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Calibrations and laminations

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Abstract

A calibration of degree $k \in \mathbb{N}$ on a Riemannian manifold M is a closed differential k-form θ such that the integral of θ over every k-dimensional, oriented submanifold N is smaller or equal to the Riemannian volume of N. A calibration θ is said to calibrate N if θ restricts to the oriented volume form of N. We investigate conditions on a calibration θ that ensure the existence of submanifolds calibrated by θ. The cases $k = 1$ and $k > 1$ turn out to be essentially different. Our main result says that, on a compact manifold M, a calibration θ calibrates a lamination if θ is simple, of class C^1, and if θ has minimal comass norm in its cohomology class.

1. *Introduction*

A *calibration of degree k* on a Riemannian manifold M is a closed k-form θ such that

$$\int_N \theta \leqslant \mathrm{vol}_k(N) \tag{1·1}$$

holds for every oriented, k-dimensional submanifold N of M. Obviously, this is equivalent to the infinitesimal condition that

$$\theta_x(v_1, \ldots, v_k) \leqslant 1 \tag{1·2}$$

whenever $x \in M$ and $v_1, \ldots, v_k \in TM_x$ have unit length. An oriented, k-dimensional submanifold N is said to be *calibrated by* θ if, restricted to N, θ is the oriented volume form of N. The importance of these concepts lies in the fact that calibrated submanifolds are $\mathbb{R}$-homologically volume-minimising. Assume e.g. that N is closed and calibrated by θ: If $\tilde{N}$ is a closed, oriented submanifold $\mathbb{R}$-homologous to N then, using Stokes' theorem, and the fact that θ is closed, and (1·1), one obtains

$$\mathrm{vol}_k(N) = \int_N \theta = \int_{\tilde{N}} \theta \leqslant \mathrm{vol}_k(\tilde{N}).$$

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Calibrations arise in two guises. On the one hand, as very special objects that only exist in very special situations, namely as parallel differential forms. Here, the most famous cases are the (normalised) exterior powers of the Kähler form of a Kähler manifold. By the Wirtinger inequality these calibrate precisely the complex submanifolds. Further important examples are studied in the groundbreaking work [**12**] by R. Harvey and H.B. Lawson. On the other hand, "calibrations" that are only measurable and weakly closed, are very common objects. If M is compact every de Rham cohomology class of comass norm one (cf. Secttion 2B) can be represented by such a weak calibration. Moreover, this calibration calibrates a non-zero closed current, cf. Section 2C, Proposition 2·2.

In this paper, we investigate conditions on k-calibrations θ ensuring the existence of submanifolds calibrated by θ in the case $k > 1$. This case is more complicated than the case $k = 1$ for (at least) the following two reasons: Distributions of dimension $k > 1$ need not be integrable. If $1 < k < \dim M - 1$ there exist non-simple (=indecomposable) k-vectors.

A natural condition on a calibration θ is that, at every point $x \in M$, the comass norm $\|\theta_x\|^*$, defined by

$$\|\theta_x\|^* = \max\{\theta_x(v_1, \ldots, v_k) \mid v_i \in TM_x \text{ and } |v_i| \leqslant 1 \text{ for } 1 \leqslant i \leqslant k\},$$

is equal to one. This is obviously necessary for the existence of a submanifold calibrated by θ through every point $x \in M$. Calibrations θ that are continuous, weakly closed, and satisfy $\|\theta_x\|^* = 1$ for all $x \in M$ will be called *calibrations of constant comass* or cc-*calibrations* for short.

Cc-calibrations of degree one are well known and fundamental objects in the one-dimensional calculus of variations, in Hamiltonian dynamics and in the study of Hamilton–Jacobi equations. This has the following reasons: suppose θ is a cc-calibration of degree one. Then:

(1) the line field calibrated by θ, defined by the condition

$$\theta(v) = |v|,$$

induces a foliation of M by $\mathbb{R}$-homologically minimising geodesics;

(2) if M is compact, the L^∞-norm of θ is minimal in its cohomology class, i.e. if $\tilde{\theta}$ is a one-form cohomologous to θ then

$$\|\tilde{\theta}\|^*_\infty = \sup_{x \in M} \|\tilde{\theta}_x\|^* \geqslant 1 = \|\theta\|^*_\infty;$$

(3) the graph of θ in the cotangent bundle is a Lagrangian submanifold invariant under the geodesic flow;

(4) if $\bar{M}$ is a covering space of M with the property that θ lifts to an exact form $\bar{\theta} = du$ on $\bar{M}$, then $u : \bar{M} \to \mathbb{R}$ is a $C^{1,1}$-solution of the Hamilton–Jacobi equation

$$|du|^* = 1.$$

We show by examples that, for $k > 1$, the pointwise condition that the calibration be of constant comass does not have consequences analogous to properties (1) and (2). However, if $k = \dim M - 1$, we will prove that a continuous calibration satisfying the (global) condition (2) will always calibrate a lamination, cf. Section 2D and Theorem 5·1. If one adds the conditions that θ be of class C^1 and simple, then this result holds for arbitrary degree k, cf. Theorem 6·1.

