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Lacunary entire functions

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Abstract

It has been observed that lacunary functions and random functions often have many properties in common (cf. [5]). The present paper has the object of showing that lacunary entire functions behave in many respects like random entire functions. Both have the property of being large except in very small neighbourhoods of their zeros and these have a uniform angular distribution.

Polya in his great paper of 1929 [13] made a study of lacunary entire functions and also of the family

$$\sum_0^\infty r_n(t) A_n z^n, \tag{1·1}$$

where $\sum A_n z^n$ was any given entire function and $r_n(t)$ were Rademacher functions. These are functions of period 1 defined for $0 \leqslant t \leqslant 1$ by $r_n(t) = r(2^n t)$, where

$$r(t) = \begin{cases} +1 & 0 < t < \frac{1}{2} \\ -1 & \frac{1}{2} < t < 1 \\ 0 & t = 0, \frac{1}{2}, 1. \end{cases}$$

Polya showed that if $\sum_0^\infty A_n z^n$ was an entire function of infinite order then almost all functions (1·1) had every direction as a Julia direction. That is, the function takes every value with one possible exception in every angle $|\arg z - \gamma| < \epsilon$. In the same paper he studied lacunary series and showed that entire functions of the form

$$f(z) = \sum_0^\infty A_n z^{\lambda n} \tag{1·2}$$

which are of infinite order and satisfy the Fabry condition

$$n/\lambda_n \to 0 \tag{1·3}$$

as n tends to infinity also have this property. This was of special interest because since then it has often been noticed (cf. [5]) that random functions and lacunary functions have similar properties.

Many more results have been added since Polya's paper but there is a sense in which it might be said that Polya's paper was the starting point. Thus Anderson and Clunie [1] have shown that if f is an entire function of finite non-zero order and if it satisfies the Fabry gap condition then every direction is a Julia direction. There are no exceptional values. Hayman [7] extended this result to functions of zero order.

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The family (1·1) was studied by Littlewood and Offord [**9**, **10**], who showed, in particular, that for functions of finite non-zero order almost all functions have every direction as a Julia direction, or indeed a Borel direction. Their results were extended to functions of zero order by Davies [**4**].

Littlewood and Offord did not study lacunary series explicitly, although their paper does contain the conclusions of Theorems 1 and 2 below in the special case for functions of finite order when

$$\frac{\log n}{\log \lambda_n} \to 0, \tag{1·4}$$

although they do not state it. The reason for this is the attitude of the authors, whose interest was in the application of probability arguments. But, if the series (1·1) is such that there are too few non-zero terms, probability arguments break down and lacunary series introduce an unwelcome diversion. The situation would be different if the coefficients A_n were subjected to some kind of normal distribution or if the Rademacher functions were replaced by Steinhaus functions. As it was, they were obliged to develop a special argument to deal with the lacunary case (cf. [**10**] p. 899, case (i)). There was a similar treatment in [**11**] lemma 7, p. 266.

The object of this paper is to study lacunary series (1·2) which satisfy the condition

$$\limsup_{n\to\infty} \frac{\log n}{\log \lambda_n} = \alpha < 1. \tag{1·5}$$

This replaces the condition

$$\limsup_{n\to\infty} \frac{\log n}{\log \lambda_n} < \frac{1}{2} \tag{1·6}$$

of [**12**]. It could mean that in some results the exceptional set or the error term is larger, but there is a distinct improvement. In particular the case when $\lambda_n = n^2$ studied by Hardy [**7**] is not now excluded. Condition (1·5) is still stronger than the Fabry condition (1·3) but on the other hand we are able to prove more. Thus Theorem 3 shows that lacunary entire functions of finite order ρ have every direction as a Borel direction. This is defined as follows. It is well known that if the zeros of $f-a$ are arranged in increasing order of modulus and if we denote the moduli by

$$r_1(a) \leqslant r_2(a) \leqslant \ldots \leqslant r_n(a) \leqslant \ldots,$$

then the series

$$\sum_0^\infty (r_n(a))^s \tag{1·7}$$

converges for $s > \rho$, and, with one possible exceptional value of a, diverges for $s < \rho$. Let us now denote by $\sum'_{\epsilon,\gamma}$ the corresponding sum taken over all zeros of $f-a$ which lie in the sector $|\arg z - \gamma| < \epsilon$. We say that the direction $\arg z = \gamma$ is a Borel direction if, for all $\epsilon > 0$ and except for one possible a, the series

$$\sum'_{\epsilon,\gamma} (r_n(a))^s \tag{1·8}$$

diverges if $s < \rho$. A Borel direction is necessarily a Julia direction. Theorem 2 asserts that if the series (1·7) diverges for one a and one γ then it and the series (1·8) diverge for every a and every γ. It would be interesting if this were still the case under the weaker Fabry gap condition.

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Theorem 1 is the main theorem of this paper and it is the analogue for lacunary functions of the results of Littlewood and Offord for almost all functions of the family (1·1). It involves the idea of a set of finite logarithmic measure (which did not occur in [**10**]) and the notion of a pit. A set E on the positive real axis has finite logarithmic measure if

$$\int_E \frac{dt}{t}$$

is finite. A *pit* is a small region of complex plane bounded by a level curve of function f such that in it f has a zero and takes every not too large value with the same multiplicity as its zeros. It was proved by Littlewood and Offord that almost all entire functions of the family (1·1) which were of finite non-zero order had the *pits property*. That is they were large except in the pits. Theorem 1 asserts that this holds for functions which satisfy (1·5). An earlier version of this theorem was given in [**12**], but it was then assumed that (1·6) was satisfied. However, the exceptional set is now larger than in [**12**], although still of finite logarithmic measure.

We use $M(r)$ for the maximum modulus of f on $|z| = r$ and $\mu(r)$ for the maximum term. We use $N(r)$ for the index of maximum term with the usual convention that at a point of discontinuity $N(r)$ is to have the largest of its possible values unless explicitly stated otherwise. Functions of zero order call for special treatment and are excluded. But, since (1·5) is only a density condition, even when f is of infinite order there may be small intervals of r where f is small relative to r. For this reason in Theorem 1 we assume that

$$\frac{\log N(r)}{\log\log r} \to \infty. \tag{1·9}$$

This is not needed in Theorem 3 and it does not appear in the hypotheses of Theorem 2. However the error terms in Theorem 2 may be large because they depend on $|z|$ and the theorem does not supply information if σ is small relative to $|z|$.

THEOREM 1. *If* $f(z) = \sum_0^\infty A_n z^{\lambda_n}$ *is an entire function such that* (1·5) *and* (1·9) *are satisfied then outside a set of finite logarithmic measure the following properties hold.*

(i) *all the zeros of* f *are in pits and to all* $\epsilon > 0$ *there exists* $r_0(\epsilon)$ *such that for all* $r > r_0$

$$|f(z)| \geqslant (M(r))^{1-\epsilon} \tag{1·10}$$

outside the pits of f.

(ii) *each pit is contained in a disc of diameter not exceeding*

$$\frac{1}{(N(r))^3}, \tag{1·11}$$

(iii) *the sum of the diameters of all pits in* $|z| \geqslant R$ *does not exceed* $1/N(R)$.

