B}_{1'}]=\omega_{1'2'}[\mathbf{1}'\mathbf{2}'\mathbf{2}''][\mathbf{2}'\mathbf{3}'\mathbf{3}'']-\omega_{22'}[\mathbf{22}'\mathbf{3}'][\mathbf{2}'\mathbf{2}''\mathbf{3}'']\end{cases}$$

$$\xrightarrow{B.2}\begin{cases}[\mathbf{232}'']\mathbf{B}_{1'}=\omega_{22'}[\mathbf{232}']\mathbf{22}''+\omega_{1'2'}[\mathbf{1}'\mathbf{2}'\mathbf{2}'']\mathbf{23},\\ \omega_{22'}([\mathbf{232}'][\mathbf{22}''\mathbf{3}''][\mathbf{2}'\mathbf{3}'\mathbf{2}'']+[\mathbf{232}''][\mathbf{22}'\mathbf{3}'][\mathbf{2}'\mathbf{2}''\mathbf{3}''])\\ \quad=\omega_{1'2'}[\mathbf{1}'\mathbf{2}'\mathbf{2}''])([\mathbf{232}''][\mathbf{2}'\mathbf{3}'\mathbf{3}'']-[\mathbf{233}''][\mathbf{2}'\mathbf{3}'\mathbf{2}'']).\end{cases}$$

The second equality in the solution has Cayley factorization

$$(\mathbf{23}\wedge\mathbf{2}'\mathbf{3}'\wedge\mathbf{2}''\mathbf{3}'')(\omega_{1'2'}[\mathbf{1}'\mathbf{2}'\mathbf{2}'']-\omega_{22'}[\mathbf{22}'\mathbf{2}''])=0.$$

Round 8:

$$\begin{cases}[\mathbf{1}\mathbf{B}_{2'}]=0\\ [\mathbf{3}\mathbf{B}_{2'}]=-\omega_{22'}[\mathbf{232}']\\ [\mathbf{1}''\mathbf{B}_{2'}]=\omega_{1'2'}[\mathbf{1}'\mathbf{2}'\mathbf{1}'']\\ [\mathbf{2}'\mathbf{3}'\mathbf{2}''][\mathbf{3}''\mathbf{B}_{2'}]=\omega_{1'2'}[\mathbf{1}'\mathbf{2}'\mathbf{2}''][\mathbf{2}'\mathbf{3}'\mathbf{3}'']-\omega_{22'}[\mathbf{22}'\mathbf{3}'][\mathbf{2}'\mathbf{2}''\mathbf{3}'']\end{cases}$$

$$\xrightarrow{B.2}\begin{cases}[\mathbf{131}'']\mathbf{B}_{2'}=\omega_{1'2'}[\mathbf{1}'\mathbf{2}'\mathbf{1}'']\mathbf{13}+\omega_{22'}[\mathbf{232}']\mathbf{11}'',\\ \omega_{22'}([\mathbf{11}''\mathbf{3}''][\mathbf{232}'][\mathbf{2}'\mathbf{3}'\mathbf{2}'']+[\mathbf{131}''][\mathbf{22}'\mathbf{3}'][\mathbf{2}'\mathbf{2}''\mathbf{3}''])\\ =\omega_{1'2'}([\mathbf{131}''][\mathbf{1}'\mathbf{2}'\mathbf{2}''][\mathbf{2}'\mathbf{3}'\mathbf{3}'']-[\mathbf{133}''][\mathbf{1}'\mathbf{2}'\mathbf{1}''][\mathbf{2}'\mathbf{3}'\mathbf{2}'']).\end{cases}$$

Solution: With new parameters $\omega_{1'2'},\omega_{22'}$, and by setting $\mathbf{B}_{3''}=0$, (4.8) is changed into

$$\begin{cases}\mathbf{B}_{3''}=0,\\ \mathbf{B}_3=\omega_{1'2'}\mathbf{1}'\mathbf{2}',\\ [\mathbf{121}'']\mathbf{B}_{3'}=\omega_{1'2'}[\mathbf{1}'\mathbf{2}'\mathbf{1}'']\mathbf{12},\\ \mathbf{B}_{1''}=\omega_{22'}\mathbf{22}',\\ [\mathbf{131}']\mathbf{B}_{2''}=\omega_{22'}[\mathbf{232}']\mathbf{11}',\\ [\mathbf{2}'\mathbf{3}'\mathbf{2}'']\mathbf{B}_1=\omega_{1'2'}[\mathbf{1}'\mathbf{2}'\mathbf{2}'']\mathbf{2}'\mathbf{3}'-\omega_{22'}[\mathbf{22}'\mathbf{3}']\mathbf{2}'\mathbf{2}'',\\ [\mathbf{1}'\mathbf{3}'\mathbf{1}'']\mathbf{B}_2=\omega_{1'2'}[\mathbf{1}'\mathbf{2}'\mathbf{1}'']\mathbf{1}'\mathbf{3}'-\omega_{22'}[\mathbf{22}'\mathbf{3}']\mathbf{1}'\mathbf{1}'',\\ [\mathbf{232}'']\mathbf{B}_{1'}=\omega_{1'2'}[\mathbf{1}'\mathbf{2}'\mathbf{2}'']\mathbf{23}+\omega_{22'}[\mathbf{232}']\mathbf{22}'',\\ [\mathbf{131}'']\mathbf{B}_{2'}=\omega_{1'2'}[\mathbf{1}'\mathbf{2}'\mathbf{1}'']\mathbf{13}+\omega_{22'}[\mathbf{232}']\mathbf{11}'',\end{cases}\tag{5.4}$$

together with

$$\begin{cases}\omega_{22'}\mathbf{11}'\wedge\mathbf{22}'\wedge\mathbf{33}'=0,\\ \omega_{1'2'}\mathbf{12}\wedge\mathbf{1}'\mathbf{2}'\wedge\mathbf{1}''\mathbf{2}''=0,\\ (\mathbf{1}'\mathbf{1}''\wedge\mathbf{2}'\mathbf{2}''\wedge\mathbf{3}'\mathbf{3}'')(\omega_{1'2'}[\mathbf{1}'\mathbf{2}'\mathbf{3}']-\omega_{22'}[\mathbf{22}'\mathbf{3}'])=0,\\ (\mathbf{23}\wedge\mathbf{2}'\mathbf{3}'\wedge\mathbf{2}''\mathbf{3}'')(\omega_{1'2'}[\mathbf{1}'\mathbf{2}'\mathbf{2}'']-\omega_{22'}[\mathbf{22}'\mathbf{2}''])=0,\\ \omega_{22'}([\mathbf{11}''\mathbf{3}''][\mathbf{232}'][\mathbf{2}'\mathbf{3}'\mathbf{2}'']+[\mathbf{131}''][\mathbf{22}'\mathbf{3}'][\mathbf{2}'\mathbf{2}''\mathbf{3}''])\\ \quad=\omega_{1'2'}([\mathbf{131}''][\mathbf{1}'\mathbf{2}'\mathbf{2}''][\mathbf{2}'\mathbf{3}'\mathbf{3}'']\\ \qquad-[\mathbf{133}''][\mathbf{1}'\mathbf{2}'\mathbf{1}''][\mathbf{2}'\mathbf{3}'\mathbf{2}'']).\end{cases}\tag{5.5}$$

6. Parametric Calotte Propagation in Scene Analysis

Previous sections show that the parametric propagation result is better than the elimination result in that there is no bifurcation and there are more factored equations. Although bifurcation will inevitably occur in further manipulation of the parametric equations (5.5), the branches will be reduced in number and interpretable geometrically, due to the factored equations.

Now consider the last equation in (5.5). Is it factorable? The answer is no if only this single equation is considered, and yes if the other four equations are taken into account. In this section, we will show that the last equation can be replaced by any of the following factored equations:

$$\begin{array}{l}(\omega_{1'2'}[\mathbf{131}'][\mathbf{1}'\mathbf{2}'\mathbf{1}'']\\ \quad-\omega_{22'}[\mathbf{11}'\mathbf{1}''][\mathbf{232}'])\mathbf{13}\wedge\mathbf{1}'\mathbf{3}'\wedge\mathbf{1}''\mathbf{3}''=0,\\ (\omega_{1'2'}[\mathbf{11}'\mathbf{3}'][\mathbf{1}'\mathbf{2}'\mathbf{1}'']\\ \quad-\omega_{22'}[\mathbf{11}'\mathbf{1}''][\mathbf{22}'\mathbf{3}'])\mathbf{13}\wedge\mathbf{1}'\mathbf{3}'\wedge\mathbf{1}''\mathbf{3}''=0.\end{array}\tag{6.1}$$

On the one hand, (6.1) is much better for further algebraic manipulation. On the other hand, it is too difficult to obtain from the equations in (5.5).

In this section, we propose an improved version of the general propagation method, called *parametric calotte propagation*. It is used specifically in polyhedral scene analysis, and can produce more factored parametric equations from the fundamental equations.

6.1. Calottes and calotte equations

Definition 6.1. A triangular face is said to be *constructed* if its three vertices are constructed. A nontriangular face is said to be *constructed* if its inhomogeneous coordinates are computed. A face is said to be *k-overconstrained*, where

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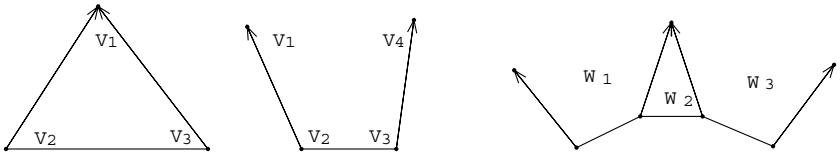


Fig. 4. Wagons and a train of wagons.

$k > 0$, if it has $k+3$ constructed vertices before it is constructed by three of them.

Definition 6.2. In the procedure of constructing a polyhedron, a *wagon* refers to either a triangular wagon or a planar wagon (see Fig. 4). A *triangular wagon* is composed of three constructed vertices V_1, V_2, V_3 connected by three edges: the edge V_2V_3 and the two directed edges V_2V_1, V_3V_1. A *planar wagon* is composed of four neighboring vertices V_1, V_2, V_3, V_4 of a constructed face connected by three edges: the edge V_2V_3 and the two directed edges V_2V_1, V_3V_4. The undirected edge is the *floor* of the wagon, and the two directed ones are the *sides*.

A *train of wagons* is a sequence of wagons $W_1, \ldots, W_l$ such that (1) two wagons have a common side if and only if they are neighbors and (2) any two planar wagons are not in the same face.

The concept "calotte" in scene analysis was first proposed by Crapo (1991). The following is a generalized version of the original definition.

