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FUNCTIONS OF CONTINUOUS CESÁRO OPERATORS

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ABSTRACT. We describe holomorphic functions and fractional powers of Cesáro operators in $L^2(\mathbb{R})$, $L^2(\mathbb{R}_+)$, and $L^2[0,1]$. Logarithms of Cesáro operators are introduced as well and their spectral properties are studied in detail. Several examples are considered.

Keywords: Hausdorff operator; Cesáro operator; holomorphic functional calculus, fractional power, Hölder mean

1. INTRODUCTION

The continuous Cesáro operator originated from the classical summation methods for integrals [17, p. 110–111], see also [7, 8, 9, 21, 22]. It is an important tool in mathematical analysis because it bridges classical integration theory with modern functional analysis and operator theory. Its importance stems from its role as a "test case" for complex mathematical phenomena and its applications in various specialized fields.

The study of discrete Cesáro operator in ℓ^2 and continuous Cesáro operators in $L^2(\mathbb{R}_+)$ and $L^2[0,1]$ was initiated by A. Brown, P.R. Halmos, and A.L. Shields [10]. On the other hand, during last two decades the theory of Hausdorff operators, the vast generalization of Cesáro operators was developed (see survey articles [11, 23, 26]).

In the last decade continuous Cesáro operators continue to attract attention of researchers (see, e.g., [2], [3], [13] and the bibliography therein).

It follows from the work of Wallen and Shields [34] that each discrete Hausdorff operator on ℓ^2 is a bounded holomorphic function of the discrete Cesáro operator on ℓ^2 (see also [19], [20, Remark 2]).

In this work, we aim to describe holomorphic functions of continuous Cesáro operators in $L^2(\mathbb{R})$, $L^2(\mathbb{R}_+)$, and $L^2[0,1]$. It turns out that the cases of the spaces $L^2(\mathbb{R}_+)$ and $L^2[0,1]$ are easily reduced to the case of $L^2(\mathbb{R})$. We also compute complex fractional powers of this operator in $L^2(\mathbb{R})$ and

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$L^2(\mathbb{R}_+)$. In both cases, the result is the well-known Hölder operator. Logarithms of Cesáro operators are introduced and their spectral properties and converse are studied in detail, as well.

We show in particular that holomorphic functions of classical Cesáro operators, and also resolvents and converse of their logarithms are a sort of Hausdorff operators. Thus, more general Hausdorff operators naturally arise as functions of continuous Cesáro operators. In this context, the paper also considers examples of several classical integral operators of harmonic analysis.

Our considerations are based on the description of normal Hausdorff operators from [28], [29] and on results on algebras and symbols of Hausdorff operators from [24], [25].

2. Preliminaries

A continuous Cesáro operator has the form

$$f \mapsto \frac{1}{x}\int_0^x f(t)dt = \int_0^1 f(ux)du. \tag{2.1}$$

This is a very special case of a one-dimensional Hausdorff operator

$$(\mathcal{H}_{K,a}f)(x) := \int_{\mathbb{R}} K(u)f(a(u)x)du, \tag{2.2}$$

where K and a are given measurable functions on $\mathbb{R}$. If $a(u) = u$ on the support of K, we shall write $\mathcal{H}_K$ instead of $\mathcal{H}_{K,a}$.

If the kernel K satisfies the condition $K(u)|a(u)|^{-\frac{1}{2}} \in L^1(\mathbb{R})$, then the operator $\mathcal{H}_{K,a}$ is bounded (by the Cauchy-Schwartz inequality) and normal in $L^2(\mathbb{R})$ and $L^2(\mathbb{R}_+)$ (see, e. g., [29]).

Note that the kernel K is uniquely determined by the operator $\mathcal{H}_K$. Indeed, let

$$\int_{\mathbb{R}} K(u)f(ux)du = \int_{\mathbb{R}} K_1(u)f(ux)du$$

for all $f \in L^2(\mathbb{R})$, $x \in \mathbb{R}$. Then, putting $f_\pm(t) = e^{-p|t|}\chi_{\mathbb{R}_\pm}(t)$ ($\mathrm{Re}(p) > 0$), where χ stands for the indicator function of the set indicated as a subscript, and $x = 1$ in this equality we get $K = K_1$ a .e. by the uniqueness theorem for the Laplace transform.

The similar uniqueness result is true for the restriction of $\mathcal{H}_K$ to $L^2(\mathbb{R}_+)$.

We shall use the following facts from [28] (see also [29]). Let $K(u)|a(u)|^{-\frac{1}{2}} \in L^1(\mathbb{R})$. The matrix valued function

$$\Phi_{\mathcal{H}_K}(s) = \begin{pmatrix} \varphi_+(s) & \varphi_-(s) \\ \varphi_-(s) & \varphi_+(s) \end{pmatrix} \ (s \in \mathbb{R}) \tag{2.3}$$

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where

$$\varphi_+(s) = \int_{(0,+\infty)} K(u)u^{-\frac{1}{2}-\imath s}du, \tag{2.4}$$

$$\varphi_-(s) = \int_{(-\infty,0)} K(u)|u|^{-\frac{1}{2}-\imath s}du$$

is called *the matrix symbol* of the Hausdorff operator (2.2) (here and below $|u|^{-\imath s} = \exp(-\imath s \log|u|)$).

According to [28] we call the function

$$\varphi(s) := \int_{\mathbb{R}} K(u)|u|^{-1/2-\imath s}du \ (s \in \mathbb{R}) \tag{2.5}$$

the scalar symbol of the Hausdorff operator $\mathcal{H}_K$ in $L^2(\mathbb{R})$.

Next, in [24] the following set of Hausdorff operators was introduced

$$\mathcal{A}_a := \left\{ \mathcal{H}_{K,a} : \frac{K(u)}{|a(u)|^{\frac{1}{2}}} \in L^1(\mathbb{R}) \right\},$$

for a fixed measurable function a on $\mathbb{R}$, $a(u) \neq 0$ for a. e. $u \in \mathbb{R}$. It is shown that this set is a non-closed commutative subalgebra of the algebra $\mathcal{L}(L^2(\mathbb{R}))$ of bounded operators on $L^2(\mathbb{R})$ without unit (for multidimensional analogs see [25]). We shall write $\mathcal{A}_u$ instead of $\mathcal{A}_a$ if $a(u) = u$.

Moreover, in [24] a holomorphic calculus for one-dimensional Hausdorff operators was considered and the next result was proved. In the sequel, $\sigma(T)$ denotes the spectrum of an operator T in $L^2(\mathbb{R})$.

Theorem 2.1. *(*[24, Theorem 3.1]*) Let $\mathcal{H}_{K,a} \in \mathcal{A}_a$. If a function F is holomorphic in a neighborhood of the set $\sigma(\mathcal{H}_{K,a}) \cup \{0\}$ and $F(0) = 0$, then $F(\mathcal{H}_{K,a}) \in \mathcal{A}_a$.*

3. Functions of the Cesáro operator in $L^2(\mathbb{R})$

In this section $\mathcal{C}$ stands for the Cesáro operator in $L^2(\mathbb{R})$. This operator is normal, and its spectrum is equal to the circle $\mathbb{T} + 1$ by [28, Theorem 2] (here and below $\mathbb{T}$ stands for the unit circle in the complex plane).

3.1. Holomorphic functions of the Cesáro operator in $L^2(\mathbb{R})$.

Recall that the Mellin transform of a function f on $(0, +\infty)$ has the form

$$(\mathcal{M}f)(z) = \int_{(0,+\infty)} f(u)u^{z-1}du.$$

The next theorem gives a full description for holomorphic functions of the Cesáro operator in $L^2(\mathbb{R})$.