This theorem is not restricted to functions of finite order, and for those of infinite order the growth of the function $N(r)$ can be very irregular. The exceptional set is of finite logarithmic measure for all $\alpha < 1$, but it naturally depends on α and is smaller with small α. In particular with $\alpha = 0$ and f of finite order we can say that for z outside the pits

$$\log|f(z)| \geqslant \log M(r) - o(r^\epsilon)$$

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and so avoid mention of an exceptional set.

The following theorem is the extension to functions satisfying (1·5) of Theorem 3 of [**12**] which was proved under condition (1·6). It should, however, be stated that this extension has only been made possible at the price of having larger error terms. The function $\kappa(r)$ which occurs will be explained in §2; it is not the same as the function $\kappa(r)$ which occurred in Theorem 3 of [**12**]. In the case of functions of finite order ρ the error terms are $o(r^{\alpha\rho+\epsilon})$. The use of lower limit $1/\kappa$ in this integral is a technical device to avoid complications which would arise if f had a zero at z.

THEOREM 2. *If f satisfies* (1·5), *if $\sigma \leqslant \frac{1}{4}r$ and if $n(t,z)$ denotes the number of zeros of f in the disc centre z and radius t then*

$$\int_{1/\kappa(r)}^{\sigma} \frac{n(t,z)}{t}\,dt = \frac{1}{2\pi}\int_0^{2\pi} \log\frac{\mu(|r+\sigma e^{i\theta}|)}{\mu(r)}\,d\theta \pm \log r \int_0^{2\pi} (\kappa(|r+\sigma e^{i\theta}|))^{\beta}\,d\theta,$$

where $\alpha < \beta < 1$.

Theorem 2 leads to

THEOREM 3. *If f satisfies* (1·5) *and is of finite order ρ then the two series* (1·7) *and* (1·8) *and the integral*

$$\int_1^{\infty} \frac{dN(\sigma)}{t^s}$$

all converge or diverge together. There are no exceptional values and if the series (1·7) *diverges for one a then it and the series* (1·8) *diverge for all a, all γ and all $\epsilon > 0$.*

Condition (1·9) is not needed for Theorems 2 and 3. In the case of functions of zero order, Theorem 3 is a consequence of Hayman's theorem that every direction is a Julia direction.

Preliminary lemmas

It has already been explained that the index $N(r)$ of the maximum term plays a key role. However its irregular behaviour may cause difficulties and it is convenient to replace it by an order type. This was done in [**11**, **12**], but now a more general function is needed. We use $\phi(t)$ for a strictly increasing continuous function which tends to infinity with t. Let us fix r and consider the functions $N(r+\theta)$ and $\phi(r/\theta)$ as functions of θ. The first is a non-decreasing step function while $\phi(r/\theta)$ as a function of θ is a strictly decreasing continuous function infinite when $\theta = 0$. Hence there will be a first value of θ, say θ_0, such that

$$N(r+\theta_0-0) \leqslant \phi\left(\frac{r}{\theta_0}\right) \leqslant N(r+\theta_0). \tag{2·1}$$

We define $\kappa(r)$ as this value of $\phi(r/\theta_0)$. Formally we have:

Definition. If $\phi(t)$ is a strictly increasing continuous function of t tending to infinity with t, then

$$\kappa(r) = \inf_{\theta>0} N(r+\theta) \vee \phi\left(\frac{r}{\theta}\right) \tag{2·2}$$

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and, if θ_0 is defined as in (2.1),

$$\kappa(r) = \phi\left(\frac{r}{\theta_0}\right). \tag{2·3}$$

The function $\kappa(r)$ used in [**11**, **12**] was the special case when $\phi(t) = t$. We write ψ for the inverse of ϕ, and, to simplify the notation, κ by itself will always mean $\kappa(r)$.

LEMMA 1. *$\kappa(r)$ is a continuous non-decreasing function of r,*

$$N(r) \leqslant \kappa(r) \leqslant N\left(r+\frac{r}{\psi(\kappa)}\right) \leqslant N\left(r+\frac{r}{\psi(N(r))}\right) \tag{2·4}$$

and

$$\kappa\left(r+\frac{r}{(\psi(\kappa))^{\frac{3}{2}}}\right) \leqslant \phi\{(\psi^{\frac{1}{2}}+1)^2\}. \tag{2·5}$$

In particular if $\phi(t) = \exp(t^{\frac{1}{2}})$ then

$$\kappa\left(r+\frac{r}{(\log\kappa)^3}\right) \leqslant e\kappa(r). \tag{2·6}$$

Proof. With θ_0 defined as in (2.1) and for $\delta < \theta_0$,

$$\kappa(r+\delta) = \inf N(r+\delta+\theta) \vee \phi\left(\frac{r+\delta}{\theta}\right)$$

$$\leqslant N(r+\theta_0-0) \vee \phi\left(\frac{r+\delta}{\theta_0-\delta}\right)$$

and the continuity of κ follows from that of ϕ. Then (2·4) follows from (2·1) and (2·3). As regards (2·5) we have

$$\kappa\left(r+\frac{r}{(\psi(\kappa))^{\frac{3}{2}}}\right) = \inf_{\theta>0} N\left(r+\frac{r}{\psi^{\frac{3}{2}}}+\theta\right) \vee \phi\left(\frac{r+r/\psi^{\frac{3}{2}}}{\theta}\right),$$

and, on setting $\theta = \theta_0 - r/\psi^{\frac{3}{2}}$ and writing ψ for $\psi(\kappa)$, this becomes

$$\kappa\left(r+\frac{r}{\psi^{\frac{3}{2}}}\right) \leqslant \phi\left(\frac{\psi^{\frac{3}{2}}+1}{\psi^{\frac{1}{2}}-1}\right) \leqslant \phi\{(\psi^{\frac{1}{2}}+1)^2\}$$

provided $\phi(\theta) > 1$.

In particular, if $\phi(t) = \exp(t^{\frac{1}{2}})$ then $\psi(t) = (\log t)^2$ and (2·6) follows. If $\phi(t) = t$ it is easy to see that

$$\kappa\left(r+\frac{r}{a\kappa}\right) \leqslant \frac{a+1}{a-1}\kappa(r).$$

The use of an order type enables us to state as lemmas standard results with a $\kappa(r)$ replacing $N(r+r/N(r))$. This is often convenient and enables the postponement of the consideration of exceptional sets.

In all that follows, unless specifically stated otherwise, we shall use κ or $\kappa(r)$ to mean $\kappa(r)$ defined by taking $\phi(t) = \exp(t^{\frac{1}{2}})$. The following lemma enables us to ignore small variations in expressions involving the maximum term, which we denote by $\mu(r)$.

