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Lipschitz classes on 0-dimensional groups

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1. *Introduction.* Let G be a compact metric 0-dimensional Abelian group. Its dual or character group Γ is a discrete countable torsion group. We denote elements of G by x, of Γ by y, the value of the character y at x by (x, y), and the Fourier transform of f by $\hat{f}$.

Our object is to prove Theorems 1 and 2 which are analogues of classical results due respectively to Bernstein and to Beurling and Pollard. See (1), Ch. 6, Theorem 3·1, or (2) Ch. 10, Theorem 1 for Theorem 1, and (2) Ch. 9, Theorem 3, Corollary 1 for Theorem 2. These theorems deduce from a hypothesis of the form $f \in \Lambda_\alpha$ (the Lipschitz class of order α) the absolute convergence of the Fourier series of f, and the fact that f satisfies spectral synthesis.

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2. *Preliminary results.* Our constructions hinge on the fact that in any locally compact 0-dimensional group, there is a base of neighbourhoods of 0 consisting of compact open subgroups of G (see (3) Theorem 2·4·4). Hence if G is in addition compact and metric, we can take a countable basic set of neighbourhoods $\{\Delta_n\}$, which are open compact subgroups of G and are ordered by inclusion,

$$G = \Delta_0 \supset \Delta_1 \supset \Delta_2 \ldots \supset \Delta_n \supset \Delta_{n+1} \supset \ldots \supset \{0\}, \quad \bigcap_{n=0}^{\infty} \Delta_n = \{0\}.$$

If G is infinite we can assume that each inclusion shown is proper.

Let $f_n(x) = p_n$ if $x \in \Delta_n$, $= 0$ if $x \notin \Delta_n$, where p_n is chosen so that

$$\int_G f_n(x)\, dx = 1.$$

Then if m is Haar measure on G, $m(G) = 1$, we have $p_n = [m(\Delta_n)]^{-1}$, and it follows that p_n is increasing, $p_0 = 1$, $p_n \to \infty$ as $n \to \infty$. Also since m is translation-invariant, p_n is the (integer) order of the finite group G/Δ_n. We denote the metric on G by

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$d(x,x') = |x-x'|$, where $|x| = \beta_n$ if $x \in \Delta_n \setminus \Delta_{n+1}$, $|0| = 0$, and β_n tends monotonically to zero. Any such sequence $\{\beta_n\}$ gives an ultra-metric on G, $|x+x'| \leqslant \max(|x|, |x'|)$, since each Δ_n is a group. We make the particular choice $\beta_n = p_{n+1}^{-1}$, motivated by the special case when $G = \prod_{i=1}^{\infty} C_{s_i}$, a countable product of finite cyclic groups of arbitrary order. Here

$$\Delta_n = \prod_{i=n+1}^{\infty} C_{s_i}, \quad p_n = s_1 s_2 \dots s_n,$$

and the usual metric is

$$|x| = \sum_{n=1}^{\infty} \frac{1}{s_1 s_2 \dots s_n} |x_n|,$$

where $|x_n|$ is the discrete metric on C_{s_n}. An equivalent metric is given by

$$|x| = (s_1 s_2 \dots s_n)^{-1} \quad \text{if} \quad x_1 = \dots = x_{n-1} = 0, \, x_n \neq 0,$$

or

$$|x| = p_n^{-1} \quad \text{if} \quad x \in \Delta_{n-1} \setminus \Delta_n.$$

Let V_n be the annihilator group of Δ_n, $V_n = \{y : y \in \Gamma, (x,y) = 1 \text{ all } x \in \Delta_n\}$. Evidently $\{0\} = V_0 \subset V_1 \subset \dots \subset V_n \subset V_{n+1} \dots \subset \Gamma$ and again all inclusions are proper. If $y \in V_n$,

