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Stochastic integrals in an arbitrary probability gauge space

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0. *Preliminary remarks*

In §7 of [1] we described how a stochastic integral of non-commuting process cesan be constructed. This was achieved via a Doob–Meyer decomposition for the square of a self-adjoint L^2-martingale. We introduced a 'condition D', derived from the 'class D' of stochastic process theory, and showed that if the square of a self-adjoint L^2-martingale satisfies this condition then it has a decomposition of the Doob–Meyer type. The purpose of this paper is to extend the results of §7 of [1] and to introduce another construction of the stochastic integral that does not employ condition D.

1. *Introduction*

1·1. In this section we shall describe the context in which we work. Let $\mathscr{A}$ be a finite von Neumann algebra acting on a Hilbert space H. Let ϕ be a faithful normal finite trace on $\mathscr{A}$ with $\phi(I) = 1$. Let $(\mathscr{A}_\alpha)$, $\alpha \in \mathbb{R}^+$ be an increasing family of von Neumann subalgebras of $\mathscr{A}$ that satisfy

(i) $(\bigcup_{\alpha \in \mathbb{R}^+} \mathscr{A}_\alpha)'' = \mathscr{A}$, (ii) $\bigcap_{\beta > \alpha} \mathscr{A}_\beta = \mathscr{A}_\alpha$,

(iii) $(\bigcup_{\beta < \alpha} \mathscr{A}_\beta)'' = \mathscr{A}_\alpha$, (iv) $\mathscr{A}_\beta$ is a von Neumann subalgebra of $\mathscr{A}_\alpha$ if $\beta < \alpha$.

For $1 \leqslant p \leqslant \infty$ and $\alpha \in \mathbb{R}^+ \cup \{\infty\}$ let $\mathbf{L}^p(\mathscr{A}_\alpha)$ denote the non-commutative analogues of the Lebesgue spaces associated with $\mathscr{A}_\alpha$ and ϕ, with the convention that $\mathscr{A} \equiv \mathscr{A}_\infty$ and $\mathbf{L}^\infty(\mathscr{A}_\alpha) = \mathscr{A}_\alpha$, [5].

A process is a family $(X_\alpha)_{\alpha \in \mathbb{R}^+}$ with $X_\alpha \in \mathbf{L}^1(\mathscr{A}_\alpha)$. For each $\alpha \in \mathbb{R}^+$ we denote by M_α the conditional expectation from $\mathbf{L}^p(\mathscr{A}_\infty)$ onto $\mathbf{L}^p(\mathscr{A}_\alpha)$. We do not distinguish between the expectations for differing values of $p \in [1, \infty)$ since each one is the appropriate extension of the expectation of $\mathscr{A}_\infty$ onto $\mathscr{A}_\alpha$. A process (X_α) is a martingale (respectively; supermartingale, submartingale) if $M_\beta(X_\alpha) = X_\beta$, $\beta \leqslant \alpha$ (respectively; $M_\beta(X_\alpha) \leqslant X_\beta$, $M_\beta(X_\alpha) \geqslant X_\beta$). If (X_α) is a supermartingale with $X_\alpha \geqslant 0$, $\alpha \in \mathbb{R}^+$, and $\phi(X_\alpha) \to 0$ as $a \to \infty$ then (X_α) is called a potential.

It follows from conditions 1·1 (i), (ii), (iii), (iv) that 'simple martingales' i.e. those of the form $X_\alpha = M_\alpha(X)$ for some $X \in \mathbf{L}^p(\mathscr{A}_\infty)$, are $\mathbf{L}^p$-continuous [2]. The properties of ϕ and $\|\cdot\|_p$ that we require are contained in [5], in particular Proposition 2·5.

2. *Class D processes*

2·1. *Definition* (7·3 *of* [1]). Let $(X_\alpha) \subseteq \mathbf{L}^1(\mathscr{A}_\infty)$ be a process. We say that (X_α) is (of) class D if and only if

$$S(X_\alpha) \overset{\text{def}}{=} \left\{ \sum_{i=1}^{n} M_{\alpha_{i-1}}(X_{\alpha_i} - X_{\alpha_{i-1}}) : 0 \leqslant \alpha_1 < \alpha_2 < \ldots < \alpha_n < \infty, n \in \mathbb{N} \right\}$$

is weakly relatively compact in $\mathbf{L}^1(\mathscr{A}_\infty)$.

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The reasons for this choice of definition are indicated in §7 of [1].

2·2. *Definition.* Let (X_α) be a submartingale. We say that (X_α) has a Doob–Meyer decomposition if we can write $X_\alpha = U_\alpha + A_\alpha$, $\alpha \in \mathbb{R}^+$, where (U_α) is a martingale and (A_α) is a positive increasing process, i.e. $0 \leqslant A_{\alpha_1} \leqslant A_{\alpha_2}$ for $0 \leqslant \alpha_1 \leqslant \alpha_2$.

In §7 of [1] it was shown that a self-adjoint $\mathbf{L}^2$-martingale whose square is class D has a Doob–Meyer decomposition. The choice of a self-adjoint martingale was for convenience as the following shows.

2·3. PROPOSITION. *Let (X_α) be an $\mathbf{L}^2$-martingale. If $(X_\alpha^* X_\alpha)$ (respectively $(X_\alpha X_\alpha^*)$) is of class D then it has a Doob–Meyer decomposition.*

Proof. As $(X_\alpha^* X_\alpha)$ (respectively $(X_\alpha X_\alpha^*)$) is a submartingale it has a *Riesz decomposition* into a martingale minus a potential. The proof of this is a straightforward rewriting of the proof from the usual theory (see [4]). This done one shows that the potential in this decomposition is of class D and then using 7·5 of [1] one obtains the result.