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Theorem 3.2. *1) Let F be a holomorphic function in a neighborhood of the spectrum $\mathbb{T}+1$ of the Cesáro operator $\mathcal{C}$ in $L^2(\mathbb{R})$ and $F(0)=0$. Then $F(\mathcal{C})=\mathcal{H}_K$ for a unique function K and the following conditions hold:*
(a) $K(u)=0$ for a.e. $u<0$,
(b) $K(u)u^{-\frac{1}{2}}\in L^1(\mathbb{R}_+)$, and
(c) $(\mathcal{M}K)(z)=F(z^{-1})$ for all z from the "critical line" $\mathrm{Re}(z)=\frac{1}{2}$.
2) If a function K meets the conditions (a), (b), and (c), where F is holomorphic in some neighborhood of $\mathbb{T}+1$, then $F(0)=0$ and $\mathcal{H}_K=F(\mathcal{C})$.

Proof. 1) Let F be a holomorphic function in a neighborhood N of $\mathbb{T}+1$, $F(0)=0$. Then $F(\mathcal{C})\in\mathcal{A}_u$ by Theorem 2.1. In other words, $F(\mathcal{C})=\mathcal{H}_K$, where K meets the condition $K(u)|u|^{-\frac{1}{2}}\in L^1(\mathbb{R})$. The uniqueness of K was proved above. By [28] the matrix symbol of $\mathcal{C}$ is

$$\Phi_{\mathcal{C}}(s)=\begin{pmatrix}(\frac{1}{2}-\imath s)^{-1} & 0\\ 0 & (\frac{1}{2}-\imath s)^{-1}\end{pmatrix},\quad s\in\mathbb{R}, \tag{3.6}$$

since $\varphi_+(s)=\varphi_+^0(s):=(\frac{1}{2}-\imath s)^{-1}$ and $\varphi_-(s)=\varphi_-^0(s):=0$ for the Cesáro operator $\mathcal{C}$.

Further, it is known (see [24]), that

$$\Phi_{F(\mathcal{C})}=\begin{pmatrix}F_1(\varphi_-^0,\varphi_+^0) & F_2(\varphi_-^0,\varphi_+^0)\\ F_2(\varphi_-^0,\varphi_+^0) & F_1(\varphi_-^0,\varphi_+^0)\end{pmatrix}$$

where

$$F_1(z_1,z_2):=\frac{1}{2\pi i}\int_\Gamma F(\lambda)\frac{\lambda-z_2}{(\lambda-z_2)^2-z_1^2}d\lambda,$$

and

$$F_2(z_1,z_2):=\frac{-z_1}{2\pi i}\int_\Gamma \frac{F(\lambda)}{(\lambda-z_2)^2-z_1^2}d\lambda.$$

Here Γ denotes the boundary of any open annulus U that contains $\mathbb{T}+1$ such that N contains its closure. Therefore,

$$\begin{aligned}\Phi_{F(\mathcal{C})}(s)&=\begin{pmatrix}F_1(0,\varphi_+^0(s)) & F_2(0,\varphi_+^0(s))\\ F_2(0,\varphi_+^0(s)) & F_1(0,\varphi_+^0(s))\end{pmatrix}\\ &=\begin{pmatrix}F_1(0,(\frac{1}{2}-\imath s)^{-1}) & F_2(0,(\frac{1}{2}-\imath s)^{-1})\\ F_2(0,(\frac{1}{2}-\imath s)^{-1}) & F_1(0,(\frac{1}{2}-\imath s)^{-1})\end{pmatrix}.\end{aligned}$$

Note that $F_2(0,z_2)=0$ and by the Cauchy's formula

$$F_1(0,z_2)=\frac{1}{2\pi i}\int_\Gamma F(\lambda)\frac{\lambda-z_2}{(\lambda-z_2)^2}d\lambda=F(z_2),$$

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as $z_2 \in U$. This yields that for all $s \in \mathbb{R}$

$$\Phi_{F(\mathcal{C})}(s) = \begin{pmatrix} F((\frac{1}{2} - \imath s)^{-1}) & 0 \\ 0 & F((\frac{1}{2} - \imath s)^{-1}) \end{pmatrix}, \tag{3.7}$$

because $(\frac{1}{2} - \imath s)^{-1} \in \mathbb{T} + 1$ as $s \in \mathbb{R}$.

On the one hand, formulas (2.3) and (2.4) hold. Since $F(\mathcal{C}) = \mathcal{H}_K$, we deduce firstly from (3.7) and (2.3) that

$$\int_{(-\infty,0)} K(u)|u|^{-\frac{1}{2}-\imath s} du = 0$$

for all $s \in \mathbb{R}$. And secondly,

$$\int_{(0,+\infty)} K(u)u^{-\frac{1}{2}-\imath s} du = F\left(\left(\frac{1}{2} - \imath s\right)^{-1}\right)$$

for all $s \in \mathbb{R}$. This implies (c).

2) Now let (a), (b), and (c) hold true for K and F, where F is a holomorphic function in some neighborhood of $\mathbb{T} + 1$. Since the Mellin transform

$$(\mathcal{M}K)\left(\frac{1}{2} + \imath t\right) = \int_0^\infty \left(K(u)u^{-\frac{1}{2}}\right) u^{\imath t} du$$

tends to zero as $t \to \infty$ by the Riemann-Lebesgue Lemma, we deduce that $F(0) = 0$.

Next, it was shown in the first part of the proof that (a), (b), and (c) along with $F(0) = 0$ imply that $\Phi_{F(\mathcal{C})}$ is given by the formula (3.7). On the other hand, since (a), we have by (2.3) that

$$\Phi_{\mathcal{H}_K}(s) = \begin{pmatrix} \varphi_+(s) & 0 \\ 0 & \varphi_+(s) \end{pmatrix} \tag{3.8}$$

where

$$\begin{aligned} \varphi_+(s) &= \int_{(0,+\infty)} K(u)u^{-\frac{1}{2}-\imath s} du \\ &= (\mathcal{M}K)\left(\frac{1}{2} - \imath s\right) = F\left(\left(\frac{1}{2} - \imath s\right)^{-1}\right) \end{aligned}$$

for all $s \in \mathbb{R}$. It follows that $\Phi_{\mathcal{H}_K} = \Phi_{F(\mathcal{C})}$ and thus $\mathcal{H}_K = F(\mathcal{C})$ (see, e.g., [24, Lemma 2.1]). □

Corollary 3.3. *Under the conditions of Theorem 3.2 one has for the spectrum* $\sigma(\mathcal{H}_K) = F(\mathbb{T} + 1)$.

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Remark 1. *Let the conditions (a), (b), and (c) hold. If K is of bounded variation in a neighborhood of a point x, then by the inverse formula for Mellin transform*

$$\frac{1}{2}(K(x+)+K(x-))=\frac{1}{2\pi\imath}v.p.\int_{\frac{1}{2}-\imath\infty}^{\frac{1}{2}+\imath\infty}F(s^{-1})x^{-s}ds.$$

Example 3.4. *Let $\mathrm{Re}(\alpha)<\frac{1}{2}$. Consider the following operator in $L^2(\mathbb{R})$:*

$$(\mathcal{P}_\alpha f)(x)=|x|^{\alpha-1}\mathrm{sign}(x)\int_0^x|v|^{-\alpha}f(v)dv$$

(Hardy operator). Then $\mathcal{P}_\alpha=\mathcal{H}_K$ where $K(u)=|u|^{-\alpha}\chi_{(0,1)}(u)$. All the conditions (a), (b), and (c) hold, because

$$(\mathcal{M}K)(z)=\int_0^1 u^{\alpha+z-1}du=\frac{1}{z-\alpha}.$$

In this case,

$$F(z)=F_\alpha(z):=\frac{z}{1-\alpha z} \tag{3.9}$$

is holomorphic in a neighborhood of $\mathbb{T}+1$, and so

$$\mathcal{P}_\alpha=F_\alpha(\mathcal{C})=\mathcal{C}\,(I-\alpha\mathcal{C})^{-1}=\alpha^{-1}((I-\alpha\mathcal{C})^{-1}-I)$$

if $\alpha\neq 0$. Putting here $\lambda=\alpha^{-1}$ we get for the resolvent of the Cesáro operator in $L^2(\mathbb{R})$

$$R(\lambda,\mathcal{C}):=(\lambda I-\mathcal{C})^{-1}=\lambda^{-2}P_{\lambda^{-1}}+\lambda^{-1}I \tag{3.10}$$

for all λ exterior to the circle $\mathbb{T}+1$.