$$\hat{f}_n(y) = \int_G f_n(x)(-x,y)\,dx = 1;$$

while if $y \notin V_n$, (x,y) is a non-constant character on Δ_n, and $\hat{f}_n(y) = 0$. Since $\hat{f} \in C_0(\Gamma)$ we deduce that V_n is finite, and that $f_n(x) = \Sigma_{V_n}(x,y)$. Since $p_n = f_n(0) = \Sigma_{V_n} 1$, it follows that V_n is a finite group of order p_n. Also V_{n+1}/V_n is a non-trivial group of order $s_{n+1} = p_{n+1}/p_n$, i.e. s_{n+1} is an integer $\geqslant 2$, and $p_n \geqslant 2^n$. Finally $\bigcup_{n=0}^{\infty} V_n = \Gamma$. For if $y \in \Gamma$, there is a neighbourhood of 0, which we take as Δ_k on which $\Re(x,y) > \frac{1}{2}$. Then

$$\Re \int_G f_k(x)(-x,y)\,dx = \Re \int_{\Delta_k} p_k(-x,y)\,dx > \tfrac{1}{2},$$

i.e.

$$\int_G f_k(x)(-x,y)\,dx = 1, \text{ and } y \in V_k.$$

LEMMA 1. *For each n, s_n may be taken prime and $\Delta_n/\Delta_{n+1} \simeq V_{n+1}/V_n \simeq C_{s_{n+1}}$.*

Proof. We know the order of V_{n+1}/V_n is $p_{n+1}/p_n = s_{n+1}$. Also p_n is the order of G/Δ_n, and $(G/\Delta_{n+1})/(\Delta_n/\Delta_{n+1}) \simeq G/\Delta_n$, and therefore the order of Δ_n/Δ_{n+1} is also s_{n+1}. By interpolating additional groups between V_n and V_{n+1}, each generated by the preceding together with one further element, we can take each factor group cyclic; by repeating the process we can take each cyclic group to be of prime order.

3. *Absolute convergence.* We denote by $A(G) = A$ the set of $f(x) = \Sigma_\Gamma \hat{f}(y)(x,y)$, where $\Sigma_\Gamma |\hat{f}(y)| < \infty$, and by $\Lambda_\alpha(G) = \Lambda_\alpha$ the set of f such that $|f(x) - f(x')| \leqslant C|x-x'|^\alpha$ for a constant C independent of x, x'. We can now prove

THEOREM 1. *If $\alpha > \frac{1}{2}$, and $f \in \Lambda_\alpha$, then $f \in A$.*

Proof. Let $f \in \Lambda_\alpha$. Then f is continuous, and in particular $f \in L^2(G)$. By Parseval's formula

$$\Sigma_\Gamma |\hat{f}(y)|^2 |(h,y) - 1|^2 = \int_G |f(x+h) - f(x)|^2 \,dx = O(|h|^{2\alpha});$$

where h is any element of G.

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Take $h \in \Delta_{n-1} \backslash \Delta_n$, $|h| = p_n^{-1}$. Then since s_n is prime, $h + \Delta_n$ generates Δ_{n-1}/Δ_n (Lemma 1).

Now suppose $y \in V_n$, and $(h, y) = 1$. Then for all $x \in \Delta_{n-1}$, there is an integer k, and $x' \in \Delta_n$ such that $x = kh + x'$. Therefore $(x, y) = (h, y)^k (x', y) = 1$, and $y \in V_{n-1}$. It follows that if $y \in V_n \backslash V_{n-1}$, $(h, y) \neq 1$. But $s_n h \in \Delta_n$, i.e. $(h, y)^{s_n} = 1$, or

$$(h, y) = \exp\left(\frac{2\pi i k}{s_n}\right) \quad \text{and} \quad |(h, y) - 1|^2 = 4 \sin^2\left(\frac{\pi i k}{s_n}\right),$$

$1 \leqslant k \leqslant s_n - 1$, and k depends on y. We now have

$$\Sigma_{T_n} |\hat{f}(y)|^2 4 \sin^2\left(\frac{\pi i k}{s_n}\right) = O(p_n^{-2\alpha}), \quad \text{where} \quad T_n = V_n \backslash V_{n-1}. \tag{i}$$

But h was arbitrarily chosen in $\Delta_{n-1} \backslash \Delta_n$, i.e. we can replace h by $2h, 3h, \ldots, (s_n - 1)h$.

