The Review of Symbolic Logic
Volume 19, Number 4, December 2026

KRIPKE COMPLETENESS FOR PREDICATE MODAL LOGICS OF BOUNDED ALTERNATIVITY: FIRST STEPS

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Abstract. We study Kripke completeness for predicate modal logics with propositional axioms of bounded alternativity corresponding to the condition on Kripke frames that every world sees at most n worlds. We prove that minimal predicate modal logics with axioms of bounded alternativity, as well as their extensions with the axiom of reflexivity, are strongly Kripke complete, whereas the addition of the axiom of transitivity results in Kripke incomplete logics, both with and without reflexivity. We show that these incomplete logics are members of an uncountable family of Kripke incomplete logics and give explicit axiomatizations of completions for some of them.

§1. Introduction. Kripke frame semantics has proven a valuable tool for the study of propositional modal logics. Despite the by now well-known phenomenon of propositional Kripke incompleteness, examples of Kripke incomplete propositional modal logics are uncommon and rather contrived. By contrast, in predicate modal logic, Kripke incompleteness is common, and the predicate counterparts of many simple Kripke complete propositional logics are Kripke incomplete. Thus, the boundaries of usefulness of Kripke frame semantics for predicate modal logics are not very well understood. Additionally, canonical models for predicate modal logics are not as well-behaved as canonical models for propositional logics: many predicate logics, despite being Kripke complete, are not canonical, that is, they are not validated by their canonical frames, which need to be modified to obtain completeness. As a result, completeness proofs in predicate modal logic are often rather nontrivial.

In this paper, we study Kripke completeness of predicate modal logics with propositional axioms of bounded alternativity, customarily called $\boldsymbol{alt}_n$. Propositional logics axiomatized by these formulas, as well as their propositional extensions, called logics of bounded alternativity, have been extensively studied. In particular, F. Bellissima showed that all monomodal logics of bounded alternativity are Kripke

Received: October 15, 2025 UTC.
2020 *Mathematics Subject Classification*: Primary 03B45.
Key words and phrases: predicate modal logic, quantified modal logic, Kripke completeness.

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doi:10.1017/S1755020326101324

https://doi.org/10.1017/S1755020326101324 Published online by Cambridge University Press

complete.[1] We consider minimal predicate logics $\mathbf{QAlt}_n$ containing axioms $\boldsymbol{alt}_n$ as well as their predicate extensions with propositional axioms of reflexivity, transitivity, and both, respectively, $\mathbf{QTAlt}_n$, $\mathbf{QK4Alt}_n$, and $\mathbf{QS4Alt}_n$. We show, using selective weak submodels [7] of canonical models, that $\mathbf{QAlt}_n$ and $\mathbf{QTAlt}_n$ are strongly Kripke complete, for every $n \geqslant 1$.[2] We also show, using Kripke bundle semantics [2, Chapter 5], that, unlike propositional logics $\mathbf{K4Alt}_n$, predicate logics $\mathbf{QK4Alt}_n$ are Kripke incomplete for every $n \geqslant 1$ (Kripke incompleteness of $\mathbf{QS4Alt}_n$, for every $n \geqslant 2$, follows easily from [3]). Lastly, we give explicit axiomatizations of Kripke completions of $\mathbf{QK4Alt}_1$ and $\mathbf{QS4Alt}_2$.

The paper is structured as follows. §2 contains preliminaries on predicate modal logics. In §3, we prove Kripke completeness of logics $\mathbf{QAlt}_n$ and $\mathbf{QTAlt}_n$. In §4, we establish Kripke incompleteness of $\mathbf{QK4Alt}_n$ and of some related logics. In §5, we axiomatize Kripke completions of $\mathbf{QK4Alt}_1$ and $\mathbf{QS4Alt}_2$. We conclude, in §6, with directions for future work.

§2. Preliminaries. This section is mostly meant to fix terminology and notation; more background on predicate modal logic can be found in [2].

The (first-order) predicate modal language contains the following symbols: a countable set *Var* of individual variables; for every $n \geqslant 0$, a countable set $\{P^n_k \mid k \geqslant 0\}$ of n-ary predicate letters (nullary predicate letters are also called *proposition letters*); the propositional constant $\bot$; the binary Boolean connective $\to$; the quantifier symbol $\forall$; and the unary modal operator $\Box$.[3] Thus, formulas are defined by the following grammar:

$$A \quad ::= \quad P^n_k(x_1, \dots, x_n) \mid \bot \mid (A \to A) \mid \forall x\, A \mid \Box A,$$

where $x, x_1, \dots, x_n \in \mathit{Var}$. We omit parentheses after proposition letters, and so write P_k instead of $P^0_k()$. The logical symbols $\top$, $\wedge$, $\vee$, $\neg$, $\leftrightarrow$, $\Diamond$, and $\exists$ are treated as standard abbreviations. We also define, for every formula A and every $n \geqslant 0$,

$$\Box^0 A \quad := \quad A, \qquad \Box^{n+1} A \quad := \quad \Box\Box^n A, \qquad \Box^{\leqslant n} A \quad := \quad \bigwedge_{i=0}^{n} \Box^i A.$$

If Γ is a set of formulas, then

$$\Box^\infty \Gamma \quad := \quad \bigcup_{n \geqslant 0} \{\Box^n A \mid A \in \Gamma\}.$$

The free variables of a formula A are referred to as *parameters* of A. We write $A(x_1, \dots, x_n)$ to mean that formula A contains parameters $x_1, \dots, x_n$, possibly among others. A formula without parameters is called a *sentence*. If Γ is a set of formulas, then $\overline{\Gamma} := \{A \in \Gamma \mid A \text{ is a sentence}\}$. The universal closure of a formula A is denoted by $\overline{\forall} A$.

[1] See, for example, [4, Theorem 3.2.12]; for related results, see [4, Chapter 9] and the references therein.

[2] Strong completeness of logics $\mathbf{QAlt}_n$ was claimed, without detailed proofs, in [6, 8, 10].

[3] In §3, we also consider, for purely technical purposes, languages enriched with individual constants; nevertheless, all the results of the paper are stated for languages without constants.

https://doi.org/10.1017/S1755020326101324 Published online by Cambridge University Press

A *propositional formula* is a formula containing neither quantifiers nor predicate letters other than proposition letters. A *closed propositional formula* is a propositional formula without proposition letters.

By a *predicate modal logic* we mean a set of formulas including both the classical predicate logic **QCL** and the minimal normal modal propositional logic **K**, and closed under Substitution (Sub), Modus Ponens (MP), Generalization (Gen), and Necessitation (Nec). Henceforth, by *logic* simpliciter we mean a predicate modal logic and by *propositional logic* we mean a normal propositional modal logic. If $\mathbf{\Lambda}$ is a propositional logic, the minimal predicate extension of $\mathbf{\Lambda}$ is denoted by $\mathbf{Q\Lambda}$. Thus, the minimal predicate modal logic is **QK**. The minimal logic including a logic L and a set Γ of formulas is denoted by $L + \Gamma$ (similarly for propositional logics and formulas).

PROPOSITION 2.1. *Let* $\mathbf{\Lambda}$ *and* $\mathbf{\Lambda}'$ *be propositional logics. Then,*

(1) $\mathbf{\Lambda} \subseteq \mathbf{\Lambda}'$ *implies* $\mathbf{Q\Lambda} \subseteq \mathbf{Q\Lambda}'$;

(2) $\mathbf{Q\Lambda} + \Gamma = \mathbf{Q}(\mathbf{\Lambda} + \Gamma)$, *for every set* Γ *of propositional formulas*;

(3) $\mathbf{Q\Lambda}$ *is conservative over* $\mathbf{\Lambda}$.

Proof. (1) and (2) follow easily from definitions; for (3), see [2, Proposition 2.15.9]. □

We next recall definitions of some propositional formulas and logics considered later on (here, $n \geqslant 1$):

$\boldsymbol{ref}$	$:= \Box P_0 \to P_0$;	$\boldsymbol{ma}$	$:= \Box\Diamond P_0 \to \Diamond\Box P_0$ (McKinsey axiom);
4	$:= \Box P_0 \to \Box\Box P_0$;	$\boldsymbol{alt}_n$	$:= \neg \bigwedge_{i=0}^{n} \Diamond(P_i \wedge \bigwedge_{j \neq i} \neg P_j)$;
K4	$:= \mathbf{K} + \mathbf{4}$;	**S4**	$:= \mathbf{K4} + \boldsymbol{ref}$;
K4.1	$:= \mathbf{K4} + \boldsymbol{ma}$;	**S4.3**	$:= \mathbf{S4} + \Box(\Box P_0 \to P_1) \vee \Box(\Box P_1 \to P_0)$;
T	$:= \mathbf{K} + \boldsymbol{ref}$;	**D4**	$:= \mathbf{K4} + \Diamond\top$;
$\mathbf{Alt}_n$	$:= \mathbf{K} + \boldsymbol{alt}_n$;	$\mathbf{TAlt}_n$	$:= \mathbf{Alt}_n + \boldsymbol{ref}$;
$\mathbf{K4Alt}_n$	$:= \mathbf{Alt}_n + \mathbf{4}$;	$\mathbf{S4Alt}_n$	$:= \mathbf{K4Alt}_n + \boldsymbol{ref}$;
$\mathbf{S4.1Alt}_n$	$:= \mathbf{S4Alt}_n + \boldsymbol{ma}$;	**SL4**	$:= \mathbf{K4Alt}_1 + \Diamond\top$;
Triv	$:= \mathbf{K} + \Box P_0 \leftrightarrow P_0$.		

A *Kripke frame* is a pair $F = (W, R)$ where W is a non-empty set of *worlds*, or *points*, and R is a binary *accessibility* relation on W. We use standard abbreviations: for all $u \in W$ and $U \subseteq W$,

$$R(u) := \{v \in W \mid uRv\} \quad \text{and} \quad R(U) := \bigcup\{R(u) \mid u \in U\}.$$

A *propositional Kripke model* over the Kripke frame $F = (W, R)$ is a pair (F, ξ) where $\xi = (\xi_u \mid u \in W)$ is a family of Boolean-valued valuations: each ξ_u is a function sending proposition letters to $\{1, 0\}$.

The definition of truth of a propositional formula A at a world u of a Kripke model M is standard (see, e.g., [1]); as usual, we write $M, u \models A$ if A is true at u in M. If F is a Kripke frame and A is a propositional formula, then we say that

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- A is *locally valid* at a world u of F, and write $F, u \models A$, if $M, u \models A$ holds for every model M over F;
- A is (globally) *valid* on F, and write $F \models A$, if $F, u \models A$ holds for every world u of F.

If F is a Kripke frame, then the set $\{A \mid F \models A\}$ is called *the propositional modal logic of* F and is denoted by $\mathbf{L}(F)$; by Soundness theorem, this set is, indeed, a propositional modal logic.

We next recall the well-known local validity conditions of some propositional formulas.

PROPOSITION 2.2. *Let $F = (W, R)$ be a Kripke frame and let $u \in W$. Then,*

(1) $F, u \models \boldsymbol{alt}_n$ *iff F is* n-alternative *at u, that is, iff* $|R(u)| \leqslant n$;
(2) $F, u \models \boldsymbol{ref}$ *iff u is reflexive, that is, iff* $u \in R(u)$;
(3) $F, u \models \mathbf{4}$ *iff F is transitive at u, that is, iff* $R(R(u)) \subseteq R(u)$.

PROPOSITION 2.3. *Let $F = (W, R)$ be a transitive Kripke frame and let $u \in W$. Then, $F, u \models \boldsymbol{ma}$ iff the world u sees a maximal world of F, that is, iff* $\exists v \in R(u)\, R(v) = \{v\}$.

A *predicate Kripke frame* is a pair $\boldsymbol{F} = (F, D)$ where $F = (W, R)$ is a Kripke frame and $D = (D_u)_{u \in W}$ is a system of non-empty domains satisfying the *expanding domains condition*: $D_u \subseteq D_v$ whenever uRv. A *predicate Kripke model* over the predicate Kripke frame $\boldsymbol{F}$ is a pair $\boldsymbol{M} = (\boldsymbol{F}, \xi)$ where ξ is a family $(\xi_u)_{u \in W}$ of valuations: each ξ_u is a function sending n-ary predicate letters to subsets of D_u^n. Since $D_u^0 = \{\varnothing\}$, it follows that, for a proposition letter P_k, either $\xi_u(P_k) = \{\varnothing\}$ or $\xi_u(P_k) = \varnothing$; we identify $\{\varnothing\}$ with 1 and $\varnothing$ with 0.