2·4. *Remark.* It is not known if this decomposition is unique (see §7 of [1] for conditions under which uniqueness is assured). This is why we speak of *a* Doob–Meyer decomposition. Given a Doob–Meyer decomposition of $(X_\alpha^* X_\alpha)$ we write $(\langle X \rangle_\alpha)$ for the positive increasing process in the decomposition. We call $(\langle X_\alpha \rangle)$ the *increasing part* of the decomposition.

We recall from §7 of [1] that a positive increasing process, (A_α), which is continuous as a map $\mathbb{R}^+ \to \mathbf{L}^1(\mathscr{A})$ defines a vector measure on the Borel σ-field of $[0, t]$ for any $t \in \mathbb{R}^+$. This measure is (the extension of) the set function dA given by

$$dA([s_1, s_2]) = A_{s_2} - A_{s_1} \quad s_1 \leqslant s_2$$

(with no change for $[s_1, s_2)$, $(s_1, s_2]$, (s_1, s_2) because (A_α) is continuous). The measure has variation $d|A|$ equal to the semivariation with respect to $\mathscr{A}$, given by $E \mapsto \|dA(E)\|_1$. The variation is a measure and hence dA satisfies Bartle's $*$-property [3]. It follows that there is a linear space of $\mathscr{A}$ valued dA-integrable functions. The increasing processes we shall be concerned with arise from Doob–Meyer decompositions. It might appear therefore that we can fix an $\mathbf{L}^2$ martingale, (X_α), with $(X_\alpha^* X_\alpha)$ of class D and for each increasing part of a Doob–Meyer decomposition obtain a linear space of $\mathscr{A}$ valued integrands. In fact these linear spaces coincide because the integrands in Bartle's theory are obtained via the semivariation of the vector measure. In our context variation and semivariation coincide for a given $(\langle X \rangle_\alpha)$. Moreover, the following lemma demonstrates that no matter which increasing part we choose $d|\langle X \rangle|$ is always the same function.

2·5. LEMMA. (i) *If (X_α) is an $\mathbf{L}^2$ martingale with $(|X_\alpha|^2)$ of class D then for $0 \leqslant s_1 \leqslant s_2$*

$$d|\langle X \rangle|([s_1, s_2]) = \phi(|X_{s_2}|^2 - |X_{s_1}|^2).$$

(ii) *If (X_α) is an $\mathbf{L}^2$ martingale with $(X_\alpha^* X_\alpha)$ and $(X_\alpha X_\alpha^*)$ of class D then*

$$d|\langle X \rangle| = d|\langle X^* \rangle|.$$

Proof. (i)

$$d|\langle X \rangle|([s_1, s_2]) = \|d\langle X \rangle([s_1, s_2])\|_1 = \|\langle X \rangle_{s_2} - \langle X \rangle_{s_1}\|_1 = \phi(\langle X \rangle_{s_2} - \langle X \rangle_{s_1}).$$

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If $|X_s|^2 = V_s + \langle X\rangle_s$, with (V_s) a martingale, then as $M_{s_1}(V_{s_2}) = V_{s_1}$ we have

$$\begin{aligned}\phi(\langle X\rangle_{s_2} - \langle X\rangle_{s_1}) &= \phi(M_{s_1}(\langle X\rangle_{s_2} - \langle X\rangle_{s_1})) = \phi(M_{s_1}(V_{s_2} + \langle X\rangle_{s_2} - (V_{s_1} + \langle X\rangle_{s_1})))\\ &= \phi(M_{s_1}(|X_{s_2}|^2 - |X_{s_1}|^2)) = \phi(|X_{s_2}|^2 - X_{s_1}|^2).\end{aligned}$$

(ii) Use (i) and the fact that ϕ is a trace.

2·6. *Definition.* Let (X_α) be an $\mathbf{L}^2$ martingale. Then the function $\alpha \to \phi(|X_\alpha|^2)$ is increasing and continuous. We shall denote the Stieltjes measure defined by this function by $d|\langle X\rangle|$.

2·7. *Remark.* It is clear from (2·5) that should $(X_\alpha^* X_\alpha)$ be of class D then the variation of $d\langle X\rangle$ is precisely this Stieltjes measure. This Stieltjes measure always exists but we do not know that $(X_\alpha^* X_\alpha)$ is always of class D. In the sequel our use of this notation, especially in the next section, does not assume the existence of a Doob–Meyer decomposition. But if a decomposition does exist the notation is consistent with that established previously. With this in mind no confusion should arise.

2·8. COROLLARY (*of* 2·6). *Let* (X_α) *be an* $\mathbf{L}^2$ *martingale. Then* $d|\langle X\rangle| = d|\langle X^*\rangle|$.

Proof. Use the fact that ϕ is a trace.

We now present a construction of a non-commutative stochastic integral that differs from that in §7 of [1]. We shall return to the question of how the construction below and that in [1] interrelate in Section 5.

3. *Non-commutative stochastic integrals*

3·1. *Definition.* (i) Let $E = [r, t)\, r < t$ and $X \in \mathbf{L}^1(\mathscr{A}_r)$. Then $f(s) = X\chi_E(s)$ is process. All processes of this form are called *elementary*.