It follows that we can replace k in (i) by $2k, \ldots, (s_n - 1)k$, i.e. for all t, $1 \leqslant t \leqslant s_n - 1$,

$$\Sigma_{T_n} |\hat{f}(y)|^2 \sin^2\left(\frac{\pi i t k}{s_n}\right) = O(p_n^{-2\alpha}).$$

Now write

$$T_{n,1} = \{y : y \in T_n, \tfrac{1}{4} < k/s_n < \tfrac{3}{4}\}$$

$$T_{n,2} = \{y : y \in T_n, \tfrac{1}{8} < k/s_n < \tfrac{1}{4} \quad \text{or} \quad \tfrac{3}{4} < k/s_n < \tfrac{7}{8}\}$$

$$\vdots$$

$$T_{n,m} = \left\{y : y \in T_n, \frac{1}{2^{m+1}} < \frac{k}{s_n} < \frac{1}{2^m} \quad \text{or} \quad 1 - \frac{1}{2^m} < \frac{k}{s_n} < 1 - \frac{1}{2^{m+1}}\right\},$$

where m is chosen so that

$$\frac{1}{2^{m+1}} < \frac{1}{s_n} \leqslant \frac{1}{2^m}, \quad \text{i.e. } m = \left[\frac{\log s_n}{\log 2}\right].$$

For $1 \leqslant j \leqslant m$, take $t_j = 2^{j-1}$. Then on $T_{n,j}$,

$$\sin^2\left(\frac{\pi i t_j k}{s_n}\right) \geqslant \tfrac{1}{2},$$

i.e.
$$\Sigma_{T_{n,j}} |\hat{f}(y)|^2 \leqslant 2\Sigma_{T_{n,j}} |\hat{f}(y)|^2 \sin^2\left(\frac{\pi i t_j k}{s_n}\right) = O(p_n^{-2\alpha}),$$

and
$$\Sigma_{T_n} |\hat{f}(y)|^2 = \sum_{j=1}^{m} \Sigma_{T_{n,j}} |\hat{f}(y)|^2 = O(m p_n^{-2\alpha}) = O(\log s_n \, p_n^{-2\alpha}).$$

But
$$(\Sigma_{T_n} |\hat{f}(y)|)^2 \leqslant (p_n - p_{n-1}) \Sigma_{T_n} |\hat{f}(y)|^2$$

by Schwarz's inequality, since T_n has $p_n - p_{n-1}$ elements, i.e.

$$\Sigma_{T_n} |\hat{f}(y)| = O(p_n^{\frac{1}{2}} (\log s_n)^{\frac{1}{2}} p_n^{-\alpha}).$$

Therefore

$$\Sigma_\Gamma |\hat{f}(y)| = |\hat{f}(0)| + \sum_{n=1}^{\infty} \Sigma_{T_n} |\hat{f}(y)| \leqslant |\hat{f}(0)| + K \sum_{n=1}^{\infty} \left[\frac{(\log s_n)^{\frac{1}{2}}}{(s_n)^{\alpha - \frac{1}{2}}}\right] p_{n-1}^{(\frac{1}{2} - \alpha)}.$$

If $\alpha > \frac{1}{2}$, the term in square brackets is bounded, also $p_{n-1} \geqslant 2^{n-1}$, and the series on the right is convergent. If follows that $\Sigma_\Gamma |\hat{f}(y)| < \infty$, or $f \in A$.

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4. *Spectral synthesis.* Suppose initially that G is simply a compact Abelian group. We say S is a pseudo-measure ($S \in PM(G)$) and write $S \sim \Sigma_\Gamma c_y(x,y)$ iff $\sup |c_y| < \infty$. (S is an object which behaves as if it had Fourier coefficients c_y; no sort of convergence of $\Sigma_\Gamma c_y(x,y)$ is implied.) If $S_1, S_2 \in PM(G)$, we define $S_1 * S_2 \sim \Sigma_\Gamma c_{1y} c_{2y}(x,y)$. If $S \in PM$, $f \in A$, then $S * f \in A$. We define the scalar product of $S \in PM$ and $f \in A$ by

$$\langle S, f \rangle = \Sigma_\Gamma c_y \bar{\gamma}_y,$$

where $f(x) = \Sigma_\Gamma \gamma_y(x,y)$.

