Differential equations v. power series

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How do you solve ordinary differential equations with variable coefficients? A typical approach is to seek their solutions in the form of a generalised power series [2]. This method is a standard topic in virtually every undergraduate course on ordinary differential equations. A power series solution produces a recurrence relation for their coefficients. In addition, we may wish to know more about the properties of these coefficients, such as their partial sums or weighted sums. Certain interesting numerical sums might be obtained from the power series by evaluation at a particular point. For example, if $f(z) = (1 + z)^n = \sum_{k \geqslant 0} \binom{n}{k} z^k$ and n is a positive integer, then putting $z = 1$, $\sum_{k \geqslant 0} \binom{n}{k} = 2^n$. If an ordinary generating function (or polynomial) satisfies a differential equation subject to initial conditions, then an exact formula for a finite sum of its coefficients might be obtained. This approach was demonstrated in [1] using a first order differential equation to verify a well-known binomial identity. We begin with a general second-order linear differential equation having two linearly independent series solutions $y_1(z)$ and $y_2(z)$, and then find a new second order differential equation having two independent solutions that generate finite sums of coefficients of $y_1(z)$ and $y_2(z)$. This is illustrated with Chebyshev's differential equation and produces, among other things, the familiar identity for the finite sum of odd integers. Our goal is to describe the interplay between differential equations and power series.

Consider the linear differential equation

$$y'' + a(z)y' + b(z)y = 0, \tag{1}$$

where $a(z)$ and $b(z)$ are continuous functions on some interval. Suppose $y(z)$ is a series solution of this equation. How are the partial sums of its coefficients related to (1)? If $y(z)$ has a Maclaurin representation $y(z) = \sum_{n \geqslant 0} c_n z^n$, then the series is known as an ordinary generating function [3] for its sequence of coefficients $\{c_n\}_{n \geqslant 0}$. The sequence $\left\{\sigma_n = \sum_{k=0}^{n} c_k\right\}$ of finite sums has generating function given by

$$S(z) = \sum_{n \geqslant 0} \sigma_n z^n = \frac{y(z)}{1 - z}. \tag{2}$$

This is true because

$$\sum_{n \geqslant 0} \left(\sum_{k=0}^{n} c_k \right) z^n = \sum_{k \geqslant 0} \sum_{n \geqslant k} c_k z^n = \sum_{k \geqslant 0} c_k \sum_{n \geqslant k} z^n = \sum_{k \geqslant 0} c_k \frac{z^k}{1 - z}$$

$$= \frac{1}{1 - z} \sum_{k \geqslant 0} c_k z^k = \frac{y(z)}{1 - z}.$$

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It is not difficult to verify that the function $S(z)$ satisfies a differential equation just like (1), namely

$$S''(z) + \left(a(z) - \frac{2}{1-z}\right)S'(z) + \left(b(z) - \frac{a(z)}{1-z}\right)S(z) = 0. \qquad (3)$$

If $y_1(z)$ and $y_2(z)$ are linearly independent solutions of (1), then $S_1(z) = y_1(z)/(1-z)$ and $S_2(z) = y_2(z)/(1-z)$ are linearly independent solutions of (3). The Wronskian $W(y_1, y_2) = y_1(z)y_2'(z) - y_2(z)y_1'(z)$ is related to that of $S_1(z)$ and $S_2(z)$ by $W(S_1, S_2) = \dfrac{W(y_1, y_2)}{(1-z)^2}$. Note that (3) does not identify the functions $S_1(z)$ and $S_2(z)$ uniquely. To do this, we specify the initial conditions: $S_k(0) = y_k(0)$, $S_k'(0) = y_k(0) + y_k'(0)$, $k = 1, 2$.