(ii) Let $f(s)$ be a process. We say $f(s)$ is *simple* on $[0, t]$ if it is a finite linear combination of elementary processes on $[0, t)$. We note that if $t > 0$ then a simple process on $[0, t]$ can always be written in the form $\sum_{i=1}^n X_i \chi_{E_i}$, where

$$E_i = [s_i, t_i)\,(1 \leqslant i \leqslant n-1),\ E_n = \{t\}$$

and

$$0 = s_1 < t_1 = s_2 < t_2 = s_3 < \ldots t_{n-1} = t.$$

(ii) Let (X_α) be an $\mathbf{L}^2$-martingale and $f(s)$ a simple process on $[0, t]$, say

$$f(s) = \sum_{i=1}^n f(s)\,\chi_{[s_i, t_i)}(s)$$

on $[0, t)$ and $[s_i, t_i)$ form a disjoint partition of $[0, t)$. Then we define the *right integral* of f by

$$\int_0^t f(s)\, dX_s = \sum_{i=1}^n f(s_i)\,(X_{t_i} - X_{s_i}),$$

and the *left integral* of f by

$$\int_0^t dX_s\, f(s) = \sum_{i=1}^n (X_{t_i} - X_{s_i}) f(s_i).$$

We note immediately that

$$\left(\int_0^t f(s)\, dX_s\right)^* = \int_0^t dX_s^*\, f(s)^*.$$

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3·2. THEOREM (*contraction property*). *Let* (X_α) *be an* $\mathbf{L}^2$*-martingale and* f *a simple process on* $[0,t]$.

(i) *If* $(X_\alpha^* X_\alpha)$ is *class D then*

$$\left\| \int_0^t dX_s f(s) \right\|_2^2 = \phi\left(\int_0^t |f^*(s)|^2 d\langle X\rangle_s \right) \leqslant \int_0^t \|f(s)\|_\infty^2 d|\langle X\rangle_s|.$$

(ii) *If* $(X_\alpha X_\alpha^*)$ *is class D then*

$$\left\| \int_0^t f(s)\, dX_s \right\|_2^2 = \phi\left(\int_0^t |f(s)|^2 d\langle X^*\rangle_s \right) \leqslant \int_0^t \|f(s)\|_\infty^2 d|\langle X\rangle_s|.$$

Proof. Consider (i). The equality follows just as in 7·15 of [1]. The inequality follows from the positivity of ϕ; if $|X_\alpha|^2 = U_s + \langle X\rangle_s$,

$$\begin{aligned}
\phi\left(\int_0^t |f^*(s)|^2 d\langle X\rangle_s \right) &= \sum_l \phi(|f_{l-1}^*|^2 (\langle X\rangle_l - \langle X\rangle_{l-1})) \\
&= \sum_l \phi(|f_{l-1}^*|^2 (|X_l|^2 - |X_{l-1}|^2)) \\
&= \sum_l \phi(\Delta X_l |f_{l-1}^*|^2 \Delta X_l^*), \\
&\leqslant \sum_l \| |f_{l-1}^*|^2 \|_\infty \phi(\Delta X_l \Delta X_l^*) \\
&= \sum_l \|f_{l-1}\|_\infty^2 \phi(|X_l|^2 - |X_{l-1}|^2) \\
&= \int_0^t \|f(s)\|_\infty^2 d|\langle X\rangle_s|,
\end{aligned}$$

where $\Delta X_l = X_l - X_{l-1}$.

3·3. *Remark.* Let (X_α) be an $\mathbf{L}^2$-martingale and let $d|\langle X\rangle|$ denote the Stieltjes measure of Definition 2·6. Then *irrespective of any satisfaction of condition D,*

$$\left\| \int_0^t dX_s f(s) \right\|_2^2 \leqslant \int_0^t \|f(s)\|_\infty^2 d|\langle X\rangle_s|, \tag{3.31}$$

$$\left\| \int_0^t f(s)\, dX_s \right\|_2^2 \leqslant \int_0^t \|f(s)\|_\infty^2 d|\langle X^*\rangle_s|. \tag{3·32}$$

Now we shall extend the stochastic integral to a wider class of integrands. The method employed in § 7 of [1] achieves the extension by using the *equality* in (3·31) and (3.32). That is, one uses the right-hand integrals in these equations to prove that the left-hand side exists and is well defined. We shall use the relations (3·31) and (3·32) to extend the integral. First we identify our integrands.

3·4. *Definition.* Let f: $\mathbb{R}^+ \to \mathscr{A}$ be a process and (X_α) an $\mathbf{L}^2$ (bounded) martingale. Let $d|\langle X\rangle|$ denote the measure derived from $s \to \phi(X_s^* X_s)$. We write $f \in \mathscr{P}\langle X\rangle$ if and only if

(i) f is the $d|\langle X\rangle|$ almost everywhere limit (in $\| \|_\infty$) of simple $\mathscr{A}$ valued processes (f_n);

(ii) $\forall \epsilon > 0$, $\exists \delta > 0$ such that if $E \subseteq \mathbb{R}^+$ is a Borel set and $d|\langle X\rangle|(E) < \delta$ then

$$\int_E \|f_n(s)\|_\infty^2 d|\langle X\rangle_s| < \epsilon \quad \text{for} \quad n = 1, 2, 3, \ldots.$$

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We say that the sequence (f_n) defines f, and that (f_n) is uniformly absolutely continuous (u, a, c) with respect to $d|\langle X\rangle|$.

3·5. LEMMA. *Let (X_α) be an $\mathbf{L}^2$-bounded martingale and $f, g \in \mathscr{P}\langle X\rangle$, $\lambda \in \mathbb{R}$. Then*

(i) $\lambda f + g \in \mathscr{P}\langle X\rangle$.

(ii) *For each $t \geqslant 0$ the function $s \to \|f(s)\|_\infty^2$ is $d|\langle X\rangle|$-integrable on $[0, t]$ and*

$$\int_0^t \|f(s)\|_\infty^2 \, d|\langle X\rangle_s| = \lim_{n\to\infty} \int_0^t \|f_n(s)\|_\infty^2 \, d|\langle X\rangle_s|.$$

(iii) $\mathscr{P}\langle X\rangle = \mathscr{P}\langle X^*\rangle$ *and* $f \in \mathscr{P}\langle X\rangle \Rightarrow f^* \in \mathscr{P}\langle X\rangle$.