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This paper is organised as follows. The basic definitions are collected in Section 2. In particular, we define precisely what we mean by a measured lamination. In Section 3 we present the strong consequences of the existence of cc-calibrations of degree one. In Section 4 we use a result from [**11**] to show that there are many cc-calibrations of degree greater than one that do not calibrate any submanifold. In Section 5 we consider continuous calibrations of codimension one. We prove that such a calibration θ calibrates a non-empty lamination by minimal hypersurfaces provided that θ minimizes the L^∞-norm in its cohomology class. On the other hand, we present an example of a cc-calibration of codimension one that calibrates a foliation, but does not minimise the L^∞-norm in its cohomology class and, hence, does not calibrate any non-zero closed current.

If one tries to generalise the result of Section 5 to calibrations of degree k with $1 < k < \dim M - 1$, one is stymied by the fact that, in contrast to the case $k = \dim M - 1$, the existence of a closed k-current T calibrated by θ does not imply that θ calibrates any submanifold, see e.g. [**10**, $4{\cdot}A_2$]. It turns out that the situation is better if the multivectorfield $\vec{T}$ of the current T is simple. In Section 6 we prove that a closed current T can be represented by a measured lamination provided its vectorfield $\vec{T}$ is the restriction to $\mathrm{spt}(T)$ of a simple C^1-vectorfield on M. This generalises a result by M. Zworski [**24**, proposition 1]. Formulated for calibrations, our result says that a simple C^1-calibration of minimal L^∞-norm in its cohomology class calibrates a measured lamination.

2. *Definitions*

Unless otherwise stated, M will denote a smooth, closed, connected, oriented, m-dimensional manifold that is equipped with a Riemannian metric g. By $S(TM)$ and $S(T^*M)$ we denote the unit tangent and cotangent bundles of (M, g).

(A) *Calibrations*

If $x \in M$ and θ is a differential k-form the *comass norm* $\|\theta_x\|^*$ of θ at x is defined by

$$\|\theta_x\|^* = \max\{\theta_x(v_1, \ldots, v_k) \mid v_1, \ldots, v_k \in S(TM_x)\}. \tag{2·1}$$

The *comass norm* $\|\theta\|^*_\infty$ of θ is the L^∞-norm induced by the pointwise comass norms

$$\|\theta\|^*_\infty = \operatorname*{ess\,sup}_{x \in M} \|\theta_x\|^*. \tag{2·2}$$

If θ is a k-form on M with coefficients in $L^1_{\mathrm{loc}}(M)$, we will say that θ is *weakly closed* if

$$\int_M d\pi \wedge \theta = 0$$

holds for all smooth $(m - k - 1)$-forms π. If θ is Lipschitz then $d\theta$ exists Lebesgue almost everywhere and θ is weakly closed if and only if $d\theta = 0$ Lebesgue almost everywhere. This is true since Stokes' Theorem holds for Lipschitz forms.

Note that every weakly closed k-form θ determines a linear functional on the de Rham cohomology vector space $H^{m-k}_{dR}(M)$, and hence a de Rham cohomology class $[\theta] \in H^k_{dR}(M)$.

Definition 2·1. (a) A calibration of degree k is a continuous, weakly closed k-form θ such that $\|\theta\|^*_\infty = 1$.

(b) A calibration of constant comass, or cc-calibration for short, is a calibration θ such that $\|\theta_x\|^* = 1$ for all $x \in M$.

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(c) An oriented linear subspace L in TM is calibrated by a calibration θ if the dimension of L equals the degree of θ, and if

$$\theta(e_1, \ldots, e_k) = 1$$

holds for every positively oriented, orthonormal basis $(e_1, \ldots, e_k)$ of L.

(d) An oriented submanifold N of M is calibrated by a calibration θ if each of its tangent spaces TN_x is calibrated by θ.

We remark that our definition of "calibration" differs from the one introduced by Harvey and Lawson in [**12**] in two respects:

(1) calibrations need only be continuous instead of smooth;
(2) we ask that $\|\theta\|^*_\infty = 1$ instead of $\|\theta\|^*_\infty \leqslant 1$. This excludes "calibrations" that do not calibrate any linear subspace.

Note that every compact domain with smooth boundary in a calibrated submanifold N is homologically volume-minimising where the homology theory is with respect to real coefficients, cf. the introduction to [**12**]. For this property we will use the term that N is $\mathbb{R}$-homologically minimising.