For the remainder of this paper, we explore the foregoing with Chebyshev's equation (in the variable x),

$$(1-x^2)y'' - xy' + p^2y = 0, \qquad (4)$$

where p is a real number. This equation has the form of (1) with $a(x) = -x/(1-x^2)$ and $b(x) = p^2/(1-x^2)$. Its solution can be expressed as a power series $y(x) = \sum_{k \geqslant 0} a_k x^k$, whose coefficients satisfy the recurrence relation $a_{k+2} = \dfrac{(k-p)(k+p)}{(k+1)(k+2)}a_k$ (see [2] or Wikipedia). A fundamental set of solutions consists of the following two functions expressed as Maclaurin series:

$$y_1(x) = 1 - \frac{p^2}{2!}x^2 + \frac{(p-2)p^2(p+2)}{4!}x^4 - \frac{(p-4)(p-2)p^2(p+2)(p+4)}{6!}x^6 + \ldots$$

$$y_2(x) = x - \frac{(p-1)(p+1)}{3!}x^3 + \frac{(p-3)(p-1)(p+1)(p+3)}{5!}x^5 - \ldots.$$

These independent solutions are obtained by choosing $a_0 = 1$, $a_1 = 0$ for $y_1(x)$ and $a_0 = 0$, $a_1 = 1$ for $y_2(x)$. When $p = n$ is a positive integer, one of these series will terminate depending on the parity of n. The resulting polynomial, when multiplied by $(-1)^{n/2}$ if n is even and $(-1)^{(n-1)/2}n$ if n is odd, is known as a Chebyshev polynomial of the first kind and is denoted $T_n(x)$. Since Chebyshev's equation (4) has regular singularities at $x = \pm 1$, another linearly independent solution cannot be a polynomial; it is usually chosen as $\sqrt{1-x^2}U_{n-1}(x)$, where $U_{n-1}(x)$ is the Chebyshev polynomial of the second kind (of degree $n-1$). This solution together with $T_n(x)$ form a fundamental set of solutions to (4) (with $p = n$).

Let us look at the sequence $\{T_n(x)\}_{n \geqslant 0}$ of Chebyshev polynomials more closely. It is a solution to the well-known defining recurrence,

$$T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x), \qquad T_0 = 1, T_1 = x, n = 2, 3\ldots. \qquad (5)$$

The next few Chebyshev polynomials are

$$T_2(x) = -1 + 2x^2, \qquad T_3(x) = -3x + 4x^3, \qquad T_4(x) = 1 - 8x^2 + 8x^4, \ldots.$$

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Note that the generating function $F(z,x) = \sum_{n\geqslant 0} T_n(x)z^n$ for these Chebyshev polynomials can be obtained from the recurrence (5). Briefly, compute

$$F(z,x) = 1 + xz + \sum_{n\geqslant 2}\left(2xT_{n-1} - T_{n-2}\right)z^n = 1 + xz + 2xz\left(F(z,x) - 1\right) - z^2F(z,x),$$

and solve for $F(z, x)$. This yields

$$F(z, x) = \sum_{n\geqslant 0} T_n(x)\, z^n = \frac{1 - xz}{1 - 2xz + z^2}. \tag{6}$$

The polynomial $T_n(x)$ can be considered as a generating function for its coefficients, which are all zero starting with index $n + 1$. Let $\sigma_{k,n}$ be the sum of all coefficients of $T_n(x)$ up to index k $(n, k \geqslant 0)$. Obviously, this sequence stabilises when k exceeds n: $\sigma_{n,n} = \sigma_{n+1,n} = \sigma_{n+2,n} = \ldots$. For example, $T_8(x) = 1 - 32x^2 + 160x^4 - 256x^6 + 128x^8$, so (suppressing the second subscript $n = 8$) we have

$$\sigma_0 = \sigma_1 = 1, \sigma_2 = \sigma_3 = -31, \sigma_4 = \sigma_5 = 129, \sigma_6 = \sigma_7 = -127, \sigma_8 = \sigma_9 = \ldots = 1.$$

In a moment we will show that $\sigma_{k,n} = 1$ for $k \geqslant n$ for any n. According to (2), the function $S_n(x) = (1 - x)^{-1}T_n(x)$ generates the sequence $\{\sigma_{k,n}\}_{k\geqslant 0}$ of finite sums, and satisfies the differential equation (3), but in the variable x. Using the expressions $a(x) = -x/(1 - x^2)$ and $b(x) = n^2/(1 - x^2)$ for Chebyshev's equation, this becomes

$$S_n''(x) - \left(\frac{2 + 3x}{1 - x^2}\right)S_n'(x) + \left(\frac{n^2(1 - x) + x}{(1 - x)(1 - x^2)}\right)S_n(x) = 0. \tag{7}$$

This equation is a variable-coefficient, linear second order differential equation with two linearly independent solutions $(1 - x)^{-1}T_n(x)$ and $(1 - x)^{-1}\sqrt{1 - x^2}U_{n-1}(x)$.