Definition 6.3. An (n, m)-*calotte* in space, where $3 \le m \le n$, is a cyclic sequence of n wagons $W_1, \ldots, W_n$ such that

(1) m wagons are planar and the rest are triangular;
(2) two wagons have a common side if and only if they are neighbors in cyclic order;
(3) any two planar wagons are not in the same face.

The cyclic sequence of vertices forming the floors of the wagons is called the *center* of the calotte (see Fig. 5). The cyclic sequence formed by the ends of the sides of the wagons is the *boundary*. The faces containing planar wagons are the *circumfaces* of the calotte. A calotte is said to be *regular* if its center lies in a plane, i.e., is the boundary of a face. An (n, n)-calotte is said to be *symmetric*, called an n-*calotte*.

Let $V_1, \ldots, V_n$ and $V_{1'}, \ldots, V_{n'}$ be the center and the boundary of a symmetric calotte such that $V_{i'}V_iV_{i+1}V_{(i+1)'}$ is a planar wagon for $1 \le i \le n$. Then the calotte can be denoted by $(V_1 \ldots V_n,\ V_{1'} \ldots V_{n'})$. For a general (n, m)-calotte with center $V_1, \ldots, V_n$ and boundary $V_{1'}, \ldots, V_{m'}$, let $k_1, \ldots, k_m$ be a subsequence of $1, \ldots, n$ such that for $1 \le i \le m$, $V_{i'}V_{k_i}V_{k_i+1}V_{(i+1)'}$ is a planar wagon, with the understanding that $V_{(m+1)'} = V_{1'}$ and $V_{n+1} = V_1$. Then the calotte can be denoted by

$$(V_{k_1}V_{k_1+1}V_{k_2}V_{k_2+1} \cdots V_{k_m}V_{k_m+1},\ V_{1'} \cdots V_{m'}).$$

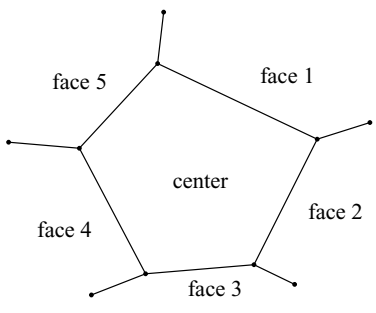
Lemma 6.1. Let $\mathbf{V}_1, \ldots, \mathbf{V}_n, \mathbf{V}, \mathbf{X}$ be n-dimensional vectors. Let $M = (\mathbf{V}_1 \cdots \mathbf{V}_n)$ be the matrix whose columns are $\mathbf{V}_1$ to $\mathbf{V}_n$. Assume that $\mathbf{V}_2, \ldots, \mathbf{V}_n$ are linearly independent. Then the equation $M\mathbf{X} = \mathbf{V}$ has a solution for $\mathbf{X}$ if and only if the following equation has a solution for x:

$$\det(M)x = \det(\mathbf{V}\ \mathbf{V}_2\ \cdots \mathbf{V}_n). \tag{6.2}$$

Proof. If $\mathbf{X} = (x_1, \ldots, x_n)$ is a solution of $M\mathbf{X} = \mathbf{V}$, then x_1 is a solution of (6.2). Conversely, if (6.2) has a solution, then if $\det(M) \neq 0$, $M\mathbf{X} = \mathbf{V}$ has the solution $\mathbf{X} = M^{-1}\mathbf{V}$, else $\det(\mathbf{V}\ \mathbf{V}_2\ \cdots \mathbf{V}_n) = 0$, or equivalently, $\mathbf{V}$ is a linear combination of $\mathbf{V}_2$ to $\mathbf{V}_n$. Let $\mathbf{V} = \lambda_2\mathbf{V}_2 + \cdots + \lambda_n\mathbf{V}_n$, then $(0, \lambda_2, \ldots, \lambda_n)^T$ is a solution of $M\mathbf{X} = \mathbf{V}$. □

Theorem 6.2. Let $(V_{k_1}V_{k_1+1}V_{k_2}V_{k_2+1} \cdots V_{k_m}V_{k_m+1}, V_{1'} \cdots V_{m'})$ be an (n, m)-calotte. Let the homogeneous coordinates of vertex V_j be $(\mathbf{j}, h_j)$, where $\mathbf{j}$ is the projection of V_j along the point or direction $\mathbf{n}$. For $1 \le i \le m$, let $\mathbf{B}_i$ be the inhomogeneous coordinates of circumface $V_{k_i}V_{k_i+1}V_{(i+1)'}V_{i'}$, and let

$$\mathbf{C}_i = \mathbf{k}_i(\mathbf{k}_i + 1), \qquad g_i = -h_{k_i}[\mathbf{i}'(i+1)'(k_i+1)] + h_{k_i+1}[\mathbf{i}'(i+1)'k_i]. \tag{6.3}$$



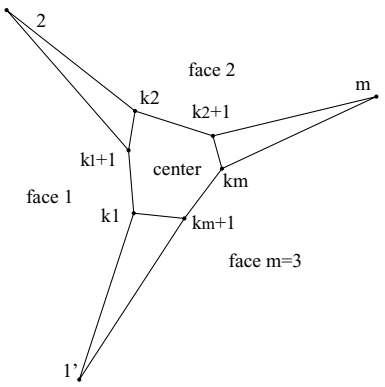


Fig. 5. A 5-calotte and a (6,3)-calotte.

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Assume that the center and $m-1$ circumfaces of the calotte have been constructed. Then the wagon in the last circumface exists, i.e., the corresponding four vertices are coplanar, if and only if the following *calotte equation* in $h_{1'}$ has a solution

$$c\,h_{1'} = \sum_{i=1}^{m} g_i \prod_{j=1}^{i-1} [\mathbf{j}'\mathbf{C}_j] \prod_{l=i+1}^{m} [(\mathbf{l}+\mathbf{1})'\mathbf{C}_l], \tag{6.4}$$

where

$$\begin{aligned} c = &-[\mathbf{1}'\mathbf{C}_1][\mathbf{2}'\mathbf{C}_2]\cdots[(\mathbf{m}-\mathbf{1})'\mathbf{C}_{m-1}][\mathbf{m}'\mathbf{C}_m] \\ &+[\mathbf{2}'\mathbf{C}_1][\mathbf{3}'\mathbf{C}_2]\cdots[\mathbf{m}'\mathbf{C}_{m-1}][\mathbf{1}'\mathbf{C}_m] \end{aligned} \tag{6.5}$$

is the *Crapo binomial* of the calotte (Crapo, 1991).

If the calotte is regular, let $\mathbf{B}_0$ be the inhomogeneous coordinates of the central face, then the calotte equation can be written as

$$\begin{aligned} &(\mathbf{B}_i - \mathbf{B}_0)c = 0, \quad \text{or equivalently,} \\ &(\mathbf{B}_{i+1} - \mathbf{B}_i)c = 0 \ \text{ for some } \ 1 \le i \le m, \end{aligned} \tag{6.6}$$

For a three-calotte, the calotte equation can be written as

$$(\mathbf{B}_2 - \mathbf{B}_1)\mathbf{11}' \wedge \mathbf{22}' \wedge \mathbf{33}' = 0. \tag{6.7}$$

Proof. The syzygy equations of the wagons in the calotte can be written as

$$\begin{pmatrix} [\mathbf{2}'\mathbf{C}_1] & -[\mathbf{1}'\mathbf{C}_1] & & & \\ & [\mathbf{3}'\mathbf{C}_2] & -[\mathbf{2}'\mathbf{C}_2] & & \\ & & \ddots & \ddots & \\ & & & [\mathbf{m}'\mathbf{C}_{m-1}] & -[(\mathbf{m}-\mathbf{1})'\mathbf{C}_{m-1}] \\ -[\mathbf{m}'\mathbf{C}_m] & & & & [\mathbf{1}'\mathbf{C}_m] \end{pmatrix} \times \begin{pmatrix} h_{1'} \\ h_{2'} \\ \vdots \\ h_{m'} \end{pmatrix} = \begin{pmatrix} g_1 \\ g_2 \\ \vdots \\ g_m \end{pmatrix}. \tag{6.8}$$

The determinant of the coefficient matrix in (6.8) is c. The determinant formed by replacing the first column of the matrix with the right-hand side of (6.8) is the right-hand side of (6.4). The subdeterminant formed by the last $m-1$ rows and $m-1$ columns is nonzero, so the hypothesis of Lemma 6.1 is satisfied. Then (6.8) has a solution for the h's if and only if (6.4) has a solution for $h_{1'}$.

If the calotte is regular, let $\mathbf{B}_0 = 0$. Then all the g's are zero. Let $\mathbf{B}_1 = \omega\mathbf{C}_1$. Then $h_{1'} = [\mathbf{1}'\mathbf{B}_1] = \omega[\mathbf{1}'\mathbf{C}_1]$, and (6.4) becomes $c\omega[\mathbf{1}'\mathbf{C}_1] = 0$. Replacing $[\mathbf{1}'\mathbf{C}_1]$ with $\mathbf{C}_1$, we get $c\omega\mathbf{C}_1 = 0$, which is just the first equation in (6.6) for $i = 1$. Let $\mathbf{B}_2 = \omega'\mathbf{C}_2$. From $[\mathbf{2}'\mathbf{B}_2] = [\mathbf{2}'\mathbf{B}_1]$, we get $\omega' = [\mathbf{2}'\mathbf{C}_1]/[\mathbf{2}'\mathbf{C}_2]$. So

$$\mathbf{B}_2 - \mathbf{B}_1 = \omega\frac{[\mathbf{2}'\mathbf{C}_1]\mathbf{C}_2 - [\mathbf{2}'\mathbf{C}_2]\mathbf{C}_1}{[\mathbf{2}'\mathbf{C}_2]},$$

which is zero if and only if $\omega = 0$. Then the two equations in (6.6) are equivalent.

If the calotte is a three-calotte, let $\mathbf{B}_1 = 0$. Let $\mathbf{B}_2 = \omega\mathbf{22}'$. Then (6.4) becomes

$$\begin{aligned} 0 &= h_3[\mathbf{121}'](-[\mathbf{131}'][\mathbf{22}'\mathbf{3}'] + [\mathbf{11}'\mathbf{3}'][\mathbf{232}']) \\ &= -\omega[\mathbf{232}'][\mathbf{121}']\mathbf{11}' \wedge \mathbf{22}' \wedge \mathbf{33}'. \end{aligned} \tag{6.9}$$

□

The calotte equation (6.4) has the benefit that if the center but no boundary vertex is constructed, the only constraint satisfied by the parameter to be introduced in the propagation from the center outward can be written without computing any circumface. Furthermore, if the calotte is regular or a three-calotte, the calotte equation is in factored form.