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LEMMA 2. *If* $0 \leqslant \delta < 1$, *then*

$$\mu\left(r+\frac{r}{\kappa^{1-\delta}}\right) \leqslant \exp(e\kappa^{\delta})\,\mu(r) \tag{2·7}$$

and if $|\epsilon| \leqslant r/\kappa$

$$\frac{1}{16} \leqslant \frac{\mu(r+\epsilon)}{\mu(r)} \leqslant 16. \tag{2·8}$$

Proof. If there are more than one maximum term at $r+r\kappa^{-1+\delta}$ it makes no difference which one we take and we can choose the one with the least index, say N_1. Then $N_1 \leqslant \kappa(r+r\kappa^{-1+\delta})$ and since

$$r\kappa^{-1+\delta} \leqslant \frac{r}{(\log\kappa)^3}$$

we shall have $N_1 \leqslant e\kappa(r)$. So denoting the coefficient of this term by A, we have

$$\begin{aligned}\mu\left(r+\frac{r}{\kappa^{1-\delta}}\right) &= A\left(r+\frac{r}{\kappa^{1-\delta}}\right)^{N_1}\\ &\leqslant \mu(r)\,(1+\kappa^{-1+\delta})^{e\kappa}\\ &\leqslant \mu(r)\exp(e\kappa^{\delta}),\end{aligned}$$

and (2·8) follows on putting $\delta = 0$.

LEMMA 3. *If* $f(z) = \sum_0^\infty B_n z^n$ *is any entire function then, with* $\phi(t) = \exp(t^{\frac{1}{3}})$,

$$|f(z)| \leqslant C\kappa(r)\,\mu(r). \tag{2·9}$$

If further $k \geqslant e^2\kappa(r)$ *then, for all* ζ *satisfying* $|\zeta - z| \leqslant r/\kappa(r)$, *we have*

$$R_k(\zeta) = \left|\sum_{2k+1}^{\infty} B_n \zeta^n\right| \leqslant C\mu(r)\exp\left(-\frac{\kappa(r)}{(\log\kappa)^3}\right). \tag{2·10}$$

Proof. This follows Valiron[**13**] p. 28ff but for convenience we reproduce it here. The proof of (2·10) will be given first. We first define the dominant function. Write

$$g_n = -\log|B_n|$$

and define a sequence G_{n_j} as follows. Let $G_0 = g_0$ and, G_{n_j} being defined, let n_{j+1} be the greatest index such that

$$G_{n_{j+1}} = \inf_{n \geqslant n_j} g_n.$$

For $n_j < n < n_{j+1}$, G_n is defined by

$$\frac{G_n - G_{n_j}}{n - n_j} = \frac{G_{n_{j+1}} - G_{n_j}}{n_{j+1} - n_j}.$$

The function $\sum_0^\infty e^{-G_n} z^n$ has the same maximum term as $f(z)$. Also, since $|\zeta - z| \leqslant r/\kappa(r)$, by Lemma 2, $\mu(|\zeta|)$ and $\mu(r)$ differ only by a constant multiplier. Also

$$\kappa(|\zeta|) \leqslant \kappa\left(r+\frac{r}{(\log\kappa)^3}\right) \leqslant e\kappa(r)$$

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and since $k \geqslant e^2\kappa(r)$

$$e^{-G_k}|\zeta|^k \leqslant \mu(|\zeta|) \leqslant C\mu(r).$$

Write

$$R_n = e^{G_n - G_{n-1}};$$

then

$$\left|\sum_{2k+1}^{\infty} e^{-G_n}\zeta^n\right| \leqslant e^{-G_k}|\zeta|^k \sum_{2k+1}^{\infty} \frac{|\zeta|^{n-k}}{R_{k+1}R_{k+2}\dots R_n}$$

$$\leqslant C\mu(r)\left(\frac{|\zeta|^k}{R_{k+1}}\right)^k \frac{|\zeta|}{R_{k+1}-|\zeta|}.$$

Let us write $|\zeta'|$ for $|\zeta|+|\zeta|/(\log\kappa(|\zeta|))^3$; then $N(|\zeta'|-0) \leqslant \kappa(|\zeta|) \leqslant e^2\kappa(r)$, and $k \geqslant e^2\kappa(r)$ so $e^{-G_k}|\zeta'|^k \geqslant e^{-G_{k+1}}|\zeta'|^{k+1}$ or $R_{k+2} \geqslant |\zeta'|$.

Hence, since $\kappa(|\zeta|) \leqslant e\kappa(r)$,

$$\left(\frac{|\zeta|}{R_{k+1}}\right)^k \frac{|\zeta|}{R_{k+1}-|\zeta|} \leqslant \left(1+\frac{1}{(\log\kappa)^3}\right)^{-k} (\log\kappa+1)^3$$

$$\leqslant \exp\left(-\frac{\kappa}{(\log\kappa)^3}\right),$$

for large r. This proves (2·10), and (2·9) follows taking $\zeta = z$ in (2·10) and using the fact that $|B_n|r^n \leqslant \mu(r)$ for $k \leqslant e^2\kappa$. This lemma could be stated for general ϕ with improved constants but as stated it is sufficient for the purposes of this paper.

It is in the next lemma that we use our hypothesis (1·5). Let $q(k)$ be the number of non-zero terms among the first k of (1·2), then by (1·5)

$$\limsup \frac{\log q}{\log k} = \alpha. \tag{2·11}$$

We choose k so that

$$e^2\kappa(r) \leqslant k \leqslant e^2\kappa(r)+1. \tag{2·12}$$

Then $\alpha < 1$ by (1.5), and so β can be found satisfying $\alpha < \beta < 1$, so that

$$q \leqslant (\kappa(r))^\beta.$$

We shall reserve the right to re-define β from time to time but always so that $\alpha < \beta < 1$.

LEMMA 4. *If f satisfies* (1·5) *and k satisfies* (2·12) *and if $\alpha < \beta < 1$ and r is large enough, we can find ζ_1 so that*

$$|\zeta_1 - z| \leqslant r/\kappa^4 \quad \textit{and} \quad |P(\zeta_1)| \geqslant \exp(-\kappa^\beta)\mu(r), \tag{2·14}$$

and ζ_2 so that

$$|\zeta_2 - z| \leqslant 1/\kappa^4 \quad \textit{and} \quad |P(\zeta_2)| \geqslant \exp(-\kappa^\beta \log r)\mu(r). \tag{2·15}$$

Proof. By definition of k, the maximum term in $P(z)$ is the same as that in $f(z)$. We denote by I the index set of all λ_n except for the λ_n corresponding to the maximum term which we denote by N. Let q be the number of terms in I so that q satisfies (2·11). As in [11] we use the operations

$$\mathfrak{D}_\lambda(P) = z^{\lambda+1}\frac{d}{dz}(z^{-\lambda}P), \quad \mathbf{D}_I(P) = \prod_{\lambda_n \in I} \mathfrak{D}_{\lambda_n} P.$$

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So $$|\mathbf{D}_I(P)| = \prod_1^q |N-\lambda_n|\,\mu(r) \geqslant ([\tfrac{1}{2}q]!)^2\,\mu(r).$$

In [11], Lemma 3 it was shown that for every regular function

$$|\mathbf{D}_I f| \leqslant \sum_0^q C^{(q)}_{q-\nu}|z^\nu f^{(\nu)}(z)|,$$

where $C^{(q)}_{q-\nu}$ is given by $$\sum_0^q C^{(q)}_{q-\nu} t^\nu = \prod_1^q (\lambda_\nu + t).$$