It follows, that the spectrum $\sigma(\mathcal{P}_\alpha)$ equals to the circle $F_\alpha(\mathbb{T}+1)$.

Example 3.5. *Let $\mathrm{Re}(\alpha)<\frac{1}{2}$. Consider the following operator in $L^2(\mathbb{R})$:*

$$(\mathcal{Q}_\alpha f)(x)=\begin{cases}x^{-\alpha}\int_x^\infty v^{\alpha-1}f(v)dv, & when\ x>0,\\ |x|^{-\alpha}\int_{-\infty}^x|v|^{\alpha-1}f(v)dv, & when\ x<0\end{cases}$$

($\mathcal{Q}_0$ is the so called Copson operator). Then $\mathcal{Q}_\alpha\in\mathcal{A}_u$ with the kernel $K(u)=\chi_{(1,\infty)}(u)|u|^{\alpha-1}$. The conditions (a) and (b) are obviously satisfied. The condition (c) holds too, because

$$(\mathcal{M}K)(z)=\int_1^\infty u^{\alpha+z-2}du=\frac{1}{1-z-\alpha},$$

if (and only if) $\mathrm{Re}(z+\alpha)-1<0$. Thus,

$$F(z)=\frac{z}{(1-\alpha)z-1} \tag{3.11}$$

is holomorphic in a neighborhood of $\mathbb{T}+1$, and so for $\alpha\neq 1$

$$\mathcal{Q}_\alpha=F(\mathcal{C})=\mathcal{C}((I-\alpha)\mathcal{C}-I)^{-1}=-(1-\alpha)^{-1}((I-(1-\alpha)\mathcal{C})^{-1}-I).$$

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Putting here $\lambda = (1-\alpha)^{-1}$ *we get for the resolvent of the Cesáro operator in* $L^2(\mathbb{R})$

$$R(\lambda, \mathcal{C}) = (\lambda I - \mathcal{C})^{-1} = -\lambda \mathcal{Q}_{1-\lambda^{-1}} + I$$

for all λ *interior to the circle* $\mathbb{T}+1$.

Thus, we conclude that

$$R(\lambda, \mathcal{C}) = \begin{cases} \lambda^{-2}\mathcal{P}_{\lambda^{-1}} + \lambda^{-1} I, when\ |\lambda - 1| > 1 \\ -\lambda \mathcal{Q}_{1-\lambda^{-1}} + I, when\ |\lambda - 1| < 1. \end{cases}$$

This is consistent with the result of D. Boyd [9] *for the resolvent of the Cesáro operator in* $L^2(\mathbb{R}_+)$.

We conclude also that the spectrum $\sigma(\mathcal{Q}_\alpha)$ *equals to the circle* $F(\mathbb{T}+1)$.

Example 3.6. *Consider the following Hausdorff operator in* $L^2(\mathbb{R})$*:*

$$(Rf)(x) = \int_0^1 \left\{\frac{1}{u}\right\} f(ux) du$$

where $\{t\} := t - \lfloor t \rfloor$ *denotes the fractional part of a real number* $t > 0$. *The kernel* $K(u) = \left\{\frac{1}{u}\right\} \chi_{(0,1)}(u)$ *of this operator satisfies conditions (a) and (b). On the other hand, it follows from the theory of the Riemann zeta function (see, e.g.* [35]*) that*

$$\zeta(s) = \frac{s}{s-1} - s\int_1^\infty \{t\} t^{-1-s} dt,\ \mathrm{Re}(s) > 0, s \neq 1.$$

Therefore,

$$(\mathcal{M}K)(s) = \int_0^1 \left\{\frac{1}{u}\right\} u^{s-1} du = \frac{1}{s-1} - \frac{1}{s}\zeta(s),\ \mathrm{Re}(s) > 0, s \neq 1.$$

Here the function

$$F(z) = \frac{z}{1-z} - z\zeta\left(\frac{1}{z}\right)$$

is not holomorphic in any neighborhood of $\mathbb{T}+1$*, since the zeta function has an essential singularity at infinity* [33]*. Thus, the operator* R *is not a holomorphic function of* $\mathcal{C}$.

3.2. **Fractional powers and the logarithm of the Cesáro operator in** $L^2(\mathbb{R})$**.** In [13] positive fractional powers of the Cesáro operator in $L^\infty(\mathbb{R}_+)$ were defined and the limits at infinity of the values of fractional powers were considered. In this section, we compute complex fractional powers of the Cesáro operator in $L^2(\mathbb{R})$. We consider the branch of the function z^α with $\mathrm{Re}(\alpha) > 0$ in the right half-plane $\{\mathrm{Re}(z) > 0\}$ given by the condition $1^\alpha = 1$ and put $0^\alpha := 0$. This function is continuous (but not holomorphic) on the spectrum $\mathbb{T}+1$ of $\mathcal{C}$. Fractional powers of the normal operator in

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$L^2(\mathbb{R})$ will be considered in the sense of the functional calculus for normal operators (see, e.g., [15], or [12]).

Recall that *the Hölder means of order* α is defined as

$$(H_\alpha f)(x) = \frac{1}{\Gamma(\alpha)} \int_0^1 \left(\log \frac{1}{u}\right)^{\alpha-1} f(ux)du.$$

This is a Hausdorff operator of the form $\mathcal{H}_K$ which is bounded in $L^2(\mathbb{R})$ if $\mathrm{Re}(\alpha) > 0$. For real α it is well known to workers in summability theory. The study of this operator was initiated by Rhoades [31], see also [21].

Theorem 3.7. *Let* $\mathrm{Re}(\alpha) > 0$. *Then*

$$\mathcal{C}^\alpha = H_\alpha. \tag{3.12}$$

Proof. According to [28, Theorem 2] (or [29, Theorem 1]) the operator $\mathcal{C}$ in $L^2(\mathbb{R})$ with matrix symbol $\Phi_\mathcal{C}$ is unitary equivalent to the operator $M_{\Phi_\mathcal{C}}$ of multiplication by the normal matrix-function $\Phi_\mathcal{C}$ in the space $L^2(\mathbb{R};\mathbb{C}^2)$ of $\mathbb{C}^2$-valued functions. This means that

$$\mathcal{C} = \mathcal{V}^{-1} M_{\Phi_\mathcal{C}} \mathcal{V}$$

for some unitary operator $\mathcal{V}$ between $L^2(\mathbb{R})$ and $L^2(\mathbb{R};\mathbb{C}^2)$. In particular, $\sigma(\mathcal{C}) = \sigma(M_{\Phi_\mathcal{C}}) =: \sigma$. Let E_{M_Φ} be the spectral measure of $M_{\Phi_\mathcal{C}}$. Since, according to the spectral theorem,

$$M_{\Phi_\mathcal{C}} = \int_\sigma \lambda dE_{M_\Phi}(\lambda),$$

we have

$$\mathcal{C} = \int_\sigma \lambda d(\mathcal{V}^{-1} E_{M_\Phi}(\lambda)\mathcal{V}).$$

It follows, that $\mathcal{V}^{-1}E_{M_\Phi}\mathcal{V}$ is the spectral measure for $\mathcal{C}$ (see, e.g., [15, Chapter X, §2, Corollary 7]), and therefore

$$\begin{aligned}
\mathcal{C}^\alpha &= \int_\sigma \lambda^\alpha d(\mathcal{V}^{-1} E_{M_\Phi}(\lambda)\mathcal{V}) \\
&= \mathcal{V}^{-1} \left(\int_\sigma \lambda^\alpha d(E_{M_\Phi}(\lambda)) \right) \mathcal{V} \\
&= \mathcal{V}^{-1} M_\Phi^\alpha \mathcal{V} \\
&= \mathcal{V}^{-1} M_{\Phi^\alpha} \mathcal{V},
\end{aligned}$$

where in view of (3.6)

$$\Phi^\alpha \equiv \Phi_\mathcal{C}^\alpha = ((\varphi_+^0)I_2)^\alpha = (\varphi_+^0)^\alpha I_2$$

(I_2 stands for the unit matrix of order two). So, $\Phi_{\mathcal{C}^\alpha} = (\varphi_+^0)^\alpha I_2$.