6.2. *Constructing calottes*

Calottes exist in many interesting cases. The following algorithm is fundamental in constructing calottes.

Algorithm: Connecting two directed edges by a train of wagons.

Input: All the constructed vertices and faces of a polyhedron; two constructed directed edges $V_1V_{1'}$, $V_2V_{2'}$; a set of constructed directed edges $calE$ containing $V_1V_{1'}$; a set of constructed faces $calF$; an integer m.

Output: $train(V_1V_{1'}, V_2V_{2'}, \mathcal{E}, \mathcal{F}, m)$, which is a train of wagons $T_1, \ldots, T_l$ using constructed vertices, edges, and faces, such that
(1) $V_1V_{1'}$ is a side of T_1 and $V_2V_{2'}$ is a side of T_l;
(2) no planar wagon belongs to any element in $\mathcal{F}$, no side belongs to $\mathcal{E} - \{V_1V_{1'}\}$;
(3) there are at least m planar wagons.

Step 1. Find all the wagons which have $V_1V_{1'}$ as a side, which do not belong to any element in $\mathcal{F}$, and the other sides of which are not in $\mathcal{E}$. Denote the set by $\mathcal{T}$.

Step 2. If there is any wagon T in $\mathcal{T}$ having $V_2V_{2'}$ as a side, then (1) if T is triangular and $m > 0$, delete T from $\mathcal{T}$;
(2) if T is planar and $m > 1$, delete T from $\mathcal{T}$;
(3) else, output $train(V_1V_{1'}, V_2V_{2'}, \mathcal{E}, \mathcal{F}, m) = T$ and exit.

Step 3. For every element T of $\mathcal{T}$, let s be the side of T other than $V_1V_{1'}$.
(1) If T is triangular, compute $t = train(s, V_2V_{2'}, \mathcal{E} \cup \{s\}, \mathcal{F}, m)$.
(2) If T is planar, let F_T be the face containing T, compute $t = train(s, V_2V_{2'}, \mathcal{E} \cup \{s\}, \mathcal{F} \cup \{F_T\}, m-1)$. (3) If t is not empty, output
$train(V_1V_{1'}, V_2V_{2'}, \mathcal{E}, \mathcal{F}, m) = T, t$ and exit.

Step 4. Output $train(V_1V_{1'}, V_2V_{2'}, \mathcal{E}, \mathcal{F}, m) =$ null.

The following algorithm is used to construct general calottes. With slight modification, it can be used to construct special calottes only, e.g., regular calottes and three-calottes.

Algorithm: Finding all calottes having F as a common circumface, but with different four-tuples of vertices in F.

Input: All the constructed vertices and faces of a polyhedron; a constructed face F which is overconstrained.

Output: A set cal of calottes each containing F as a circumface.

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Initialization: Set *cal* to be the empty set.
Step 1. Find all planar wagons in F. Denote the set by *calT*.
Step 2. For each element $T = V_1V_2V_3V_4$ in $\mathcal{T}$, do the following.
(1) Compute $t = train(V_3V_4, V_2V_1, \{V_3V_4\}, \{F\}, 2)$. If t is not empty then put it into *cal*.
(2) If F has only four vertices, then output *cal* and exit.

6.3. *Improving the syzygy equations with factored calotte equations*

Any three neighboring vertices of a face (or wagon) are called a *frame* of the face (or wagon). The *syzygy equation* of vertex V_i with respect to a frame V_1, V_2, V_3 refers to

$$h_i[\mathbf{123}] - h_1[\mathbf{23i}] + h_2[\mathbf{13i}] - h_3[\mathbf{12i}] = 0. \qquad (6.10)$$

If a face has $k+3$ constructed vertices before it is constructed by a frame composed of three of them, the expression of the inhomogeneous coordinates of the face will be substituted into the fundamental equations of the other k vertices, and the result is the k syzygy equations of the k vertices with respect to the frame. If there is a calotte containing the face as a circumface, then if the calotte is regular or a three-calotte, the calotte equation will be in factored form, which is very beneficial to further manipulation of the parametric equations. On the other hand, the syzygy equation (6.10) is usually not factorable. Thus, some syzygy equations should be replaced by equivalent but factored calotte equations.

Definition 6.4. Let there be an oriented face with some wagons and a frame. Assume that the vertices in any wagon and the frame follow the orientation of the face. For any two different wagons (or a wagon and the frame) W_1, W_2, define the *gap* from W_1 to W_2 by

$$gap(W_1, W_2) = \begin{cases} -3, & \text{if the last three vertices of } W_1 \text{ are the first three vertices of } W_2; \\ -2, & \text{if the last two vertices of } W_1 \text{ are the first two vertices of } W_2; \\ -1, & \text{if the last vertex of } W_1 \text{ is the first vertex of } W_2; \\ 0, & \text{if the last vertex of } W_1 \text{ is not in } W_2. \end{cases}$$

A *link* (or *loop*) of wagons is a sequence (or loop) of wagons, or wagons and the frame, in which every two neighbors have gap -3. It is assumed that the elements in any link follow the orientation of the face. For two links L_1, L_2, the *gap* from L_1 to L_2 refers to the gap from the last element of L_1 to the first element of L_2. The gap between a link and an empty link is defined to be zero.

Algorithm: Improving the syzygy equations with factored calotte equations.
Input: All the constructed vertices and faces of a polyhedron; the last constructed face F which is k-overconstrained and which is constructed by frame M. An orientation of F is fixed.
Output: *cond*, which is a set of k calotte equations and syzygy equations with respect to M and which guarantees the existence of F.

In the following procedure, any syzygy equation is with respect to M.
Initialization: Set *cond* to be the empty set.
Step 1. Finding calottes: Find l regular calottes and three-calottes having F as a circumface, but with different four-tuples of vertices in F. Here, $0 \le l \le k+3$. Denote the set of wagons of the calottes in face F by $\mathcal{W}$.
If $l = 0$, then output the k syzygy equations and exit.
Step 2. Building links: Group the elements in $\mathcal{W} \cup \{M\}$ into links of wagons. Let the links be $L_0, \ldots, L_m$ following the orientation of the face. Here $0 \le m \le l$, and L_0 contains M.
Let n be the number of vertices in $\mathcal{W} \cup M$. If either $m = 0$ or there is a link of wagons having at least $n-3$ wagons, then (1) put the calotte equations corresponding to the first $n-3$ wagons into *cond*, (2) put the syzygy equations of the vertices not in $\mathcal{W} \cup \{M\}$ into *cond*, (3) output *cond* and exit.
Step 3. Deleting extra wagons: For i from 0 to m, let the gap from L_i to L_{i+1} be g_i, where $L_{m+1} = L_0$.
(1) If $g_i = 0$, then set $I_+ = i$, $I_- = i+1$, go to Step 4.
(2) If $g_i = -1$ is maximal among $g_0, \ldots, g_m$, then
(a) set $I_+ = i$, $I_- = i+1$.
(b) If $M \in L_i$, delete the first element in L_{i+1}, else delete the last element in L_i.
(c) Go to Step 4.
If $g_0 = \cdots = g_m = -2$, then
(1) for j from 1 to m, if L_j contains only one element, then delete the element, set $I_+ = j-1$, $I_- = j+1$, go to Step 4.
(2) Delete the last two elements in L_m. Set $I_+ = m$, $I_- = m+1$.
Step 4. Connecting links: For i from 0 up to $I_+ - 1$, let W be the first element in L_{i+1}.
(1) If $gap(L_i, L_{i+1}) = -2$, put the syzygy equation of the third vertex of W into *cond*.
(2) If $gap(L_i, L_{i+1}) = -1$, put the syzygy equations of the second and third vertices of W into *cond*.
(3) If $gap(L_i, L_{i+1}) = 0$, put the syzygy equations of the first three vertices of W into *cond*.
Step 5. Connecting links reversely: For i from $m+1$ down to $I_- + 1$, let W be the last element in L_{i-1}.
(1) If $gap(L_{i-1}, L_i) = -2$, put the syzygy equation of the second vertex of W into *cond*.
(2) If $gap(L_{i-1}, L_i) = -1$, put the syzygy equations of the second and thirrd vertices of W into *cond*.
(3) If $gap(L_{i-1}, L_i) = 0$, put the syzygy equations of the last three vertices of W into *cond*.
Step 6. Collecting pedestrians and wagons: (1) Find all the constructed vertices of F which do not belong to $\mathcal{W} \cup \{M\}$. Put their syzygy equations into *cond*. (2) For every wagon in $L_0 \cup \ldots \cup L_m$, put its corresponding calotte equation into *cond*.

Proof of the correctness. (1) If there is only one link (or loop), it must have a link of at least $n-3$ wagons. Any link of $n-3$ wagons covers all the n vertices, while any link of less than $n-3$ wagons does not. The calotte equations of the $n-3$ wagons guarantee that the n vertices are in

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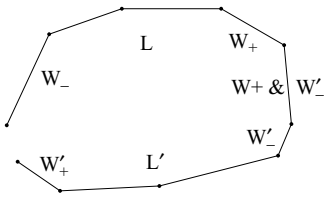


Fig. 6. The case $gap(L, L') = 0$ but $W_+ \cap W'_-$ is not empty.

F. The syzygy equations of the other vertices guarantee that the other vertices are in F. This justifies Step 2.

(2) Suppose that no link of wagons covers all the n vertices. There must be at least two links. Let L be the link containing M. Then all the vertices in L are in the plane determined by M. Any other link must be pasted to the plane by fixing either its first or its last three vertices to the plane.

First consider the case that there are only two links L and L'. Let the first and the last elements in L, L' be, respectively, W_+, W_- and W'_+, W'_-.

Case 2.1. If $gap(L, L') = 0$, let there be i vertices in $W_+ \cap W'_-$ (see Fig. 6). The last $3 - i$ vertices in $W'_- - W_+$ can be pasted to F by assuming that their syzygy equations hold. Then all the vertices in L' are in F.

Case 2.2. If $gap(L, L') = gap(L', L) = -2$, then if $W'_- = W'_+$, the number of wagons in L must be $n - 3$. After removing L', we are back to the case in Step 2. If $W'_- \neq W'_+$, then every vertex in $(W'_- \cap W_+) \cup (W'_+ \cap W_-)$ is covered by three wagons (or two wagons and the frame), and at least two wagons must be removed. By deleting the first two wagons in L', we have $gap(L, L') = 0$, and we are back to Case 2.1.

Case 2.3. If $gap(L, L') = -1 \geq gap(L', L)$, then every vertex in $(W'_- \cap W_+) \cup (W'_+ \cap W_-)$ is covered by two or three wagons, one of which may be the frame. At least one wagon must be removed. By deleting W'_+, we have $gap(L, L') = 0$, and we are back to Case 2.1.