Let $\boldsymbol{M} = (W, R, D, \xi)$ be a predicate Kripke model and let $u \in W$. A *D_u-sentence* is an expression obtained from a formula by replacing each parameter by (a syntactic copy of) an element of D_u. The *truth relation* $\models$ between a world u of a model $\boldsymbol{M}$ and a D_u-sentence A is defined by recursion:

- $\boldsymbol{M}, u \models P_k^n(a_1, \ldots, a_n)$ if $(a_1, \ldots, a_n) \in \xi_u(P_k^n)$;
- $\boldsymbol{M}, u \not\models \bot$;
- $\boldsymbol{M}, u \models A \to B$ if $\boldsymbol{M}, u \not\models A$ or $\boldsymbol{M}, u \models B$;
- $\boldsymbol{M}, u \models \forall x\, A(x)$ if $\boldsymbol{M}, u \models A(a)$ whenever $a \in D_u$;
- $\boldsymbol{M}, u \models \Box A$ if $\boldsymbol{M}, v \models A$ whenever $v \in R(u)$.

We say that a predicate formula A is

- *locally true* at a world u of a model $\boldsymbol{M}$ (also written $\boldsymbol{M}, u \models A$) if $\boldsymbol{M}, u \models \overline{\forall} A$;
- (globally) *true* in $\boldsymbol{M}$ (also written $\boldsymbol{M} \models A$) if $\boldsymbol{M}, u \models A$ holds for every world u of $\boldsymbol{M}$;
- *locally valid* at a world u of a predicate Kripke frame $\boldsymbol{F}$ (also written $\boldsymbol{F}, u \models A$) if $\boldsymbol{M}, u \models A$ holds for every model $\boldsymbol{M}$ over $\boldsymbol{F}$;
- (globally) *valid* on $\boldsymbol{F}$ (also written $\boldsymbol{F} \models A$) if $\boldsymbol{F}, u \models A$ holds for every world u of $\boldsymbol{F}$;
- *valid* on a class $\mathscr{C}$ of predicate Kripke frames if $\boldsymbol{F} \models A$ holds for every $\boldsymbol{F} \in \mathscr{C}$.

The following is well known (see, e.g., [2, Lemma 3.2.23]) and easy to check.

PROPOSITION 2.4. *Let $\boldsymbol{F} = (F, D)$ be a predicate Kripke frame and let A be a propositional formula. Then, $\boldsymbol{F} \models A$ iff $F \models A$.*

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If $\mathscr{C}$ is a class of predicate Kripke frames, then the set of all predicate formulas valid on every frame from $\mathscr{C}$ is called the *logic of* $\mathscr{C}$ and denoted by $\mathbf{L}(\mathscr{C})$. The following is well known.

THEOREM 2.5 (Soundness theorem). *For every class $\mathscr{C}$ of predicate Kripke frames, the set $\mathbf{L}(\mathscr{C})$ is a predicate modal logic.*

If L is a propositional logic and F is a Kripke frame such that $F \models L$, then F is said to be an *L-frame*; similarly for predicate logics and frames.

DEFINITION 2.6. *A predicate formula A is a* logical consequence in Kripke semantics *of a logic L, also written $L \models_{\mathcal{K}} A$, if every predicate L-frame validates A.*

Let L be a predicate logic. If $L = \mathbf{L}(\mathscr{C})$, for some class $\mathscr{C}$ of predicate Kripke frames, then L is called *Kripke complete*.[4] The smallest Kripke complete logic extending a logic L is called the *Kripke completion* of L and is denoted by $\widehat{L}$. It should be clear that $\widehat{L} = \{A \mid L \vDash_{\mathcal{K}} A\}$.

The following fact is an easy consequence of Propositions 2.1 (3) and 2.4.

PROPOSITION 2.7. *Let $\mathbf{\Lambda}$ be a propositional logic. If $\mathbf{Q\Lambda}$ is Kripke complete, then $\mathbf{\Lambda}$ is Kripke complete.*

We next recall the definitions of subframes and submodels:

- a Kripke frame $F' = (W', R')$ is a *subframe* of $F = (W, R)$ (in symbols, $F' \subseteq F$) if $W' \subseteq W$ and $R' = R \restriction W'$; if, additionally, $R(W') \subseteq W'$, then F' is a *generated subframe* of F (in symbols, $F' \sqsubseteq F$);
- a propositional Kripke model $M' = (W', R', \xi')$ is a *submodel* of $M = (W, R, \xi)$ (in symbols, $M' \subseteq M$) if $(W', R') \subseteq (W, R)$ and $\xi' = \xi \restriction W'$; if, additionally, $(W', R') \sqsubseteq (W, R)$, then M' is a *generated submodel* of M (in symbols, $M' \sqsubseteq M$).

We use analogous notions and notation for predicate Kripke frames and models:

- a predicate frame $\boldsymbol{F}' = (W', R', D')$ is a *subframe* of $\boldsymbol{F} = (W, R, D)$ (in symbols, $\boldsymbol{F}' \subseteq \boldsymbol{F}$) if $(W', R') \subseteq (W, R)$ and $D' = D \restriction W'$; if also $(W', R') \sqsubseteq (W, R)$, then $\boldsymbol{F}'$ is a *generated subframe* of $\boldsymbol{F}$ (in symbols, $\boldsymbol{F}' \sqsubseteq \boldsymbol{F}$);
- a predicate Kripke model $\boldsymbol{M}' = (W', R', D', \xi')$ is a *submodel* of $\boldsymbol{M} = (W, R, D, \xi)$ (in symbols, $\boldsymbol{M}' \subseteq \boldsymbol{M}$) if $(W', R', D') \subseteq (W, R, D)$ and $\xi' = \xi \restriction W'$; if also $(W', R') \sqsubseteq (W, R)$, then $\boldsymbol{M}'$ is a *generated submodel* of $\boldsymbol{M}$ (in symbols, $\boldsymbol{M}' \sqsubseteq \boldsymbol{M}$).

The smallest generated subframe of a frame F containing a world u is denoted by $F{\uparrow}u$; similar notation is used for predicate frames and models.

The following is well known (see, e.g., [2, Lemma 3.3.18]).

PROPOSITION 2.8.

(1) *Let $\boldsymbol{M}$ and $\boldsymbol{M}'$ be predicate Kripke models such that $\boldsymbol{M}' \sqsubseteq \boldsymbol{M}$. Then, for every world* u *of $\boldsymbol{M}'$ and every D_u-sentence A,*

$$\boldsymbol{M}', u \models A \iff \boldsymbol{M}, u \models A.$$

[4] We remind the reader that Kripke completeness for propositional logics is defined analogously, but in terms of classes of (propositional) Kripke frames.

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(2) *Let* $\boldsymbol{F}$ *and* $\boldsymbol{F}'$ *be predicate Kripke frames such that* $\boldsymbol{F}' \sqsubseteq \boldsymbol{F}$. *Then, for every predicate formula* A,

$$\boldsymbol{F} \models A \implies \boldsymbol{F}' \models A.$$

We also recall that a *p-morphism* of a Kripke frame $F = (W, R)$ onto a Kripke frame $F' = (W', R')$ is a surjection $f\colon W \to W'$ such that $f(R(u)) = R(f(u))$, for every $u \in W$; if there exists a p-morphism of F onto F', we write $F \twoheadrightarrow F'$. It is well known that $F \twoheadrightarrow F'$ implies $\mathbf{L}(F) \subseteq \mathbf{L}(F')$.

§3. Kripke completeness of $\mathbf{QAlt}_n$ and $\mathbf{QTAlt}_n$. To prove Kripke completeness of logics $\mathbf{QAlt}_n$ and $\mathbf{QTAlt}_n$, we use selective weak submodels [7, Section 6] of canonical models [2, Chapter 6]. These resemble both selective filtrations [5] in propositional modal logic and elementary submodels in classical model theory.

In this section, we work with languages expanded with individual constants that are used as individuals of canonical models. All the constants are assumed to come from some universal countable set $\mathcal{U}$. Since the set $\mathcal{U}$ is fixed throughout this section, it is mentioned only when necessary. A subset $\mathcal{C}$ of $\mathcal{U}$ is *small* if the set $\mathcal{U} - \mathcal{C}$ is infinite.[5] A *theory* is a set of sentences, possibly with constants from $\mathcal{U}$.

The set of constants occurring in a theory Γ is denoted by D_Γ; the set of all sentences with constants from D_Γ is called the *language of* Γ and denoted by $\mathcal{L}(\Gamma)$. A theory Γ is *small* if D_Γ is small.

Let L be a logic. We say that a formula B is *L-derivable*, and write $\vdash_L B$, if there exists a formula $A(x_1, \dots, x_n) \in L$ and constants $c_1, \dots, c_n$ such that $B = A(c_1, \dots, c_n)$. If Γ is a theory and A is a formula possibly containing constants, we say that A is *L-derivable from* Γ, and write $\Gamma \vdash_L A$, if there exists a derivation of A using L-derivable formulas and members of Γ as axioms and (MP) and (Gen) as rules of inference (note that neither (Sub) nor (Nec) is allowed).

If L is a logic and Γ is a theory, then, as usual, Γ is *L-consistent* if $\Gamma \nvdash_L \bot$; if, additionally, Γ is maximal among L-consistent theories included in $\mathcal{L}(\Gamma)$, then Γ is *L-maximal*. The following facts are well known.

LEMMA 3.1. *Let* Γ *be an* L*-consistent theory. Then,* Γ *is* L*-maximal if and only if, for every* $A \in \mathcal{L}(\Gamma)$, *exactly one of* A *and* $\neg A$ *belongs to* Γ.

LEMMA 3.2. *Let* Γ *be an* L*-maximal theory. Then,*

(1) $A \in \mathcal{L}(\Gamma)$ *and* $\Gamma \vdash_L A$ *imply* $A \in \Gamma$. (*In particular,* $\overline{L} \subseteq \Gamma$.[6])
(2) Γ *is conjunction-closed* (*i.e.*, $A \wedge B \in \Gamma$ *iff both* $A \in \Gamma$ *and* $B \in \Gamma$).

If Γ is a theory, we define $\Box^- \Gamma := \{A \mid \Box A \in \Gamma\}$ and $\Box\Gamma := \{\Box B \mid B \in \Gamma\}$. Then, Lemma 3.1 and definition of $\Diamond$ imply the following.

LEMMA 3.3. *Let* Γ *and* Δ *be* L*-maximal theories such that* $\mathcal{L}(\Gamma) = \mathcal{L}(\Delta)$ *and* $\Box^- \Gamma \subseteq \Delta$. *Then, for every* D_Γ*-sentence* A,

$$A \in \Delta \implies \Diamond A \in \Gamma.$$

[5] The construction used here can be modified for a universal set $\mathcal{U}$ of constants of an arbitrary infinite cardinality κ; then, a set $\mathcal{C}$ of constants is small if $|\mathcal{U} - \mathcal{C}| = \kappa$.

[6] Recall that $\overline{L}$ denotes the set of all sentences from L.

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The following well-known fact (see, e.g., [2, Lemma 6.1.16]) is a simple consequence of Lindenbaum lemma.

LEMMA 3.4 (Existence lemma). *Let* Γ *be an L-consistent theory such that* $\Diamond A \in \Gamma$. *Then, there exists an L-maximal theory* Δ *such that* $D_\Gamma = D_\Delta$ *and* $\Box^- \Gamma \cup \{A\} \subseteq \Delta$.

The following definition is well known from classical predicate logic.

DEFINITION 3.5. *A theory* Γ *is* Henkin *if, for every formula* $\exists x\, A(x) \in \mathcal{L}(\Gamma)$, *there exists* $c \in D_\Gamma$ *such that* $(\exists x A(x) \to A(c)) \in \Gamma$.

We next define theories used as points of canonical models.