Proof. (i) Obvious.

(ii) Since (f_n) converges pointwise $d|\langle X\rangle|$ almost everywhere (in $\|\ \|_\infty$) to f and $(\mathbb{R}^+, d|\langle X\rangle|)$ is a totally finite measure space then $s \to \|f(s)\|_\infty$ is measurable. So therefore is $s \to \|f(s)\|_\infty^2$ and, moreover, $\|f_n(s)\|_\infty^2 \to \|f(s)\|_\infty^2$ as $n \to \infty$ for $d|\langle X\rangle|$ almost every $s \in \mathbb{R}^+$. We now apply (for example) Theorem 10 of [3] to conclude that (ii) holds.

(iii) 2·6 and the fact that $\|\ \|_\infty$ is a C^* norm.

3·6. THEOREM. *Let (X_α) be an $\mathbf{L}^2$-bounded martingale. Let (f_n) be a sequence of simple process that define $f \in \mathscr{P}\langle X\rangle$. Then for $t \geqslant 0$ the sequences*

$$\left(\int_0^t dX_s \, f_n(s)\right), \quad \left(\int_0^t f_n(s)\, dX_s\right)$$

are Cauchy in $\mathbf{L}^2(\mathscr{A})$. Their limits which we denote by

$$\int_0^t dX_s f(s) \quad \text{and} \quad \int_0^t f(s)\, dX_s$$

respectively (the left, respectively right, integral of f with respect to (X_α)) are independent of the sequence (f_n) defining f.

Proof. Consider the right integral. From Theorem 3·2

$$\left\| \int_0^t (f_n(s) - f_m(s))\, dX_s \right\|_2^2 \leqslant \int_0^t \|f_n(s) - f_m(s)\|_\infty^2 \, d|\langle X\rangle_s|.$$

Now let $\epsilon > 0$ and choose $\delta > 0$ so that $d|\langle X\rangle|(E) < \delta$ gives

$$\int_E \|f_n(s)\|_\infty^2 \, d|\langle X\rangle_s| < \frac{\epsilon}{6} \quad \text{for} \quad n = 1, 2, \ldots.$$

Now on $[0, t]$ (f_n) converges $d|\langle X\rangle|$ almost uniformly to f hence there is a Borel set $F \subseteq [0, t]$ with $d|\langle X\rangle|([0, t] \setminus F) < \delta$ and $f_n \to f$ uniformly on F. Set $d|\langle X\rangle|([0, t]) = U$ and $F' = [0, t] \setminus F$. By choosing $N(\epsilon)$ large enough we can ensure that

$$\|f(s) - f_n(s)\|_\infty < \frac{1}{2}\sqrt{\frac{\epsilon}{2U}} \quad \text{for} \quad s \in F \quad \text{and} \quad n \geqslant N(\epsilon)$$

(if $d|\langle X\rangle|([0, t]) = 0$ then (X_t) is constant on $[0, t]$ and the theory is trivial). It follows that for $m, n \geqslant N(\epsilon)$, $s \in F$

$$\|f_n(s) - f_m(s)\|_\infty^2 \leqslant \left(\frac{1}{2}\sqrt{\frac{\epsilon}{6U}} + \frac{1}{2}\sqrt{\frac{\epsilon}{6U}}\right)^2 = \frac{\epsilon}{6U}.$$

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Hence

$$\int_F \|f_n(s)-f_m(s)\|_\infty^2 d|\langle X\rangle_s| < \frac{\epsilon}{6U} d|\langle X\rangle|(F) \leqslant \frac{\epsilon}{6}.$$

Moreover

$$I_{F'} = \int_{F'} \|f_m(s)-f_n(s)\|_\infty^2 d|\langle X\rangle_s| \leqslant 2\int_{F'} \|f_m(s)\|_\infty^2 d|\langle X\rangle_s| + 2\int_{F'} \|f_n(s)\|_\infty^2 d|\langle X\rangle_s|$$

but $d|\langle X\rangle|(F') < \delta$, so $I_{F'} < \frac{1}{3}\epsilon + \frac{1}{3}\epsilon$. Thus

$$\int_0^t \|f_n(s)-f_m(s)\|_\infty^2 d|\langle X\rangle_s| < \frac{5\epsilon}{6} < \epsilon.$$

Suppose now that (f_n) and (g_n) define f. Then the compound sequence

$$h_n = \begin{cases} f_n & \text{if } n = 2r \quad r\in\mathbb{N}^+ \\ g_n & \text{if } n = 2r+1 \quad r\in\mathbb{N}^+ \end{cases}$$

also defines f and so $\left(\int_0^t h_n(s)\,dX_s\right)$ is Cauchy in $\mathbf{L}^2$. Clearly

$$\left(\int_0^t f_n(s)\,dX_s\right) \quad \text{and} \quad \left(\int_0^t g_n(s)\,dX_s\right)$$

converge to the same limit as $\left(\int_0^t h_n(s)\,dX_s\right)$, showing that the limit is independent of (f_n).