(B) *The comass norm on cohomology*

The Riemannian metric on M induces a natural norm $\| \ \|^*$ called *comass norm*, on the de Rham cohomology vector spaces $H^k_{dR}(M)$. If $\alpha \in H^k_{dR}(M)$ one defines

$$\|\alpha\|^* = \inf\{\|\theta\|^*_\infty \mid \theta \text{ is a closed } k\text{-form representing } \alpha\}. \tag{2·3}$$

To see that $\| \ \|^*$ is indeed a norm, one has to prove that $\|\alpha\|^* > 0$ if $\alpha \neq 0$. This follows from inequality (2·7) below. In general, the infimum in (2·3) will not be attained by a smooth k-form. However, there always exists a weakly closed k-form θ with measurable coefficients that represents α and such that $\|\theta\|^*_\infty = \|\alpha\|^*$. For details see [**23**, IX, theorem 5A] or [**6**, 4·1·19]. Thus, for every cohomology class α with $\|\alpha\|^* = 1$ there exists a (generally non-unique) form θ representing α and of minimal comass norm in α. Then θ satisfies all the requirements for calibrations except for the continuity. In view of this it seems natural to generalise the notion "calibration" so as to include weakly closed L^∞-forms of comass one, as is done in [**3**, definition 3·1]. Here we refrained to do so since this would necessitate additional arguments in order to replace pointwise statements like (2·7) and (2·9).

(C) *Currents and the stable norm on homology*

Here we present the theory dual to differential forms and comass. First note that the *mass norm* $\| \ \|$ on the spaces of multivectors $\Lambda_k(TM_x)$ is dual to the comass norm defined in (2·1), i.e. if $v \in \Lambda_k(TM_x)$ then

$$\|v\| = \max\{\langle \pi, v\rangle \mid \pi \in \Lambda_k(TM^*_x), \|\pi\|^* \leqslant 1\}.$$

A *k-current of finite mass* is a linear functional T acting on the vector space $\Omega^k M$ of smooth k-forms that is continuous with respect to the comass norm $\| \ \|^*_\infty$. Its operator norm

$$\mathbf{M}(T) = \sup\{T(\omega) \mid \omega \text{ is a } k\text{-form with } \|\omega\|^*_\infty \leqslant 1\} \tag{2·4}$$

is called the *mass of* T. As a consequence of the Riesz representation theorem every k-current T of finite mass is *representable by integration*, i.e. there exist a Radon measure μ_T on M, called the *total variation measure* of T, and a μ_T-measurable k-vectorfield $\vec{T}$ such

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that $\|\vec{T}(x)\| = 1$ for μ_T-almost every $x \in M$, that are characterised by the equality

$$T(\omega) = \int_M \langle \omega(x), \vec{T}(x) \rangle d\mu_T(x) \tag{2·5}$$

for all k-forms ω. Note that $\mathbf{M}(T) = \mu_T(M)$. Moreover, (2·5) allows one to extend T to k-forms with coefficients in $L^1(\mu)$. A k-current T is *closed* if

$$T(d\pi) = 0$$

holds for all $(k-1)$-forms π. A closed k-current T defines a linear functional on the k'th de Rham cohomology $H^k_{dR}(M)$ and, hence, a homology class

$$[T] \in H_k(M, \mathbb{R}) \equiv H^k_{dR}(M)^*.$$

Note that every homology class $h \in H_k(M, \mathbb{R})$ can be represented by a closed current of finite mass. This can be seen as follows. By singular homology theory there exists a Lipschitz k-cycle $\Sigma r_i \sigma_i$ with real coefficients representing h, and $\Sigma r_i \sigma_i$ can be considered a closed k-current of finite mass by setting

$$(\sum r_i \sigma_i)(\omega) = \sum r_i \int_{\Delta^k} \sigma_i^* \omega$$

for every k-form ω. Hence one can define the *stable norm* on $H_k(M, \mathbb{R})$ by

$$\|h\| = \min\{\mathbf{M}(T) \mid T \text{ is a closed } k\text{-current representing } h\}. \tag{2·6}$$

The existence of the minimum in (2·6) follows from the Banach–Alaoglu theorem. If T is a closed k-current of finite mass and $[T] = h \in H_k(M, \mathbb{R})$, and if θ is a closed k-form representing $\alpha \in H^k_{dR}(M)$, then (2·4) and (2·5) imply

$$\langle \alpha, h \rangle = T(\theta) = \int_M \langle \theta(x), \vec{T}(x) \rangle d\mu_T(x) \leqslant \|\theta\|^*_\infty \mathbf{M}(T) \tag{2·7}$$

and, in particular,

$$\langle \alpha, h \rangle \leqslant \|\alpha\|^* \|h\|.$$

It is important, but not completely obvious, that the stable norm $\| \ \|$ on $H_k(M, \mathbb{R})$ and the comass norm $\| \ \|^*$ on $H^k_{dR}(M)$ are dual norms. The difficulty is that in Definition 2·4 the forms ω need not be closed, while in Definition 2·3 the forms θ are closed. The duality of stable norm and comass norm is a consequence of the Hahn–Banach theorem, and was first proved by H. Federer in [**7**, 4·10], see [**9**, 4·35] or [**3**, theorem 3·8] for versions that may be more easily accessible.