DEFINITION 3.6. *Let L be a predicate logic. An* L-place *is a small L-maximal Henkin theory.*

The following is well known (see, e.g., [2, Lemma 6.1.9]).

LEMMA 3.7 (Henkin–Lindenbaum lemma). *Every small L-consistent theory is included into an L-place.*

DEFINITION 3.8. *The* canonical predicate Kripke frame *for a predicate logic L is the tuple* $\boldsymbol{F}_L := (W_L, R_L, D_L)$ *where*

- W_L *is the set of all L-places*;
- $\Gamma\, R_L\, \Delta := \Box^- \Gamma \subseteq \Delta$;
- $D_L(\Gamma) = D_\Gamma$.

It is not hard to see that $\boldsymbol{F}_L$ is a predicate Kripke frame: if $\Gamma\, R_L\, \Delta$ and $c \in D_\Gamma$, then, by Lemma 3.2 (1), $\Box(P(c) \to P(c)) \in \Gamma$, and so $P(c) \to P(c) \in \Delta$, meaning that $c \in D_\Delta$.

DEFINITION 3.9. *The* canonical predicate Kripke model *for a predicate logic L is the Kripke model* $\boldsymbol{M}_L := (\boldsymbol{F}_L, \xi_L)$ *where* $\boldsymbol{F}_L$ *is the canonical predicate frame for L and* $(\xi_L)_\Gamma(P_k^n) := \{\mathbf{c} \in D_\Gamma^n \mid P_k^n(\mathbf{c}) \in \Gamma\}$, *for each* n*-ary predicate letter* P_k^n.

Note that, since the individuals of $\boldsymbol{M}_L$ are constants, if Γ is an L-place, then $D_L(\Gamma)$-sentences are precisely the members of $\mathcal{L}(\Gamma)$. The following is well known (see, e.g., [2, Theorem 6.1.21]).

THEOREM 3.10 (Canonical model theorem). *Let L be a predicate logic. Then,*

(1) *For every* $\Gamma \in W_L$ *and every* $A \in \mathcal{L}(\Gamma)$,

$$\boldsymbol{M}_L, \Gamma \models A \iff A \in \Gamma.$$

(2) *For every formula B,*

$$\boldsymbol{M}_L \models B \iff B \in L.$$

DEFINITION 3.11. *A predicate logic L is* canonical *if it is valid on its canonical predicate Kripke frame, that is, if* $\boldsymbol{F}_L \models L$.

If a theory Γ contains no occurrences of constants from $\mathcal{U}$, we say that Γ is a *theory without constants*.

DEFINITION 3.12. *A theory* Γ *without constants is* satisfiable *in a predicate Kripke model* $\boldsymbol{M}$ *if there exists a world* u *of* $\boldsymbol{M}$ *such that* $\boldsymbol{M}, u \models \Gamma$.

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DEFINITION 3.13. *A logic L is* strongly Kripke complete *if every L-consistent theory without constants is satisfiable in a model over an L-frame.*

REMARK 3.14. Analogously to the situation in propositional logic,[7] every canonical logic is strongly Kripke complete [2, Lemma 6.1.24].[8] However, canonical predicate modal logics are scarce [2, Section 6.1]. In particular, logics $\mathbf{QAlt}_n$ and $\mathbf{QTAlt}_n$ are not canonical since in their canonical predicate frames every place Γ containing $\Diamond\top$ sees infinitely many places with constants not in D_Γ: indeed, since D_Γ is small, there exists an infinite set $\mathcal{C} = \{c_i \mid i \in \omega\}$ of constants with $D_\Gamma \cap \mathcal{C} = \varnothing$. Since $\Diamond\top \in \Gamma$, by Existence and Henkin–Lindenbaum lemmas, Γ sees an L-place, say Δ; then, each theory

$$\Delta_i \quad := \quad \Delta \cup \{P(c_i) \to P(c_i)\}$$

is L-consistent and small, and so is included into an L-place Δ_i' that can be constructed not to contain any constants from $\mathcal{C} - \{c_i\}$ (this might require an infinite set of additional constants, which is available to us due to cardinality considerations). Then, Γ sees each Δ_i', and all of them are distinct.

We next show that, despite not being canonical, $\mathbf{QAlt}_n$ and $\mathbf{QTAlt}_n$ are strongly Kripke complete.

DEFINITION 3.15. *A Kripke frame (W', R') is a* weak subframe *of a Kripke frame (W, R) if $W' \subseteq W$ and $R' \subseteq R$. A predicate Kripke model $\boldsymbol{M}' = (W', R', D', \xi')$ is a* weak submodel *of a predicate Kripke model $\boldsymbol{M} = (W, R, D, \xi)$ if (W', R') is a weak subframe of (W, R) and both $D' = D \restriction W'$ and $\xi' = \xi \restriction W'$; if, additionally, for every $u \in W'$ and every D_u-sentence A,*

$$\boldsymbol{M}, u \models \Diamond A \quad \Longrightarrow \quad \exists v \in R'(u)\, \boldsymbol{M}, v \models A, \tag{sel}$$

then $\boldsymbol{M}'$ is a selective *weak submodel of $\boldsymbol{M}$, also written as $\boldsymbol{M}' \Subset \boldsymbol{M}$.*

Note that weak subframes differ from subframes: in weak subframes it is possible that $R' \subset R \restriction W'$, while in subframes $R' = R \restriction W'$. The following is easy to check.

LEMMA 3.16. *If $\boldsymbol{M}' \sqsubseteq \boldsymbol{M}$, then $\boldsymbol{M}' \Subset \boldsymbol{M}$.*

Due to the Canonical model theorem, in the particular case when $\boldsymbol{M}$ is a canonical model, the condition (*sel*) can be rewritten as follows:

$$\Diamond A \in u \quad \Longrightarrow \quad \exists v \in R'(u)\, A \in v. \tag{cm-sel}$$

LEMMA 3.17. *Let a Kripke model $\boldsymbol{M}' = (W', R', D', \xi')$ be a weak submodel of $\boldsymbol{M} = (W, R, D, \xi)$. Then, the following conditions are equivalent*:

(1) $\boldsymbol{M}' \Subset \boldsymbol{M}$;
(2) *for every $u \in W'$ and every D_u-sentence A,*

$$\boldsymbol{M}, u \models A \quad \Longleftrightarrow \quad \boldsymbol{M}', u \models A.$$

[7] Of course, in propositional logic (strong) Kripke completeness and canonicity are defined in terms of propositional rather than predicate Kripke frames.

[8] Here we use the notion of strong Kripke completeness from [2]; the claim, however, also holds for strong Kripke completeness with constants considered in [7].

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Proof. (1) ⇒ (2): We proceed by induction on A, written using only logical symbols $\bot$, $\to$, $\forall$, and $\Diamond$. We only consider the case $A = \Diamond B$, leaving the rest to the reader.

First, suppose that $u \in W'$ and $\boldsymbol{M}, u \models \Diamond B$. Since $\boldsymbol{M}' \Subset \boldsymbol{M}$ and $u \in W'$, there exists $v \in R'(u)$ such that $\boldsymbol{M}, v \models B$. Then, by inductive hypothesis, $\boldsymbol{M}', v \models B$, and so $\boldsymbol{M}', u \models \Diamond B$.

Next, suppose that $\boldsymbol{M}', u \models \Diamond B$. Then there exists $v \in R'(u)$ such that $\boldsymbol{M}', v \models B$. Since $\boldsymbol{M}'$ is a weak submodel of $\boldsymbol{M}$, we know that $R' \subseteq R$, and so $v \in R(u)$. By inductive hypothesis, $\boldsymbol{M}, v \models B$. Hence, $\boldsymbol{M}, u \models \Diamond B$.

(2) ⇒ (1): We show that (*sel*) holds for $\boldsymbol{M}$ and $\boldsymbol{M}'$. Let $u \in W'$ and let A be a D_u-sentence such that $\boldsymbol{M}, u \models \Diamond A$. Then, by (2), $\boldsymbol{M}', u \models \Diamond A$, and so $\exists v \in R'(u)\, \boldsymbol{M}', v \models A$, which, by (2), is equivalent to $\exists v \in R'(u)\, \boldsymbol{M}, v \models A$. □

LEMMA 3.18. *Let $\boldsymbol{M}$, $\boldsymbol{M}'$, and $\boldsymbol{M}''$ be predicate Kripke models. Then,*

(1) *$\boldsymbol{M}'' \Subset \boldsymbol{M}'$ and $\boldsymbol{M}' \Subset \boldsymbol{M}$ imply $\boldsymbol{M}'' \Subset \boldsymbol{M}$;*
(2) *$\boldsymbol{M}'{\uparrow}u \Subset \boldsymbol{M}$ implies $\boldsymbol{M}'{\uparrow}u \Subset \boldsymbol{M}{\uparrow}u$.*

Proof. (1) Follows from Lemma 3.17.

(2) If $\boldsymbol{M}' \uparrow u \Subset \boldsymbol{M}$, that $\boldsymbol{M}'{\uparrow}u$ is a weak submodel of $\boldsymbol{M}{\uparrow}u$. Also, if v is a world from $\boldsymbol{M}'{\uparrow}u$ (and hence from $\boldsymbol{M}{\uparrow}u$), then, by Lemma 3.17 and Proposition 2.8(1),

$$\boldsymbol{M}'{\uparrow}u, v \models A \iff \boldsymbol{M}, v \models A \iff \boldsymbol{M}{\uparrow}u, v \models A.$$

Hence, by Lemma 3.17, $\boldsymbol{M}'{\uparrow}u \Subset \boldsymbol{M}{\uparrow}u$. □

DEFINITION 3.19. *A* quasi-canonical model *for a predicate logic L is a selective weak submodel of the canonical model for L.*

DEFINITION 3.20. *A predicate logic L is* quasi-canonical *if, for every L-place Γ, there exists a quasi-canonical model over an L-frame containing Γ.*

THEOREM 3.21. *Every quasi-canonical logic is strongly Kripke complete.*

Proof. Let L be a quasi-canonical logic and let Γ_0 be an L-consistent theory without constants. By Henkin–Lindenbaum lemma, there exists an L-place Γ such that $\Gamma_0 \subseteq \Gamma$. Hence, by the Canonical model theorem, $\boldsymbol{M}_L, \Gamma \models \Gamma_0$.

Since L is quasi-canonical, there exists a weak selective submodel $\boldsymbol{M} = (W, R, D, \xi)$ of $\boldsymbol{M}_L$ such that $(W, R, D) \models L$ and $\Gamma \in W$. By Lemma 3.17, $\boldsymbol{M}, \Gamma \models \Gamma_0$. Hence, L is strongly Kripke complete. □

We now turn to proving our main result in this section. If Γ is an L-place, we set

$$C(\Gamma) \;:=\; \{\Delta \mid \Delta \text{ is an } L\text{-maximal theory } \&\; D_\Delta = D_\Gamma \;\&\; \Box^{-}\Gamma \subseteq \Delta\}.$$

Due to Henkin–Lindenbaum lemma, for each $\Delta \in C(\Gamma)$, we can choose some $\Delta^\circ \in W_L$ with $\Delta \subseteq \Delta^\circ$; then, we put

$$C^\circ(\Gamma) \;:=\; \{\Delta^\circ \mid \Delta \in C(\Gamma)\}.$$

We emphasise that $C^\circ(\Gamma)$ contains exactly one $\Delta^\circ \in W_L$ for each $\Delta \in C(\Gamma)$; this implies that $|C^\circ(\Gamma)| \leqslant |C(\Gamma)|$. Note that $C^\circ(\Gamma)$ is not uniquely determined by $C(\Gamma)$, since we choose each Δ° arbitrarily. Also note that $C^\circ(\Gamma) \subseteq R_L(\Gamma)$.

LEMMA 3.22. *For every D_Γ-sentence A,*

$$\Diamond A \in \Gamma \implies \exists \Delta^\circ \in C^\circ(\Gamma)\; A \in \Delta^\circ.$$

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Proof. By Existence lemma and definition of $C(\Gamma)$, there exists $\Delta \in C(\Gamma)$ such that $A \in \Delta$; since $\Delta \subseteq \Delta^\circ$, surely $A \in \Delta^\circ$. □

We now prove the main result of this section.