3·7. THEOREM. *Let (X_α) be an $\mathbf{L}^2$-bounded martingale. Then $\left(\int_0^t f(s)\,dX_s\right)$ and $\left(\int_0^t dX_s\,f(s)\right)$ are martingales. Moreover the relations of Theorem 3·2 extend to $f\in\mathscr{P}\langle X\rangle$, i.e.*

$$\left\|\int_0^t dX_s\,f(s)\right\|_2^2 \leqslant \int_0^t \|f(s)\|_\infty^2 d|\langle X\rangle_s| \quad \textit{and} \quad \left\|\int_0^t f(s)\,dX_s\right\|_2^2 = \leqslant \int_0^t \|f(s)\|_\infty^2 d|\langle X\rangle_s|.$$

Proof. Let $f\in\mathscr{P}\langle X\rangle$ be defined by (f_n). It is easily verified that for each $n\in\mathbb{N}$

$$\left(\int_0^t f_n(s)\,dX_s\right) \quad \text{and} \quad \left(\int_0^t dX_s\,f_n(s)\right)$$

are martingales. For $t\in\mathbb{R}^+$, the conditional expectation operator M_t is $\|\ \|_2$ continuous, so for $t_1 \leqslant t_2$,

$$\begin{aligned} M_{t_1}\left(\int_0^{t_2} f(s)\,dX_s\right) &= M_{t_1}\left(\|\cdot\|_2 - \lim_n \int_0^{t_2} f_n(s)\,dX_s\right) \\ &= \|\cdot\|_2 - \lim_n M_{t_1}\left(\int_0^{t_2} f_n(s)\,dX_s\right) \\ &= \|\cdot\|_2 - \lim_n \int_0^{t_1} f_n(s)\,dX_s \\ &= \int_0^{t_1} f(s)\,dX_s. \end{aligned}$$

Similarly for the left integral.

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The inequalities required follow from Lemma 3·5(ii), Theorem 3·2 and Theorem 3·6. We conclude this section with a differentiation theorem.

3·8. THEOREM. *Let $f, g \in \mathscr{P}\langle X\rangle$ be defined by (f_n) and (g_n) and $Y_\alpha = \int_0^\alpha f(s)\,dX_s$. Suppose that*

(i) *$g_n f_m, m, n \in \mathbb{N}$ are u.a.c. with respect to $d|\langle X\rangle|$ and*

(ii) *$g_n, n \in \mathbb{N}$ are u.a.c. with respect to $d|\langle Y\rangle|$.*

Then $gf \in \langle \mathscr{P}X\rangle$, $g \in \mathscr{P}\langle Y\rangle$ and $\int_0^\alpha g(s)\,dY_s = \int_0^\alpha g(s)f(s)\,dX_s$.

Proof. By ignoring the union of two $d|\langle X\rangle|$ null sets we have $g_n f_n \to gf$ pointwise, $g_n f_n$ simple, and $g_n f_n$, $n \in \mathbb{N}$, u.a.c. with respect to $d|\langle X\rangle|$ and so $gf \in \mathscr{P}\langle X\rangle$. Since for $s \leqslant t$,

$$\phi(|Y_t|^2 - |Y_s|^2) = \phi(|Y_t - Y_s|^2) = \left\| \int_s^t f\,dX \right\|_2^2 \leqslant \int_s^t \|f\|_\infty^2\, d|\langle X\rangle|.$$

Then any $d|\langle X\rangle|$ null set is $d|\langle Y\rangle|$ null. So $g_n \to g - d|\langle Y\rangle|$ a.e. and this with (ii) shows $g \in \mathscr{P}\langle Y\rangle$ and that (g_n) define g in $\mathscr{P}\langle Y\rangle$. So $\int_0^\alpha g_n\,dY \to \int_0^\alpha g\,dY$ in $\|\cdot\|_2$. However, if $h(s)$ is elementary, $h(s) = h\chi_{(r,u)}(s)$ with $h \in \mathscr{A}_r$, $0 \leqslant r \leqslant u \leqslant \alpha$, and (f_n) are simple processes defining f. We have

$$\int_0^\alpha h(s)\,dY_s = h(Y_u - Y_r) = h\int_r^u f(s)\,dX_s = h\lim_n \int_r^u f_n(s)\,dX_s.$$

But multiplication by h is $\|\cdot\|_2$ continuous. Bearing in mind that $\int_r^u f_n(s)\,dX_s$ is just a finite sum we have

$$h\cdot\lim_n \int_r^u f_n(s)\,dX_s = \lim_n \int_r^u h f_n(s)\,dX_s = \lim_n \int_0^\alpha h(s)f_n(s)\,dX_s.$$

Now $h(s)f_n(s)$ is a sequence of simple processes converging $d|\langle X\rangle|$ almost everywhere to $h(s)f(s)$ and $(h(s)f_n(s))$ are u.a.c. with respect to $d|\langle X\rangle|$. So

$$hf \in \mathscr{P}\langle X\rangle \quad \text{and} \quad \lim_n \int_0^\alpha h(s)f_n(s)\,dX_s = \int_0^\alpha h(s)f(s)\,dX_s$$

by Theorem 3·6.

The result for a simple h follows by linearity. If g is not simple and (g_n) are the simple processes (defining g) of the hypothesis then $g_n f \in \mathscr{P}\langle X\rangle$ and $g_n f \to gf\, d|\langle X\rangle|$ a.e. But $g_n f = \lim_m g_n f_m - d|\langle X\rangle|$ a.e. and $(g_n f_m)_{m=1}^\infty$ are u.a.c. with respect to $d|\langle X\rangle|$. It follows from Theorem 10 of [3] that

$$\int \|g_n f\|_\infty^2\, d|\langle X\rangle| = \lim_m \int \|g_n f_m\|_\infty^2\, d|\langle X\rangle|.$$

Thus $(g_n f)$ are u.a.c. with respect to $d|\langle X\rangle|$. Now

$$\left\| \int_0^\alpha g_n f\,dX - \int_0^\alpha gf\,dX \right\|_2^2 \leqslant \int_0^\alpha \|g_n f - gf\|_\infty^2\, d|\langle X\rangle|.$$