A closed k-current T of finite mass is said to be *calibrated* by a calibration θ if

$$T(\theta) = \mathbf{M}(T). \tag{2·8}$$

According to (2·7) this is equivalent to the condition that

$$\langle \theta(x), \vec{T}(x) \rangle = 1 \tag{2·9}$$

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μ_T-almost everywhere. Moreover, it follows from (2·7) that if T is calibrated by θ, then T is a current of minimal mass in its homology class, i.e. $\mathbf{M}(T) = \|[T]\|$. The duality between stable norm and comass norm has the following simple consequence:

PROPOSITION 2·2. *Suppose θ is a calibration of degree k representing the cohomology class $\alpha \in H^k_{dR}(M)$. Then the following statements are equivalent*:

(i) *the calibration θ has minimal comass in its cohomology class, i.e.* $\|\alpha\|^* = \|\theta\|^*_\infty (= 1)$;
(ii) *there exists a closed k-current $T \neq 0$ of finite mass calibrated by θ.*

Proof. If (i) holds we can use the duality between stable norm and comass norm to find a homology class $h \in H_k(M, \mathbb{R}) \setminus \{0\}$ such that

$$\langle \alpha, h \rangle = \|\alpha\|^* \|h\| = \|h\|.$$

According to (2·6) there exists a closed k-current $T \neq 0$ representing h such that

$$\mathbf{M}(T) = \|h\|.$$

Using (2·7) and the preceding equations we obtain

$$\|h\| = \langle \alpha, h \rangle = T(\theta) \leqslant \|\theta\|^*_\infty \mathbf{M}(T) = \mathbf{M}(T) = \|h\|,$$

and hence $T(\theta) = \mathbf{M}(T)$, i.e. T is calibrated by θ.

Conversely, if $T \neq 0$ is a closed k-current such that $T(\theta) = \mathbf{M}(T)$, then the homology class h of T satisfies $\|\alpha\|^* \|h\| \geqslant \langle \alpha, h \rangle = T(\theta) = \mathbf{M}(T) \geqslant \|h\|$. Since $\mathbf{M}(T) > 0$ and $\|\alpha\|^* \leqslant \|\theta\|^*_\infty = 1$, we conclude that

$$\|\alpha\|^* = 1 = \|\theta\|^*_\infty.$$

(D) *Measured laminations*

In [**19**], D. Ruelle and D. Sullivan introduced a special type of closed current baptised "geometric current". Using different terms one could say that these are currents that are given by a measured lamination. In Section 6 we will prove that every closed current that is calibrated by a simple calibration of class C^1 is of this type. Although the notion "measured lamination" is widely used in the literature we could not find a comprehensive treatment of its basics. So, below we will present precise definitions of the notions that we will use.

(D_1) *Laminations*

Let $m \geqslant 1$ and $0 \leqslant k \leqslant m$ and $r \geqslant 1$ be integers, and let $S \neq \varnothing$ be a closed subset of an m-dimensional manifold M. We will define the notion of a *k-dimensional C^r-lamination structure with support S*. A *k-dimensional C^r*-lamination chart $\varphi : S^\varphi \to D^k \times T^\varphi$ *for S* is a homeomorphism from an open subset S^φ of S onto a product $D^k \times T^\varphi$ where D^k denotes the open ball in $\mathbb{R}^k$ and T^φ is a subset of $\mathbb{R}^{m-k}$, satisfying the following properties:

$$\text{for every } y \in T^\varphi \text{ the map } \varphi^{-1} \mid D^k \times \{y\} \text{ is a } C^r\text{-embedding (into } M); \tag{2·10}$$

$$\text{all partial derivatives of } \varphi^{-1} \text{ with respect to } x \in D^k \text{ and of order } s \leqslant r \text{ are continuous as functions on } D^k \times T^\varphi \subseteq \mathbb{R}^m. \tag{2·11}$$

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Two C^r-lamination charts φ and ψ for S are *compatible* if every point $(x_0, y_0) \in \varphi(S^\varphi \cap S^\psi)$ has a product neighbourhood $A \times B$ in $\varphi(S^\varphi \cap S^\psi) \subseteq D^k \times T^\varphi$ such that there exist maps $F : A \times B \to D^k$, $G : B \to T^\psi$ such that

$$\psi \circ \varphi^{-1}(x, y) = (F(x, y), G(y)) \tag{2·12}$$

holds for all $(x, y) \in A \times B$.