THEOREM 3.23. *Let* $L \in \{\mathbf{QAlt}_n, \mathbf{QTAlt}_n\}$, *with* $n \geqslant 1$. *Then*, L *is quasi-canonical and, hence, strongly Kripke complete.*

Proof. Let $\boldsymbol{M}_L = (W_L, R_L, D_L, \xi_L)$ be the canonical model for L and let $\Gamma_0 \in W_L$. We obtain a selective weak submodel of $\boldsymbol{M}_L$ over an L-frame containing Γ_0. First, we prove the following. □

LEMMA 3.24. $|C(\Gamma)| \leqslant n$, *for every* $\Gamma \in W_L$.

Proof. Suppose, for contradiction, that $\Delta_0, \ldots, \Delta_n$ are distinct theories from $C(\Gamma)$.

Since $D_{\Delta_i} = D_\Gamma$, (that is, $\mathcal{L}(\Delta_i) = \mathcal{L}(\Gamma)$, for all $i \leqslant n$), and since all these theories are maximal, we know that, for all $i, j \in \{0, \ldots, n\}$ with $i \neq j$, there exists $A_{ij} \in \mathcal{L}(\Gamma)$ such that $A_{ij} \in \Delta_i$ and $A_{ij} \notin \Delta_j$. Let $B_i := \bigwedge\limits_{j \neq i} A_{ij}$. Since every Δ_i is maximal, it follows, by Lemma 3.2(2), that $B_i \in \Delta_j$ iff $i = j$, and thus, by Lemma 3.1, $(B_i \wedge \bigwedge\limits_{j \neq i} \neg B_j) \in \Delta_i$, for all $i \leqslant n$. Since $\mathcal{L}(\Delta_i) = \mathcal{L}(\Gamma)$, we know that $(B_i \wedge \bigwedge\limits_{j \neq i} \neg B_j) \in \mathcal{L}(\Gamma)$, and so, by Lemma 3.3, $\Diamond(B_i \wedge \bigwedge\limits_{j \neq i} \neg B_j) \in \Gamma$.

Since Γ is L-maximal, it follows, by Lemma 3.2(2), that $\bigwedge\limits_{i=0}^{n} \Diamond(B_i \wedge \bigwedge\limits_{j \neq i} \neg B_j) \in \Gamma$. On the other hand, the sentence $\neg(\bigwedge\limits_{i=0}^{n} \Diamond(B_i \wedge \bigwedge\limits_{j \neq i} \neg B_j))$ is a substitution instance of $\boldsymbol{alt}_n$; hence, it is L-derivable, and so, by Lemma 3.2 (1), belongs to Γ. This contradicts L-consistency of Γ. □

We now proceed with the proof of the theorem, distinguishing two cases.

Case $L = \mathbf{QAlt}_n$: We construct the frame (W, R) of a sought selective weak submodel $\boldsymbol{M}$ of $\boldsymbol{M}_L$ by recursion. To that end, we define an increasing sequence $(W_k, R_k)_{k<\omega}$ of weak subframes of the canonical frame (W_L, R_L).

Steps 0 and 1: Put $W_0 := \varnothing$, $W_1 := \{\Gamma_0\}$, and $R_0 := R_1 := \varnothing$.

Step$(k+1)$, *with* $k > 0$: Suppose the frames (W_k, R_k) and (W_{k-1}, R_{k-1}) have been defined. Call the worlds from the set $W_k - W_{k-1}$ *new* at this step. Put

$$W_{k+1} \quad := \quad W_k \cup \bigcup_{\Gamma \in W_k - W_{k-1}} C^\circ(\Gamma).$$

Next, for every $\Gamma \in W_{k+1}$, put

$$R_{k+1}(\Gamma) \quad := \quad \begin{cases} R_k(\Gamma) & \text{if } \Gamma \in W_{k-1}; \\ C^\circ(\Gamma) & \text{if } \Gamma \in W_k - W_{k-1}; \\ \varnothing & \text{if } \Gamma \in W_{k+1} - W_k. \end{cases} \tag{3.1}$$

Then, $W_k \subseteq W_{k+1} \subseteq W_L$ and $R_k \subseteq R_{k+1} \subseteq R_L$. In particular, if there are no new worlds, that is, if $W_k = W_{k-1}$, then $W_{k+1} = W_k$ and $R_{k+1} = R_k$.

Lastly, we put

$$W \quad := \quad \bigcup_{k<\omega} W_k; \qquad R \quad := \quad \bigcup_{k<\omega} R_k; \qquad D := D_L \restriction W; \qquad \xi := \xi_L \restriction W,$$

and $\boldsymbol{M} := (W, R, D, \xi)$.

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We next prove that $\boldsymbol{M}$ is a weak selective submodel of $\boldsymbol{M}_L$ based on an L-frame.

First, we show that $(W, R, D) \models L$. It follows from the definition of our sequence of weak frames that, if Γ is new at step $k+1$, then $R_{k+1}(\Gamma) = C^\circ(\Gamma)$ and $R_{i+1}(\Gamma) = R_i(\Gamma)$, for every $i \geqslant k+1$. Hence, if $\Gamma \in W$, then $R(\Gamma) = C^\circ(\Gamma)$, and so $|R(\Gamma)| = |C^\circ(\Gamma)| \leqslant |C(\Gamma)|$. Thus, by Lemma 3.24, $|R(\Gamma)| \leqslant n$, which is equivalent, by Proposition 2.2(1), to $(W, R) \models \boldsymbol{alt}_n$ and so, by Proposition 2.4, and Soundness theorem, to $(W, R, D) \models L$.

Next, since $R(\Gamma) = C^\circ(\Gamma)$ for every $\Gamma \in W$, it follows from Lemma 3.22 that (*cm-sel*) holds for all $\Gamma \in W$ and $A \in \mathcal{L}(\Gamma)$. Hence, $\boldsymbol{M} \Subset \boldsymbol{M}_L$.

Thus, $\boldsymbol{M}$ is a quasi-canonical model for L over an L-frame containing Γ_0. Therefore, L is quasi-canonical.

Case $L = \mathbf{QTAlt}_n$: As in the previous case, we construct the sequence (W_k, R_k) of weak subframes of (W_L, R_L) by recursion. In this case, R_L is reflexive, and we need to ensure reflexivity of R. The recursion proceeds similarly to the previous case, with one modification: since $\Box^- \Gamma \subseteq \Gamma$, it follows that $\Gamma \in C(\Gamma)$, and so we can choose Γ° to be Γ itself, thus making sure that $\Gamma \in C^\circ(\Gamma) (= R(\Gamma))$. Then, by Propositions 2.2(2) and 2.4, and Soundness theorem, $(W, R, D) \models L$, and so $\boldsymbol{M}$ is a quasi-canonical model for L based on an L-frame. Hence, L is quasi-canonical. □

We next show that strong Kripke completeness is preserved if we add to the logics from Theorem 3.23 closed propositional formulas (i.e., propositional formulas without proposition letters). First, we recall the following [2, Lemma 6.1.25(1) and (2)].

PROPOSITION 3.25. *Let L and L' be logics such that $L \subseteq L'$. Then,*

(1) $W_{L'} = \{\Gamma \in W_L \mid \overline{L'} \subseteq \Gamma\}$;
(2) $\boldsymbol{M}_{L'} \sqsubseteq \boldsymbol{M}_L$.

Next, we prove that quasi-canonicity is preserved under the addition of closed propositional formulas.

THEOREM 3.26. *If L is a quasi-canonical logic and X is a set of closed propositional formulas, then the logic $L + X$ is quasi-canonical.*

Proof. Let $L' := L + X$ and suppose that $\Gamma_0 \in W_{L'}$. By Proposition 3.25(1), $\Gamma_0 \in W_L$. Since L is quasi-canonical, there exists a model $\boldsymbol{M} = (W, R, D, \xi)$ such that $\boldsymbol{M} \Subset \boldsymbol{M}_L$, $(W, R, D) \models L$ and $\Gamma_0 \in W$. We show that $\boldsymbol{M}{\uparrow}\Gamma_0$ is a weak selective submodel of $\boldsymbol{M}_{L'}$ based on an L'-frame, which proves that L' is quasi-canonical.

In view of Lemma 3.16, $\boldsymbol{M}{\uparrow}\Gamma_0 \Subset \boldsymbol{M} \Subset \boldsymbol{M}_L$; hence, by Lemma 3.18, $\boldsymbol{M}{\uparrow}\Gamma_0 \Subset \boldsymbol{M}_L{\uparrow}\Gamma_0$. Since, by Proposition 3.25(2), $\boldsymbol{M}_{L'} \sqsubseteq \boldsymbol{M}_L$ and since $\Gamma_0 \in W_{L'}$, it follows that $\boldsymbol{M}_L{\uparrow}\Gamma_0 \sqsubseteq \boldsymbol{M}_{L'}$. Hence, by Lemma 3.18(1), $\boldsymbol{M}{\uparrow}\Gamma_0 \Subset \boldsymbol{M}_{L'}$.

Since, by Canonical model theorem, $\boldsymbol{M}_{L'} \models X$, it then follows by Lemma 3.17, that $\boldsymbol{M}{\uparrow}\Gamma_0 \models X$. Since X contains only closed propositional formulas, whose truth does not depend on valuations, it follows that $\boldsymbol{F}{\uparrow}\Gamma_0 \models X$, where $\mathrm{F} = (\mathrm{W}, \mathrm{R}, \mathrm{D})$ is the frame underlying $\boldsymbol{M}$.

Moreover, since $\boldsymbol{F} \models L$, it follows, by Proposition 2.8(2), that $\boldsymbol{F}{\uparrow}\Gamma_0 \models L$. Thus, by Soundness theorem, $\boldsymbol{F}{\uparrow}\Gamma_0 \models L + X\ (= L')$. □

COROLLARY 3.27. *Let X be a set of closed propositional formulas. Then, for every $n \geqslant 1$, logics $\mathbf{QAlt}_n + X$ and $\mathbf{QTAlt}_n + X$ are quasi-canonical, and hence strongly Kripke complete.*

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We conclude this section with an observation that will be used in §5.

PROPOSITION 3.28. *The intersection of two quasi-canonical logics is quasi-canonical.*

Proof. Let L_1 and L_2 be quasi-canonical logics and let $L := L_1 \cap L_2$. Suppose that $\Gamma \in W_L$. We obtain a selective weak submodel of $\boldsymbol{M}_L$ over an L-frame containing Γ.

First, note that $\Gamma \in W_{L_1} \cup W_{L_2}$: otherwise, there exists a finite subset X of Γ that is both L_1-inconsistent and L_2-inconsistent; hence, $\vdash_{L_1} \neg \bigwedge X$ and $\vdash_{L_2} \neg \bigwedge X$, and so $\vdash_L \neg \bigwedge X$, in contradiction with L-consistency of Γ.

Now, suppose that $\Gamma \in W_{L_1}$ (if $\Gamma \in W_{L_2}$, the argument is similar). Since L_1 is quasi-canonical, there exists a model $\boldsymbol{M} \Subset \boldsymbol{M}_{L_1}$ based on an L_1-frame containing Γ. By Proposition 3.25, $\boldsymbol{M}_{L_1} \sqsubseteq \boldsymbol{M}_L$, and so, by Lemma 3.16, $\boldsymbol{M}_{L_1} \Subset \boldsymbol{M}_L$. Hence, by Lemma 3.18(1), $\boldsymbol{M} \Subset \boldsymbol{M}_L$. Since the frame of $\boldsymbol{M}$ validates L_1 and $L \subseteq L_1$, it surely validates L. Thus, $\boldsymbol{M}$ is a sought selective weak submodel of $\boldsymbol{M}_L$, and so L is quasi-canonical. □

§4. Kripke incompleteness of QK4Alt$_n$ and some related logics. In this section, we prove Kripke incompleteness of some logics $\mathbf{Q}\boldsymbol{\Lambda}$ where $\boldsymbol{\Lambda}$ is either a propositional extension of $\mathbf{K4Alt}_n$ or a closely related logic.

***4.1.* QS4Alt$_n$.** We begin with the simple case of logics $\mathbf{QS4Alt}_n$ with $n \geqslant 2$. Recall that a propositional logic is *locally tabular* if, for every $n \geqslant 0$, it contains only finitely many pairwise non-equivalent formulas in n variables.