Applied to the function $(\zeta-z)^{-1}$ this gives

$$\left|\mathbf{D}_I \frac{1}{\zeta-z}\right| \leqslant \frac{q!}{|\zeta-z|}\prod_1^q\left(\lambda_\nu + \left|\frac{z}{\zeta-z}\right|\right). \tag{2·16}$$

So if we take the contour $|\zeta-z| = r/\kappa^4$ and choose ζ_1 so that $|P(\zeta_1)|$ is the maximum of $|P(\zeta)|$ on this contour, we shall have, by Cauchy's integral formula,

$$|\mathbf{D}_I P| \leqslant q!\,|P(\zeta_1)|\prod_0^q(\lambda_\nu+\kappa^4) \leqslant q!\,|P(\zeta_1)|\,(\kappa+1)^{4q},$$

whence $$|P(\zeta_1)| \geqslant 2^{-q}\kappa^{-4q}\mu(r) \geqslant \exp(-\kappa^\beta)\mu(r),$$

by (2·12) and (2·13), on redefining β.

Similarly, taking the contour $|\zeta-z| = 1/\kappa^4$, we can find ζ_2 such that $|\zeta_2-z| = 1/\kappa^4$ and

$$\left|\mathbf{D}_I\left(\frac{1}{\zeta-z}\right)\right| \leqslant \frac{q!}{|\zeta-z|}\prod_1^q(\lambda_\nu+r\kappa^4)$$

$$\leqslant \frac{q!}{|\zeta-z|}\exp(\log r\kappa^\beta),$$

and (2·15) follows as before.

LEMMA 5. *If* (1·5) *is satisfied and* r *is large then there is a point* ζ_1 *satisfying* $|\zeta_1-z| = r/\kappa^4$ *such that*

$$|f(\zeta_1)| \geqslant \exp(-\kappa^\beta)\mu(r), \tag{2·17}$$

and a point ζ_2 *satisfying* $|\zeta_2-z| \leqslant 1/\kappa^4$ *such that*

$$|f(\zeta_2)| \geqslant \exp(-\kappa^\beta \log r)\mu(r). \tag{2·18}$$

Further, the number of zeros of f *in the disc* $|\zeta-z| \leqslant \frac{1}{2}r\kappa^{\alpha-1}$ *does not exceed* κ^β *where* $\alpha < \beta < 1$.

Proof. For $i = 1, 2$ we have

$$|f(\zeta_i)| \geqslant |P(\zeta_i)| - R(\zeta_i),$$

and (2·17) and (2·18) follow from Lemmas 3 and 4, when r is large. Let $n(t, \zeta_1)$ be the number of zeros of f in the disc $|\zeta-\zeta_1| \leqslant t$ and let us write $\sigma_1 = \frac{1}{2}r\kappa^{\alpha-1} + r/\kappa^3$ and $\sigma_2 = r\kappa^{\alpha-1} - r/\kappa^3$. Then, by Jensen's theorem,

$$n(\sigma_1, \zeta_1) \leqslant \left(\log\frac{\sigma_2}{\sigma_1}\right)^{-1}\int_{\sigma_1}^{\sigma_2}\frac{n(t_1,\zeta_1)}{t}dt \leqslant C\frac{1}{2\pi}\int_0^{2\pi}\log\frac{|f(\zeta_1+\sigma_2 e^{i\theta})|}{|f(\zeta_1)|}d\theta.$$

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By (2·9) and (2·7)

$$|f(\zeta_1+\sigma_2 e^{i\theta})| \leqslant C\kappa(|\zeta_1|+\sigma_2)\,\mu(|\zeta_1|+\sigma_2)$$
$$\leqslant C\kappa(r)\exp(e\kappa^{\alpha})\,\mu(r),$$

and, since the disc $|\zeta-z| \leqslant \frac{1}{2}\kappa^{\alpha-1}$ lies in $|\zeta-z| \leqslant \sigma_1$, the desired result follows.

In the following lemma we make use of the Blaschke product for the zeros $\{a_\nu\}$ of f in the disc centre z and radius σ. This is defined as

$$\prod(\zeta; a_\nu) = \prod(\zeta; z, \sigma, \alpha_j)$$
$$= \prod_{\nu=1}^{s} \frac{\sigma(\zeta-a_j)}{s^2-(z-z)\,\overline{(a_j-z)}}.$$

It has the property that $|\prod| = 1$ if $|\zeta-z| = \sigma$ and $|\prod| < 1$ for $|\zeta-z| < \sigma$. We also use the property that if $\zeta = \rho e^{i\theta}+z$, where $\rho < \sigma$, then

$$\frac{1}{2\pi}\int_0^{2\pi} \log\left|\frac{\sigma(\zeta-a_\nu)}{\sigma^2-(\zeta-z)\,\overline{(a_\nu-z)}}\right| d\theta = \log\left(\frac{\rho}{\sigma} \vee \frac{|a_\nu-z|}{\sigma}\right). \tag{2·19}$$

Lemma 6. *If* (1·5) *is satisfied and* $\prod(\zeta; a_\nu)$ *is the Blaschke product for the zeros of* f *in* $|\zeta-z| \leqslant \sigma$, *where*

$$r/\kappa^4 \leqslant \sigma \leqslant r\kappa^{\alpha-1}, \tag{2·20}$$

then $f(\zeta) = \chi(\zeta)\prod(\zeta; a_\nu)$, *where in* $|\zeta-z| \leqslant \frac{1}{4}\sigma$ *we have* $|\chi(\zeta)| \geqslant \exp(-\kappa^{\beta})\,\mu(r)$. *If, in addition,* (1·9) *is satisfied,* (2·20) *can be replaced by*

$$1/\kappa^4 \leqslant \sigma \leqslant r\kappa^{\alpha-1}.$$

Proof. The hypothesis (2·20) permits us, in virtue of Lemma 2, to ignore small variations of $|\zeta|$ in $\mu(|\zeta|)$. By Lemma 5 we can choose ζ_1 so that $|\zeta_1-z| \leqslant \frac{1}{8}\sigma$. Then

$$|\chi(\zeta_1)| \geqslant |f(\zeta_1)| \geqslant \exp(-\kappa^{\beta})\,\mu(r)$$

and so, on $|\zeta-z| = \sigma$,

$$\left|\frac{\chi(\zeta)}{\chi(\zeta_1)}\right| \leqslant \exp(\kappa^{\beta})\frac{|f(\zeta)|}{\mu(r)} \leqslant \exp(\kappa^{\beta}),$$

by Lemma 3. But $\chi(\zeta)/\chi(\zeta_1)$ has no zeros in the disc $|\zeta-\zeta_1| \leqslant \frac{7}{8}\sigma$, so $q(\zeta) = \log \chi(\zeta)/\chi(\zeta_1)$ has a branch which is regular in this disc and

$$\operatorname{Re} q(\zeta) = \log\left|\frac{\chi(\zeta)}{\chi(\zeta_1)}\right| \leqslant \kappa^{\beta}.$$