Note that $G : B \subseteq T^\varphi \to G(B) \subseteq T^\psi$ is a homeomorphism and, due to (2·10) and (2·11), for all $y \in B$ the map $x \in A \to F(x, y) \in D^k$ is a C^r-diffeomorphism from $A \subseteq D^k$ onto its image in D^k. All partial derivatives of F with respect to $x \in A$ and of order $s \leqslant r$ depend continuously on $(x, y) \in A \times B$. Moreover, since $S \subseteq M$ is closed, the set $T^\varphi \subseteq \mathbb{R}^{m-k}$ is locally compact.

Now a *k-dimensional C^r-lamination* atlas $\mathcal{A}$ for $S \subseteq M$ is a set of pairwise compatible k-dimensional C^r-lamination charts for S such that $\bigcup_{\varphi \in \mathcal{A}} S^\varphi = S$. A *$k$-dimensional C^r-lamination structure* $\mathcal{L}$ with support $\operatorname{spt}(\mathcal{L}) = S$ is a maximal such atlas.

We will use the terminology that is standard for foliations also for laminations $\mathcal{L}$. So, if $\varphi \in \mathcal{L}$ is a lamination chart and $y \in T^\varphi$, then the k-dimensional submanifold $\varphi^{-1}(D^k \times \{y\})$ of M will be called a *plaque* of φ. The set of all plaques of all $\varphi \in \mathcal{L}$ is a basis of a topology on S, called the *leaf topology*. The leaf topology will often be strictly finer than the topology induced on S from M. The connected components of S with respect to the leaf topology are the *leaves of the lamination* $\mathcal{L}$. As in the case of a foliation one can show that each leaf L of a lamination $\mathcal{L}$ carries a unique manifold structure such that the inclusion $i : L \to M$ is an immersion. According to (2·11) the tangent spaces to the leaves of $\mathcal{L}$ depend continuously on their footpoints in $S = \operatorname{spt}(\mathcal{L})$. Hence they define a continuous *k-dimensional distribution* $\mathcal{D}_\mathcal{L}$ over S, i.e., if $x \in S$ and if L is the leaf of $\mathcal{L}$ containing x, then $\mathcal{D}_\mathcal{L}|_x = i_*(TL_x) \subseteq TM_x$. An oriented lamination structure is a lamination structure $\mathcal{L}$ together with a (continuous) orientation of $\mathcal{D}_\mathcal{L}$.

At this point many questions can be asked concerning the notions defined above, in particular concerning their relation to objects with the same name that already exist in the literature. Due to limitations of space, in the following remarks we will only address those points which we find most urgent.

Remark 2·3. There are various possibilities to relax or strengthen the analytic properties in the definition of laminations. For example, one can relax condition (2·11) by requiring the continuity of the partial derivatives with respect to x only up to some fixed order $s < r$. On the other hand, one can strengthen the definition by requiring that the lamination charts are bilipschitz or by requiring that the partial derivatives in (2·11) are Lipschitz (or Hölder). Here, Lipschitz continuity seems to be a particularly natural condition, cf. the results in [**15**, **21**, **25**].

Remark 2·4. One can consider the concept "lamination" as a generalization of the notion "submanifold". However, while the dimension and differential structure of an embedded submanifold is uniquely determined by the underlying subset of M, this is in general not the case for laminations.

Remark 2·5. Our definition of lamination structure involves the surrounding manifold M via (2·10) and (2·11). Note, however, if $S \subseteq N \subseteq M$ and N is a submanifold of M then every lamination structure $\mathcal{L}$ on S as a subset of N is also a lamination structure on S as a subset of M, and vice versa.

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Remark 2·6. In [**15**] J.N. Mather requires that the charts of a lamination be defined on open subsets of M, while D. Ruelle and D. Sullivan [**19**] are somewhat hesitant on this point. It is clear that not every chart in our sense can be extended to a homeomorphism of an open set in M, since, in the codimension one case, our charts might not respect the natural (local) order of the leaves. On the other hand, one might expect that any lamination structure in our sense contains an atlas of charts that can be extended to open sets in M. We are able to prove this for lamination structures $\mathcal{L}$ whose distribution $\mathcal{D}_{\mathcal{L}}$ is locally Lipschitz. Here we can achieve extensions that are bilipschitz, but satisfy (2·11) only for the first derivatives.

The Frobenius theorem clarifies the relation between foliations and distributions. Under appropriate assumptions, its proof carries over to laminations, see Proposition 2·7 below. We let $\pi : G_k M \to M$ denote the Grassmann bundle of k-dimensional linear subspaces in TM. The following proposition will be used in the proof of Theorems 6·1 and 6·8.