The treatment of any other case is similar, and we have justified Step 3 under the assumption that there are only two links.

(3) Assume that there are multiple links.

Case 3.1. If there are neighboring links L_i, L_{i+1} satisfying $gap(L_i, L_{i+1}) = 0$, we split the sequence of links into two groups, the first being from L_0 to L_i, the second being from L_{i+1} to L_0. For the first group, we fix to the plane the first three vertices of every link other than L_0. If such a vertex belongs to a previous link, then it is already fixed, else it can be fixed by its syzygy equation. For the second group, the fixing procedure goes from L_0 to L_{i+1} in the reverse orientation. This explains Step 4 and Step 5.

Case 3.2. If no two neighboring links have gap zero, let there be two links L_i, L_{i+1} having gap -1. By removing a wagon as in Case 2.3, the gap between the two links becomes zero, and we are back to Case 3.1.

Case 3.2. If every two neighboring links have gap -2, then if there is a link having only one element and the element is a wagon, then its two neighboring links have gap zero, so removing the single link changes the situation to Case 3.1, else we need to remove the last two wagons from L_m to be back to Case 3.1.

Now we have finished the justification of Step 3 by proving that the number of deleted wagons is minimal.

(4) A calotte equation fixes the last (or first) vertex to the plane determined by the other three vertices in the wagon, depending on whether the link containing the wagon is fixed to F by starting from its first vertex or reversely. All the other vertices are fixed to F by their syzygy equations. Thus, there are exactly k equations in *cond* and they guarantee the existence of F.

6.4. Parametric calotte propagation

When using parametric propagation in scene analysis, the combinatorial nature should be taken into consideration. This lies in two aspects: first, the direction of propagation should be determined by the incidence relations instead of the algebraic equations; second, the parametric equations should be improved by factored calotte equations.

Definition 6.5. The *original face of construction* is set to have *construction level* zero. All vertices on the origin face of construction are set to have *construction level* zero.

For subsequent constructions, the *construction level* of a vertex is the minimum of all the construction levels of the constructed faces incident to it. The *construction level* of a face equals one plus the maximum of all the construction levels of the constructed vertices used in constructing the face. The *construction difficulty* of a face is the sum of the construction levels of the constructed vertices used in constructing the face.

For example, In Fig. 3 of Example 1, assume that face $F_{3''}$ is the original face of construction, and then, faces F_3, $F_{3'}$, $F_{1''}$, $F_{2''}$ are constructed sequentially. Vertices V_1, V_2, $V_{1'}$, $V_{2'}$ each have construction level 0, because they are on the original face of construction. Face F_3 is constructed from $V_{1'}$, $V_{2'}$; it has construction difficulty $0 + 0 = 0$ and construction level $1 + 0 = 1$. Then vertices $V_{1''}$, $V_{2''}$ each have construction level 1, because they are both on face F_3, and are constructed with the face at the same time.

After F_3 is constructed, face $F_{3'}$ has construction level $1 + 1 = 2$, because two level-0 vertices V_1, V_2 and a level-1 vertex $V_{1''}$ or $V_{2''}$ are used to construct it. The construction difficulty of face $F_{3'}$ is $0 + 0 + 1 = 1$.

Similarly, $F_{1''}$ is constructed from $V_2, V_{2'}$, it has construction difficulty 0 and construction level 1. Vertices V_3, $V_{3'}$ on face $F_{1''}$ each have construction level 1. After $F_{1''}$ is constructed, face $F_{2''}$ has construction level 2 and construction difficulty 1.

Algorithm: Parametric calotte propagation

Input: The incidence structure of a connected polyhedron, the 2D coordinates of the vertices, the origin of propagation.

Output: A set $para$ of parameters, the equations $soln$ satisfied by the parameters, a set $expr$ of explicit expressions of the inhomogeneous coordinates of the faces in the set $\mathcal{B}$ of all the nontriangular faces not in the origin.

Initialization: Set $para, soln, expr$ to be empty.

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Procedure. For i from 3 to 1, do the following.
Find a face F in $\mathcal{B}$ which has at least i constructed vertices. If there are more than one such faces then choose one with minimal construction difficulty. If such a face is found then
Step 1. If F is k-overconstrained, then
(1) construct F by three neighboring vertices whose sum of construction levels is minimal. Put the expression of the coordinates $\mathbf{B}$ of F into $expr$;
(2) find k syzygy equations and factored calotte equations. Put them into $soln$.
Step 2. If F is not overconstrained, then
(1) construct F by introducing at most $3-i$ parameters $\omega(F)$;
(2) put $\omega(F)$ into $para$, put the expression of $\mathbf{B}$ into $expr$.
Step 3. Remove F from $\mathcal{B}$. If $\mathcal{B}$ is empty, then exit, else go to the beginning of the Procedure.

Below we use parametric calotte propagation to solve Example 1. The original face of construction, called the *origin* henthforth, is $F_{3''}$. The following are different rounds of the Procedure.

Round 1. The origin has four neighboring faces each sharing one edge with it. They all have zero construction difficulty. The first propagation is toward face F_3.
Round 2. Only face $F_{3'}$ has more than two constructed vertices. It is constructed in this round. The factored calotte equation of $(V_1V_{1'}V_{1''}, V_2V_{2'}V_{2''})$ occurs.
Rounds 3 and 4. The six remaining faces each have two constructed vertices. Faces $F_{1''}, F_{2''}$ each have zero construction difficulty. They are constructed in the third and fourth rounds. The factored calotte equation of $(V_1V_2V_3, V_{1'}V_{2'}V_{3'})$ occurs.
Rounds 5–8. Faces $F_1, F_2, F_{1'}, F_{2'}$ have the same construction difficulty $0+1+1=2$, because they each contain a level-0 vertex and two level-1 vertices. When the propagation goes in this order, the factored calotte equations of the following three calottes occur:

1. $(V_{1'}V_{2'}V_{3'},\ V_{1''}V_{2''}V_{3''})$;
2. $(V_2V_{2'}V_{2''},\ V_3V_{3'}V_{3''})$;
3. $(V_1V_{1'}V_{1''},\ V_3V_{3'}V_{3''})$ or $(V_1V_2V_3,\ V_{1''}V_{2''}V_{3''})$.

The propagation sequence is

$$\mathbf{B}_{3''}=0 \xrightarrow{\omega_{1'2'}} \mathbf{B}_3 \xrightarrow{B.2} \mathbf{B}_{3'} \xrightarrow{\omega_{22'}} \mathbf{B}_{1''} \xrightarrow{B.2} \mathbf{B}_{2''} \xrightarrow{B.1} \mathbf{B}_1 \xrightarrow{B.2} \mathbf{B}_2,\ \mathbf{B}_{1'},\ \mathbf{B}_{2'}.$$

The parametric equations are all of the form (6.7)

$$\begin{cases} \omega_{1'2'}\mathbf{12}\wedge\mathbf{1'2'}\wedge\mathbf{1''2''}=0,\\ \omega_{22'}\mathbf{11'}\wedge\mathbf{22'}\wedge\mathbf{33'}=0,\\ (\mathbf{B}_1-\mathbf{B}_3)\mathbf{1'1''}\wedge\mathbf{2'2''}\wedge\mathbf{3'3''}=0,\\ (\mathbf{B}_1-\mathbf{B}_{1''})\mathbf{23}\wedge\mathbf{2'3'}\wedge\mathbf{2''3''}=0,\\ (\mathbf{B}_2-\mathbf{B}_{2''})\mathbf{13}\wedge\mathbf{1'3'}\wedge\mathbf{1''3''}=0. \end{cases} \tag{6.11}$$

Substituting (5.4) into (6.11), and using (2.6), we can change (6.11) into

$$\begin{cases} \omega_{1'2'}\mathbf{12}\wedge\mathbf{1'2'}\wedge\mathbf{1''2''}=0,\\ \omega_{22'}\mathbf{11'}\wedge\mathbf{22'}\wedge\mathbf{33'}=0,\\ (\omega_{1'2'}[\mathbf{1'2'3'}]-\omega_{22'}[\mathbf{22'3'}])\mathbf{1'1''}\wedge\mathbf{2'2''}\wedge\mathbf{3'3''}=0,\\ (\omega_{1'2'}[\mathbf{1'2'2''}]-\omega_{22'}[\mathbf{22'2''}])\mathbf{23}\wedge\mathbf{2'3'}\wedge\mathbf{2''3''}=0,\\ (\omega_{1'2'}[\mathbf{131'}][\mathbf{1'2'1''}]\mathbf{1'3'}-\omega_{22'}[\mathbf{131'}][\mathbf{22'3'}]\mathbf{1'1''}\\ \quad -\omega_{22'}[\mathbf{1'3'1''}][\mathbf{232'}]\mathbf{11'})\mathbf{13}\wedge\mathbf{1'3'}\wedge\mathbf{1''3''}=0. \end{cases} \tag{6.12}$$

Remark. A propagation sequence also induces a bileveled lexicographic order among the $\mathbf{B}$'s, in which the primary level is the construction level, the secondary level is the construction difficulty. The reverse of this order is a nice order for elimination in that bifurcation occurs only in the end. The order (4.9) is such an example.

6.5. *Further analysis of the propagation result*

In scene analysis, the result of parametric propagation is a set of explicit expressions of the bivectors and a system of linear homogeneous equations satisfied by the new parameters. After eliminating these parameters, we obtain a variety of systems of bracket polynomials formed by the 2D coordinates of vertices. *Further analysis* of the propagation result includes (1) solving the system of new parameters, (2) solving the systems of bracket polynomials, and (3) explaining the results geometrically. In general, the last step needs Cayley factorization.

As an example, let us analyze the result (6.12) of Example 1. The last equality has a scalar factor and a bivector factor, denoted by A, which is a multiple of $\mathbf{1'3'}$. A can be replaced by $[A\mathbf{V}]$ for any vector $\mathbf{V}$ as long as $[A\mathbf{V}]\neq 0$. Here $\mathbf{V}$ can be any of $\mathbf{1}, \mathbf{3}, \mathbf{1''}, \mathbf{3''}$. Vectors $\mathbf{1}, \mathbf{1''}$ each change A into a bracket binomial, which are given in (6.1); vectors $\mathbf{3}, \mathbf{3''}$ each change A into a three-termed bracket polynomial. If necessary, A can be replaced by either of its two shortest scalar forms.