Consider the functions $P_n(x)$ defined by the relation $T_{n+1}(x) = P_n(x)(1 - x) + 1$. Recurrence (5) for the Chebyshev sequence yields the following non-homogeneous difference equation for the sequence $\{P_n(x)\}_{n\geqslant 0}$:

$$P_n(x) = 2xP_{n-1}(x) - P_{n-2}(x) - 2, \qquad P_0 = 0, P_1 = -1, n = 2.3, \ldots,$$

and shows $P_n(x)$ to be a polynomial of degree $n - 1$ $(n > 0)$. Since

$$S_n(x) = (1-x)^{-1}T_n(x) = P_{n-1}(x) + (1-x)^{-1} = P_{n-1}(x) + \sum_{n\geqslant 0} x^n,$$

we extract kth coefficients and find that $\sigma_{k,n} = 1$ for $k \geqslant n$, that is, the sum $T_k(1)$ of the first k coefficients of $T_n(x)$ is 1 for $k \geqslant n$. A similar computation, working with the other solution $\sqrt{1 - x^2}U_{n-1}(x)$ of Chebyshev's equation (4) mentioned above, shows that the sum of coefficients of the Chebyshev polynomial $U_{n-1}(x)$ of the second kind is $U_{n-1}(1) = n$. To prove this, one may use the representation $U_{n-1}(x) = Q_{n-2}(x)(1 - x) + n$, where $Q_{n-2}(x)$ is a polynomial of degree $n - 2$. The details are left to the reader.

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Let us pursue the sequence $\{P_n(x)\}_{n\geqslant 0}$ and find its generating function. Multiplying $P_n(x) = (1-x)^{-1}\big(T_{n+1}(x)-1\big)$ by z^n and summing with respect to n, we obtain

$$\sum_{n\geqslant 0} P_n(x)z^n = \sum_{n\geqslant 0}\frac{T_{n+1}(x)-1}{1-x}\,z^n = \frac{1}{z(1-x)}\sum_{n\geqslant 0}T_{n+1}z^{n+1} - \frac{1}{1-x}\sum_{n\geqslant 0}z^n.$$

Now $\sum_{n\geqslant 0} T_{n+1}z^{n+1}$ is equal to $F(z, x) - 1$. Subtracting 1 from $F(z, x)$ in (6) and substituting the result into the preceding displayed expression yields the following generating function for $\{P_n(x)\}_{n\geqslant 0}$:

$$P(z, x) = \sum_{n\geqslant 1} P_n(x)\,z^n = -\frac{z(1+z)}{\big(1-2xz+z^2\big)(1-z)}.$$

Using l'Hôpital's rule together with the identity $U_{n-1}(1) = n$ and the well-known derivative formula $T'_{n+1}(x) = (n+1)U_n(x)$, we find that

$$\lim_{x\to 1} P_n(x) = \lim_{x\to 1}\frac{T_{n+1}(x)-1}{1-x} = -(n+1)^2.$$

Thus the function $-P(z, x)$ with $x = 1$ generates the square integers. Now $q(z) = z(1+z)/(1-z)^2$ generates the odd integers, as can be seen by writing

$$\frac{z(1+z)}{(1-z)^2} = 1 + \frac{3z}{(1-z)^2} - \frac{1}{(1-z)^2} = 1 + \sum_{n\geqslant 0} 3nz^n - \sum_{n\geqslant 0}(n+1)z^n = \sum_{n\geqslant 0}(2n-1)z^n.$$

In the preceding line, notice that $\sum_{n\geqslant 0}(n+1)z^n = 1/(1-z)^2$ is the derivative of $1/(1-z)$. Since $-P(z, 1) = q(z)/(1-z)$ generates finite sums of the coefficients of $q(z)$, we compare coefficients and obtain the familiar induction exercise $\sum_{k=1}^{n}(2k-1) = n^2$.