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On the other hand, the operator H_α has the kernel

$$K(u) = \frac{1}{\Gamma(\alpha)} \left(\log \frac{1}{u} \right)^{\alpha-1} \chi_{(0,1)}(u),$$

and thus its matrix symbol Φ_{H_α} has the form (3.8), where

$$\varphi_+(s) = \int_{(0,+\infty)} K(u) u^{-\frac{1}{2}-\imath s} du \tag{3.13}$$

$$= \frac{1}{\Gamma(\alpha)} \int_0^1 \left(\log \frac{1}{u} \right)^{\alpha-1} u^{-1/2-\imath s} du$$

$$= \left(\frac{1}{2} - \imath s \right)^{-\alpha} = (\varphi_+^0)^\alpha(s).$$

Above we used the formula

$$\int_0^1 x^{\alpha-1} \log^\sigma \frac{1}{x} dx = \alpha^{-\sigma-1} \Gamma(\sigma+1) \tag{3.14}$$

which is valid for $\mathrm{Re}(\alpha) > 0, \mathrm{Re}(\sigma) > -1$ (see, e.g., [30, p. 38, 4.37]). It follows that $\Phi_{H_\alpha} = \Phi_{\mathcal{C}^\alpha}$, which completes the proof. □

Remark 2. *P. I. Delgado, C. Lennard, and J. Sivek* [13] *used* (3.12) *as a definition of $\mathcal{C}^\alpha$ on the space $L^\infty(\mathbb{R}_+)$ for $\alpha > 0$ and proved the semigroup property.*

Corollary 3.8. *Let* $\mathrm{Re}(\alpha) > 0$. *Then*

$$\sigma(H_\alpha) = \sigma(\mathcal{C}^\alpha) = \{z^\alpha : z \in \mathbb{T} + 1\},$$

and

$$\|H_\alpha\| = \|\mathcal{C}^\alpha\| = \left(\frac{2\mathrm{Re}(\alpha)}{|\alpha|} \right)^{\mathrm{Re}(\alpha)} e^{\mathrm{Im}(\alpha) \arg(\alpha)}.$$

In particular, if $\alpha > 0$, then $\sigma(H_\alpha) = \sigma(\mathcal{C}^\alpha)$ is an ark

$$\left\{ ((2\cos \alpha t)^\alpha \cos \alpha t, (2\cos \alpha t)^\alpha \sin \alpha t) : t \in \left[-\frac{\pi}{2}, \frac{\pi}{2} \right] \right\},$$

and

$$\|H_\alpha\| = \|\mathcal{C}^\alpha\| = 2^\alpha.$$

Proof. The first statement follows from the spectral mapping theorem. Further, since H_α is normal, we have

$$\|H_\alpha\| = \sup\{|z^\alpha| : z \in \mathbb{T} + 1\} = \sup\{|z|^{\mathrm{Re}(\alpha)} e^{-\mathrm{Im}(\alpha) \arg(z)} : z \in \mathbb{T} + 1\}.$$

Let $\alpha = a + \imath b, a > 0$, $z = 1 + e^{\imath\theta}, \theta \in [-\pi, \pi]$. Since $\arg(1 + e^{\imath\theta}) = \theta/2$, and $|1 + e^{\imath\theta}| = 2\cos\theta/2$, one has

$$\|H_\alpha\| = \max_{\theta \in [-\pi,\pi]} \left(2\cos \frac{\theta}{2} \right)^a e^{-b\frac{\theta}{2}} = 2^a \max_{x \in [-\pi/2, \pi/2]} (\cos x)^a e^{-bx}.$$

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The function $(\cos x)^a\, e^{-bx}$ attains its maximum on $[-\pi/2, \pi/2]$ at the point $x = -\tan^{-1}(b/a) = -\arg(\alpha)$ and the statement for $\|H_\alpha\|$ follows.

The formulas for the special case $\alpha > 0$ follow from the fact that if $z \in \mathbb{T} + 1$ and $\arg z = t$, then $|z| = 2\cos t/2$. □

Corollary 3.9. *If* $\mathrm{Re}(\alpha), \mathrm{Re}(\beta) > 0$*, then* $(H_\alpha)^\beta = H_{\alpha\beta}$.

Corollary 3.10. *The operator* H_α *is a holomorphic function of* $\mathcal{C}$ *if and only if* $\alpha \in \mathbb{Z}_+$.

Corollary 3.11. *If we put* $H_0 := I$*, then* H_α *(as a family of operators in* $L^2(\mathbb{R})$*) is a semigroup which is holomorphic in the right half-plane* $\{\alpha : \mathrm{Re}(\alpha) > 0\}$*. The generator of this semigroup is a normal operator whose spectrum lies in some half-plane* $\{z : \mathrm{Re}(z) \le \gamma\}$ *with* $\gamma < \infty$*. The restriction of this semigroup to the semiaxis* $\{\alpha \ge 0\}$ *is a positive semigroup.*

We shall see below that one can take $\gamma = \log 2$ and this value is sharp.

Proof. Indeed, it was shown in the previous theorem that H_α is unitary equivalent to the multiplication semigroup $M_{(\varphi_+^0)^\alpha}$ which is a holomorphic semigroup in the right half-plane. The next statement follows from [32, Theorem 13.37]. The last statement is obvious. □

Since $0 \in \mathcal{C}$, the general approach to operator logarithms does not work in the case of $\mathcal{C}$. Nevertheless, we accept the following definition (cf. [27, Proposition 10.1.2, Theorems 10.1.3, 10.1.4]).

Definition 3.12. *We denote the generator of the semigroup* $H_\alpha = \mathcal{C}^\alpha$ *by* $\log \mathcal{C}$.

Corollary 3.13. *The semigroup* $T(t) = 2^{-t} H_t$ *is uniformly stable, i. e.*

$$\lim_{t \to +\infty} 2^{-t} \|H_t f\| = 0$$

for all $f \in L^2(\mathbb{R})$.