THEOREM 4.1. *If* $\boldsymbol{\Lambda}$ *is a locally tabular propositional logic such that* $\mathbf{S4} \subseteq \boldsymbol{\Lambda}$ *and* $\mathbf{Q}\boldsymbol{\Lambda}$ *is Kripke complete, then* $\mathbf{S5} \subseteq \boldsymbol{\Lambda}$.

Proof. Since $\mathbf{S4} \subseteq \boldsymbol{\Lambda}$ and $\mathbf{Q}\boldsymbol{\Lambda}$ is Kripke complete, it follows, by [3, Theorem 7.3], that either $\boldsymbol{\Lambda} \subseteq \mathbf{S4.3}$ or $\mathbf{S5} \subseteq \boldsymbol{\Lambda}$. Since $\mathbf{S4.3}$ is not locally tabular, and local tabularity is preserved for extensions, surely $\boldsymbol{\Lambda} \not\subseteq \mathbf{S4.3}$. Hence, $\mathbf{S5} \subseteq \boldsymbol{\Lambda}$. □

COROLLARY 4.2. *For every* $n \geqslant 2$, *the logic* $\mathbf{QS4Alt}_n$ *is Kripke incomplete.*

Proof. By Segerberg's theorem [1, Theorem 12.21], $\mathbf{S4Alt}_n$ is locally tabular[9] Since $\mathbf{S5}$ is refuted on a two-element chain, which validates $\mathbf{S4Alt}_n$, surely $\mathbf{S5} \not\subseteq \mathbf{S4Alt}_n$. Hence, by Theorem 4.1, $\mathbf{QS4Alt}_n$ is Kripke incomplete. □

***4.2. Logics between* QK4Alt$_n$ *and* QSL4.** For incompleteness proofs in this section we use Kripke bundle semantics, which we now briefly recall (for more details, consult [2, Chapter 5]).

A *Kripke bundle* is a tuple $\mathbb{F} = (F, D, \rho)$ where $F = (W, R)$ is a Kripke frame, $D = (D_u \mid u \in W)$ is a family of non-empty disjoint domains, and $\rho = (\rho_{uv} \mid (u, v) \in R)$ is a family of *inheritance relations* $\rho_{uv} \subseteq D_u \times D_v$ satisfying the constraint that $\rho_{uv}(a) \neq \varnothing$ whenever uRv and $a \in D_u$ (i.e., every individual has at least one inheritor in each accessible world).

A *model* over a Kripke bundle $\mathbb{F}$ is a pair $\mathbb{M} = (\mathbb{F}, \xi)$ where ξ is a family $(\xi_u)_{u \in W}$ of valuations: each ξ_u is a function assigning to each n-ary predicate letter a subset of D_u^n. The *truth relation* $\models$ between a world u of a model $\mathbb{M}$ and a D_u-sentences A is defined by recursion:

[9] By Chagrov's criterion [1, Theorem 12.1], it is even tabular.

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- $\mathbb{M}, u \models P(a_1, \dots, a_n)$ if $(a_1, \dots, a_n) \in \xi_u(P)$;
- $\mathbb{M}, u \not\models \bot$;
- $\mathbb{M}, u \models A \to B$ if $\mathbb{M}, u \not\models A$ or $\mathbb{M}, u \models B$;
- $\mathbb{M}, u \models \forall x\, A(x)$ if $\mathbb{M}, u \models A(a)$ whenever $a \in D_u$;
- $\mathbb{M}, u \models \Box A(a_1, \dots, a_n)$, with distinct $a_1, \dots, a_n \in D_u$, if

$$\forall v \in R(u)\, \forall b_1 \in \rho_{uv}(a_1) \dots \forall b_n \in \rho_{uv}(a_n)\ \mathbb{M}, v \models A(b_1, \dots, b_n).$$

Notice that, if a constant a occurs in a D_u-sentence A several times, then determining whether $\mathbb{M}, u \models \Box A$ involves substituting, for every $v \in R(u)$, the same v-inheritor for every occurrence of a in A.

A formula is (globally) *true* in a Kripke bundle model if its universal closure is true at every world of the model. A formula A is *strongly valid* on a Kripke bundle $\mathbb{F}$, also written $\mathbb{F} \models^+ A$, if every substitution instance of A is true in every model over $\mathbb{F}$ (notice that this notion differs from that of validity on Kripke frames).

The following is well known [2, Proposition 5.2.12].

PROPOSITION 4.3. *For every Kripke bundle $\mathbb{F}$, the set $\{A \mid \mathbb{F} \models^+ A\}$ is a predicate modal logic.*

DEFINITION 4.4. *With a Kripke bundle $\mathbb{F} = (W, R, D, \rho)$, we associate a family $\{F_k \mid k < \omega\}$ of propositional Kripke frames, in the following way: set $D_0 := W$ and $R_0 := R$, and, for every $k > 0$, set*

$$\begin{aligned} D_k &:= \bigcup \{D_u^k \mid u \in W\}; \\ R_k &:= \{(\mathbf{a}, \mathbf{b}) \in D_k \times D_k \mid \forall i\ a_i \rho b_i\ \&\ \forall i, j\ (a_i = a_j \Rightarrow b_i = b_j)\}; \\ F_k &:= (D_k, R_k). \end{aligned}$$

We will be using the following well-known [2, Proposition 5.3.7] criterion of strong validity of propositional formulas on Kripke bundles.

PROPOSITION 4.5. *Let $\mathbb{F}$ be a Kripke bundle and A a propositional modal formula. Then $\mathbb{F} \models^+ A$ iff $F_k \models A$, for every $k < \omega$.*

We now prove that, if $n \geqslant 1$, then every logic between $\mathbf{QK4Alt}_n$ and $\mathbf{QSL4}$ is Kripke incomplete. To that end, we use the following sentences:

$$\begin{aligned} \forall ref &:= \forall x\, (\Box P(x) \to P(x)); \\ A_n &:= \Diamond^{n+1} \top \to \Diamond \forall ref. \end{aligned}$$

We recall that $\models_{\mathcal{K}}$ denotes consequence in Kripke semantics.

LEMMA 4.6. *If $n \geqslant 1$, then $\mathbf{QK4Alt}_n \models_{\mathcal{K}} A_n$.*

Proof. Fix $n \geqslant 1$ and suppose that $\boldsymbol{F} = (W, R, D)$ is a $\mathbf{QK4Alt}_n$-frame. By Proposition 2.4, $(W, R) \models \mathbf{K4Alt}_n$, and so, by Proposition 2.2(1) and (3), R is transitive and $|R(u)| \leqslant n$, for every $u \in W$. To prove that $\boldsymbol{F} \models A_n$, we assume that $\boldsymbol{F}, u_0 \models \Diamond^{n+1} \top$ to show that $\boldsymbol{F}, u_0 \models \Diamond \forall ref$, for an arbitrary u_0 from $\boldsymbol{F}$. Since $\boldsymbol{F}, u_0 \models \Diamond^{n+1} \top$, there exist $u_1, \dots, u_{n+1} \in W$ such that $u_0 R u_1 \dots u_n R u_{n+1}$; hence, by transitivity, $u_0 R u_i$, for every $i \in \{1, \dots, n+1\}$. Since $|R(u_0)| \leqslant n$, there exist $i, j \in \{1, \dots, n+1\}$ such that $i \neq j$ and $u_i = u_j$. Since R is transitive, it follows that u_i is reflexive. Hence, $\boldsymbol{M}, u_i \models \Box P(a) \to P(a)$, for every model $\boldsymbol{M}$ over $\boldsymbol{F}$ and every $a \in D_{u_i}$. Thus, $\boldsymbol{F}, u_i \models \forall ref$, and so $\boldsymbol{F}, u_0 \models \Diamond \forall ref$. □

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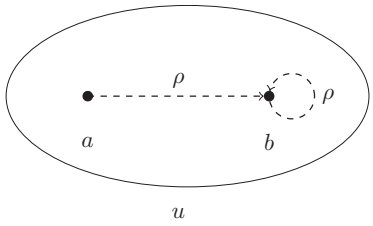


Figure 1. Kripke bundle $\mathbb{F}_0$.

LEMMA 4.7. **QSL4** $\nvdash A_n$.

Proof. In view of Proposition 4.3, it suffices to obtain a Kripke bundle strongly validating **SL4**, but not A_n. To that end, we use a Kripke bundle from [7] (see Figure 1):

$$W := \{u\},\ \ R := \{(u,u)\},\ \ D_u := \{a,b\},\ \ \rho_{uu} := \{(a,b),(b,b)\},\ \ \mathbb{F}_0 := (W,R,D,\rho).$$

To see that $\mathbb{F}_0 \not\models^+ A_n$, consider the model $\mathbb{M}_b = (\mathbb{F}_0, \xi)$ where $\xi_u(P) = \{b\}$. Since $R(u) = \{u\}$, $\rho_{uu}(a) = \{b\}$, and $\mathbb{M}_b, u \models P(b)$, it follows that $\mathbb{M}_b, u \models \Box P(a)$. Since also $\mathbb{M}_b, u \not\models P(a)$, it follows that $\mathbb{M}_b, u \not\models \Box P(a) \to P(a)$ and so $\mathbb{M}_b, u \not\models \forall ref$. Since u sees only itself, $\mathbb{M}_b, u \not\models \Diamond \forall ref$. On the other hand, since R is serial, $\mathbb{M}_b, u \models \Diamond^{n+1}\top$. Hence, $\mathbb{M}_b, u \not\models A_n$, and so $\mathbb{F}_0 \not\models^+ A_n$.

That $\mathbb{F}_0 \models^+ \mathbf{SL4}$ was observed in [7, Proof of Lemma 5.12]: by Proposition 4.5, it suffices to check that, for every propositional frame $F_k = (D_k, R_k)$ associated with $\mathbb{F}_0$ (see Definition 4.4), $F_k \models \mathbf{SL4}$; this holds since every R_k is transitive and functional (and, hence, 1-alternative). □

THEOREM 4.8. *Let $n \geqslant 1$. Then, every logic between* $\mathbf{QK4Alt}_n$ *and* **QSL4** *is Kripke incomplete.*

Proof. Let L be a logic such that $\mathbf{QK4Alt}_n \subseteq L \subseteq \mathbf{QSL4}$. By Lemma 4.6, $L \models_{\mathcal{K}} A_n$, and, by Lemma 4.7, $L \nvdash A_n$. The statement of the theorem follows. □

***4.3. Extensions of* QK4.1.** For the proofs in this section, we use the sentence

$$B_1 \quad := \quad \Diamond\Box\exists x P(x) \to \Diamond\exists x \Box P(x).$$

LEMMA 4.9. $\mathbf{QK4.1} \models_{\mathcal{K}} B_1$.

Proof. Suppose that $\boldsymbol{F} = (W,R,D)$ is a **QK4.1**-frame, and so, by Proposition 2.4, $(W,R) \models \mathbf{K4.1}$. Then, by Propositions 2.2(3) and 2.3, R is transitive and every world from W sees a maximal world.

To prove that $\boldsymbol{F} \models B_1$, we consider a model over $\boldsymbol{F}$ and, assuming that, in that model, $u \models \Diamond\Box\exists x P(x)$, show that $u \models \Diamond\exists x\Box P(x)$. Since $u \models \Diamond\Box\exists x P(x)$, there exists $v \in R(u)$ such that $v \models \Box\exists x P(x)$. Let w be a maximal world accessible from v. Then, $w \models \exists x P(x)$, and so $w \models P(a)$, for some $a \in D_w$. Since w is maximal, surely $w \models \Box P(a)$, and so $w \models \exists x \Box P(x)$. Since R is transitive, $w \in R(u)$; hence, $u \models \Diamond\exists x\Box P(x)$. □

LEMMA 4.10. *Let* $\boldsymbol{\Lambda}$ *be a Kripke complete propositional extension of* **K4.1** *such that* $\boldsymbol{\Lambda} \neq \mathbf{Triv}$. *Then, either* $\boldsymbol{\Lambda} \subseteq \mathbf{SL4}$ *or* $\boldsymbol{\Lambda} \subseteq \mathbf{S4.1Alt}_2$.