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But

$$\|g_n f - gf\|_\infty^2 \to 0 \, d|\langle X\rangle| \text{ a.e.}$$

and

$$\int_0^\alpha \|g_n f - gf\|_\infty^2 \, d|\langle X\rangle| \leqslant 2\int_0^\alpha \|g_n f\|_\infty^2 \, d|\langle X\rangle + 2\int_0^\alpha \|gf\|_\infty^2 \, d|\langle X\rangle|.$$

So $(g_n f - gf)$ is u.a.c. with respect to $d|\langle X\rangle|$. By Theorem 10 of [3],

$$\int_0^\alpha \|g_n f - gf\|_\infty^2 \, d|\langle X\rangle| \to 0 \quad \text{as} \quad n \to \infty.$$

Thus

$$\int_0^\alpha g(s)\, dY_s = \lim_n \int_0^\alpha g_n(s)\, dY_s = \lim_n \int_0^\alpha g_n(s) f(s)\, dX_s = \int_0^\alpha g(s) f(s)\, dX_s. \blacksquare$$

4. *Extending the integral to Borel sets*

So far our stochastic integral is only defined for intervals. By additivity we extend to sets that are finite disjoint unions of intervals. We wish to extend the integral to any element of the σ-field generated by sets which are finite unions of intervals. Our first step is to extend the relations to sets which are finite unions of intervals.

4·1. LEMMA. *Let $\mathscr{F}$ be the ring of subsets of $\mathbb{R}^+$ comprising sets that are finite unions of intervals. Then for each $\mathbf{L}^2$-bounded martingale, (X_α), $f \in \mathscr{P}\langle X\rangle$, and $E \in \mathscr{F}$ we have*

$$\left\| \int_E f(s)\, dX_s \right\|_2^2 \leqslant \int_E \|f(s)\|_\infty^2 \, d|\langle X\rangle_s|$$

and

$$\left\| \int_E dX_s f(s) \right\|_2^2 \leqslant \int_E \|f(s)\|_\infty^2 \, d|\langle X\rangle_s|.$$

Proof. This just uses the fact that

$$Y_\alpha = \int_0^\alpha f(s)\, dX_s$$

is a martingale. Let $I_1, \ldots, I_n$ be disjoint intervals with $E = \cup_{i=1}^n I_i$ and, say, I_i has end-points a_i, b_i, $a_i \leqslant b_i$.

Now

$$\left\| \int_E f(s)\, dX_s \right\|_2^2 = \left\| \sum_{i=1}^n \int_{I_i} f(s)\, dX_s \right\|_2^2 = \left\| \sum_{i=1}^n (Y_{b_i} - Y_{a_i}) \right\|_2^2$$

$$= \sum_{i=1}^n \sum_{j=1}^n \phi((Y_{b_i} - Y_{a_i})^* (Y_{b_j} - Y_{a_j}))$$

taking M_{a_i} if $a_i \geqslant b_j$ or M_{a_j} if $a_j \geqslant b_i$ zero's the $i \neq j$ terms leaving

$$\left\| \int_E f(s)\, dX_s \right\|_2^2 = \sum_{i=1}^n \phi(|Y_{b_i} - Y_{a_i}|^2) \leqslant \sum_{i=1}^n \int_{a_i}^{b_i} \|f(s)\|_\infty^2 \, d|\langle X\rangle_s| = \int_E \|f(s)\|_\infty^2 \, d|\langle X\rangle_s|.$$

4·2. THEOREM. *The set function $\nu : \mathscr{F} \to \mathbf{L}^2(\mathscr{A})$ given by*

$$E \to \int_E f(s)\, dX_s,$$

where (X_α) is an $\mathbf{L}^2$-bounded martingale and $f \in \mathscr{P}\langle X\rangle$, is finitely additive and has a unique countably additive extension to $\sigma(\mathscr{F})$, the σ-ring generated by $\mathscr{F}$.

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Proof. The finite additivity of ν is clear. Let

$$\lambda(E) = \int_E \|f(s)\|_\infty^2 \, d|\langle X\rangle_s| \quad \text{for} \quad E \in \sigma(\mathscr{F}).$$

Recalling the construction of λ from $\lambda \restriction \mathscr{F}$, we can see that there are outer sets F_n with $F_n \supseteq E$ and $\lambda(F_n) \downarrow \lambda(E)$ as $n \to \infty$. Since each outer set is the union of a sequence of disjoint intervals then there are sets $G_n \in \mathscr{F}$ with $G_n \subseteq F_n$ and $\lambda(F_n \setminus G_n) \to 0$ as $n \to \infty$. Now as $G_n \setminus E \subseteq F_n \setminus E$ then $\lambda(G_n \setminus E) \to 0$ as $n \to \infty$. So as $\lambda(G_n) = \lambda(G_n \setminus E) + \lambda(G_n \cap E)$ and the left-hand side converges to $\lambda(E)$ we have $\lim_{n\to\infty} \lambda(G_n \cap E) = \lambda(E)$. Hence $\lambda(E \setminus G_n) = \lambda(E) - \lambda(G_n \cap E) \to 0$ as $n \to \infty$. So $\lambda(E \Delta G_n) \to 0$ as $n \to \infty$. Since

$$\begin{aligned} G_n \setminus G_m &= ((G_n \cap E) \cup G_n \setminus E) \setminus G_m \\ &= (G_n \cap E) \setminus G_m \cup (G_n \setminus E) \setminus G_m \\ &\subseteq E \setminus G_m \cup G_n \setminus E. \end{aligned}$$