Proof. Since $\forall t \ge 0\ \|T(t)\| = 1$, the semigroup T is Cesáro ergodic in $L^2(\mathbb{R})$ (see, e.g., [4, Corollary 4.3.5]). This means that for each $f \in L^2(\mathbb{R})$ the limit $f_1 = \lim_{t\to+\infty} 2^{-t} H_t f$ exists. Further, by [4, Proposition 4.3.1] for each $f \in L^2(\mathbb{R})$ the vector f_1 belongs to the kernel $\mathrm{Ker}(A)$, where A is the generator of T. Assume that $f_1 \ne 0$ for some $f \in L^2(\mathbb{R})$. Since $A = \log \mathcal{C} - (\log 2) I$, it follows that $\log 2$ is an eigenvalue of $\log \mathcal{C}$. On the other hand, the point spectrum

$$\sigma_p(\log \mathcal{C}) = \varnothing, \tag{3.15}$$

and we get a contradiction. To prove the equality (3.15), note that according to [28, Theorem 1], the point spectrum of H_t equals to the point spectrum

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of the operator M'_{φ_+} of multiplication by φ_+ in $L^2(\mathbb{R})$, where $\varphi_+(s) = (1/2 - \imath s)^{-t}$ is the scalar symbol of H_t by (3.13). But $\sigma_p(M'_{\varphi_+}) = \varnothing$ and so, $\sigma_p(H_t) = \varnothing$, too. It remains to apply the spectral mapping theorem for the point spectrum that states that $\sigma_p(H_t)\setminus\{0\} = \exp(t\sigma_p(\log \mathcal{C}))$ (see, e.g., [16, p. 277]), and the result follows. □

In the following

$$\nu(y,-1) = \int_{-1}^{\infty} \frac{y^s}{\Gamma(s+1)} ds = \int_0^{\infty} \frac{y^{t-1}}{\Gamma(t)} dt$$

(see, e.g., [6]).

Recall that for any linear operator A its *spectral bound* is defined by

$$s(A) := \sup\{\mathrm{Re}(\lambda) : \lambda \in \sigma(A)\}.$$

The next theorem describes spectral properties of the operator $\log \mathcal{C}$.

Theorem 3.14. *The following assertions hold true.*

(i) Let $\mathrm{Re}(\lambda) > \log 2$. *Then the resolvent of* $\log \mathcal{C}$ *is a Hausdorff operator of the form*

$$R(\lambda, \log \mathcal{C}) f(x) = e^{-\lambda} \int_0^1 \nu(e^{-\lambda} \log \frac{1}{u}, -1) f(ux) du, \ f \in L^2(\mathbb{R}). \tag{3.16}$$

(ii) Let $\mathrm{Re}(\lambda) > \log 2$. *Then the spectrum of* $R(\lambda, \log \mathcal{C})$ *is*

$$\sigma(R(\lambda, \log \mathcal{C})) = \left\{ \frac{1}{\lambda + \log(\frac{1}{2} - \imath s)} : s \in \mathbb{R} \right\} \cup \{0\}.$$

(iii) The spectrum of $\log \mathcal{C}$ *equals to its continuous spectrum and is a curve*

$$\sigma(\log \mathcal{C}) = \left\{ \log(2\cos y) + \imath y : y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \right\}. \tag{3.17}$$

In particular, the spectral bound $s(\log \mathcal{C}) = \log 2$.

(iv) Each $\lambda \in \mathbb{R}$, $\lambda \neq \log 2$ *is a regular point for* $\log \mathcal{C}$ *and we have for the norm of the resolvent*

$$\|R(\lambda, \log \mathcal{C})\| = \frac{1}{\lambda - \log 2} \text{ for } \lambda > \log 2, \lim_{\lambda \to -\infty} \|R(\lambda, \log \mathcal{C})\| = \frac{2}{\pi}. \tag{3.18}$$

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Proof. (i) For $\mathrm{Re}(\lambda) > \log 2$, $f \in L^2(\mathbb{R})$, and $x \in \mathbb{R}$ one has by the Fubini's theorem

$$
\begin{aligned}
R(\lambda, \log \mathcal{C}) f(x) &= \int_0^\infty e^{-\lambda\alpha} \mathcal{C}^\alpha f(x) d\alpha \qquad (3.19)\\
&= \int_0^\infty e^{-\lambda\alpha} H_\alpha f(x) d\alpha \\
&= \int_0^\infty e^{-\lambda\alpha} \int_0^1 \frac{1}{\Gamma(\alpha)} (\log \frac{1}{u})^{\alpha-1} f(ux) du d\alpha \\
&= \int_0^1 \left(\int_0^\infty e^{-\lambda\alpha} \frac{1}{\Gamma(\alpha)} (\log \frac{1}{u})^{\alpha-1} d\alpha \right) f(ux) du \\
&= \int_0^1 \left(\int_{-1}^\infty e^{-\lambda(s+1)} \frac{1}{\Gamma(s+1)} (\log \frac{1}{u})^{s} ds \right) f(ux) du \\
&= e^{-\lambda} \int_0^1 \nu(e^{-\lambda} \log \frac{1}{u}, -1) f(ux) du.
\end{aligned}
$$

To justify the application of the Fubini's theorem in (3.19) and to guarantee the convergence of the last integral, note that for each integer N

$$\nu(y, -1) = \exp(y) + O(|y|^{-1-N}), y \to +\infty$$

(see, e.g., [6, §18.3]), and consider the function

$$
\begin{aligned}
g(u) :&= \int_{-1}^\infty \left| e^{-\lambda s} \frac{1}{\Gamma(s+1)} (\log \frac{1}{u})^s \right| ds \\
&= \int_{-1}^\infty e^{-s\mathrm{Re}(\lambda)} \frac{1}{\Gamma(s+1)} (\log \frac{1}{u})^s ds \\
&= \nu(e^{-\mathrm{Re}(\lambda)} \log \frac{1}{u}, -1) \\
&= \exp(e^{-\mathrm{Re}(\lambda)} \log \frac{1}{u}) + O\left(\left| e^{-\mathrm{Re}(\lambda)} \log \frac{1}{u} \right|^{-1-N} \right) (u \to 0+) \\
&= \frac{1}{u^{e^{-\mathrm{Re}(\lambda)}}} + O\left(\left(e^{-\mathrm{Re}(\lambda)} \log \frac{1}{u} \right)^{-1-N} \right) (u \to 0+).
\end{aligned}
$$

Since $2e^{-\mathrm{Re}(\lambda)} < 1$, the first summand in the last expression belongs to $L^2(0,1)$. If $N = -1$, the second summand belongs to $L^2(0,1)$, too. Thus, $g(u) \in L^2(0,1)$. Since $f(ux) \in L^2(0,1)$ for all x, the application of Fubini's theorem in (3.19) is correct.

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(ii) It follows from [28, Theorem 1] and the fact that by (i) for $\mathrm{Re}(\lambda) > \log 2$ the scalar symbol of $R(\lambda, \log \mathcal{C})$ is

$$\varphi(s) = \frac{1}{\lambda + \log(\frac{1}{2} - \imath s)}. \tag{3.20}$$

To prove this equality, note that in the case under consideration the kernel

$$K(u) = e^{\lambda}\nu(e^{-\lambda}\log\frac{1}{u}, -1)\chi_{(0,1)}(u).$$

Then according to (2.5)

$$\begin{aligned}
\varphi(s) &= e^{-\lambda}\int_0^1 \nu(e^{-\lambda}\log\frac{1}{u}, -1)u^{-\frac{1}{2}-\imath s}du \\
&= e^{-\lambda}\int_0^1\int_0^\infty \frac{\left(e^{-\lambda}\log\frac{1}{u}\right)^{t-1}}{\Gamma(t)}dtu^{-\frac{1}{2}-\imath s}du \\
&= \int_0^\infty \left(\int_0^1 u^{-\frac{1}{2}-\imath s}\left(\log\frac{1}{u}\right)^{t-1}du\right)\frac{e^{-\lambda t}}{\Gamma(t)}dt \\
&= \frac{1}{\lambda + \log(\frac{1}{2} - \imath s)}.
\end{aligned}$$

Above we used the formula (3.14) ones more.