Note that if S is actually a measure or integrable function with c_y as Fourier coefficient, then

$$\langle S, f \rangle = \Sigma_\Gamma \bar{\gamma}_y \int_G (-x,y)\, d\mu(x) = \int_G \bar{f}(x)\, d\mu(x)$$

which is the usual scalar product. If

$$d_y = \sum_{y_1} c_{y_1} \gamma_{y-y_1} = \langle S, (x,y) \bar{f}(x) \rangle,$$

then the product $S.f$ is the pseudo-measure given by $S.f \sim \Sigma_\Gamma d_y(x,y)$.

Note that $S.(fg) = (S.f).g$ by associativity of convolution on Γ. We say S is null on an open set $\Omega \subset G$ iff $S.f = 0$ for all $f \in A$ with support in Ω, or equivalently $\langle S, f \rangle = 0$ for all such f.

LEMMA 2. *If S is null on each of a family $(\Omega\alpha)\, \alpha \in B$ of open sets it is null on their union.*

Proof. Let $\Omega = \bigcup_{\alpha \in B} \Omega\alpha$. Let $f \in A$, $\operatorname{supp} f = F \subset \Omega$.

For each $x \in F$, there is an $f_x \in A$, such that $0 \leqslant f_x \leqslant 1$, $f_x = 1$ in a neighbourhood of x, and $\operatorname{supp} f_x$ is contained in some $\Omega\alpha$. Since F is compact, we can take $f_1, \ldots, f_N$; $\Omega_1, \ldots, \Omega_N$ such that for each $x \in F$, $f_i = 1$ in a neighbourhood U_i of x for some $1 \leqslant i \leqslant N$ and $\operatorname{supp} f_i \subset \Omega_i$.

Then $f = f[1 - (1-f_1)(1-f_2) \ldots (1-f_N)]$, since some f_i is 1 at every point of F, and f is 0 outside F.

Then $S.f = S.f[1 - (1-f_1)(1-f_2) \ldots (1-f_N)] = 0$ since $S.f_i = 0$ each i, and S is null on Ω.

Lemma 2 allows us to define the support of S as the complement of the largest open set on which S is null. If S is a measure or integrable function, this is in accordance with the usual notion of support.

LEMMA 3. *If* $\operatorname{supp} S = \varnothing$, *then* $c_y = 0$ *all* y, *i.e.* $S = 0$.

Proof. $\Sigma_\Gamma c_y \bar{\gamma}_y$ must $= 0$ for all $\Sigma_\Gamma |\gamma_y| < \infty$.

LEMMA 4. *If* $S_1, S_2 \in PM$, *then* $\operatorname{supp}(S_1 * S_2) \subset \operatorname{supp} S_1 + \operatorname{supp} S_2$.

Proof. The result is obvious when S_1, S_2 are functions in A, since then

$$(f_1 * f_2)(x) = \Sigma_\Gamma \gamma_y \delta_y(x,y) = \Sigma_\Gamma \delta_y \int_G f_1(x')(-x', y)\, dx'(x,y)$$
$$= \int_G f_1(x') \Sigma_\Gamma \delta_y(x - x', y)\, dx' = \int_G f_1(x') f_2(x - x')\, dx'.$$

For any $S \in PM$, $S \sim \Sigma_\Gamma c_y(x,y)$, define $S^\dagger \sim \Sigma_\Gamma \bar{c}_y(x,y)$. Then for any

$$S_1, S_2 \in PM, \quad f \in A, \quad \langle S_1 * S_2, f \rangle = \langle S_1, S_2^\dagger * f \rangle.$$

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Now $\operatorname{supp} S^\dagger = -\operatorname{supp} S$, since

$$\langle S^\dagger, f\rangle = \Sigma_\Gamma \bar{c}_\gamma \bar{\gamma}_\gamma = 0 \text{ iff } \Sigma_\Gamma c_\gamma \gamma_\gamma = 0,$$

i.e. iff $\langle S, f(-x)\rangle = 0$. Also for any sets A, B, C, $(A+B)\cap C = \varnothing$ iff $A \cap (-B+C) = \varnothing$. Take first the case where $S_1 \in PM$, $S_2 = f_2 \in A$, $f \in A$ has support disjoint from the sum of the supports of S_1, f_2. Then $\langle S_1 * f_2, f\rangle = \langle S_1, f_2^\dagger * f\rangle$ and $\operatorname{supp} f_2^\dagger * f$ is contained in $-\operatorname{supp} f_2 + \operatorname{supp} f$ which is disjoint from $\operatorname{supp} S_1$, i.e. $\langle S_1 * f_2, f\rangle = 0$.