Proof. First, we recall that $\mathbf{K} + \boldsymbol{ma} \vdash \Diamond\top$: since $\mathbf{K} \vdash \Box\top$, it follows that $\mathbf{K} + \Box\Diamond\top \to \Diamond\Box\top \vdash \Box\Diamond\top \to \Diamond\top$; since $\mathbf{K} \vdash \Box\bot \to \Box\Diamond\top$, it then follows that

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Figure 2. Kripke frames G_1 (left) and G_2 (right).

$\mathbf{K} + \boldsymbol{ma} \vdash \Box\bot \to \Diamond\top$, and so $\mathbf{K} + \boldsymbol{ma} \vdash \Diamond\top$. Thus, every Kripke frame validating $\mathbf{\Lambda}$ is serial.

By assumption, there exists a Kripke frame F such that $\mathbf{\Lambda} = \mathbf{L}(F)$. As we have seen, F is serial; moreover, since $\mathbf{\Lambda} \neq \mathbf{Triv}$, this frame is not a singleton. Hence, there exists a nontrivial rooted generated subframe $G = (W, R)$ of F, and so, by Generation lemma (the propositional version of Proposition 2.8(2)), $\mathbf{\Lambda} = \mathbf{L}(F) \subseteq \mathbf{L}(G)$.

Now, consider Kripke frames (see Figure 2)

$$G_1 := (\{a, b\}, \{(a, b), (b, b)\}) \quad \text{and} \quad G_2 := (\{c, d\}, \{(c, c), (c, d), (d, d)\}).$$

One can show, using standard modal logic arguments, that $\mathbf{L}(G_1) = \mathbf{SL4}$ and $\mathbf{L}(G_2) = \mathbf{S4.1Alt}_2$. (The simplest way is to use Bellissima's completeness theorem mentioned in the introduction and consider rooted frames validating these logics: there are only a few possibilities.)

Next, we show that either $G \twoheadrightarrow G_1$ or $G \twoheadrightarrow G_2$.

Let u_0 be the root of G (thus, $W = \{u_0\} \cup R(u_0)$) and C_0 be the cluster of G containing u_0. Since G is nontrivial and $G \models \boldsymbol{ma}$, it follows, by Proposition 2.3, that G is not a cluster; hence, $C_0 \neq W$. Now, if u_0 is irreflexive, then there exists a p-morphism of G onto G_1 sending u_0 to a, and all other points to b. If, on the other hand, u_0 is reflexive, then there exists a p-morphism of G onto G_2 mapping C_0 to c and $W - C_0$ to d.

Thus, either $G \twoheadrightarrow G_1$ or $G \twoheadrightarrow G_2$, and so either $\mathbf{L}(G) \subseteq \mathbf{L}(G_1)(= \mathbf{SL4})$ or $\mathbf{L}(G) \subseteq \mathbf{L}(G_2)(= \mathbf{S4.1Alt}_2)$, which proves the lemma. □

Remark 4.11. In fact, Lemma 4.10 holds for arbitrary, maybe Kripke incomplete, nontrivial extensions of **K4.1**; we give here a sketch of the proof.

First, consider a propositional logic $\mathbf{\Lambda}$ in the language with a fixed finite set of proposition letters (such logics are called *weak*) containing $\boldsymbol{ma}$ and **4**. Let M be the weak canonical model of $\mathbf{\Lambda}$ and let F be its underlying frame. In F, all maximal clusters are singletons. If $\mathbf{\Lambda} \neq \mathbf{Triv}$, then F must contain points at the next level (i.e., points of depth 2); see, e.g., [1, Section 8.7]. Let u be such a point. Then, $M{\uparrow}u$ is finite, and all its points are definable by formulas. By Canonical model theorem, $M \models \mathbf{\Lambda}$; hence, by Generation lemma, $M{\uparrow}u \models \mathbf{\Lambda}$. Since all subsets of $M{\uparrow}u$ are definable, it follows that $F{\uparrow}u \models \mathbf{\Lambda}$. The proof of Lemma 4.10 shows that either G_1 or G_2 is a p-morphic image of $F{\uparrow}u$; therefore, either $G_1 \models \mathbf{\Lambda}$ or $G_2 \models \mathbf{\Lambda}$.

Now, let $\mathbf{\Lambda}$ be a logic in countably many proposition letters and let $\mathbf{\Lambda} \restriction n$ be its fragment in n proposition letters. Then, as we have seen, either $\mathbf{\Lambda} \restriction n \subseteq \mathbf{SL4}$ or $\mathbf{\Lambda} \restriction n \subseteq \mathbf{S4.1Alt}_2$. If infinitely many of these weak logics are included into **SL4**, then $\mathbf{\Lambda} \subseteq \mathbf{SL4}$. Otherwise, $\mathbf{\Lambda} \restriction n \subseteq \mathbf{S4.1Alt}_2$, for every sufficiently large n, which implies that $\mathbf{\Lambda} \subseteq \mathbf{S4.1Alt}_2$.

Another proof is based on the (probably, folklore) fact that the lattice of extensions of **K4** is partitioned into slices numbered $1, 2, \ldots, \omega$. In our case, $\mathbf{\Lambda}$ is not in slice 1, so it is included into a maximal logic of slice 2. The latter is a logic of a two-element

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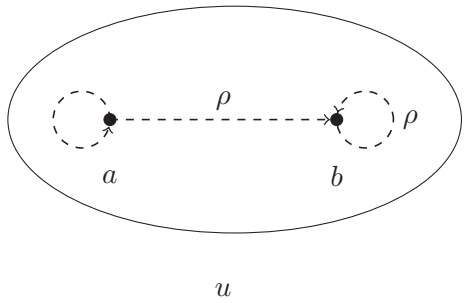


Figure 3. Kripke bundle $\mathbb{F}_1$.

chain containing reflexive or irreflexive points. Since $\mathbf{\Lambda} \supseteq \mathbf{K4.1}$, this chain is isomorphic either to G_1 or to G_2.

LEMMA 4.12. $\mathbf{QSL4} \nvdash B_1$.

Proof. In view of Proposition 4.3, it suffices to obtain a Kripke bundle strongly validating **QSL4**, but not B_1. We use the Kripke bundle $\mathbb{F}_0$ from the proof of Lemma 4.7. We know that $\mathbb{F}_0 \models^+ \mathbf{QSL4}$, so it remains to show that $\mathbb{F}_0 \not\models^+ B_1$.

To that end, consider the model $\mathbb{M}_a = (\mathbb{F}_0, \theta)$ where $\theta_u(P) = \{a\}$. Since, in this model, $u \models P(a)$, surely $u \models \exists x P(x)$, and so, by triviality of R, also $u \models \Diamond\Box\exists x P(x)$. On the other hand, since $u \not\models P(b)$, it follows that $u \not\models \Box P(a)$ and $u \not\models \Box P(b)$, and so $u \not\models \exists x \Box P(x)$. Hence, $u \not\models \Diamond\exists x\Box P(x)$. □

LEMMA 4.13. $\mathbf{QS4.1Alt}_2 \nvdash B_1$.

Proof. The proof is similar to that of the previous lemma. We use the Kripke bundle defined as follows (see Figure 3):

$$W := \{u\}, \quad R := \{(u,u)\}, \quad D_u := \{a,b\}, \quad \rho_{uu} := \{(a,a),(a,b),(b,b)\}, \quad \mathbb{F}_1 := (W,R,D,\rho).$$

To prove that $\mathbb{F}_1 \models^+ \mathbf{QS4.1Alt}_2$, we consider the associated family $\{F_k = (D_k, R_k) \mid k < \omega\}$ of Kripke frames and show that $F_k \models \mathbf{S4.1Alt}_2$, for every $k < \omega$.

Since F_1 is isomorphic to the frame G_2 from the proof of Lemma 4.10, we know that $F_1 \models \mathbf{S4.1Alt}_2$.

Now, let $k \geqslant 1$ and let $\mathbf{b} = (b, \ldots, b) \in D_k$. We show that, for every $\mathbf{c} \in D_k$, either both $\mathbf{c} = \mathbf{b}$ and $R_k(\mathbf{c}) = \{\mathbf{c}\}$, or else both $\mathbf{c} \neq \mathbf{b}$ and $R_k(\mathbf{c}) = \{\mathbf{c}, \mathbf{b}\}$.

If $\mathbf{c} = \mathbf{b}$, then, since $\rho(b) = \{b\}$, surely $R_k(\mathbf{c}) = R_k(\mathbf{b}) = \{\mathbf{b}\} = \{\mathbf{c}\}$.

Now, let $\mathbf{c} \neq \mathbf{b}$ and suppose that $\mathbf{e} \in R_k(\mathbf{c})$. Then, by definition of ρ, for every $i \in \{1, \ldots, k\}$,

$$c_i = b \implies e_i = b\,(= c_i),$$

and either $c_i = a \implies e_i = a\,(= c_i)$ for every i, or $c_i = a \implies e_i = b$ for every i. In the former case, $\mathbf{e} = \mathbf{c}$, while in the latter, $\mathbf{e} = \mathbf{b}$. Thus, $R_k(\mathbf{c}) = \{\mathbf{c}, \mathbf{b}\}$.

To show that $\mathbb{F}_1 \not\models^+ B_1$, we use the model $\mathbb{M}_1 = (\mathbb{F}_1, \theta)$ with $\theta_u(P) = \{a\}$. In this model, $u \models \exists x P(x)$, and so $u \models \Diamond\Box\exists x P(x)$. On the other hand, $u \not\models \Box P(a)$ and $u \not\models \Box P(b)$, and so $u \not\models \Diamond\exists x\Box P(x)$. Thus, $\mathbb{F}_1 \not\models^+ B_1$. □

THEOREM 4.14. *If* $\mathbf{\Lambda}$ *is a propositional extension of* **K4.1** *such that* $\mathbf{Q\Lambda}$ *is Kripke complete, then* $\mathbf{\Lambda} = \mathbf{Triv}$.

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Proof. Assume, for contradiction, that $\mathbf{\Lambda} \neq \mathbf{Triv}$. By Proposition 2.7, $\mathbf{\Lambda}$ is Kripke complete. Hence, by Lemma 4.10 and Proposition 2.1(1), either $\mathbf{QK4.1} \subseteq \mathbf{Q\Lambda} \subseteq \mathbf{QSL4}$ or $\mathbf{QK4.1} \subseteq \mathbf{Q\Lambda} \subseteq \mathbf{QS4.1Alt}_2$. Thus, by Lemma 4.9, $\mathbf{Q\Lambda} \models_{\mathcal{K}} B_1$, while, by Lemmas 4.12 and 4.13, $\mathbf{Q\Lambda} \not\vdash B_1$, in contradiction with Kripke completeness of $\mathbf{Q\Lambda}$. □

***4.4. Extensions of* QD4.** For the proofs in this section, we use the sentence

$$B_2 \quad := \quad \Diamond\Box\Diamond\exists x P(x) \to \Diamond\exists x\Box\Diamond P(x).$$

LEMMA 4.15. *If* $\mathbf{\Lambda}$ *is a locally tabular extension of* **K4**, *then* $\mathbf{Q\Lambda} \models_{\mathcal{K}} B_2$.

Proof. Let $\boldsymbol{F} = (W, R, D)$ be a $\mathbf{Q\Lambda}$-frame. Then, by Proposition 2.4, $(W, R) \models \mathbf{\Lambda}$. By Segerberg–Maksimova theorem [1, Theorem 12.21], $\mathbf{\Lambda}$ has a finite transitive depth. Hence, every point from W sees a maximal cluster.

To prove that $\boldsymbol{F} \models B_2$, we consider a model over $\boldsymbol{F}$ and, assuming that, in that model, $u \models \Diamond\Box\Diamond\exists x P(x)$, show that $u \models \Diamond\exists x\Box\Diamond P(x)$.