Let us turn to the sum $s_n(x) = \sum_{k=0}^{n} T_k(x)$ of the first $n+1$ Chebyshev polynomials. It can be shown that the generating function $F(z, x)$ in (6) satisfies the differential equation (1) where $a(z) = 2c'(z)/c(z)$, $b(z) = c''(z)/c(z)$ and $c(z) = \big(1-2xz+z^2\big)$. Thus the generating function for the sum,

$$S(z, x) = \sum_{n\geqslant 0} s_n(x)\,z^n = \frac{1-xz}{c(z)(1-z)}, \tag{8}$$

is a solution of the exact second order linear differential equation (3) (in the variable z):

$$\big(1-2xz+z^2\big)(1-z)S''(z,x) - 2\big(1+2x-2z-4xz+3z^2\big)S'(z,x)$$
$$+\,2(1-3z+2x)S(z,x) = 0,$$

subject to the initial conditions $S(0, x) = s_0(x) = 1$ and $S'(0, x) = s_1(x) = 1+x$.

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We conclude with an outline of the steps for finding a closed-form formula for the sum $s_n(x)$. The process begins with such a formula for $T_n(x)$. This is obtained through the partial fraction decomposition of its generating function (6). Since the equation $z^2 - 2xz + 1 = 0$ has two conjugate roots

$$z_1 = z_1(x) = x + \sqrt{x^2 - 1} \qquad \text{and} \qquad z_2 = z_2(x) = x - \sqrt{x^2 - 1},$$

computation shows that

$$F(z, x) = -\frac{1}{2}\left(\frac{z_1}{z - z_1} + \frac{z_2}{z - z_2}\right) = \frac{1}{2}\left(\frac{1}{1 - \frac{z}{z_1}} + \frac{1}{1 - \frac{z}{z_2}}\right). \tag{9}$$

Using the relation $z_1^{-1} = z_2$, we express (9) as a sum of two geometric series, extract coefficients of z^n, and obtain the explicit formula:

$$T_n(x) = \tfrac{1}{2}\left(z_1^{-n} + z_2^{-n}\right) = \tfrac{1}{2}\left(z_1^n + z_2^n\right).$$

Next, we turn to the partial fraction decomposition of the generating function $S(z, x)$ in (8). Computation shows that

$$\frac{F(z, x)}{1 - z} = \frac{1}{2}\left(\frac{1}{1 - z} + \frac{z_1}{(z - z_1)(z_1 - 1)} + \frac{z_2}{(z - z_2)(z_2 - 1)}\right).$$

By extracting coefficients, we obtain the formula

$$s_n(x) = \sum_{k=0}^{n} T_k(x) = \frac{1}{2}\left(1 + \frac{z_2^n}{1 - z_1} + \frac{z_1^n}{1 - z_2}\right) = \frac{1}{2} + \frac{T_n(x) - T_{n+1}(x)}{2(1 - x)}.$$

Concluding remark

We have seen that an ordinary generating function $G(z,x) = \sum_{n \geqslant 0} y_n(x) z^n$ like the one in (6) depends on two variables, the parameter x and the variable z. When this generating function is a rational function of z, its denominator reflects the recurrence relation satisfied by the corresponding sequence of coefficients $y_n(x)$. This is illustrated with the Chebyshev recurrence (5) and the denominator in (6). The same is true of the generating function for the sequence of finite sums. Moreover, each of these generating functions is a solution to a second-order differential equation. For example, $F(z, x)$ satisfies the differential equation (1) in the variable z. Thus generating functions (that is, power series) of this type, are associated with differential equations at two levels: one is the differential equation with respect to z that they satisfy and the other is the differential equation with respect to x associated with the difference equation satisfied by the sequence of generated coefficients.

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