(iii) Let $\lambda > \log 2$. It is well known that

$$\sigma(R(\lambda, \log \mathcal{C})) \setminus \{0\} = \left\{\frac{1}{\lambda - z} : z \in \sigma(\log \mathcal{C})\right\}$$

(see, e.g., [16, Theorem 1.13]). Comparing this with (ii) we have

$$\left\{\frac{1}{\lambda + \log(\frac{1}{2} - \imath s)} : s \in \mathbb{R}\right\} = \left\{\frac{1}{\lambda - z} : z \in \sigma(\mathcal{C})\right\}.$$

Therefore the spectrum of $\log \mathcal{C}$ is the curve

$$\sigma(\log \mathcal{C}) = \{-\log(\frac{1}{2} - \imath s) : s \in \mathbb{R}\}$$

with the parametric equations

$$x = -\log\sqrt{\frac{1}{4} + s^2} \text{ and } y = \tan^{-1}(2s),\ s \in \mathbb{R}.$$

It remains to note that $y \in (-\frac{\pi}{2}, \frac{\pi}{2})$ and

$$2\cos y = 2\sqrt{\cos^2(\tan^{-1}(2s))} = \frac{2}{\sqrt{1 + \tan^2(\tan^{-1}(2s))}} = \frac{1}{\sqrt{\frac{1}{4} + s^2}}.$$

Thus, the spectrum of $\log \mathcal{C}$ is the graph of the function $x = \log(2\cos y)$ ($y \in (-\frac{\pi}{2}, \frac{\pi}{2})$), and (3.17) follows.

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Finally, the point spectrum of $\log \mathcal{C}$ is empty (see above), and it is known that for any linear normal operator in a Hilbert space the residual spectrum is empty.

(iv) By [28, Theorem 1]

$$\|R(\lambda, \log \mathcal{C})\| = \sup_{s\in\mathbb{R}} |\varphi(s)| = \frac{1}{d(\lambda)},$$

where

$$d(\lambda) = \min_{s\in\mathbb{R}} |\lambda + \log(\frac{1}{2} - \imath s)| = \operatorname{dist}(\lambda, \sigma(\log \mathcal{C}))$$

(see the proof of (iii)). Further, it follows from (iii) that $\sigma(\log \mathcal{C}) \cap \mathbb{R} = \{\log 2\}$. Therefore, each $\lambda \in \mathbb{R}, \lambda \neq \log 2$ is a regular point for $\log \mathcal{C}$, and it is obvious that $d(\lambda) = \lambda - \log 2$ for $\lambda > \log 2$. It remains to note that (iii) implies that $\lim_{\lambda\to-\infty} d(\lambda) = \pi/2$, since the lines $y = \pm\pi/2$ are horizontal asymptotes for the curve $\sigma(\log \mathcal{C})$ as $y \to \pm\pi/2$. The proof is complete. □

Now we are in a position to compute the inverse of the operator $\log \mathcal{C}$.

Theorem 3.15. *The operator $\log \mathcal{C}$ has a bounded inverse $(\log \mathcal{C})^{-1}$ that belongs to the algebra $\mathcal{A}_u$. Moreover, this inverse is of the following form*

$$(\log \mathcal{C})^{-1} f(x) = -\int_1^\infty u^{-1} \nu(\log u, -1) f(ux) du \quad (f \in L^2(\mathbb{R}).$$

Proof. We put $L := \log \mathcal{C}$ and $R_1 := R(1, \log \mathcal{C})$ for short. First note that 0 and 1 are regular points for L by Theorem 3.14 (iii) above, and the assertion (ii) of this theorem implies that 1 is a regular point for R_1. On the other hand, it is easy to verify that for any operator L

$$L^{-1} = (I - L)^{-1}((I - L)^{-1} - I)^{-1}$$

if all inverse operators exist. It follows that

$$(\log \mathcal{C})^{-1} = F(R_1),$$

where $F(z) = z/(z - 1)$ is a holomorphic function on a neighborhood of the spectrum of R_1 by Theorem 3.14 (ii), and $F(0) = 0$. Since $R_1 \in \mathcal{A}_u$ by Theorem 3.14 (i), we conclude that $(\log \mathcal{C})^{-1}$ belongs to $\mathcal{A}_u$ by Theorem 2.1. Theorem 3.14 (i) shows that the operator R_1 satisfies all the conditions of [28, Theorem 1]. By this theorem, this operator is unitary equivalent to the operator of coordinate-wise multiplication by its scalar symbol (see formula (3.20))

$$\varphi_1(s) = \frac{1}{1 + \log(\frac{1}{2} - \imath s)}$$

in the space $L^2(\mathbb{R}, \mathbb{C}^2)$ of $\mathbb{C}^2$-valued functions. Thus, the matrix symbol of R_1 is $\Phi_1 = \operatorname{diag}(\varphi_1, \varphi_1)$. It follows that the matrix symbol or $F(R_1) =$

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$(\log \mathcal{C})^{-1}$ is $\Phi = F(\Phi_1) = \operatorname{diag}(F(\varphi_1), F(\varphi_1))$ (see the proof of Theorem 3.1 in [24]). So, the scalar symbol of $(\log \mathcal{C})^{-1}$ is

$$\varphi(s) = F(\varphi_1)(s) = -\frac{1}{\log(\frac{1}{2} - \imath s)}.$$

On the other hand, for the kernel

$$K(u) := -u^{-1}\nu(\log u, -1)\chi_{(1,\infty)}(u),$$

the scalar symbol of the Hausdorff operator $\mathcal{H}_K$ is

$$\varphi_{\mathcal{H}_K}(s) = \int_1^\infty K(u)u^{-\frac{1}{2}-\imath s}du = \int_0^\infty K(e^t)e^{-(\frac{1}{2}-\imath s)t}dt.$$

The right-hand side here is the Laplace transform (at the point $p = \frac{1}{2} - \imath s$) of the function $K(e^t) = -\nu(t, -1)$. Let $\nu(t) = \int_0^\infty t^s/\Gamma(s+1)ds$. Then $\nu(t, -1) = \nu'(t)$. Since by [5, Section 4.26, p. 203] the Laplace transform of ν is $1/p \log p$ (for $\operatorname{Re}(p) > 0$), the Laplace transform of ν' is $1/\log p$. Thus, we get $\varphi = \varphi_{\mathcal{H}_K}$ for the scalar symbols, and therefore $(\log \mathcal{C})^{-1} = \mathcal{H}_K$, because by [28, Theorem 1] these operators are uniquely determined by their scalar symbols. The proof is complete. □

The following corollary of Theorem 3.15 describes the operator $\log \mathcal{C}$ in terms of the Mellin transform.

Corollary 3.16. *The following statements are equivalent:*
(i) $f = (\log \mathcal{C})g,\ f \in L^2(\mathbb{R})$;
(ii) $(\mathcal{M}f)(z) = -(\log z)(\mathcal{M}g)(z),\ \operatorname{Re}(z) > 0, |z| > 1$,
where $\log 1 := 0$.

Proof. Let $g := (\log \mathcal{C})^{-1}f$. Then (i) is equivalent to

$$g(x) = \int_0^\infty u^{-1}N(u)f(ux)du, \tag{3.21}$$

where $N(u) := -\chi_{(1,+\infty)}(u)\nu(\log u, -1)$. An application of the Mellin transform to both sides of (3.21) implies (see, e. g., [30, p. 12, 1.14]) that (3.21) is equivalent to

$$(\mathcal{M}g)(z) = (\mathcal{M}f)(z)(\mathcal{M}N)(-z),\ \operatorname{Re}(z) > 0, |z| > 1. \tag{3.22}$$

Further, by Fubini theorem

$$\begin{aligned}(\mathcal{M}N)(-z) &= -\int_1^\infty u^{-z-1}\nu(\log u, -1)du \\ &= -\int_1^\infty u^{-z-1}\left(\int_0^\infty \frac{(\log u)^{t-1}}{\Gamma(t)}dt\right)du \\ &= -\int_0^\infty \frac{dt}{\Gamma(t)}\int_1^\infty u^{-z-1}(\log u)^{t-1}du.\end{aligned}$$

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Since formula (3.14) shows that

$$\int_1^\infty u^{-z-1}(\log u)^{t-1}du$$

$$= \int_0^1 v^{z-1}\left(\log\frac{1}{v}\right)^{t-1}dv = z^{-t}\Gamma(t),$$

we have

$$(\mathcal{M}N)(-z) = -\int_0^\infty z^{-t}dt = -\frac{1}{\log z},$$

and thus formula (3.22) is equivalent to (ii).