Hence the function

$$Q(\zeta) = \frac{q(\zeta)}{2\kappa^{\beta}-q(\zeta)}$$

is regular and satisfies $|Q(\zeta)| \leqslant 1$ in this disc. Therefore, by Schwarz's lemma

$$|Q(\zeta)| \leqslant \frac{|\zeta-\zeta_1|}{\frac{7}{8}\sigma}$$

in the disc; but this entails

$$|q(\zeta)| \leqslant \frac{2\kappa^{\beta}|\zeta-\zeta_1|}{\frac{7}{8}\sigma-|\zeta-\zeta_1|}.$$

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Taking $|\zeta-z| \leqslant \frac{1}{4}\sigma$ we have $|\zeta-\zeta_1| \leqslant \frac{3}{8}\sigma$ and so $|q(\zeta)| \leqslant \frac{3}{2}\kappa^\beta$, for $|\zeta-z| \leqslant \frac{1}{4}\sigma$, whence $\operatorname{Re} q(\zeta) \geqslant -\frac{3}{2}\kappa^\beta$, and so, on re-defining β, $|\chi(\zeta)| \geqslant \exp(-\kappa^\beta)\mu(r)$ for $|\zeta-z| \leqslant \frac{1}{4}\sigma$.

LEMMA 7. *If f satisfies* (1·5) *and if* $1/\kappa^4 \leqslant \rho \leqslant \frac{1}{4}r$ *then*

$$\frac{1}{2\pi}\int_0^{2\pi} \log|f(z+\rho e^{i\theta})|\,d\theta \geqslant \frac{1}{2\pi}\int_0^{2\pi} \log\ \mu(|z+\rho e^{i\theta}|)\,d\theta - C\int_0^{2\pi} (\kappa(|z+\rho e^{i\theta}|))^\beta\,d\theta. \quad (2\cdot21)$$

Proof. It is convenient to distinguish two cases: (i) when

$$\rho \leqslant \frac{r-\rho}{\kappa(r-\rho)}$$

and (ii) when $$\frac{r-\rho}{\kappa(r-\rho)} \leqslant \rho \leqslant \tfrac{1}{4}r.$$

We deal with case (i) first. It is easy to see that in this case $e^{-1}\kappa(r-\rho) \leqslant e\kappa(r) \leqslant e\kappa(r+\rho)$ and by Lemma 2 small variations in $|\zeta|$ when $|\zeta-z| \leqslant 4\rho$ can be ignored. This means we can apply Lemma 6 to the disc $|\zeta-z| \leqslant 4\rho$. Using (2·19) we have

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi} \log|\prod(\zeta; z, 4\rho, a_j)|\,d\theta &= \frac{1}{2\pi}\sum_{j=1}^{s}\int_0^{2\pi} \log\left|\frac{4\rho(\zeta-a_j)}{16\rho^2-(\zeta-z)\overline{(a_j-z)}}\right| d\theta \\ &= \sum_{j=1}^{s} \log\left(\frac{1}{4} \vee \frac{|a_j-z|}{4\rho}\right) \\ &\geqslant -C(\kappa(r))^\beta\end{aligned}$$

and, for case (i),

$$\frac{1}{2\pi}\int_0^{2\pi} \log|f(z+\rho e^{i\theta})|\,d\theta \geqslant \log\mu(r) - C(\kappa(r))^\beta,$$

from which (2·21) follows.

In case (ii) we first define $w_1 = z+\rho e^{i\theta_1}$, where $\theta_1 = \arg z$ and $|w_1| = |z|+\rho$; also

$$\rho\tau_1 = \frac{r}{(\kappa(w_1))^2}.$$

We then choose θ_ν on the semi-circumference $\arg z \leqslant \theta \leqslant \arg z+\pi$ in the following way. When θ_ν and τ_ν are found we consider the curves defined by the functions

$$g_1(\theta) = (\theta-\theta_\nu-\tau_\nu)\rho, \quad g_2(\theta) = \frac{r}{(\kappa(|z+\rho e^{i\theta}|))^2}.$$

Since on the semi-circumference $|z+\rho e^{i\theta}|$ decreases with θ, both these functions are non-decreasing and $g_2(\theta)$ is bounded by $r/(\kappa(|r-\rho|))^2$. We define $\theta_{\nu+1}$ as the first intersection of the two curves so that

$$(\theta_{\nu+1}-\theta_\nu-\tau_\nu)\rho = \frac{r}{(\kappa(|z+\rho e^{i\theta}|))^2},$$

and write $$w_{\nu+1} = z+\rho e^{i\theta_{\nu+1}}, \quad \tau_{\nu+1} = \theta_{\nu+1}-\theta_\nu-\tau_\nu.$$

We have then for all ν

$$\rho\tau_\nu = \frac{r}{(\kappa(|w_\nu|))^2} \quad \text{and} \quad \frac{\kappa(r-\rho)}{(\kappa(r+\rho))^2} \leqslant \tau_\nu \leqslant \frac{1}{\kappa(r-\rho)}. \quad (2\cdot22)$$

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In this way the semi-circumference $\arg z \leqslant \theta \leqslant \arg z + \pi$ is divided into a finite number of adjacent non-overlapping intervals $(\theta_\nu - \tau_\nu, \theta_\nu + \tau_\nu)$. On the semi-circumference $-\pi + \arg z \leqslant \theta \leqslant \arg z$ we take the points $2\pi - \theta_\nu$.

We apply Lemma 6 to a disc centre w_ν and radius

$$\frac{4|w_\nu|}{(\kappa(|w_\nu|))^2} = \sigma \tag{2·23}$$

and consider for $\zeta = z + \rho e^{i\theta}$, where $\theta_\nu - \tau_\nu \leqslant \theta \leqslant \theta_\nu + \tau_\nu$ and $\rho\tau_\nu = \frac{1}{4}\sigma$,

$$\prod_{j=1}^{s} (\zeta; w_\nu, \sigma, a_j) = \prod_{j=1}^{s} \frac{\sigma(\zeta - a_j)}{\sigma^2 - (\zeta - w_\nu)\overline{(a_j - w_\nu)}} = \prod_{j=1}^{s} I_{\nu,j}.$$

Since $|\zeta - w_\nu| \leqslant \frac{1}{4}\sigma$ and $|a_j - w_\nu| \leqslant \sigma$

$$|I_{\nu,j}| \geqslant \frac{4|\zeta - a_j|}{5\sigma}.$$

But, writing $\arg a_j = \gamma$,

$$|\zeta - a_j| \geqslant \rho \sin |\theta - \gamma| \geqslant (2/\pi)\rho|\theta - \gamma|.$$

So

$$\int_{\theta_\nu - \tau_\nu}^{\theta_\nu + \tau_\nu} \log |I_{\nu,j}|\, d\theta \geqslant \int_{\theta_\nu - \tau_\nu}^{\theta_\nu + \tau_\nu} \log \frac{8\rho}{5\pi\sigma} |\theta - \gamma|\, d\theta$$

$$\geqslant 2\tau_\nu \log \frac{\rho\tau_\nu}{\sigma} \geqslant -4\tau_\nu \log \kappa(|w_\nu|),$$

by (2·23). Whence, on re-defining β,

$$\int_{\theta_\nu - \tau_\nu}^{\theta_\nu + \tau_\nu} \log \prod(\zeta; w_\nu, \sigma, a_j)\, d\theta \geqslant -\int_{\theta_\nu - \tau_\nu}^{\theta_\nu + \tau_\nu} (\kappa(|z + \rho e^{i\theta}|))^\beta\, d\theta$$

and the desired result now follows on summing with respect to ν.