Since $u \models \Diamond\Box\Diamond\exists x P(x)$, there exists $v \in R(u)$ such that $v \models \Box\Diamond\exists x P(x)$. Let $C \subseteq R(v)$ be a maximal cluster. Then, there exist $w \in C$ and $a \in D_w$ such that $w \models P(a)$. But then $w \models \Box\Diamond P(a)$: if $t \in C$, then $D_t = D_w$ and so $a \in D_t$, which together with $w \in R(t)$ gives us $t \models \Diamond P(a)$; since $R(w) = C$, it then follows that $w \models \Box\Diamond P(a)$. Thus, $w \models \exists x\Box\Diamond P(x)$, and so, by transitivity, $u \models \Diamond\exists x\Box\Diamond P(x)$. □

LEMMA 4.16. *Let* $\mathbf{\Lambda}$ *be a locally tabular extension of* **D4** *such that* $\mathbf{S5} \not\subseteq \mathbf{\Lambda}$. *Then, either* $\mathbf{\Lambda} \subseteq \mathbf{SL4}$ *or* $\mathbf{\Lambda} \subseteq \mathbf{S4Alt}_2$.

Proof. The argument is similar to that of Lemma 4.10.

It is well known that every locally tabular logic has the finite model property, and so is complete with respect to a class of rooted finite frames. Since $\mathbf{S5} \not\subseteq \mathbf{\Lambda}$, there exists a rooted finite $\mathbf{\Lambda}$-frame $G = (W, R)$ that is not a cluster. Since $\mathbf{D} \subseteq \mathbf{\Lambda}$, the frame G is serial. Let u be the root of G and C the cluster containing u.

We now argue similarly to the proof of Lemma 4.10. If u is irreflexive, there exists a p-morphism from G onto G_1 sending u to a and all other points to b. If, on the other hand, u is reflexive, then there exists a p-morphism from G onto G_2 mapping C to c and $W - C$ to d. Thus, either $\mathbf{\Lambda} \subseteq \mathbf{L}(G) \subseteq \mathbf{L}(G_1) (= \mathbf{SL4})$ or $\mathbf{\Lambda} \subseteq \mathbf{L}(G) \subseteq \mathbf{L}(G_2) (= \mathbf{S4Alt}_2)$. □

REMARK 4.17. Local tabularity of $\mathbf{\Lambda}$ is not essential for the previous lemma; the argument is similar to one at the end of Remark 4.11: if $\mathbf{\Lambda}$ is not in the first slice, then it is included into the logic of a two-element chain with reflexive or irreflexive points; since $\mathbf{\Lambda} \vdash \Diamond\top$, this chain is isomorphic either to G_1 or to G_2.

LEMMA 4.18. $\mathbf{QSL4} \not\vdash B_2$.

Proof. Similar to the proof of Lemma 4.12; we use the Kripke bundle model $\mathbb{M}_a$ defined there. Since we already know that the underlying Kripke bundle validates **QSL4**, we only need to show that $\mathbb{M}_a, u \not\models B_2$.

Since, in this model, $u \models \exists x P(x)$ and R is trivial, it follows that $u \models \Diamond\Box\Diamond\exists x P(x)$. On the other hand, since $u \not\models P(b)$, it follows that $u \not\models \Diamond P(b)$; hence, $u \not\models \Box\Diamond P(a)$ and $u \not\models \Box\Diamond P(b)$, and so $u \not\models \exists x\Box\Diamond P(x)$. Thus, $u \not\models \Diamond\exists x\Box\Diamond P(x)$, due to triviality of R. □

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LEMMA 4.19. $\mathbf{QS4Alt}_2 \nvdash B_2$.

Proof. Similar to the proof of Lemma 4.13; we use the Kripke bundle $\mathbb{F}_1$ and the model $\mathbb{M}_1$ defined there. We only need to show that $\mathbb{M}_1, u \not\models B_2$.

In this model, $u \models \exists x P(x)$, and so $u \models \Diamond\Box\Diamond\exists x P(x)$. On the other hand, $u \not\models \Diamond P(b)$, and so $u \not\models \Box\Diamond P(a)$ and $u \not\models \Box\Diamond P(b)$. Hence, $u \not\models \exists x\Box\Diamond P(x)$, and so $u \not\models \Diamond\exists x\Box\Diamond P(x)$. □

THEOREM 4.20. *Let* $\mathbf{\Lambda}$ *be a locally tabular extension of* **D4** *such that* $\mathbf{Q\Lambda}$ *is Kripke complete. Then* $\mathbf{S5} \subseteq \mathbf{\Lambda}$.

Proof. Similar to the proof of Theorem 4.14. By Lemmas 4.15, 4.18, and 4.19, $\mathbf{\Lambda} \not\subseteq \mathbf{SL4}$ and $\mathbf{\Lambda} \not\subseteq \mathbf{S4Alt}_2$. Hence, by Lemma 4.16, $\mathbf{S5} \subseteq \mathbf{\Lambda}$. □

REMARK 4.21. Note that Theorem 4.20 generalizes Theorem 4.1. There exists a continuum of locally tabular extensions of **D4** that are not extensions of **S4** (this can be shown, e.g., by constructing a continuum of **GL**-frames of height 3 not reducible to each other and then making their maximal points reflexive). Hence, Theorem 4.20 implies the existence of a continuum of Kripke incomplete modal predicate logics not covered by Ghilardi's theorem [3, Theorem 7.3].

§5. Kripke completions. In this section, we axiomatize Kripke completions for some of the Kripke incomplete logics considered in §4. We recall the following notation introduced in §§2 and 3: the Kripke completion of logic L is denoted by $\widehat{L}$; if L is a logic, Γ is a set of formulas, and A is a formula, then $\Gamma \vdash_L A$ means that A is derivable from Γ using theorems of L and rules (MP) and (Gen).

***5.1. Completion of* QSL4.** Let

$$L_0 \quad := \quad \mathbf{QSL4} + \Box\forall \mathit{ref}.$$

LEMMA 5.1. $\mathbf{QSL4} \models_{\mathcal{K}} \Box\forall \mathit{ref}$.

Proof. It is not hard to see that the propositional logic **SL4** has only two Kripke frames: a reflexive singleton and a frame isomorphic to the frame G_1 from the proof of Lemma 4.10. So, in every **QSL4**-frame, starting from any world, in a single step we reach a reflexive world; this world validates $\forall \mathit{ref}$. □

The following result was stated, without proof, in [7, Section 6].[10]

THEOREM 5.2. $\widehat{\mathbf{QSL4}} = L_0$.

Proof. In view of Lemma 5.1, it suffices to show that L_0 is Kripke complete. Due to Theorem 3.21, it is enough to prove that L_0 is quasi-canonical. Let $C(\Gamma)$ be defined as in §3.

First, we notice that, if $\Gamma \in W_{L_0}$, then $C(\Gamma)$ is a singleton: since $L_0 \vdash \Diamond\top$, we know that $\Diamond\top \in \Gamma$ and so, by Existence lemma, $C(\Gamma) \neq \varnothing$; also, $L_0 \vdash \boldsymbol{alt}_1$, and so, by Lemma 3.24, $|C(\Gamma)| \leqslant 1$.

Now, suppose that $\Gamma \in W_{L_0}$. To prove the existence of a quasi-canonical model containing Γ, we consider two cases.

Case $C(\Gamma) = \{\Gamma\}$: In this case, $\boldsymbol{M}_{L_0} \restriction \{\Gamma\}$ is a weak selective submodel of $\boldsymbol{M}_{L_0}$ since $\Diamond A \to \Box A \in \Gamma$, for every $A \in \mathcal{L}(\Gamma)$, and $\Box^{-}\Gamma \subseteq \Gamma$. Also, the base predicate

[10] The proof was not given there, since it is analogous to other proofs from that section.

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frame $\boldsymbol{F}$ validates L_0: since the underlying Kripke frame F is a reflexive singleton, surely $F \models \mathbf{SL4}$, and so, by Proposition 2.4, $\boldsymbol{F} \models \mathbf{QSL4}$; moreover, by Lemma 5.1, $\mathbf{QSL4} \models_{\mathcal{K}} \Box\forall ref$, and so $\boldsymbol{F} \models \Box\forall ref$.

Case $C(\Gamma) = \{\Delta\}$, *with* $\Delta \neq \Gamma$: We start by showing that, in this case, $\Delta R_{L_0}\Delta$.

First, we note that $\mathbf{SL4} \vdash \Box\boldsymbol{ref}$, which can be shown either semantically, using Kripke completeness of **SL4**, or syntactically, using

$$\mathbf{SL4} \vdash \Box(\Box P \to P) \leftrightarrow (\Box\Box P \to \Box P) \quad \text{and}$$
$$\mathbf{SL4} \vdash (\Box\Box P \to \Box P) \leftrightarrow (\Diamond\Diamond P \to \Diamond P).$$

Second, by [7, Proposition 3.10], $L_0 \vdash \Box A$ whenever $A \in \mathbf{QT}$; in particular, $L_0 \vdash \Box\overline{\forall}(\Box B \to B)$, for every formula B. Thus, by Lemma 3.2(1), $\Box\overline{\forall}(\Box B \to B) \in \Gamma$, and so, since $\Gamma R_{L_0}\Delta$,

$$\overline{\forall}(\Box B \to B) \in \Delta. \tag{5.1}$$

Now, to prove that $\Delta R_{L_0}\Delta$, we assume that $\Box C(\mathbf{a}) \in \Delta$, where $\mathbf{a}$ is a list of constants, to show that $C(\mathbf{a}) \in \Delta$. Since, by (5.1),

$$\forall\mathbf{x}(\Box C(\mathbf{x}) \to C(\mathbf{x})) \in \Delta,$$

it follows that

$$\Box C(\mathbf{a}) \to C(\mathbf{a}) \in \Delta,$$

and so, by (MP), $C(\mathbf{a}) \in \Delta$.

Since $\Delta R_{L_0}\Delta$, we know that $\Delta \in C(\Delta)$, and so, since $C(\Delta)$ is a singleton, $C(\Delta) = \{\Delta\}$. It follows that $\boldsymbol{M}_{L_0} \restriction \{\Gamma, \Delta\}$ is a selective weak submodel of $\boldsymbol{M}_{L_0}$. The underlying Kripke frame $\boldsymbol{F} := \boldsymbol{F}_{L_0} \restriction \{\Gamma, \Delta\}$ is isomorphic to the frame G_1 from the proof of Lemma 4.10, and so $\boldsymbol{F} \models \mathbf{SL4}$. Then, by Proposition 2.4, Lemma 5.1, and Soundness theorem, $\boldsymbol{F} \models L_0$. □

5.2. Completion of $\mathbf{QK4Alt}_1$**.** Let A_1 be as defined in §4.2 and recall that, by Lemma 4.6,

$$A_1\big(= \Diamond^2\top \to \Diamond\forall ref\big) \in \widehat{\mathbf{QK4Alt}}_1. \tag{5.2}$$

THEOREM 5.3. $\widehat{\mathbf{QK4Alt}}_1 = \mathbf{QK4Alt}_1 + A_1$.

Proof. Let $L_1 := \mathbf{QK4Alt}_1 + A_1$. In view of (5.2), it suffices to prove that L_1 is Kripke complete. Again, we show that L_1 is quasi-canonical. To this end, we prove a series of lemmas.

LEMMA 5.4. $L_1 + \Diamond^2\top = L_0$.

Proof. Since $\mathbf{K} + \Diamond^2\top \vdash \Diamond\top$, it follows that $L_1 + \Diamond^2\top \supseteq \mathbf{QSL4}$. Also, due to A_1,

$$L_1 + \Diamond^2\top \vdash \Diamond\forall ref,$$

and so, due to $\boldsymbol{alt}_1$,

$$L_1 + \Diamond^2\top \vdash \Box\forall ref.$$

Hence, $L_0 \subseteq L_1 + \Diamond^2\top$.