The above calculations shows also that the integral

$$\int_0^\infty \frac{dt}{\Gamma(t)}\int_1^\infty |u^{-z-1}(\log u)^{t-1}|du$$

converges and therefore the application of Fubini theorem is correct. □

4. Functions of the Cesáro operator in $L^2(\mathbb{R}_+)$

In this section, we briefly discuss how the results from the previous sections can be applied to the Cesáro operator in $L^2(\mathbb{R}_+)$.

4.1. **Holomorphic functions of the Cesáro operator in $L^2(\mathbb{R}_+)$.** We consider $L^2(\mathbb{R}_+)$ as a subspace of $L^2(\mathbb{R})$, and for an operator $(A, D(A))$ in $L^2(\mathbb{R})$ we denote by A_+ the restriction $A|(D(A)\cap L^2(\mathbb{R}_+))$. $L^2(\mathbb{R}_+)$ is a closed invariant subspace for an operator $\mathcal{H}_K$ if K is supported in $\mathbb{R}_+$ and $K(u)|u|^{-\frac{1}{2}} \in L^1(\mathbb{R})$.

Since $\sigma(\mathcal{C}_+) = \sigma(\mathcal{C}) = \mathbb{T}+1$ [10], [28] the expression $F(\mathcal{C}_+)$ makes sense for each function F holomorphic in a neighborhood of the circle $\mathbb{T}+1$. A preliminary lemma is required.

Lemma 4.17. *Let F be a holomorphic function in a neighborhood N of the circle $\mathbb{T}+1$. Then*

$$F(\mathcal{C}_+) = F(\mathcal{C})_+.$$

Proof. One can assume that $F(0) = 0$. Let $\lambda \notin \sigma(\mathcal{C}_+) = \sigma(\mathcal{C})$. For each $g \in L^2(\mathbb{R}_+)$ the equality $f = R(\lambda, \mathcal{C})_+ g$ means that $f \in L^2(\mathbb{R})$ is the unique solution to the equation $(\lambda I - \mathcal{C})f = g$. Similarly, the equality $h = R(\lambda, \mathcal{C}_+)g$ means that $h \in L^2(\mathbb{R}_+) \subset L^2(\mathbb{R})$ is the unique solution to the equation $(\lambda I - \mathcal{C})h = g$. Thus, $f = h$ and therefore $R(\lambda, \mathcal{C})_+ = R(\lambda, \mathcal{C}_+)$. Now let Γ denotes the boundary of any open annulus U that contains $\mathbb{T}+1$

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such that N contains its closure. Then

$$\begin{aligned} F(\mathcal{C})_+ &:= \frac{1}{2\pi i}\left(\int_\Gamma F(\lambda)R(\lambda,\mathcal{C})d\lambda\right)_+ \\ &= \frac{1}{2\pi i}\int_\Gamma F(\lambda)R(\lambda,\mathcal{C})_+ d\lambda \\ &= \frac{1}{2\pi i}\int_\Gamma F(\lambda)R(\lambda,\mathcal{C}_+)d\lambda \\ &= F(\mathcal{C}_+). \end{aligned}$$

□

As a simple corollary of Theorem 3.2 and Lemma 4.17 we get the following result.

Theorem 4.18. *Let F be a holomorphic function in a neighborhood of the circle $\mathbb{T}+1$, $F(0)=0$. Then there is such function K, that $F(\mathcal{C}_+)=\mathcal{H}_{K+}$, and conditions (a), (b), and (c) hold.*

Conversely, if a function K meets the conditions (a), (b), and (c), where F is holomorphic in some neighborhood of $\mathbb{T}+1$, then $F(0)=0$ and $\mathcal{H}_{K+}=F(\mathcal{C}_+)$.

Proof. Indeed, if F is a holomorphic function in a neighborhood of the circle $\mathbb{T}+1$, $F(0)=0$, then $F(\mathcal{C})=\mathcal{H}_K$ and (a), (b), and (c) hold by Theorem 3.2. It follows by Lemma 4.17 that $F(\mathcal{C}_+)=F(\mathcal{C})_+=\mathcal{H}_{K+}$.

Conversely, if a function K meets the conditions (a), (b), and (c), where F is holomorphic in some neighborhood of $\mathbb{T}+1$, then $F(0)=0$ and $F(\mathcal{C})=\mathcal{H}_K$ and therefore $\mathcal{H}_{K+}=F(\mathcal{C})_+=F(\mathcal{C}_+)$. □

Corollary 4.19. *Under the conditions of Theorem 4.18 one has for the spectrum $\sigma(\mathcal{H}_{K+})=F(\mathbb{T}+1)$.*

Example 4.20. *Let $\mathrm{Re}(\alpha)<\frac{1}{2}$. Consider the averaging operator of Boyd in $L^2(\mathbb{R}_+)$* [7], [8]

$$P_\alpha f(x)=x^{\alpha-1}\int_0^x t^{-\alpha}f(t)dt=\int_0^1 u^{-\alpha}f(ux)\,du,$$

where $P_0=\mathcal{C}_+$ is the continuous Cesáro operator. Then $P_\alpha=\mathcal{P}_{\alpha+}$ (see Example 3.4). Therefore by Lemma 4.17

$$P_\alpha=\alpha^{-1}((I-\alpha\mathcal{C}_+)^{-1}-I)$$

It follows also, that the spectrum $\sigma(P_\alpha)$ equals to the circle $F_\alpha(\mathbb{T}+1)$, where F_α is given by (3.9).

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Example 4.21. *Let* $\mathrm{Re}(\alpha) < \frac{1}{2}$. *Consider the other averaging operator of Boyd in* $L^2(\mathbb{R}_+)$ [7], [8]

$$Q_\alpha f(x) = x^{-\alpha} \int_x^\infty t^{\alpha-1} f(t)dt = \int_1^\infty u^{\alpha-1} f(ux)\, du.$$

Then $Q_\alpha = \mathcal{Q}_{\alpha+}$ *(see Example 3.5), and by Lemma 4.17*

$$Q_\alpha = \mathcal{C}_+((I-\alpha)\mathcal{C}_+ - I)^{-1}.$$

We conclude also, that the spectrum $\sigma(Q_\alpha)$ *equals to the circle* $F(\mathbb{T}+1)$, *where* F *is given by* (3.11).

Example 4.22. *Let* $\alpha > 0$. *Consider the generalized Cesáro operator of order* $\alpha > 0$ *in* $L^2(\mathbb{R}_+)$

$$(\mathcal{C}_{\alpha+} f)(x) = \alpha x^{-\alpha} \int_0^x (x-t)^{\alpha-1} f(t)dt.$$

Then $\mathcal{C}_{\alpha+} = \mathcal{H}_{K+}$ *where* $K(u) = \alpha(1-u)^{\alpha-1}\chi_{(0,1)}(u)$. *In this case,* $(\mathcal{M}K)(z) = \alpha \mathrm{B}(\alpha, z) = F(z^{-1})$ *where*

$$F(z) = \frac{\Gamma(\alpha+1)\Gamma(z^{-1})}{\Gamma(\alpha+z^{-1})}.$$

Since this function is not holomorphic in any neighborhood of $\mathbb{T}+1$, *the operator* $\mathcal{C}_{\alpha+}$ *is not a holomorphic function of* $\mathcal{C}_+$.