PROPOSITION 2·7. *Let $S \neq \varnothing$ be a closed subset of M, and let $\mathcal{D} : M \to G_k M$ be a distribution of class C^r, $1 \leqslant r \leqslant \infty$. Suppose the following conditions are satisfied for all C^r-vector fields X, Y with values in $\mathcal{D}$:*

(a) *if $p \in S$ then $[X, Y]_p \in \mathcal{D}_p$;*
(b) *S is invariant under the flow of X.*

Then there is a unique C^r-lamination structure $\mathcal{L}$ on S such that its tangent distribution $\mathcal{D}_{\mathcal{L}}$ coincides with $\mathcal{D}|_S$.

As will be explained below we can choose a lamination atlas in $\mathcal{L}$ whose charts are restrictions to S of C^r-diffeomorpisms defined an open subset of M.

Proof. We start by constructing lamination charts for S. We choose coordinates for M in a neighbourhood of p such that p corresponds to $0 \in \mathbb{R}^m$ and such that $\mathcal{D}_{(x,y)}$ projects bijectively onto $\mathbb{R}^k \times \{0\} \subseteq \mathbb{R}^m$ for all (x, y) in a neighbourhood $U \times V \subseteq \mathbb{R}^k \times \mathbb{R}^{m-k}$ of 0. Then there are unique vector fields $Y_1, \ldots, Y_k$ on $U \times V$ generating $\mathcal{D}$ such that the standard projection $\mathbb{R}^k \times \mathbb{R}^{m-k} \to \mathbb{R}^k$ maps each Y_i, $1 \leqslant i \leqslant k$, to the constant vector field E_i on U. Now a simple calculation shows that:

(a) implies that, for all $(x, y) \in S \cap (U \times V)$, we have $[Y_i, Y_j]_{(x,y)} = 0$ for $1 \leqslant i, j \leqslant k$. Let $\Phi^1, \ldots, \Phi^k$ denote the local flows of $Y_1, \ldots, Y_k$.

There exist $\epsilon > 0$ and a neighbourhood $V_0 \subseteq V$ of $0 \in \mathbb{R}^{m-k}$ such that the map

$$F : (t, y) \in U_0 \times V_0 \longrightarrow \Phi^k_{t_k} \circ \cdots \circ \Phi^1_{t_1}(0, y) \in U \times V$$

is defined, where $U_0 = \{(t_1, \ldots, t_k) \mid |t_i| < \epsilon \text{ for } 1 \leqslant i \leqslant k\}$. Using (b) and the fact that $[Y_i, Y_j]_{(x,y)} = 0$ for $(x, y) \in S \cap (U \times V)$, the standard proof shows that, for every permutation σ of $\{1, \ldots, k\}$, we have

$$\phi^{\sigma(k)}_{t_{\sigma(k)}} \circ \cdots \circ \Phi^{\sigma(1)}_{t_{\sigma(1)}}(0, y) = \Phi^k_{t_k} \circ \cdots \circ \Phi^1_{t_1}(0, y) = F(t, y), \qquad (*)$$

whenever $t \in U_0$ and $(0, y) \in S \cap (\{0\} \times V_0)$. Since F is regular at $(0, 0) \in U_0 \times V_0$ we can assume that F is a C^r-diffeomorphism from $U_0 \times V_0$ onto an open subset W of $U \times V$. We consider $\varphi := F^{-1} : W \to U_0 \times V_0$. Using (b) and the definition of F we see that $\varphi(W \cap S) = U_0 \times T^{\varphi}$, where $T^{\varphi} \subseteq \mathbb{R}^{m-k}$ is defined by $\{0\} \times T^{\varphi} = (\{0\} \times V_0) \cap S$. Moreover, equation $(*)$ implies that $D\varphi_{(x,y)}(\mathcal{D}_{(x,y)}) = \mathbb{R}^k \times \{0\}$ whenever $(x, y) \in W \cap S$. This shows that, up to composition with a diffeomorphism between U_0 and D^k, φ is a k-dimensional lamination chart for S that is the restriction to S of a C^r-chart for M. This

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implies that (2·10) and (2·11) are satisfied in class C^r. Finally if φ and ψ are two lamination charts of this type defined in neighbourhoods S^φ, S^ψ of $p \in S$ then

$$D(\psi \circ \varphi^{-1})_{(x,y)}(\mathbb{R}^k \times \{0\}) = \mathbb{R}^k \times \{0\}$$

for all $(x, y) \in \varphi(S^\varphi \cap S^\psi)$. This implies that φ and ψ are compatible. Hence, the collection of all these lamination charts defines a C^r-lamination atlas for S. On the other hand, two lamination structures $\mathcal{L}$ and $\mathcal{L}'$ on S coincide if their tangent distributions $\mathcal{D}_\mathcal{L}$ and $\mathcal{D}_{\mathcal{L}'}$ are Lipschitz and $\mathcal{D}_\mathcal{L} = \mathcal{D}_{\mathcal{L}'}$. This follows from the uniqueness of the solutions of initial value problems for ordinary differential equations that are Lipschitz.