Hence the result is proved when $S_2 \in A$.

In the general case, if $\operatorname{supp} f$ is disjoint from the sum of the supports of S_1, S_2, then $\langle S_1 * S_2, f\rangle = \langle S_1, S_2^\dagger * f\rangle$, and the support of $S_2^\dagger * f$ is contained in $-\operatorname{supp} S_2 + \operatorname{supp} f$ which is disjoint from $\operatorname{supp} S_1$, and $S_2^\dagger * f \in A$.

Then $\langle S_1 * S_2, f\rangle = 0$, and the lemma is proved.

Suppose that $S \in PM$; E is a compact subset of G; $f \in A$; $\operatorname{supp} S \subset E \subset f^{-1}(0)$. (ii) We say S (respectively E, resp. f) satisfies spectral synthesis iff for all E, f (resp. for all S, f; resp. for all S, E) satisfying (ii), we have $\langle S, f\rangle = 0$.

Then when G is metric and 0-dimensional we have the following theorem. Notation as in section 2.

THEOREM 2. *If $f \in A \cap \Lambda_{\frac{1}{2}}$, then f satisfies spectral synthesis.*

Proof. Suppose (ii) is satisfied.

Define $E_h = \{x : d(x, E) \leqslant h\}$.

If E is compact, $m(E_h \backslash E) \to 0$ as $h \to 0$.

Let $S_n(x) = S * f_n(x)$. Then $S_n \in A$, $\operatorname{supp} S_n \subset E + \Delta_n$ (Lemma 4), and $E + \Delta_n \subset E_h$ when $h = p_{n+1}^{-1}$.

Then $\langle S_n, f\rangle = \Sigma_{V_n} c_\gamma \bar{\gamma}_\gamma \to \Sigma_\Gamma c_\gamma \bar{\gamma}_\gamma = \langle S, f\rangle$ as $n \to \infty$, since $\bigcup_{n=1}^{\infty} V_n = \Gamma$, and the right-hand side is absolutely convergent. For spectral synthesis it is sufficient to show $\langle S_n, f\rangle \to 0$ as $n \to \infty$. Now

$$\langle S_n, f\rangle = \int_G S_n(x)\bar{f}(x)\,dx = \int_{E_h \backslash E} S_n(x)\bar{f}(x)\,dx, \quad \text{since } \operatorname{supp} S_n \subset E_h$$

and

$$f = 0 \quad \text{on} \quad E.$$

Therefore

$$|\langle S_n, f\rangle|^2 \leqslant \int_{E_h \backslash E} |S_n(x)|^2\,dx \int_{E_h \backslash E} |f(x)|^2\,dx.$$

$$\begin{aligned}\int_{E_h \backslash E} |f(x)|^2\,dx &= O(h)\, m(E_h \backslash E), \quad \text{since} \quad f \in \Lambda_{\frac{1}{2}}, \quad \text{and} \quad f = 0 \quad \text{on} \quad E,\\ &= o(p_{n+1}^{-1}), \quad \text{since} \quad h = p_{n+1}^{-1} \to 0 \quad \text{and} \quad m(E_h \backslash E) \to 0 \quad \text{as} \quad n \to \infty.\end{aligned}$$

$$\int_{E_h \backslash E} |S_n(x)|^2\,dx \leqslant \int_G |S_n(x)|^2\,dx = \Sigma_{V_n} |c_\gamma|^2 \leqslant \sup |c_\gamma|^2 \, \Sigma_{V_n} 1,$$

i.e.

$$\int_{E_h \backslash E} |S_n(x)|^2\,dx = O(p_n).$$

Hence $|\langle S_n, f\rangle|^2 = O(p_n)\, o(p_{n+1}^{-1}) = o(s_{n+1}^{-1})$ as $n \to \infty$, as required.

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Note. That Theorem 1 contains the best value of α for which the result is true for all 0-dimensional groups, follows from a construction analogous to that in (2), Ch. 10, No. 5, in the case where $G = \prod_{n=1}^{\infty} C_p$. It is not known if the value $\frac{1}{2}$ can be improved in Theorem 2, either in the classical, or the 0-dimensional case.

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