Equations (6.12) represent a set of linear equations in factored form. Below we discuss all the branches of the solutions of $\omega_{1'2'}, \omega_{22'}$.
Case 1. If $\omega_{1'2'}=\omega_{22'}=0$, then (6.12) is satisfied. Since all the $\mathbf{B}$'s are zero, the corresponding 3D realization is trivial in that all faces are coplanar.
Case 2. If $\omega_{1'2'}=0\neq\omega_{22'}$, then $\mathbf{B}_3=\mathbf{B}_{3'}=\mathbf{B}_{3''}$, i.e., faces $F_3, F_{3'}, F_{3''}$ are coplanar. The set (6.12) becomes

$$\begin{cases} \mathbf{11'}\wedge\mathbf{22'}\wedge\mathbf{33'}=0,\\ \mathbf{1'1''}\wedge\mathbf{2'2''}\wedge\mathbf{3'3''}=0,\\ [\mathbf{22'2''}]\mathbf{23}\wedge\mathbf{2'3'}\wedge\mathbf{2''3''}=0,\\ ([\mathbf{131'}][\mathbf{22'3'}]\mathbf{1'1''}+[\mathbf{1'3'1''}][\mathbf{232'}]\mathbf{11'})\\ \quad \mathbf{13}\wedge\mathbf{1'3'}\wedge\mathbf{1''3''}=0. \end{cases} \tag{6.13}$$

The bivector factor A can be replaced by $[\mathbf{11'1''}]$, if either $\mathbf{1}$ or $\mathbf{1''}$ is used to change it to a scalar factor.

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Lemma 6.3. (1) If

$$\begin{cases} \mathbf{12} \wedge \mathbf{1'2'} \wedge \mathbf{1''2''} = 0, \\ \mathbf{23} \wedge \mathbf{2'3'} \wedge \mathbf{2''3''} = 0, \\ \mathbf{11'} \wedge \mathbf{22'} \wedge \mathbf{33'} = 0, \\ \mathbf{1'1''} \wedge \mathbf{2'2''} \wedge \mathbf{3'3''} = 0, \end{cases} \tag{6.14}$$

then

$$\mathbf{13} \wedge \mathbf{1'3'} \wedge \mathbf{1''3''} = 0, \ \mathbf{11''} \wedge \mathbf{22''} \wedge \mathbf{33''} = 0. \tag{6.15}$$

(2) If

$$\begin{cases} \mathbf{12} \wedge \mathbf{1'2'} \wedge \mathbf{1''2''} = 0, \\ \mathbf{13} \wedge \mathbf{1'3'} \wedge \mathbf{1''3''} = 0, \\ \mathbf{23} \wedge \mathbf{2'3'} \wedge \mathbf{2''3''} = 0, \\ \mathbf{11''} \wedge \mathbf{22''} \wedge \mathbf{33''} = 0, \end{cases} \tag{6.16}$$

then under the condition $[\mathbf{1'2'3'}] \neq 0$, we have

$$\mathbf{11'} \wedge \mathbf{22'} \wedge \mathbf{33'} = 0, \ \mathbf{1'1''} \wedge \mathbf{2'2''} \wedge \mathbf{3'3''} = 0. \tag{6.17}$$

Proof. We only prove the first part of (6.15). The other conclusions can be proved similarly. The first and the third equalities in (6.14) can be written as

$$[\mathbf{12}(\mathbf{1'2'} \wedge \mathbf{1''2''})] = 0, \ [\mathbf{11'}(\mathbf{22'} \wedge \mathbf{33'})] = 0.$$

This system is of type **V.1**. It has the solution

$$\mathbf{1} = \mathbf{2}(\mathbf{1'2'} \wedge \mathbf{1''2''}) \wedge \mathbf{1'}(\mathbf{22'} \wedge \mathbf{33'}). \tag{6.18}$$

We need to prove that the right-hand side of (6.18) is nonzero. First, $\mathbf{1'2'} \wedge \mathbf{1''2''} = [\mathbf{1'2'2''}]\mathbf{1''} - [\mathbf{1'2'1''}]\mathbf{2''} \neq 0$, and since $\mathbf{2}, \mathbf{1'}, \mathbf{2'}$ are not collinear, $\mathbf{2}(\mathbf{1'2'} \wedge \mathbf{1''2''}) \neq 0$. Similarly, $\mathbf{1'}(\mathbf{22'} \wedge \mathbf{33'}) \neq 0$. If the right-hand side of (6.18) equals zero, then $\mathbf{2}, \mathbf{1'}, \mathbf{22'} \wedge \mathbf{33'}$ are collinear, i.e., lines $\mathbf{21'}, \mathbf{22'}, \mathbf{33'}$ concur at point $\mathbf{2}$. Since this is impossible because $\mathbf{2}, \mathbf{3}, \mathbf{3'}$ are not collinear, (6.18) must be nonzero.

In the same way, we obtain the following nonzero solution of $\mathbf{3''}$ from the second and the fourth equalities in (6.14):

$$\mathbf{3''} = \mathbf{2''}(\mathbf{23} \wedge \mathbf{2'3'}) \wedge \mathbf{3'}(\mathbf{1'1''} \wedge \mathbf{2'2''}). \tag{6.19}$$

Substituting (6.18) and (6.19) into the expansion $[\mathbf{133'}][\mathbf{1'1''3''}] - [\mathbf{131'}][\mathbf{3'1''3''}]$ of $\mathbf{13} \wedge \mathbf{1'3'} \wedge \mathbf{1''3''}$, we get zero by expanding the four brackets. □

Case 2.1. If both $\mathbf{13} \wedge \mathbf{1'3'} \wedge \mathbf{1''3''} = 0$ and $\mathbf{23} \wedge \mathbf{2'3'} \wedge \mathbf{2''3''} = 0$, then by Lemma 6.3, the six tuples

$$\begin{matrix} (\mathbf{12}, \mathbf{1'2'}, \mathbf{1''2''}), & (\mathbf{13}, \mathbf{1'3'}, \mathbf{1''3''}), & (\mathbf{23}, \mathbf{2'3'}, \mathbf{2''3''}), \\ (\mathbf{11'}, \mathbf{22'}, \mathbf{33'}), & (\mathbf{11''}, \mathbf{22''}, \mathbf{33''}), & (\mathbf{1'1''}, \mathbf{2'2''}, \mathbf{3'3''}) \end{matrix} \tag{6.20}$$

are all concurrent lines. The corresponding configuration is called the *triple Desargues configuration* (Fig. 7). By Desargues Theorem, the six intersections $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{A'}, \mathbf{B'}, \mathbf{C'}$ of the six tuples lie on two lines, with each line passing through three intersections. The 2D configuration can be lifted to seven different planes in space: two triangular tubes protruding from the same spatial plane onto which $\mathbf{1}, \mathbf{1'}, \mathbf{1''}, \mathbf{2}, \mathbf{2'}, \mathbf{2''}$ are lifted.

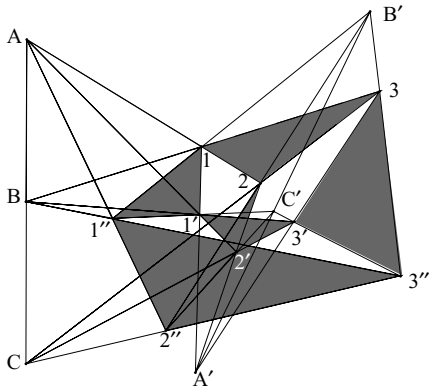


Fig. 7. Triple Desargues configuration.

Case 2.2. If $[\mathbf{11'1''}] = [\mathbf{22'2''}] = 0$, then $\mathbf{3}, \mathbf{3'}, \mathbf{3''}$ must be collinear. Lines $\mathbf{11''}, \mathbf{22''}, \mathbf{33''}$ concur. It can be easily deduced that

$$\mathbf{B}_1 = \mathbf{B}_{1'} = \mathbf{B}_{1''}, \quad \mathbf{B}_2 = \mathbf{B}_{2'} = \mathbf{B}_{2''}. \tag{6.21}$$

Furthermore,

$$\mathbf{B}_{1''} - \mathbf{B}_{2''} = \omega_{22'} \left(\mathbf{22'} - \frac{[\mathbf{232'}]}{[\mathbf{131'}]} \mathbf{11'} \right). \tag{6.22}$$

If (6.22) equals zero, i.e., $[\mathbf{232'}]\mathbf{11'} = [\mathbf{131'}]\mathbf{22'}$, then make the outer product of both sides with $\mathbf{1}$, we get $[\mathbf{131'}][\mathbf{122'}] = 0$, which is against the Practical Assumption. Similarly it can be proved that $\mathbf{B}_{1''} \neq \mathbf{B}_{3''}$ and $\mathbf{B}_{2''} \neq \mathbf{B}_{3''}$. The 2D configuration can be lifted to three different planes in space, each plane containing a flat image of a triangular tube.

Case 2.3. If $[\mathbf{11'1''}] = 0 \neq [\mathbf{22'2''}]$, then $\mathbf{3}, \mathbf{3'}, \mathbf{3''}$ are not collinear. For two nondegenerate triangles $\mathbf{22'2''}, \mathbf{33'3''}$, since lines $\mathbf{23}, \mathbf{2'3'}, \mathbf{2''3''}$ concur, by Desargues Theorem, the three distinct intersections $\mathbf{22'} \cap \mathbf{33'}, \mathbf{22''} \cap \mathbf{33''}, \mathbf{2'2''} \cap \mathbf{3'3''}$ are collinear, i.e., $\mathbf{11''} \wedge \mathbf{22''} \wedge \mathbf{33''} = 0$. By (5.4), $\mathbf{B}_2 = \mathbf{B}_{2'} = \mathbf{B}_{2''}$, i.e., faces $F_2, F_{2'}, F_{2''}$ are coplanar. The 2D configuration can be lifted to five different planes in space, with two flat tubes and one stereo tube.

Case 2.4. If $[\mathbf{11'1''}] \neq 0 = [\mathbf{22'2''}]$, the case is similar to Case 2.3.

Case 3. If $\omega_{22'} = 0 \neq \omega_{1'2'}$, then $\mathbf{B}_{1''} = \mathbf{B}_{2''} = \mathbf{B}_{3''}$, and (6.12) becomes

$$\begin{cases} \mathbf{12} \wedge \mathbf{1'2'} \wedge \mathbf{1''2''} = 0, \\ [\mathbf{1'2'3'}](\mathbf{1'1''} \wedge \mathbf{2'2''} \wedge \mathbf{3'3''}) = 0, \\ \mathbf{23} \wedge \mathbf{2'3'} \wedge \mathbf{2''3''} = 0, \\ \mathbf{13} \wedge \mathbf{1'3'} \wedge \mathbf{1''3''} = 0. \end{cases} \tag{6.23}$$

Case 3.1. If $\mathbf{11''} \wedge \mathbf{22''} \wedge \mathbf{33''} = 0$ but $[\mathbf{123}] \neq 0$, by Lemma 2, the corresponding configuration is the triple Desargues configuration, and the 3D lift lies in seven planes.