The converse inclusion $L_1 + \Diamond^2\top \subseteq L_0$ holds since $\mathbf{K4Alt}_1 \subseteq \mathbf{SL4}$, $\mathbf{SL4} \vdash \Diamond^2\top$, and $\mathbf{QSL4} \vdash \Box\forall ref \to \Diamond\forall ref$. □

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LEMMA 5.5. *For every* $n \geqslant 0$,

$$\mathbf{Alt}_1 + \Diamond^2\top \to \Box(\Box P \to P) \quad \vdash \quad \Diamond^2\top \to \Box^n\Diamond^2\top.$$

Proof. Let

$$\Lambda_0 \quad := \quad \mathbf{Alt}_1 + \Diamond^2\top \to \Box(\Box P \to P).$$

Then, by substitution,

$$\Lambda_0 \vdash \Diamond^2\top \to \Box(\Box^2\bot \to \Box\bot),$$

and so, by **K**,

$$\Lambda_0 \vdash \Diamond^2\top \to (\Diamond\Box^2\bot \to \Diamond\Box\bot).$$

Hence, by $\boldsymbol{alt}_1$,

$$\Lambda_0 \vdash \Diamond^2\top \to (\Diamond\Box^2\bot \to \Box^2\bot),$$

and so, by classical logic,

$$\Lambda_0 \vdash \Diamond^2\top \to \Box\Diamond^2\top. \tag{5.3}$$

Hence, by monotonicity of $\Box$, for every $k \geqslant 0$,

$$\Lambda_0 \vdash \Box^k\Diamond^2\top \to \Box^{k+1}\Diamond^2\top. \tag{5.4}$$

Eventually, by induction on n, using (5.3) and (5.4), we obtain that, for every $n \geqslant 0$,

$$\Lambda_0 \vdash \Diamond^2\top \to \Box^n\Diamond^2\top.$$

□

LEMMA 5.6. *For every formula* B,

$$L_0 \vdash B \quad \Longrightarrow \quad L_1 \vdash \Diamond^2\top \to B.$$

Proof. We use deduction theorems from [2]. By Lemma 5.4, $L_0 = L_1 + \Diamond^2\top$. So assume that $L_1 + \Diamond^2\top \vdash B$. Then, by [2, Theorem 2.8.4],

$$\Box^\infty\Diamond^2\top \vdash_{L_1} B.$$

Thus, for some $n \geqslant 0$,

$$\Diamond^2\top, \Box\Diamond^2\top, \ldots, \Box^n\Diamond^2\top \vdash_{L_1} B. \tag{5.5}$$

Due to $\boldsymbol{alt}_1$ and the substitution into A_1,

$$L_1 \vdash \Diamond^2\top \to \Box(\Box P \to P).$$

Hence, by Lemma 5.5, for every $n \geqslant 0$,

$$L_1 \vdash \Diamond^2\top \to \Box^n\Diamond^2\top,$$

and so, by (5.5),

$$\Diamond^2\top \vdash_{L_1} B.$$

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Hence, by Deduction theorem,

$$L_1 \vdash \Diamond^2\top \to B.$$

□

Now, put

$$L_2 \quad := \quad \mathbf{QAlt}_1 + \Box^2\bot.$$

LEMMA 5.7. $L_1 + \Box^2\bot = L_2$.

Proof. Since $\mathbf{K} \vdash \Box^2\bot \to \Box^2 P$, and so $\mathbf{K} \vdash \Box^2\bot \to (\Box P \to \Box^2 P)$, it follows that

$$\mathbf{K4} \subseteq \mathbf{K} + \Box^2\bot.$$

Also, by classical logic,

$$\mathbf{QK} + \Box^2\bot \vdash A_1.$$

Hence, $L_1 + \Box^2\bot \subseteq L_2$. The converse inclusion is trivial. □

LEMMA 5.8. *For every formula B*,

$$L_2 \vdash B \quad \Longrightarrow \quad L_1 \vdash \Box^2\bot \to B.$$

Proof. By Lemma 5.7, $L_2 = L_1 + \Box^2\bot$. So assume that $L_1 + \Box^2\bot \vdash B$. It follows, as in the proof of Lemma 5.6, that, for some $n \geqslant 2$,

$$\Box^2\bot, \dots, \Box^n\bot \vdash_{L_1} B.$$

Since, for every $m \geqslant 2$,

$$\mathbf{K} \vdash \Box^2\bot \to \Box^m\bot,$$

it follows that $\Box^2\bot \vdash_{L_1} B$, and hence, $L_1 \vdash \Box^2\bot \to B$. □

LEMMA 5.9. $L_1 = L_0 \cap L_2$.

Proof. The inclusion $L_1 \subseteq L_0 \cap L_2$ follows immediately from Lemmas 5.4 and 5.7.

For the converse, assume that $A \in L_0 \cap L_2$. Then, by Lemma 5.6, $L_1 \vdash \Diamond^2\top \to A$ and, by Lemma 5.8, $L_1 \vdash \Box^2\bot \to A$. Since $\mathbf{K} \vdash \Diamond^2\top \vee \Box^2\bot$, it follows that $L_1 \vdash A$. □

We now complete the proof of Theorem 5.3: By Theorems 3.23 and 3.26, L_2 is quasi-canonical. As we have seen in the proof of Theorem 5.2, L_0 is also quasi-canonical. Hence, by Lemma 5.9 and Proposition 3.28, L_1 is quasi-canonical. □

***5.3. Completion of* QS4Alt$_2$.** Define, for every $n \geqslant 1$,

$$C^{+n} \quad := \quad \Diamond P_1^n(x_1, \dots, x_n) \to P_1^n(x_1, \dots, x_n).$$

Now, define

$$L_3 \quad := \quad \mathbf{QS4Alt}_2 + \{q \vee \Box(q \to \overline{\forall} C^{+n}) \mid n \geqslant 1\}.$$

LEMMA 5.10. $\mathbf{QS4Alt}_2 \models_{\mathcal{K}} q \vee \Box(q \to \overline{\forall} C^{+n})$.

Proof. Let $\boldsymbol{F} = (W, R, D)$ be a $\mathbf{QS4Alt}_2$-frame and let $\boldsymbol{M}$ be a model over $\boldsymbol{F}$. Supposing that $\boldsymbol{M}, u \not\models q$, we show that $\boldsymbol{M}, u \models \Box(q \to \overline{\forall} C^{+n})$, for every $n \geqslant 1$.

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Assume that uRv and $\boldsymbol{M}, v \models q$. Then, $u \neq v$. Since $(W, R) \models \mathbf{S4Alt}_2$, it follows that $R(u) = \{u, v\}$ and $R(v) = \{v\}$. Hence, for all $a_1, \ldots, a_n \in D_v$,

$$\boldsymbol{M}, v \models \Diamond P^n(a_1, \ldots, a_n) \to P^n(a_1, \ldots, a_n),$$

and so $\boldsymbol{M}, v \models \overline{\forall} C^{+n}$. Since $R(u) = \{u, v\}$ and $\boldsymbol{M}, u \not\models q$, it follows that $\boldsymbol{M}, u \models \Box(q \to \overline{\forall} C^{+n})$. □

THEOREM 5.11. $\widehat{\mathbf{QS4Alt}}_2 = L_3$.

Proof. In view of Lemma 5.10, it suffices to prove that L_3 is Kripke complete; to that end, we show that L_3 is quasi-canonical.

Let Γ be an L_3-place. Since $L_3 \vdash \Box P \to P$, the canonical frame $\boldsymbol{F}_{L_3}$ is reflexive. Arguing as in the proof of Lemma 3.24, we see that $|C(\Gamma)| \leqslant 2$; moreover, due to reflexivity, we may assume that $\Gamma \in C^\circ(\Gamma)$. We consider two cases.

Case $C^\circ(\Gamma) = \{\Gamma\}$: In this case, $\boldsymbol{M}_{L_3} \upharpoonright \{\Gamma\}$ is a selective submodel of $\boldsymbol{M}_{L_3}$. The underlying predicate Kripke frame $\boldsymbol{F}$ is a reflexive singleton with a domain; hence, $\boldsymbol{F} \models \mathbf{QTriv}$, and so, since $\mathbf{QS4Alt}_2 \subseteq \mathbf{QTriv}$, also $\boldsymbol{F} \models \mathbf{QS4Alt}_2$. Thus, by Lemma 5.10, $\boldsymbol{F} \models L_3$.

Case $C^\circ(\Gamma) = \{\Gamma, \Delta\}$, *with* $\Delta \neq \Gamma$: In this case, there exists a D_Γ-sentence A such that $A \notin \Gamma$ and $A \in \Delta$. Then, by Lemma 3.3, $\Diamond A \in \Gamma$. By substitution, for every formula B,

$$L_3 \vdash A \vee \Box(A \to \overline{\forall}(\Diamond B \to B)),$$

and so, since $A \notin \Gamma$, we know that $\Box(A \to \overline{\forall}(\Diamond B \to B)) \in \Gamma$. Then, since $\Box^- \Gamma \subseteq \Delta$, it follows that $A \to \overline{\forall}(\Diamond B \to B) \in \Delta$.

Now, let Θ be an L_3-place containing Δ. Then, $A, A \to \overline{\forall}(\Diamond B \to B) \in \Theta$, and so, by (MP), for every formula B,

$$\overline{\forall}(\Diamond B \to B) \in \Theta. \tag{5.6}$$

But then $\boldsymbol{M} := \boldsymbol{M}_{L_3} \upharpoonright \{\Gamma, \Theta\}$ is selective: if $\Diamond B(\mathbf{a}) \in \Theta$, for a D_Θ-sentence $B(\mathbf{a})$, then, due to (5.6), $\Diamond B(\mathbf{a}) \to B(\mathbf{a}) \in \Theta$, and so $B(\mathbf{a}) \in \Theta$. That $\boldsymbol{M}$ is based on an L_3-frame is proved as in the preceding case. □

§6. Final remarks. First, we note that adding the Barcan formula to some logics studied in this paper, such as $\mathbf{QAlt}_n$, $\mathbf{QTAlt}_n$, $\mathbf{QK4Alt}_n$, $\mathbf{QS4Alt}_n$, $\mathbf{QK4.1Alt}_n$, and $\mathbf{QS4.1Alt}_n$ yields Kripke completeness. This follows from the Tanaka–Ono theorem on Kripke completeness of minimal constant-domain extensions of universal propositional logics [2, Theorem 7.4.7],[11] [11].

Second, our results can be transferred to logics with equality: by Theorem 3.8.7 from [2], strong Kripke completeness with respect to Kripke frames with equality is preserved for minimal extensions with equality.

Still, the results of this paper can be viewed as a small part of the study of Kripke completeness in first-order modal logic. Although the boundaries of Kripke completeness can hardly be identified, it seems reasonable to find them for special kinds of logics, such as logics $\mathbf{Q}\Lambda$ where

[11] The statement of Theorem 7.4.7 from [2] should mention that the logic Λ is Kripke complete.

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(1) $\mathbf{\Lambda}$ is tabular;

(2) $\mathbf{\Lambda}$ is a logic of a finite tree;

(3) $\mathbf{\Lambda}$ is a logic of a finite **GL**-tree;

(4) $\mathbf{\Lambda}$ is a logic of a finite **S4**-tree;

(5) $\mathbf{Alt}_1 \subseteq \mathbf{\Lambda}$.

We conjecture that, for all logics $\mathbf{\Lambda}$ from (2) and (3), $\mathbf{Q\Lambda}$ are Kripke complete.

This paper also touched on the problem of axiomatizing Kripke completions. Proposition 3.12.8 from [2] implies that if a propositional logic $\mathbf{\Lambda}$ is elementary, then the predicate logic $\widehat{\mathbf{Q\Lambda}}$ is recursively axiomatizable; in particular, so are completions of logics mentioned in (1) through (5) above. However, finding explicit axiomatizations of these logics may be hard. In particular, the problem remains unresolved for the following logics related to this paper: $\widehat{\mathbf{QS4Alt}_n}$ with $n > 2$ and $\widehat{\mathbf{QK4Alt}_n}$ with $n > 1$. Other important logics for which this question is open are the Kripke completions of $\mathbf{QS4} + \boldsymbol{bd}_n$ and $\mathbf{QGrz.3} + \boldsymbol{bd}_n$, where $\boldsymbol{bd}_n$ is the axiom of transitive depth n.

Another direction for future research concerns determining whether Kripke completions are finitely axiomatizable. This problem is open, in particular, for $\widehat{\mathbf{QS4Alt}_2}$ and for many logics from [7].

Acknowledgments. We are grateful to anonymous reviewers for their comments and suggestions. Preliminary results presented here have been published in [9].

Funding. The work of the first author is an output of the research project (HSE-BR-2025-84) implemented as part of the Basic Research Program at HSE University.

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https://doi.org/10.1017/S1755020326101324 Published online by Cambridge University Press