LEMMA 8. *If* (1·5) *and* (1·9) *are satisfied and if z is any given point for which $|z|$ is large, then σ can be found satisfying $\sigma \leqslant 1/\kappa^3$ such that on the circumference centre z and radius σ,*

$$|f(\zeta)| \geqslant \exp(-\kappa^\beta)\, \mu(r). \tag{2·24}$$

If (1·5) *but not* (1·9) *is satisfied then σ can be found satisfying $\sigma \leqslant r/\kappa^3$ such that*

$$|f(\zeta)| \geqslant \exp(-\kappa^\beta - \log r)\, \mu(r) \tag{2·25}$$

on a circumference centre z and radius σ.

Proof. In Lemma 6 we choose for σ the smaller of the two numbers $r\kappa^{-1+\alpha}$ and $\frac{1}{2}$. This is possible when (1·9) is satisfied, and with this value of σ

$$\left|\prod_1^s (\zeta; a_j)\right| \geqslant \prod_1^s |\zeta - a_j|.$$

Cartan's lemma (cf. [3] p. 75) states that the points ζ for which

$$\prod_1^s |\zeta - a_j| \leqslant h^s$$

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can be enclosed in s discs the sum of whose diameters is equal to $4eh$. This implies that a disc centre z of radius $8eh$ must contain a circumference on which

$$\prod_1^s |\zeta - a_j| \geqslant \exp(-\tfrac{1}{2}\kappa^\beta)$$

and (2·24) follows.

If condition (1·9) is not satisfied then the smallest we can take σ is r/κ^3; this leads to

$$\left|\prod_1^s (\zeta; a_j)\right| \geqslant \left(\frac{1}{\sigma}\right)^s \prod_1^s |\zeta - a_j|$$

which leads to

$$\left|\prod_1^s (\zeta; a_j)\right| \geqslant \exp(-\kappa^\beta \log r)$$

on a circumference centre z, and so by Lemma 6 to (2·25).

§3 *Proof of Theorem* 1

The inequality (2·24) may not always give useful information, especially if $N(r)$ behaves irregularly. In fact, if α is allowed to have any value in the range $0 < \alpha < 1$, we shall need

$$\log \kappa(r) \leqslant (1+\epsilon) \log\log \mu(r) \tag{3·1}$$

which would entail $\log |f(z)| \geqslant \log \mu(r) - (\log \mu(r))^\beta$ for some $\beta < 1$.

It is indeed possible to show that except for a set of finite logarithmic measure (3·1) holds. There is a general theorem on the growth of increasing functions which probably goes back to Borel (cf. 2·12) and in the following form is due to Hayman [**6**], p. 324.

LEMMA 9. *Suppose $N(r)$ is a positive non-decreasing function which is continuous on the right. If $|h(r)| = (\log N(r))^{-2}$ then*

$$e^{-1} N(r) \leqslant N(r e^{h(r)}) \leqslant eN(r) \tag{3·2}$$

for all r outside a set of finite logarithmic measure.

Hayman in fact uses $|h(r)| = (N(r))^{-\alpha}$ where $\alpha > 0$ and e is replaced by $1+\alpha$. For convenience we reproduce the proof of the second inequality.

Let r_0 be any given real number and when r_n is defined we write $r_n' = r_n \exp(|h(r_n)|)$ and $r_n'' = r_n' \exp(h(r_n))$. Then let r_{n+1} be the least number (if any) such that $r_{n+1} \geqslant r_n''$ and

$$N(r_{n+1} \exp(|h(r_{n+1})|)) \geqslant eN(r_{n+1}).$$

Since $N(r)$ is continuous on the right, the lower bound of such r is attained. Then

$$N(r_n') \geqslant eN(r_n) \geqslant eN(r_{n-1}') \geqslant \ldots \geqslant e^n N(r_0)$$

and so

$$\sum_1^\infty \log \frac{r_n''}{r_n} = 2\sum_1^\infty |h(r_n)| = 2\sum_1^\infty \frac{1}{(\log N(r_n))^2}$$

$$\leqslant 2\sum_1^\infty \frac{1}{\{\log N(r_0) + n\}^2} \leqslant \frac{2}{\log N(r_0)}.$$

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Hence the union of all intervals (r_n, r_n'') has finite logarithmic measure. The proof of the first inequality in (3·2) is the same as that of Hayman's.

With $\kappa(r)$ defined as in lemma 1 and with $\phi(t) = \exp(t^{\frac{1}{2}})$ by (2·4)

$$\kappa(r) \leqslant N\left(r + \frac{r}{(\log N(r))^2}\right).$$

So, if r is not in the exceptional set,

$$\kappa(r) \leqslant eN(r).$$

Again
$$\log \mu(r) = \int_1^r \frac{N(t)}{t}\,dt \geqslant \int_{re^{-h}}^r \frac{N(t)}{t}\,dt \geqslant N(re^{-h})\,h(r).$$

So
$$\frac{\log \kappa(r)}{\log \log \mu(r)} \leqslant \frac{\log N(r) + 1}{\log N(r) - 1 + \log |h(r)|},$$

and this tends to 1 as r tends to infinity so (3·1) holds.

Let us denote the complement of the exceptional set by E and say that z is in E if $|z|$ is in E. If z is in E, (3·1) holds and, in particular, by Lemma 3

$$M(r) \leqslant C\kappa(r)\,\mu(r) \leqslant (\mu(r))^{1+\epsilon}. \tag{3·3}$$

Now by Lemma 8 $|f(\zeta)| \geqslant \exp(-\kappa^\beta)\,\mu(r)$ on the circumference $|\zeta - z| = 1/(\kappa(r))^3$, and so, by (3·1) and (3·3), on this circumference

$$|f(\zeta)| \geqslant (M(r))^{1-\epsilon}. \tag{3·4}$$

But by Rouché's theorem this means that either (3·4) holds throughout the disc $|\zeta - z| \leqslant 1/\kappa^3$ or that in this disc f has a zero and takes every value not exceeding $(M(r))^{1-\epsilon}$. If the latter occurs we say that it is a pit of f. Lemma 8 then asserts that the diameter of each pit does not exceed $2/\kappa^3$ and outside these pits (but still in E) (3·4) holds. We have therefore proved (i) and (ii) of Theorem 1.

In proving (iii) a little care is needed because Theorem 1 is not restricted to functions of finite order and for functions of infinite order (3·3) need not hold outside E.