(D_2) *Invariant measures*

The notion of an invariant measure of a lamination is defined in analogy to the well-known special case of a foliation. We assume that $\mathcal{L}$ is a k-dimensional lamination structure whose support is a closed subset $S \neq \varnothing$ of M. An *invariant measure* μ for $\mathcal{L}$ is an assignment of a Radon measure μ^φ on T^φ to every lamination chart $\varphi : S^\varphi \to D^k \times T^\varphi$, that satisfies the following invariance property:

If $\varphi, \psi \in \mathcal{L}$ and if

$$\psi \circ \varphi^{-1}(x, y) = (F(x, y), G(y))$$

for all (x, y) in a product $A \times B \subseteq \varphi(S^\varphi \cap S^\psi) \subseteq D^k \times T^\varphi$, cf. (2·12), then

$$\mu^\psi(G(C)) = \mu^\varphi(C) \tag{2·13}$$

holds for all Borel subsets C of $B \subseteq T^\varphi$.

It suffices that the μ^φ are given and satisfy (2·13) for all charts φ of a lamination atlas. Then there exists a unique invariant measure of the corresponding lamination structure extending the assignment $\varphi \to \mu^\varphi$. As in the case of foliations, there exists a non-zero invariant measure if M is compact and if there exists a leaf of subexponential growth, cf. [**17**]. A *measured lamination* $(\mathcal{L}, \mu)$ is a lamination structure $\mathcal{L}$ together with a non-zero invariant measure μ. As explained in [**19**], every oriented, k-dimensional, measured lamination $(\mathcal{L}, \mu)$ in a compact manifold M induces a closed k-current $T = T(\mathcal{L}, \mu)$ as follows: As before, let $S = \mathrm{spt}(\mathcal{L})$. If $\omega \in \Omega^k M$ satisfies

$$\mathrm{spt}\omega \cap S \subseteq S^\varphi$$

for some $\varphi \in \mathcal{L}$, we define

$$T(\omega) = \int_{T^\varphi} \left(\int_{\varphi^{-1}(D^k \times \{y\})} \omega \right) d\mu^\varphi(y), \tag{2·14}$$

where the orientation of the plaque $\varphi^{-1}(D^k \times \{y\})$ is determined by the orientation of $\mathcal{L}$. Using (2·12) and (2·13) one sees that this definition is independent of the choice of $\varphi \in \mathcal{L}$ with $\mathrm{spt}(\omega) \cap S \subseteq S^\varphi$. Since M is compact, we can decompose every $\omega \in \Omega^k M$ into a finite sum $\omega = \sum_{i=1}^{l} \omega_i$, where $\omega_i \in \Omega^k M$ and $\mathrm{spt}(\omega_i) \cap S \subseteq S^{\varphi_i}$ for some $\varphi_i \in \mathcal{L}$. Then we define

$$T(\omega) = \sum_{i=1}^{l} T(\omega_i),$$

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and this is independent of the choice of the decomposition. Applying Stokes' Theorem to (2·14) we see that $T(d\pi) = 0$ for every $\pi \in \Omega^{k-1}M$, i.e. T is a closed current. Next we calculate the Radon measure μ_T and the k-vectorfield $\vec{T}$ associated to a current $T = T(\mathcal{L}, \mu)$.

Fact 2·8. If $(\mathcal{L}, \mu)$ is an oriented, k-dimensional, measured lamination on a compact Riemannian manifold (M, g), then $T = T(\mathcal{L}, \mu)$ is a closed k-current of finite mass with $\mathrm{spt}(T) \subseteq \mathrm{spt}(\mathcal{L}) = S$. $\vec{T}$ is the simple, unit k-vectorfield on $\mathrm{spt}(T)$ defined by the oriented distribution $\mathcal{D}_{\mathcal{L}} \lfloor_{\mathrm{spt}(T)}$. For continuous functions $f : M \to \mathbb{R}$ with support in an open set U, such that $U \cap S \subseteq S^{\varphi}$ for some $\varphi \in \mathcal{L}$, we have

$$\int_M f d\mu_T = \int_{T^{\varphi}} \left(\int_{D^k \times \{y\}} f \circ \varphi^{-1}(x, y) d\mathrm{vol}^g_{\varphi,y}(x) \right) d\mu^{\varphi}(y), \tag{2·15}$$

where $d\mathrm{vol}^g_{\varphi,y}(x)$ denotes the Riemannian k-volume induced by $(\varphi^{-1})^* g$ on $D^k \times \{y\}$.

Fact 2·8 is a simple consequence of the definitions, cf. (2·5) and (2·14).

Remark 2·9. Compactness of M is not essential in this section. If M is not compact, an oriented measured lamination on M defines a closed current that is locally of finite mass.