4.2. **Fractional powers of the Cesáro operator in** $L^2(\mathbb{R}_+)$**.** The operator $\mathcal{C}_+$ is normal and the analog of Theorem 3.7 is valid for $\mathcal{C}_+$, too. The case $\alpha \in \mathbb{N}$ in (4.23) below was proved by Boyd [8]. For another proof of Boyd's formula see [25].

Theorem 4.23. *Let* $\mathrm{Re}(\alpha) > 0$. *Then*

$$(\mathcal{C}_+)^\alpha = H_{\alpha+}. \tag{4.23}$$

Proof. Note that the operator $\mathcal{C}$ has the form (2.2) with $a(u) = u\chi_{[0,1]}(u)$ and thus $a(u) \geq 0$ for all u. In this case the Corollary 4 in [28] is applicable. According to this corollary the operator $\mathcal{C}_+$ is unitary equivalent to the operator $M_{\varphi_+^0}$ of multiplication by φ_+^0 in $L^2(\mathbb{R}_+)$. Thus, one can in fact to repeat the proof of Theorem 3.7 simply replacing the operator $M_{\Phi_\mathcal{C}}$ by the operator $M_{\varphi_+^0}$. □

Corollary 4.24. *Let* $\log(\mathcal{C}_+)$ *stands for the generator of the semigroup* $(\mathcal{C}_+)^\alpha = H_{\alpha+}$. *Then* $\log(\mathcal{C}_+) = (\log \mathcal{C})_+$.

This follows, e.g., from [16, Subsection II.2.3, Corollary on p. 61].

Corollary 4.25. *Analogs of corollaries 3.8—3.11 and Theorem 3.14 are valid for* $H_{\alpha+}$ *as well as for* H_α.

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Proof. Indeed, since the restriction of H_α to the subspace $L^2(\mathbb{R})\ominus L^2(\mathbb{R}_+) = L^2(\mathbb{R}_-)$ is zero, the semigroup $H_{\alpha+}$ has the same spectral properties as H_α. □

5. Holomorphic functions of the Cesáro operator in $L^2[0,1]$

We consider $L^2[0,1]$ as a subspace of $L^2(\mathbb{R})$, and for an operator A on $L^2(\mathbb{R})$ we denote by A_1 the bi-restriction (restriction and co-restriction) of A to the space $L^2[0,1]$. This means that the domain of A_1 is $L^2[0,1]$ and $A_1 f = (Af)\chi_{(0,1)}$ for all $f \in L^2[0,1]$.

Since $\sigma(\mathcal{C}_1) = \overline{\mathbb{D}} + 1$ [10], the expression $F(\mathcal{C}_1)$ makes sense for each function F which is holomorphic in a neighborhood of the closed disc $\overline{\mathbb{D}}+1$. Next, the following analog of Lemma 4.17 holds true.

Lemma 5.26. *Let F be a holomorphic function in a neighborhood N of the disc $\overline{\mathbb{D}} + 1$. Then*

$$F(\mathcal{C}_1) = F(\mathcal{C})_1.$$

Proof. One can assume that $F(0) = 0$. Let $\lambda \notin \sigma(\mathcal{C}_1) = \overline{\mathbb{D}} + 1$ and $f \in L^2[0,1]$. The equality $R(\lambda, \mathcal{C}_1)f = h$ means that $h \in L^2[0,1]$ and h is the unique solution of the equation $(\lambda - \mathcal{C}_1)h = f$.

On the other hand, the equality $R(\lambda, \mathcal{C})_1 f = g$ means that $g \in L^2[0,1]$ and g is the unique solution of the equation $(\lambda - \mathcal{C})g = f$. It follows that $\mathcal{C}g = \lambda g - f \in L^2[0,1]$ and therefore $\mathcal{C}g = \mathcal{C}_1 g$. Thus, $(\lambda - \mathcal{C}_1)g = f$. This yields $g = h$, and we get that $R(\lambda, \mathcal{C}_1) = R(\lambda, \mathcal{C})_1$.

If Γ denotes the boundary of any open disc U that contains $\overline{\mathbb{D}} + 1$ such that N contains its closure. Then for each $f \in L^2[0,1]$ one has

$$\begin{aligned} F(\mathcal{C})_1 f &:= \frac{1}{2\pi i}\left(\int_\Gamma F(\lambda) R(\lambda, \mathcal{C}) f d\lambda\right)\chi_{(0,1)} \\ &= \frac{1}{2\pi i}\int_\Gamma F(\lambda) R(\lambda, \mathcal{C}) f \chi_{(0,1)} d\lambda \\ &= \frac{1}{2\pi i}\int_\Gamma F(\lambda) R(\lambda, \mathcal{C})_1 f d\lambda \\ &= \frac{1}{2\pi i}\int_\Gamma F(\lambda) R(\lambda, \mathcal{C}_1) f d\lambda \\ &= F(\mathcal{C}_1) f. \end{aligned}$$

□

Now we have the following analog of Theorem 4.18.

Theorem 5.27. *Let F be a holomorphic function in a neighborhood of the disk $\overline{\mathbb{D}} + 1$, $F(0) = 0$. Then then there is such a function K, that $F(\mathcal{C}_1) = (\mathcal{H}_K)_1$, and conditions (a), (b), and (c) hold.*

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Conversely, if a function K meets the conditions (a), (b), and (c), where F is holomorphic in some neighborhood of $\overline{\mathbb{D}}+1$, then $F(0)=0$ and $(\mathcal{H}_K)_1 = F(\mathcal{C}_1)$.

Proof. Indeed, if F is a holomorphic function in a neighborhood of the disc $\overline{\mathbb{D}}+1$, $F(0)=0$, then $F(\mathcal{C}) = \mathcal{H}_K$ and (a), (b), and (c) hold by Theorem 3.2. It follows by Lemma 5.26 that $F(\mathcal{C}_1) = F(\mathcal{C})_1 = (\mathcal{H}_K)_1$.

Conversely, if a function K meets the conditions (a), (b), and (c), where F is holomorphic in some neighborhood of $\overline{\mathbb{D}}+1$, then $F(0)=0$ as in the proof of Theorem 3.2, $F(\mathcal{C}) = \mathcal{H}_K$ and therefore $(\mathcal{H}_K)_1 = F(\mathcal{C})_1 = F(\mathcal{C}_1)$. □

Corollary 5.28. *We have for the resolvent of $\mathcal{C}_1$*

$$R(\lambda, \mathcal{C}_1) = \lambda^{-2}(P_{\lambda^{-1}})_1 + \lambda^{-1} I_1$$

for all regular λ, i.e., for all λ exterior to the disc $\overline{\mathbb{D}}+1$.

Proof. This follows from (3.10) and Lemma 5.26, since the spectrum of $\mathcal{C}_1$ equals $\overline{\mathbb{D}}+1$.

□

6. Conclusion

The theory of the continuous Cesáro operator is important because it serves as a "bridge" between classical analysis, semigroup theory, and modern operator dynamics. For example in Summability and Convergence Theory the Cesáro operator is a natural generalization of the method of arithmetic means for functions. It helps to find the values of divergent integrals (Cesáro summability) and estimate the rate of convergence of orbits of functions to their limits, which is critical for numerical methods and ergodic theory. Moreover the Cesáro operator is a classic object for testing hypotheses about chaotic behavior. From this points of view the results obtained will make it possible to further develop the theory of the continuous Cesáro operator and discover connections between this operator and other important operators of harmonic and functional analysis such as Hausdorff, Hölder, Boyd, and Coposon operators. Moreover, since our description of holomorphic functions of the continuous Cesáro operator deals with the Mellin trunsform on the critical line, our results are connected with the theory of Riemann zeta function, as well. We also introduce the logarithm of continuous Cesáro operator although the general approach adopted in operator theory does not allow this to be done.

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8. DATA AVAILABILITY AND CONFLICT OF INTEREST

The authors confirm that all data generated or analyzed during this study are included in this article. This work does not have any conflict of interest.

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