The function $\kappa(r)$ defined by (2·6) of Lemma 1 is a continuous non-decreasing function and we can choose a sequence $\{r_n\}$ such that

$$\kappa(r_{n+1}) = 2\kappa(r_n) = 2\kappa_n.$$

If $n(r_n)$ is the number of zeros of f in $r_{n-1} \leqslant r \leqslant r_n$ and if

$$r' = r_n' = r_n + r_n(\kappa(r_n))^{-\frac{1}{2}}$$

then
$$n(r_n) \leqslant (\kappa(r_n))^{\frac{1}{2}} \int_{r_r}^{r'} \frac{n(t)}{t}\,dt \leqslant (\kappa_n)^{\frac{1}{2}} \log M(r').$$

By Jensen's theorem, and by lemma 3,

$$M(r') \leqslant C\kappa(r')\,\mu(r').$$

Applying Lemma 2 with $\delta = \frac{1}{2}$ we see that this does not exceed a constant multiple of $(\kappa_n)^{\frac{1}{2}} \exp(\kappa_n^{\frac{1}{2}})\,\mu(r_n)$, so

$$n(r_n) \leqslant C(\kappa_n + \kappa_n^{\frac{1}{2}} \log \mu(r_n)).$$

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But $$\log \mu(r) = \int_1^r \frac{N(t)}{t} dt \leqslant N(r) \log r \leqslant (\kappa(r))^{1+\epsilon}$$

by (1·8), which implies $n(r_n) \leqslant C(\kappa(r_n))^2$.

So far we have not restricted the zeros to E. If now we estimate the sum of the diameters of all pits in $r_{n-1} \leqslant r \leqslant r_n$ which are also in E, then this sum does not exceed $C/\kappa(r_n)$ and we have

$$\sum_{r_n \geqslant R} \frac{1}{\kappa(r_n)} \leqslant \frac{C}{\kappa(R)} \sum_0^\infty 2^{-n} \leqslant C/\kappa(R)$$

as desired.

We now give an example of a function of order 1 for which (1·10) fails to hold in a sequence of regions lying in the sector $|\arg z| \leqslant \pi/12$. This does not mean that there are fewer zeros in these regions, only that they do not lie in small pits which satisfy (1·11).

We write $$f(z) = \sum_0^\infty A_n z^{\lambda_n} P_n(z), \tag{3·5}$$

where

$$\lambda_n = 2^{2^n} \quad \text{so} \quad \lambda_n = (\lambda_{n-1})^2, \quad A_n = \lambda_n^{-\lambda_n}, \quad \text{and} \quad P_n(z) = \left(1 - \frac{z}{\lambda_n}\right)^{c_n}$$

and where $c_n = [\lambda_n^\alpha]$, $\frac{1}{2} < \alpha < 1$.

Then for $|\arg z| \leqslant \pi/12$ and

$$\lambda_n + \lambda_{n-1}(\log \lambda_n)^2 \leqslant |z| \leqslant \lambda_n + \lambda_n^\alpha, \tag{3·6}$$

we have $$\frac{\log |f(z)|}{\log \mu(r)} \leqslant \frac{C}{\log r}. \tag{3·7}$$

It is convenient to use logarithms to base 2, but this does not affect the result when (3·7) is conventionally interpreted. We define r_n by

$$A_{n-1} r_n^{\lambda_{n-1}} = A_n r_n^{\lambda_n}.$$

So $$\log r_n = \log \lambda_n + O\left(\frac{\log \lambda_{n-1}}{\lambda_{n-1}}\right),$$

whence $\lambda_n \leqslant r_n \leqslant \lambda_n + \lambda_{n-1}$, and for z satisfying (3·6)

$$\mu(r_n) \geqslant A_n r^{\lambda_n}. \tag{3·8}$$

If now $z = (\lambda_n + \rho) e^{i\theta}$, where $\lambda_{n-1}(\log \lambda_{n-1})^2 \leqslant \rho \leqslant \lambda_n^\alpha$, we have $|1 - z/\lambda_n| \leqslant \frac{3}{2}|\theta|$. So if $|\theta| \leqslant \pi/12$

$$\log |P_n(z)| \leqslant c_n \log \pi/8 \leqslant -\tfrac{4}{3}\lambda_n^\alpha, \tag{3·9}$$

for z in (3·6). If

$$|f(z)| \leqslant \sum_{\nu=1}^{n-1} A_\nu r^{\lambda_\nu} |P_\nu(z)| + A_n r^{\lambda_n} |P_n(z)| + \sum_{\nu=n+1}^{\infty} A_\nu r^{\lambda_n} |P_\nu(z)|$$
$$= \Sigma_1 + \Sigma_2 + \Sigma_3,$$

we have $$\log \Sigma_2 \leqslant \lambda_n \log \frac{r}{\lambda_n} - \tfrac{4}{3}\lambda_n^\alpha \leqslant \rho - \tfrac{4}{3}\lambda_n^\alpha \leqslant -\tfrac{1}{3}\lambda_n^\alpha.$$

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From a short calculation, $\log \sum_3 \leqslant -\lambda_{n-1}$, and

$$\begin{aligned}\log \sum_1 &\leqslant \log (2A_{r-1} r^{\lambda_{n-1}} |P_{n-1}(z)|)\\ &\leqslant \lambda_{n-1} \log \frac{|z|}{\lambda_{n-1}} + c_{n-1} \log \frac{2\lambda_n}{\lambda_{n-1}}\\ &\leqslant 2\lambda_{n-1} \log \lambda_{n-1}.\end{aligned}$$

While from (3·8) $$\log \mu(r) \geqslant \lambda_n \log \frac{r}{\lambda_n} \geqslant \rho.$$

Hence $$\frac{\log |f(z)|}{\log \mu(r)} \leqslant \frac{2\lambda_{n-1} \log \lambda_{n-1}}{\rho} = O\left(\frac{1}{\log \lambda_{r-1}}\right),$$

and (3·7) follows.

It will be observed that if $\alpha < \frac{1}{2}$ the interval (3·6) disappears and in fact in this case the conclusions of Theorem 1 apply to the function (3·5) and there is no exceptional set. Indeed if

$$\log \log \mu(r) \geqslant (\alpha - \epsilon) \log \kappa(r) \tag{3·10}$$

then the inequality (2·18) provides the information we need. And in the case of the function (3·5) $\lambda_{n+1} = \lambda_n^2$ and the A_n are such that $\kappa(r) \leqslant (N(r))^2$ and (1·10) holds when $\alpha < \frac{1}{2}$.