Then $$\lambda(G_n \setminus G_m) \leqslant \lambda(E \setminus G_m) + \lambda(G_n \setminus E)$$

and so $$\lambda(G_n \Delta G_m) = \lambda(G_n \setminus G_m) + \lambda(G_m \setminus G_n) \to 0 \quad \text{as} \quad m, n \to \infty.$$

Now let (I_n) and (H_n) be sequences of sets in $\mathscr{F}$ for which $\lambda(I_n \Delta H_m) \to 0$ as $m, n \to \infty$. Then

$$\begin{aligned} \|\nu(I_n) - \nu(H_m)\|_2 &= \|\nu(I_n \setminus H_m) - \nu(H_m \setminus I_n)\|_2 \\ &\leqslant \|\nu(I_n \setminus H_m)\|_2 + \|\nu(H_m \setminus I_n)\|_2. \end{aligned}$$

Using Theorem 3·2,

$$\begin{aligned} \|\nu(I_n \setminus H_m)\|_2^2 + \|\nu(H_m \setminus I_n)\|_2^2 &\leqslant \lambda(I_n \setminus H_m) + \lambda(H_m \setminus I_n) \\ &= \lambda(I_n \Delta H_m) \\ &\to 0 \quad \text{as} \quad m, n \to \infty. \end{aligned}$$

Thus $(\nu(I_n))$ and $(\nu(H_n))$ converge or diverge together. Putting $G_n = I_n = H_n$ shows that $(\nu(G_n))$ is Cauchy in $\mathbf{L}^2(\mathscr{A})$. If $(H_n) \subseteq \mathscr{F}$ and $\lim_n \lambda(E \Delta H_n) = 0$ then (as above)

$$\begin{aligned} H_n \setminus G_m &= (H_n \cap E) \setminus G_m \cup (H_n \setminus E) \setminus G_m \\ &\subseteq E \setminus G_m \cup H_n \setminus E. \end{aligned}$$

So $\lambda(H_n \setminus G_m) + \lambda(G_m \setminus H_n) = \lambda(G_m \Delta H_n) \to 0$ as $m, n \to \infty$ hence $(\nu(G_n))$ and $(\nu(H_n))$ are both Cauchy in $\mathbf{L}^2(\mathscr{A})$ and with the same limit.

For $E \in \sigma(\mathscr{F})$ we define $\nu(E)$ as $\lim_n \nu(G_n)$, where $(G_n) \subseteq \mathscr{F}$ and $\lim_n \lambda(E \Delta G_n) \to 0$ as $n \to \infty$. From the work above we see that $\nu(E)$ is well defined.

Since $|\lambda(E) - \lambda(G_n)| = |\lambda(E \setminus G_n) - \lambda(G_n \setminus E)| \leqslant \lambda(E \Delta G_n)$ we have that,

$$\|\nu(E)\|_2^2 = \lim_n \|\nu(G_n)\|_2^2 \leqslant \lim_n \lambda(G_n) = \lambda(E) \quad E \in \sigma(\mathscr{F}). \tag{*}$$

To see that ν is countably additive we note that finite additivity with the relation (*) above will show that ν is continuous from above at $\varnothing$. The rest is a standard argument. To see that ν is finitely additive, let $F, E \in \sigma(\mathscr{F})$ with (F_n) and (E_n) sequences in $\mathscr{F}$ such that $\lim_n \lambda(F \Delta F_n) = \lim_n \lambda(E \Delta E_n) = 0$. Suppose further that $E \cap F = \varnothing$.

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Let $G_n = E \cap E_n$ and $H_n = F \cap F_n$. Then $G_n \cap H_n = \varnothing$ and $\lambda(G_n) \to (E)$, $\lambda(F_n) \to \lambda(F)$. So by relation $(*)$ $\nu(G_n) \to \nu(E)$ and $\nu(F_n) \to \nu(F)$ as $n \to \infty$. Similarly $\nu(G_n \cup H_n) \to (E \cup F)$. Thus

$$\nu(E \cup F) = \lim_n \nu(G_n \cup H_n) = \lim_n \nu(G_n) + \lim_n \nu(H_n) = \nu(E) + \nu(F).$$

The uniqueness of the extension follows from the fact that ν is defined by its values on $\mathscr{F}$.

4·3. COROLLARY. *Let (X_α) be an $\mathbf{L}^2$-martingale. If $\nu_X: \mathscr{F} \to \mathbf{L}^2(\mathscr{A})$ is the set function given by $\nu_X([a,b)) = X_b - X_a$, $a \leqslant b$, then ν_X has a unique countably additive extension to $\sigma(\mathscr{F})$.*

Proof. Put $g(s) = I$ for $s \in \mathbb{R}^+$ and consider $\left(\int_0^\alpha g(s)\, dX_s \right)$.

5. *Stochastic integrals via a Doob–Meyer decomposition*

5·1. We describe briefly the stochastic integral constructed in §7 of [1]. Let (X_α) be a self-adjoint $\mathbf{L}^2$-martingale for which $(|X_\alpha|^2)$ is a class D process. Accordingly we have (Proposition 2·3) for $\alpha \in \mathbb{R}^+$, $|X_\alpha|^2 = U_\alpha + \langle X \rangle_\alpha$, where (U_α) is a martingale and $(\langle X \rangle_\alpha)$ a positive increasing process null at zero. It is easily verified that for each $\mathscr{A}$-valued simple process g

$$\left\| \int_0^\alpha g(s)\, dX_s \right\|_2^2 = \phi \left(\int_0^\alpha |g(s)|^2 d\langle X \rangle_s \right). \tag{5·11}$$

We call this relation the *isometry property*. The integral with respect to (X_α) is now extended to $g \in \bar{\mathscr{P}}(X)$; $\bar{\mathscr{P}}(X)$ is the class of maps g defined by

(i) $g: \mathbb{R}^+ \to \mathscr{A}$ is a process.