Case 3.2. If $\mathbf{11''} \wedge \mathbf{22''} \wedge \mathbf{33''} = 0$, and $[\mathbf{123}] = 0$, then $\mathbf{B}_{1'} = \mathbf{B}_{2'} = \mathbf{B}_{3'}$. If furthermore $[\mathbf{1'2'3'}] = 0$, then

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$\mathbf{B}_1 = \mathbf{B}_2 = \mathbf{B}_3$. The 2D configurations are lifted to five and three spatial planes, respectively.

Case 3.3. If $[\mathbf{1'2'3'}] = 0$, then $\mathbf{B}_1 = \mathbf{B}_2 = \mathbf{B}_3$. If furthermore $[\mathbf{123}] = 0$, then $\mathbf{B}_{1'} = \mathbf{B}_{2'} = \mathbf{B}_{3'}$. The 2D configurations are lifted to five and three spatial planes, respectively.

Case 4. Suppose that $\omega_{1'2'}$, $\omega_{22'}$ are both nonzero. Among the following three equalities,

$$\omega_{1'2'}[\mathbf{131'}][\mathbf{1'2'1''}]\mathbf{1'3'} - \omega_{22'}([\mathbf{131'}][\mathbf{22'3'}]\mathbf{1'1''} + [\mathbf{1'3'1''}][\mathbf{232'}]\mathbf{11'}) = 0, \tag{6.24}$$

$$\omega_{1'2'}[\mathbf{1'2'2''}] - \omega_{22'}[\mathbf{22'2''}] = 0, \tag{6.25}$$

$$\omega_{1'2'}[\mathbf{1'2'3'}] - \omega_{22'}[\mathbf{22'3'}] = 0, \tag{6.26}$$

if (6.25) and (6.26) hold, then

$$[\mathbf{1'2'3'}][\mathbf{22'2''}] - [\mathbf{1'2'2''}][\mathbf{22'3'}] = [\mathbf{21'2'}][\mathbf{2'3'2''}] = 0,$$

which violates the Practical Assumption. If (6.24) and (6.26) hold, then by substituting (6.26) into the second of the following two shortest scalar forms of (6.24),

$$\omega_{1'2'}[\mathbf{131'}][\mathbf{1'2'1''}] - \omega_{22'}[\mathbf{11'1''}][\mathbf{232'}] = 0,$$
$$\omega_{1'2'}[\mathbf{11'3'}][\mathbf{1'2'1''}] - \omega_{22'}[\mathbf{11'1''}][\mathbf{22'3'}] = 0,$$

we get

$$[\mathbf{11'3'}][\mathbf{1'2'1''}] - [\mathbf{11'1''}][\mathbf{1'2'3'}] = [\mathbf{11'2'}][\mathbf{1'3'1''}] = 0,$$

which violates the Practical Assumption. Similarly, (6.24) and (6.25) cannot hold at the same time. Thus, at most one of (6.24), (6.25), (6.26) holds.

Case 4.1. If (6.26) is satisfied but $[\mathbf{1'2'3'}] \neq 0$, or if one of (6.24), (6.25) is satisfied, then by Lemma 1, the only possible configuration is the triple Desargues configuration. The 2D configuration can be lifted to nine different spatial planes. This is the only nondegenerate lifting, i.e., the result is a 2D manifold.

Case 4.2. If the configuration is not triple Desargues, then (6.26) is satisfied and $[\mathbf{1''2''3''}] = 0$, so $\mathbf{B}_{1'} = \mathbf{B}_{2'} = \mathbf{B}_{3'}$ and $\mathbf{B}_1 = \mathbf{B}_2 = \mathbf{B}_3$. The lift lies in five spatial planes.

From the above analysis, we get the rank classification mentioned in Section 1.

6.6. Linear construction sequence

According to Sugihara (1999), a *resolvable sequence* of a polyhedron is defined to be a permutation of all its vertices and faces such that (1) when a vertex occurs, it is incident to at most three previous faces; (2) when a face occurs, it is incident to at most three previous vertices; (3) when two faces share three or more vertices, they appear earlier than the third of the common vertices; (4) when two vertices are incident to three or more faces, they appear earlier than the third of the common faces.

Sugihara proved that all spherical polyhedra have resolvable sequences. Ros and Thomas (2002) used such sequences to parameterize the 2D coordinates of the vertices in a correct line drawing, which in turn enables them to choose a subset of numerical data that is nonredundant and that is still enough to specify the polyhedron unambiguously.

In the fundamental equations, when all the 3D coordinates of the vertices and faces are taken as variables, then under any prescribed order among the variables, a triangulation procedure will produce a characteristic set (Wu, 1984). Under some suitable order, the characteristic set may be *irreducible*, and even *linearly irreducible*, i.e., its components are irreducible polynomials and are linear with respect to their leading variables. A linear irreducible characteristic set, when its polynomials are arranged ascendingly according to the order of the leading variables, can be explained geometrically. The geometric interpretation of a linear irreducible characteristic set is called a *linear construction sequence*.

A resolvable sequence is nothing but a linear construction sequence in which the constructions are among the given incidence relations. In a general linear construction sequence, the constructions are not restricted to the given incidence relations, but also include some derived incidence relations. All the properties of resolvable sequences needed in applications are given in the form of linear construction sequences.

Theorem 6.4. In Example 1, under the assumption that (1) no three neighboring vertices in a face are collinear and (2) no two faces sharing a common edge are coplanar, the torus has linear construction sequences.

Proof. By projecting the torus to a plane in such a way that no face is projected into a line, the Practical Assumption is fulfilled. Since no two faces sharing a common edge are coplanar, by the classification of the rank of the syzygy equations, the system allows only the triple Desargues configuration. The following is a linear construction sequence for this configuration: let $\mathbf{2}, \mathbf{3}, \mathbf{1'}, \mathbf{2'}, \mathbf{3'}, \mathbf{1''}, \mathbf{2''}$ be free points in the plane, let $\mathbf{B}_{3''}$ be a free bivector in the plane and let $\omega_{1'2'}$, $\omega_{22'}$ be free parameters, then

$$\begin{cases}
\mathbf{1} = \mathbf{2}(\mathbf{1'2'} \wedge \mathbf{1''2''}) \wedge \mathbf{1'}(\mathbf{22'} \wedge \mathbf{33'}), \\
\mathbf{3''} = (\mathbf{23} \wedge \mathbf{2'3'})\mathbf{2''} \wedge (\mathbf{1'1''} \wedge \mathbf{2'2''})\mathbf{3'}, \\
\mathbf{B}_3 = \mathbf{B}_{3''} + \omega_{1'2'}\mathbf{1'2'}, \\
\mathbf{B}_{3'} = \mathbf{B}_{3''} + \omega_{1'2'}([\mathbf{1'2'1''}]/[\mathbf{121''}])\mathbf{12}, \\
\mathbf{B}_{1''} = \mathbf{B}_{3''} + \omega_{22'}\mathbf{22'}, \\
\mathbf{B}_{2''} = \mathbf{B}_{3''} + \omega_{22'}([\mathbf{232'}]/[\mathbf{131'}])\mathbf{11'}, \\
\mathbf{B}_1 = \mathbf{B}_{3''} + \omega_{1'2'}([\mathbf{1'2'2''}]/[\mathbf{2'3'2''}])\mathbf{2'3'} \\
\qquad - \omega_{22'}([\mathbf{22'3'}]/[\mathbf{2'3'2''}])\mathbf{2'2''}, \\
\mathbf{B}_2 = \mathbf{B}_{3''} + \omega_{1'2'}([\mathbf{1'2'1''}]/[\mathbf{1'3'1''}])\mathbf{1'3'} \\
\qquad - \omega_{22'}([\mathbf{22'3'}]/[\mathbf{1'3'1''}])\mathbf{1'1''}, \\
\mathbf{B}_{1'} = \mathbf{B}_{3''} + \omega_{1'2'}([\mathbf{1'2'2''}]/[\mathbf{232''}])\mathbf{23} \\
\qquad + \omega_{22'}([\mathbf{232'}]/[\mathbf{232''}])\mathbf{22''}, \\
\mathbf{B}_{2'} = \mathbf{B}_{3''} + \omega_{1'2'}([\mathbf{1'2'1''}]/[\mathbf{131''}])\mathbf{13} \\
\qquad + \omega_{22'}([\mathbf{232'}]/[\mathbf{131''}])\mathbf{11''}, \\
h_1 = [\mathbf{1B}_{3''}], \quad h_2 = [\mathbf{2B}_{3''}], \quad h_3 = [\mathbf{3B}_{1''}], \\
h_{1'} = [\mathbf{1'B}_{3''}], \quad h_{2'} = [\mathbf{2'B}_{3''}], \quad h_{3'} = [\mathbf{3'B}_{1''}], \\
h_{1''} = [\mathbf{1''B}_3], \quad h_{2''} = [\mathbf{2''B}_3], \quad h_{3''} = [\mathbf{3''B}_1].
\end{cases} \tag{6.27}$$

□

The linear construction sequence is certainly not unique. For example, $\mathbf{3}, \mathbf{1''}$ can also be solved from (6.14) explicitly. Example 5 in the next section is on a polyhedron composed

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of two such tori. It also has multiple linear construction sequences.

7. Further Examples

The parametric calotte propagation algorithm is implemented with Maple 8, and has been tested by 20 examples. The examples include both trihedral and nontrihedral, both spherical and nonspherical, polyhedral scenes. In the experiment, no bifurcation occurs during the propagation. However, bifurcation occurs immediately after the elimination is invoked to solve for the new parameters. This suggests that by fewer parameters, no matter if they are independent or not, only part of the solution space may be parameterized. That the number of new parameters is minimal is also supported by the fact that with any fewer parameters the propagation cannot proceed. The minimal number of new parameters has to be a combinatorial invariant. By now we are still unable to prove any formula relating it to the incidence structure.

Example 2. A spherical polyhedron has 14 vertices and 12 faces. Each face has four vertices and each vertex is in four faces. In Fig. 8, face F_{12} is composed of vertices $V_{11}, V_{12}, V_{13}, V_{14}$.