Proof of Theorem 2. If $\sigma < r/\kappa$, by Lemma 2 the first term on the right reduces to a constant and by Lemma 5 the number of zeros in $|\zeta - z| < r/\kappa$ is at most κ^β, hence

$$\int_{1/\kappa}^{r/\kappa} \frac{n(t,z)}{t}\, dt \leqslant \kappa^\beta \log r.$$

There is therefore nothing to prove in this case. When $r/\kappa \leqslant \sigma \leqslant \frac{1}{4}r$ the theorem follows from Lemmas 2, 3 and 7 and Jensen's theorem; thus

$$\begin{aligned}\int_{1/\kappa}^{\sigma} \frac{n(t,z)}{t}\, dt &\geqslant \int_{r/\kappa}^{\sigma} \frac{n(t,z)}{t}\, dt\\ &= \frac{1}{2\pi} \int_0^{2\pi} \log \left| \frac{f(z+\sigma e^{i\theta})}{f(z+r/\kappa e^{i\theta})} \right| d\theta\\ &\geqslant \frac{1}{2\pi} \int_0^{2\pi} \log \frac{\mu(|r+\sigma e^{i\theta}|)}{\mu(r)}\, d\theta - C \int_0^{2\pi} (\kappa(|r+\sigma e^{i\theta}|))^\beta\, d\theta.\end{aligned}$$

Again using Lemmas 3 and 7 for $\sigma \geqslant r/\kappa$

$$\frac{1}{2\pi} \int_0^{2\pi} \log \left| \frac{f(z+\sigma e^{i\theta})}{f(z+(r/\kappa) e^{i\theta})} \right| \leqslant \frac{1}{2\pi} \int_0^{2\pi} \log \frac{\mu(|z+\sigma e^{i\theta}|)}{\mu(r)}\, d\theta + C(\kappa(r))^\beta.$$

§4. *Functions of finite order and the proof of Theorem* 3

Throughout this section we confine ourselves to functions of finite order and by tradition order is denoted by ρ so we shall not now use ρ as alternative to r. We use $n(t, z; a)$ for the number of zeros of $f(\zeta) - a$ in the disc $|\zeta - z| \leqslant t$ and write

$$J(r,\sigma) = \frac{1}{2\pi} \int_0^{2\pi} \log \frac{\mu(|z+\sigma e^{i\theta}|)}{\mu(r)}\, d\theta \tag{4·1}$$

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Then by Theorem 2 applied to the function $f(\zeta)-a$

$$\int_{1/\kappa}^{\sigma} \frac{n(t,z;a)}{t}\,dt = J(r,\sigma)\pm O(r^{\beta\rho}). \tag{4·2}$$

The function $J(r,\sigma)$ was used in [9] and [10] from which we quote for some of its properties. To derive the required properties we first replace (4·1) by a slightly modified form. We denote by $\mu(r,|z+\sigma e^{i\theta}|)$ the value at $|z+\sigma e^{i\theta}|$ of the maximum term at $|z|$. Since the corresponding term has no zeros in $|\zeta-z|=\sigma$

$$\frac{1}{2\pi}\int_0^{2\pi} \log \mu(r,|z+\sigma e^{i\theta}|)\,d\theta = \log \mu(r)$$

and so

$$J(r,\sigma) = \frac{1}{2\pi}\int_0^{2\pi} \log \frac{\mu(|z+\sigma e^{i\theta}|)}{\mu(r,|z+\sigma e^{i\theta}|)}\,d\theta.$$

The inequalities for $J(r,\sigma)$ given in Lemma 17 of [10], (p. 933) are

$$J(r,\sigma) \leqslant \frac{\sigma}{r}(N(r+\sigma-0)-N(r-\sigma+0)),$$

$$J(r,\sigma) \geqslant \frac{(1-h)^{\frac{3}{2}}}{4\pi}\frac{\sigma}{r}(N(r+h\sigma)-N(r-h\sigma)),$$

for any h satisfying $0<h<1$, and

$$J(r,\tfrac{1}{32}\sigma) \leqslant \tfrac{1}{2}J(r,\sigma).$$

From (4·2) we deduce

$$\begin{aligned} n(\sigma,z;a) &\geqslant (\log 32)^{-1}\int_{\sigma/32}^{\sigma} \frac{n(t,z;a)}{t}\,dt \\ &\geqslant (\log 32)^{-1}\{J(r,\sigma)-J(r,\tfrac{1}{32}\sigma)\}-O(r^{\beta\rho}) \\ &\geqslant C(1-h)^{\frac{3}{2}}\frac{\sigma}{r}\{N(r+h\sigma)-N(r-h\sigma)\}-O(r^{\beta\rho}). \end{aligned} \tag{4·3}$$

Consider now the zeros of $f-a$ in the sector $|\arg z-\gamma|<\epsilon$. In the portion $2^k \leqslant |z| \leqslant 2^{k+1}$ of this sector we construct two families of discs all of which have the same radius $\sigma = 2^k \sin\frac{1}{2}\epsilon$. Discs of the first family have their centres at the points $2^k+2j\sigma$ where $0 \leqslant j \leqslant \frac{1}{4}\sin\frac{1}{2}\epsilon$. Those of the second have centres at the points $2^k+(2j-1)\sigma$ where $1 \leqslant j \leqslant \frac{1}{4}\sin\frac{1}{2}\epsilon$. In (4·3) we take $h=\frac{2}{3}$ and $\sigma = 2^k\sin\frac{1}{2}\epsilon$ so that for the disc centre at $r_j e^{i\gamma}$ say

$$n(\sigma,z_j;a) \geqslant C\sin\tfrac{1}{2}\epsilon\,(N(r_j+\tfrac{2}{3}\sigma)-N(r_j-\tfrac{2}{3}\sigma)).$$

Since $2^k+(2j+1)\sigma-\frac{2}{3}\sigma < 2^k+2j\sigma+\frac{2}{3}\sigma$, the intervals of one family overlap with those of the other. And the number of a-values in the sector must be at least half of the sum of the number of a-values in each of the two sets of discs. Therefore this number must be at least

$$C\sin\tfrac{1}{2}\epsilon(N(2^{k+1})-N(2^k))-C2^{k\beta\rho}/2\sin\tfrac{1}{2}\epsilon.$$

Therefore if $r_n(a)$ is the modulus of the nth zero of $f-a$ and if $\sum'_\gamma$ stands for the sum over all pits in the sector $|\arg z-\gamma| \leqslant \epsilon$, then

$$\sum'_{\gamma\ r_n\leqslant R} (r_n)^{-s} \geqslant C\epsilon\int_1^R \frac{dN(t)}{t^S} - \frac{C}{\epsilon}\sum_{2^k<R} 2^{-k(s-\beta\rho)}. \tag{4·4}$$

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Therefore if $s > \beta\rho$ the divergence of the integral will imply that of the series and, since f is of order ρ,

$$\limsup_{r\to\infty} \frac{\log N(r)}{\log r} = \rho$$

and the integral must therefore diverge if $s < \rho$; we infer that for $s < \rho$ the series $\sum_\gamma'$ diverges for all a. There are no exceptional values.

Using (4.2) and considering the zeros of $f-a$ in the annulus $r-\sigma \leqslant |z| \leqslant r+\sigma$ one can get a similar inequality to (4.4), namely

$$\sum_{r_n<R} r_n^{-s} \leqslant C\int_1^R \frac{dN(t)}{t^s} + C\sum_{2^k<R} 2^{-k(s-\beta\rho)}. \tag{4·5}$$

But it is well known that the series on the left converges for all a, if $s > \rho$, so (4·5) adds little to what is already known. However in the case $s = \rho$ (4·4) and (4·5) tell us that the two series and the integral all converge or diverge together and that if one of the series converges or diverges for one particular value a then it and the other series does so for every value of a.

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