(ii) g is the $d\langle X \rangle$-a.e. limit of (in $\|\cdot\|_\infty$) a sequence of simple processes (g_n).

(iii) $\operatorname{ess\,sup}_{s \in [0,t]} \|g(s)\|_\infty^2 < \infty$.

One uses Bartle's bilinear vector integral theory, which applies to $\int_0^\alpha |g(s)|^2 d\langle X \rangle_s$, and the isometry property (5·11) to define the integral of $g \in \bar{\mathscr{P}}(X)$ with respect to (X_α). The argument is, essentially, that one can choose the g_n in (ii) to be uniformly bounded and since they converge $d\langle X \rangle$ almost everywhere to g on $[0, \alpha]$ then

$$\int_0^\alpha |g_n(s) - g_m(s)|^2 d\langle X \rangle_s$$

converges to zero in $\mathbf{L}^1$. One uses 5·11 to show that $\left(\int_0^\alpha g_n(s)\, dX_s \right)$ is Cauchy in $\mathbf{L}^2(\mathscr{A})$ and that the limit is independent of the *uniformly bounded* sequence (g_n). This limit is $\int_0^\alpha g(s)\, dX_s$. The left integral can be constructed in the same way by noting that $g \in \bar{\mathscr{P}}(X) \Rightarrow g^* \in \bar{\mathscr{P}}(X)$ and as

$$\left(\int_0^\alpha dX_s g(s) \right)^* = \int_0^\alpha g(s)^*\, dX_s$$

for simple g and $\|A\|_2 = \|A^*\|_2$ for $A \in \mathbf{L}^2(\mathscr{A})$ then the isometry property takes the form

$$\left\| \int_0^\alpha dX_s g(s) \right\|_2^2 = \phi \left(\int_0^\alpha |g(s)^*|^2 d\langle X \rangle_s \right). \tag{5·12}$$

The extension to $g \in \bar{\mathscr{P}}(X)$ is achieved as with the right integral.

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When we are dealing with a martingale which is not necessarily self-adjoint we cannot obtain the left integral from the right integral (and vice versa) in such a straightforward fashion. The following theorem is a summary of what we know to hold.

5·2. THEOREM. *Let* (X_α) *be an* $\mathbf{L}^2$ *martingale.*

(i) *If* $(|X_\alpha|^2)$ *is a class D process and* $g \in \bar{\mathscr{P}}(X)$ *then for* $\alpha > 0$, $\int_0^\alpha dX_s\, g(s)$ *and* $\int_0^\alpha g(s)\, dX_s^*$ *exist (in the sense described above), and*

$$\left(\int_0^\alpha dX_s\, g^*(s)\right)^* = \int_0^\alpha g(s)\, dX_s^*.$$

(ii) *If* $(|X^*|^2)$ *is a class D process and* $g \in \bar{\mathscr{P}}(X^*)$ *then* $\int_0^\alpha g(s)\, dX_s$ *and* $\int_0^\alpha dX_s^*\, g(s)$ *exist and*

$$\left(\int_0^\alpha g^*(s)\, dX_s\right)^* = \int_0^\alpha dX_s^*\, g(s).$$

Proof. Use the method of (7·17) of [1] with the appropriate isometry property (5·11) or (5·12).

It is natural to question the relationship between the integral defined in §3 above and the integral defined via $d\langle X\rangle_s$. Since the former uses a contraction property and the latter an isometry property we shall refer to the 'contraction integral' and the 'isometry integral'.

5·3. THEOREM. (i) *Let* (X_α) *be an* $\mathbf{L}^2$*-bounded martingale such that* $(|X_\alpha|^2)$ *(resp.* $|X_\alpha^*|^2$*) is a class D process. Then* $\bar{\mathscr{P}}(X) \subseteq \mathscr{P}\langle X\rangle$ *(resp.* $\bar{\mathscr{P}}(X^*) \subseteq \mathscr{P}\langle X^*\rangle$*) and the isometry integral of* $g \in \bar{\mathscr{P}}(X)$ *(resp.* $\bar{\mathscr{P}}(X^*)$*) is identical with the contraction integral of* g.

Proof. A sequence of functions (f_n) converges $d\langle X\rangle$-a.e. to f if there is a Borel set with zero $d\langle X\rangle$-semivariation (with respect to $\mathscr{A}$) off which it converges pointwise (see [3]). But the variation of $d\langle X\rangle$ is identical with its semivariation and by (2·5) they are therefore both equal to the Stieltjes measure $d|\langle X\rangle|$. It is clear that a $d|\langle X\rangle|$ essentially bounded function satisfies condition 3·4(ii) so $\bar{\mathscr{P}}(X) \subseteq \mathscr{P}\langle X\rangle$. Lemma 2·5 shows also that the variation of $d\langle X^*\rangle$ is $d|\langle X\rangle|$. By the same arguments we have $\bar{\mathscr{P}}\langle X^*\rangle \subseteq \mathscr{P}\langle X^*\rangle$. Now

$$\text{(contraction)} \quad \int g\, dX = \mathbf{L}^2 - \lim \int f_n\, dX = \text{(isometry)} \int g\, dX.$$

So the contraction integral and the isometry integral of g coincide.

Remarks. Theorem 5·3 indicates that for the purposes of our stochastic integration the theory of class D processes is unnecessary. Note that we do *not* claim (yet) that in case $\mathscr{A}$ is commutative the contraction integral and the usual stochastic integral of a (suitable) process coincide. We hope to clarify this relationship in a forthcoming paper.

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