In the following, the image of vertex V_i is denoted by $\mathbf{V}_i$. The propagation sequence is

$$\mathbf{B}_1 = 0 \xrightarrow{\omega_{12}} \mathbf{B}_2 \xrightarrow{B.1} \mathbf{B}_7 \xrightarrow{\omega_{23}} \mathbf{B}_4 \xrightarrow{B.1} \mathbf{B}_5 \xrightarrow{B.1} \mathbf{B}_3 \xrightarrow{B.1} \mathbf{B}_6,$$
$$\mathbf{B}_8,\ \mathbf{B}_9 \xrightarrow{B.2} \mathbf{B}_{10},\ \mathbf{B}_{11} \xrightarrow{B.2} \mathbf{B}_{12}.$$

Two parameters ω_{12}, ω_{23} are added. They satisfy three equalities:

$$\begin{aligned}
&\omega_{12}[\mathbf{V}_3\mathbf{V}_4\mathbf{V}_9]f_1(6,2) = \omega_{23}[\mathbf{V}_1\mathbf{V}_4\mathbf{V}_5]f_2(6,6),\\
&\omega_{12}[\mathbf{V}_3\mathbf{V}_4\mathbf{V}_9]f_3(6,6) = \omega_{23}[\mathbf{V}_1\mathbf{V}_4\mathbf{V}_5]f_4(6,2),\\
&\omega_{12}[\mathbf{V}_3\mathbf{V}_4\mathbf{V}_9]f_5(7,6) = \omega_{23}[\mathbf{V}_1\mathbf{V}_4\mathbf{V}_5]f_6(7,6).
\end{aligned} \tag{7.28}$$

Here $f_i(p, q)$ denotes a bracket polynomial of degree p in brackets and of q terms.

The three equations can all be replaced by calotte equations. In fact, f_1 and f_4 are two Crapo binomials. The three calottes are

1. $(V_1V_2V_8V_9V_{10}V_4,\ V_5V_6V_{12}V_{13}V_{14}V_7)$;
2. $(V_5V_6V_2V_3V_4V_7,\ V_{11}V_{12}V_8V_9V_{10}V_{14})$;

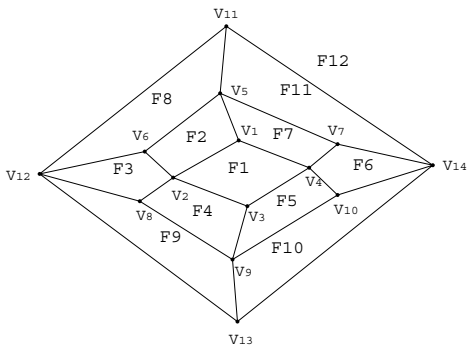


Fig. 8. Example 2.

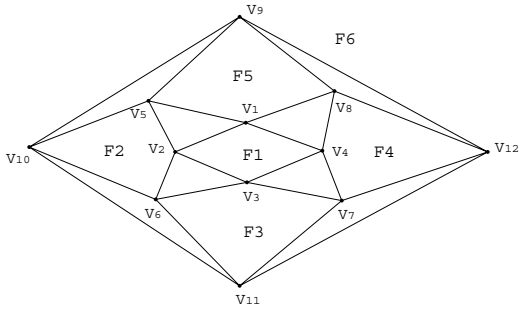


Fig. 9. Example 3.

3. $(V_{11}V_5V_1V_2V_8V_{12},\ V_{14}V_7V_4V_3V_9V_{13})$ or $(V_{11}V_5V_1V_4V_{10}V_{14},\ V_{12}V_6V_2V_3V_9V_{13})$.

Example 3. A spherical polyhedron has 12 vertices (see fig. 9). After removing eight triangular faces, there are six faces each with four vertices.

Propagation: $\mathbf{B}_1 = 0 \xrightarrow{\omega_{25},\omega_{26}} \mathbf{B}_2 \xrightarrow{\omega_{36}} \mathbf{B}_3 \xrightarrow{\omega_{15}} \mathbf{B}_5 \xrightarrow{B.1} \mathbf{B}_4 \xrightarrow{B.2} \mathbf{B}_6$. Four parameters $\omega_{25}, \omega_{26}, \omega_{36}, \omega_{15}$ are introduced. They satisfy

$$\begin{aligned}
&\omega_{26}[\mathbf{V}_4\mathbf{V}_7\mathbf{V}_8][\mathbf{V}_3\mathbf{V}_6\mathbf{V}_7]([\mathbf{V}_1\mathbf{V}_8\mathbf{V}_9][\mathbf{V}_2\mathbf{V}_5\mathbf{V}_6][\mathbf{V}_{10}\mathbf{V}_{11}\mathbf{V}_{12}]\\
&-[\mathbf{V}_1\mathbf{V}_5\mathbf{V}_8][\mathbf{V}_2\mathbf{V}_6\mathbf{V}_{10}][\mathbf{V}_9\mathbf{V}_{11}\mathbf{V}_{12}])\\
&+\omega_{25}[\mathbf{V}_1\mathbf{V}_5\mathbf{V}_8][\mathbf{V}_4\mathbf{V}_7\mathbf{V}_8]([\mathbf{V}_2\mathbf{V}_5\mathbf{V}_6][\mathbf{V}_3\mathbf{V}_7\mathbf{V}_{11}][\mathbf{V}_9\mathbf{V}_{10}\mathbf{V}_{12}]\\
&-[\mathbf{V}_2\mathbf{V}_5\mathbf{V}_{10}][\mathbf{V}_3\mathbf{V}_6\mathbf{V}_7][\mathbf{V}_9\mathbf{V}_{11}\mathbf{V}_{12}])\\
&+\omega_{36}[\mathbf{V}_1\mathbf{V}_5\mathbf{V}_8][\mathbf{V}_2\mathbf{V}_5\mathbf{V}_6]([\mathbf{V}_3\mathbf{V}_6\mathbf{V}_7][\mathbf{V}_4\mathbf{V}_8\mathbf{V}_{12}][\mathbf{V}_9\mathbf{V}_{10}\mathbf{V}_{11}]\\
&-[\mathbf{V}_3\mathbf{V}_6\mathbf{V}_{11}][\mathbf{V}_4\mathbf{V}_7\mathbf{V}_8][\mathbf{V}_9\mathbf{V}_{10}\mathbf{V}_{12}])\\
&+\omega_{15}[\mathbf{V}_3\mathbf{V}_6\mathbf{V}_7][\mathbf{V}_2\mathbf{V}_5\mathbf{V}_6]([\mathbf{V}_1\mathbf{V}_5\mathbf{V}_9][\mathbf{V}_4\mathbf{V}_7\mathbf{V}_8][\mathbf{V}_{10}\mathbf{V}_{11}\mathbf{V}_{12}]\\
&-[\mathbf{V}_1\mathbf{V}_5\mathbf{V}_8][\mathbf{V}_4\mathbf{V}_7\mathbf{V}_{12}][\mathbf{V}_9\mathbf{V}_{10}\mathbf{V}_{11}]) = 0.
\end{aligned} \tag{7.29}$$

The equation can be replaced by the calotte equation of either $(V_5V_1\ V_4V_7V_{11}V_{10},\ V_9V_8V_{12})$ or the other three similar ones. The four binomials in (7.29) are the corresponding Crapo binomials.

Example 4. A spherical polyhedron has 24 vertices (see Fig. 10). After removing all triangular faces, there are 18 faces each with four vertices.

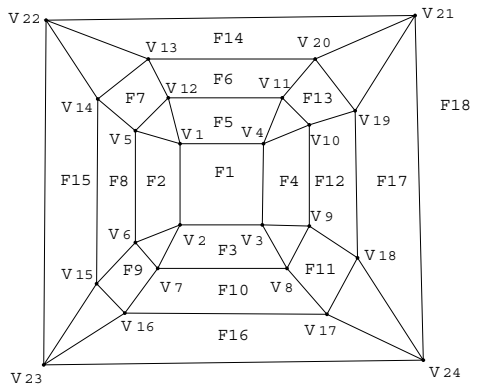


Fig. 10. Example 4.

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Propagation:

$$\begin{aligned}\mathbf{B}_1 = 0 &\xrightarrow{\omega_{12}} \mathbf{B}_2 \quad \xrightarrow{\omega_{23}} \mathbf{B}_3 \xrightarrow{\omega_{34}} \mathbf{B}_4 \quad \xrightarrow{\omega_{14}} \mathbf{B}_5\\ &\xrightarrow{\omega_{56}} \mathbf{B}_8 \quad \longrightarrow \quad \xrightarrow{B.1} \mathbf{B}_7,\ \mathbf{B}_9 \xrightarrow{B.1} \mathbf{B}_6,\ \mathbf{B}_{10}\\ &\xrightarrow{B.1} \mathbf{B}_{11},\ \mathbf{B}_{13} \xrightarrow{B.2} \mathbf{B}_{12} \xrightarrow{\omega_{14,15}} \mathbf{B}_{15} \quad \longrightarrow\\ &\xrightarrow{B.1} \mathbf{B}_{14},\ \mathbf{B}_{16} \xrightarrow{B.2} \mathbf{B}_{18} \xrightarrow{B.2} \mathbf{B}_{17}.\end{aligned}$$

Six parameters $\omega_{12}, \omega_{23}, \omega_{34}, \omega_{14}, \omega_{56}$, and $\omega_{14,15}$ are introduced. They satisfy three equations:

$$\begin{pmatrix} f_{11}(8,4) & f_{12}(8,4) & f_{13}(8,4) & f_{14}(8,4) & f_{15}(8,2) & 0\\ f_{21}(8,12) & f_{22}(8,4) & f_{23}(8,4) & 0 & f_{25}(8,8) & f_{26}(8,4)\\ f_{31}(10,14) & f_{32}(10,8) & f_{33}(10,8) & f_{34}(10,2) & f_{35}(10,8) & f_{36}(10,2)\end{pmatrix} \times \begin{pmatrix}\omega_{12}\\ \omega_{14}\\ \omega_{23}\\ \omega_{34}\\ \omega_{56}\\ \omega_{14,15}\end{pmatrix} = 0.$$

The equations can all be replaced by calotte equations. The three calottes are

1. $(V_5V_{12}V_{11}V_{10}V_9V_8V_7V_6,\ V_{14}V_{13}V_{20}V_{19}V_{18}V_{17}V_{16}V_{15})$;
2. $(V_{22}V_{13}V_{12}V_1V_2V_7V_{16}V_{23},\ V_{21}V_{20}V_{11}V_4V_3V_8V_{17}V_{24})$;
3. $(V_{20}V_{11}V_4V_3V_8V_{17}V_{24}V_{21},\ V_{19}V_{10}V_9V_{18})$ or
   $(V_{13}V_{20}V_{19}V_{18}V_{17}V_{16}V_{15}V_{14},\ V_{22}V_{21}V_{24}V_{23})$ or
   $(V_{22}V_{14}V_5V_1V_4V_{10}V_{19}V_{21},\ V_{23}V_{15}V_6V_2V_3V_9V_{18}V_{24})$.

Example 5. Two tori in Example 1 are merged together by identifying two of their outward faces and then deleting that common face. As in Fig. 11, such a polyhedron has 14 vertices and 16 faces. Each face has four vertices. Ten vertices are each on four faces and four vertices are each on six faces. Vertices V_1, V_3, V_8, V_{10} are no longer coplanar.

Propagation:

$$\begin{aligned}\mathbf{B}_1 = 0 &\xrightarrow{\omega_{57}} \mathbf{B}_2 \xrightarrow{\omega_{15}} \mathbf{B}_5 \xrightarrow{B.2} \mathbf{B}_8 \xrightarrow{B.1} \mathbf{B}_6 \xrightarrow{B.2} \mathbf{B}_7,\ \mathbf{B}_9,\ \mathbf{B}_{10}\\ &\xrightarrow{\omega_{13}} \mathbf{B}_3 \xrightarrow{B.2} \mathbf{B}_4 \xrightarrow{\omega_{1,13}} \mathbf{B}_{14} \xrightarrow{B.2} \mathbf{B}_{11} \xrightarrow{B.1} \mathbf{B}_{12} \xrightarrow{B.2} \mathbf{B}_{13},\\ &\mathbf{B}_{15}, \mathbf{B}_{16}.\end{aligned}$$

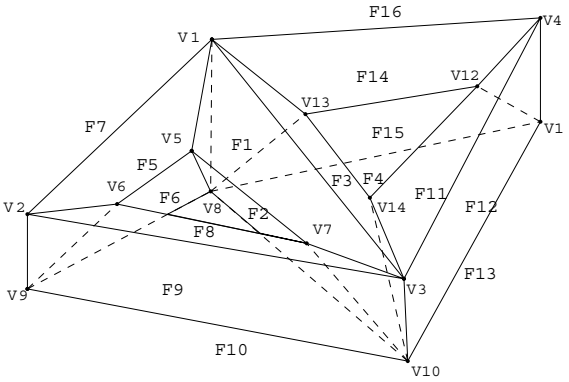


Fig. 11. Example 5.

Four parameters $\omega_{13}, \omega_{15}, \omega_{1,13}, \omega_{57}$ are introduced. They satisfy nine equations, among which only eight are algebraically independent

$$\begin{cases}\omega_{15}\mathbf{V}_1\mathbf{V}_5 \wedge \mathbf{V}_2\mathbf{V}_6 \wedge \mathbf{V}_3\mathbf{V}_7 = 0,\\ \omega_{1,13}\mathbf{V}_1\mathbf{V}_{13} \wedge \mathbf{V}_3\mathbf{V}_{14} \wedge \mathbf{V}_4\mathbf{V}_{12} = 0,\\ (\omega_{15}[\mathbf{V}_1\mathbf{V}_5\mathbf{V}_8] - \omega_{57}[\mathbf{V}_5\mathbf{V}_7\mathbf{V}_8])\mathbf{V}_1\mathbf{V}_2 \wedge \mathbf{V}_5\mathbf{V}_6 \wedge \mathbf{V}_8\mathbf{V}_9 = 0,\\ (\omega_{15}[\mathbf{V}_1\mathbf{V}_5\mathbf{V}_6] + \omega_{57}[\mathbf{V}_5\mathbf{V}_6\mathbf{V}_7])\mathbf{V}_5\mathbf{V}_8 \wedge \mathbf{V}_6\mathbf{V}_9 \wedge \mathbf{V}_7\mathbf{V}_{10} = 0,\\ \omega_{57}\mathbf{V}_5\mathbf{V}_7 \wedge \mathbf{V}_8\mathbf{V}_{10} \wedge \mathbf{V}_{13}\mathbf{V}_{14} - \omega_{13}\mathbf{V}_1\mathbf{V}_3 \wedge \mathbf{V}_8\mathbf{V}_{10} \wedge\\ \quad \mathbf{V}_{13}\mathbf{V}_{14} = 0,\\ (\omega_{57}[\mathbf{V}_5\mathbf{V}_7\mathbf{V}_8] - \omega_{13}[\mathbf{V}_1\mathbf{V}_3\mathbf{V}_8] + \omega_{113}[\mathbf{V}_1\mathbf{V}_8\mathbf{V}_{13}])\mathbf{V}_1\mathbf{V}_4 \wedge\\ \quad \mathbf{V}_8\mathbf{V}_{11} \wedge \mathbf{V}_{12}\mathbf{V}_{13} = 0,\\ (\omega_{15}[\mathbf{V}_1\mathbf{V}_5\mathbf{V}_6][\mathbf{V}_3\mathbf{V}_7\mathbf{V}_{10}] + \omega_{57}[\mathbf{V}_3\mathbf{V}_6\mathbf{V}_7][\mathbf{V}_5\mathbf{V}_7\mathbf{V}_{10}])\mathbf{V}_2\mathbf{V}_3 \wedge\\ \quad \mathbf{V}_6\mathbf{V}_7 \wedge \mathbf{V}_9\mathbf{V}_{10} = 0,\\ (\omega_{57}[\mathbf{V}_5\mathbf{V}_7\mathbf{V}_{10}][\mathbf{V}_{12}\mathbf{V}_{13}\mathbf{V}_{14}] + \omega_{1,13}[\mathbf{V}_1\mathbf{V}_{12}\mathbf{V}_{13}][\mathbf{V}_{10}\mathbf{V}_{13}\mathbf{V}_{14}]\\ \quad + \omega_{13}[\mathbf{V}_1\mathbf{V}_3\mathbf{V}_{10}][\mathbf{V}_{12}\mathbf{V}_{13}\mathbf{V}_{14}])\mathbf{V}_8\mathbf{V}_{13} \wedge \mathbf{V}_{10}\mathbf{V}_{14} \wedge \mathbf{V}_{11}\mathbf{V}_{12} = 0,\\ (\omega_{57}\mathbf{V}_1\mathbf{V}_4 \wedge \mathbf{V}_5\mathbf{V}_7 \wedge \mathbf{V}_8\mathbf{V}_{10} - \omega_{1,13}[\mathbf{V}_1\mathbf{V}_4\mathbf{V}_{13}][\mathbf{V}_1\mathbf{V}_8\mathbf{V}_{10}]\\ \quad - \omega_{13}[\mathbf{V}_1\mathbf{V}_3\mathbf{V}_4][\mathbf{V}_1\mathbf{V}_8\mathbf{V}_{10}])\mathbf{V}_3\mathbf{V}_4 \wedge \mathbf{V}_{10}\mathbf{V}_{11} \wedge \mathbf{V}_{12}\mathbf{V}_{14} = 0.\end{cases}$$

In the list, there are eight factored calotte equation. The other equation can also be replaced by a calotte equation. The nine calottes are

1. $(V_1V_2V_3,\ V_5V_6V_7)$;
2. $(V_1V_5V_8,\ V_2V_6V_9)$;
3. $(V_2V_6V_9,\ V_3V_7V_{10})$;
4. $(V_5V_6V_7,\ V_8V_9V_{10})$;
5. $(V_{12}V_{13}V_{14},\ V_1V_3V_4)$;
6. $(V_3V_{10}V_{14},\ V_4V_{11}V_{12})$;
7. $(V_4V_{12}V_{13},\ V_8V_{10}V_{11})$;
8. $(V_4V_{11}V_{12},\ V_1V_8V_{13})$;
9. $(V_1V_5V_8V_{13},\ V_3V_7V_{10}V_{14})$.

Example 6. Two trihedral spherical polyhedra $\mathcal{S}_1$ and $\mathcal{S}_2$ are connected through two tubes $\mathcal{T}_1, \mathcal{T}_2$ at two pairs of five-vertexed faces (see Fig. 12). The two pairs of faces are then removed so that the corresponding vertices are no longer coplanar. Thus, $\mathcal{S}_1$ has 60 vertices and 30 faces, among which 20 faces are six-vertexed and 10 faces are five-vertexed. $\mathcal{S}_1$ has 20 vertices and 10 faces, and each face has five vertices. Both tubes are composed of three columns, and each column is composed of five quadrangle faces. All together, there are 100 vertices and 70 faces.

The propagation can start from any face and run throughout the polyhedron. Seven parameters are introduced. They satisfy 40 equations, all of which can be replaced by calotte ones.

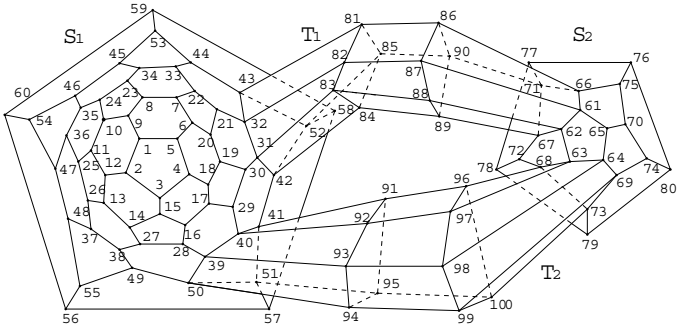


Fig. 12. Example 6.

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An alternative approach is to decompose the polyhedron into four smaller parts $\mathcal{S}_1$, $\mathcal{S}_2$, $\mathcal{T}_1$, and $\mathcal{T}_2$, run parametric calotte propagation in each part, then assemble the polyhedron by joining the parts at the junction loops.

Step 1. Start from a face of $\mathcal{S}_1$, run parametric propagation in $\mathcal{S}_1$. One parameter is introduced, which satisfies 24 equations. The equations can be replaced by calotte ones, among which 22 are in factored form.

Step 2. Start from the loop $\mathcal{S}_1 \cap \mathcal{T}_1$, run parametric propagation in $\mathcal{T}_1$. Three parameters are introduced, which satisfy three nonfactored calotte equations.

Step 3. Start from the loop $\mathcal{S}_1 \cap \mathcal{T}_2$, run parametric propagation in $\mathcal{T}_2$. Three parameters are introduced, which satisfy three nonfactored calotte equations.

Step 4. Start from the two loops $\mathcal{S}_2 \cap \mathcal{T}_1$ and $\mathcal{S}_2 \cap \mathcal{T}_2$, run parametric propagation in $\mathcal{S}_2$. No parameter is introduced. There are 10 parametric equations, 2 of which are factored calotte equations. After the four parts are put together, the remaining eight equations can be replaced by eight calotte equations in nonfactored form.

8. Conclusion

In this paper, we propose a general method, called parametric calotte propagation, to solve realization and parameterization problems in polyhedral scene analysis at the same time. Starting with the fundamental equations of Sugihara in the form of bivector equations, we can parameterize all the bivectors by introducing new parameters. The realization conditions are implied in the scalar equations satisfied by the new parameters, and can be derived by further analysis of the propagation result. The propagation procedure generally does not bifurcate, and the result often contains equations in factored form, thus making further algebraic manipulation easier. In applications, parametric propagation can be used to find linear construction sequences for nonspherical polyhedra.

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