If $(\mathcal{L}, \mu)$ is an oriented, measured lamination and $\tilde{S} \subseteq \mathrm{spt}(\mathcal{L}) = S$ is a closed subset that is a union of leaves of $\mathcal{L}$, there is an obvious way to restrict $(\mathcal{L}, \mu)$ to $\tilde{S}$ to obtain an oriented, measured lamination $(\tilde{\mathcal{L}}, \tilde{\mu})$ with $\mathrm{spt}(\tilde{\mathcal{L}}) = \tilde{S}$ such that the leaves of $\tilde{\mathcal{L}}$ are the leaves of $\mathcal{L}$ contained in $\tilde{S}$, and such that $T(\tilde{\mathcal{L}}, \tilde{\mu})(\omega) = T(\mathcal{L}, \mu)(\omega)$ if ω is a bounded, measurable k-form with $\mathrm{spt}(\omega) \cap S \subseteq \tilde{S}$. Now, if $T = T(\mathcal{L}, \mu)$ and $\mathrm{spt}(T) \neq \mathrm{spt}(\mathcal{L})$ we can restrict $(\mathcal{L}, \mu)$ to $\mathrm{spt}(T)$, and obtain a measured lamination $(\tilde{\mathcal{L}}, \tilde{\mu})$ such that $T = T(\tilde{\mathcal{L}}, \tilde{\mu})$ and $\mathrm{spt}(T) = \mathrm{spt}(\tilde{\mathcal{L}})$. We will say that a closed k-current T is *represented by* the k-dimensional measured lamination $(\mathcal{L}, \mu)$ if $T = T(\mathcal{L}, \mu)$ and $\mathrm{spt}(T) = \mathrm{spt}(\mathcal{L})$.

Note that Fact 2·8 implies that there is at most one measured lamination representing a given current. On the other hand, it is well-known that for $1 < k < \dim M - 1$ there exist closed k-currents of finite mass that cannot be represented by measured laminations. A simple example for this is the closed 2-current T on $\mathbb{R}^4/\mathbb{Z}^4$ with $\vec{T} = (E_1 \wedge E_2 + E_3 \wedge E_4)/\sqrt{2}$ and μ_T induced from Lebesgue measure on $\mathbb{R}^4$. However, this T is the sum of two currents that can be represented by measured laminations, and so the natural question is, if any closed current of finite mass is a sum or an integral of currents representable by measured laminations. The examples in [**24**] and in [**10**, $4{\cdot}A_2$], show that the answer to this question is "no".

Remark 2·10. As shown in [**2**] and [**4**], every closed, locally minimising current T of codimension one can be represented by an oriented, measured "lamination" $(\mathcal{L}, \mu)$ the leaves of which are locally minimising hypersurfaces. If $m = \dim M \geqslant 8$, these minimal hypersurfaces may have closed singular sets of Hausdorff dimension smaller or equal to $m - 8$. So, if one of these singular sets is non-empty, then the "lamination" $\mathcal{L}$ does not satisfy the definition of lamination given in this section. On the other hand, if all the leaves of the "lamination" $\mathcal{L}$ are regular, then it follows from [**6**, theorem 5·3·14], that there exist local coordinates in a neighbourhood of every point such that all the leaves intersecting this neighbourhood can be represented as graphs of functions, see also [**21**, proof of theorem 1·1]. From this one can easily construct an atlas of lamination charts.

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Finally, we note that if the closed, codimension one current T of finite mass is calibrated by a calibration θ, then all leaves of the corresponding lamination $\mathcal{L}$ are regular and smooth. This is a consequence of the assumed continuity of θ, and can be seen as follows. Since θ is continuous, the field of oriented tangent hyperplanes of every leaf of $\mathcal{L}$ can be continuously extended from the regular part of the leaf to the whole leaf. Locally, every leaf is the boundary of a set of finite perimeter which we can assume to be a set E in $\mathbb{R}^m$. Using the concepts from [**8**, chapter 3], we conclude that the reduced boundary $\partial^* E$ of E coincides with ∂E. Moreover, the generalised (euclidean) normal $\nu(x)$ exists for every $x \in \partial E$, and depends continuously on $x \in \partial E$. Now [**8**, theorem 4·10] shows that that ∂E is a C^1-hypersurface. This proves that the leaves of $\mathcal{L}$ are locally C^1-hypersurfaces. Finally, one can use [**13**, chapter 4, theorem 6·3] to conclude that smoothness of the Riemannian metric implies smoothness of the leaves of $\mathcal{L}$.

3. *One-dimensional calibrations of constant comass*

In this section we exhibit the relations between cc-calibrations of degree one, solutions of Hamilton–Jacobi equations and Aubry–Mather theory. This is the starting point for our investigation of calibrations of degree greater than one in the remaining sections. We will also treat regularity properties of cc-calibrations of degree one. Since these are local properties, closedness of M does not really play a role. To be consistent with previous definitions, we nevertheless assume M to be closed and oriented.

Definition 3·1. A measurable one-form θ is called locally exact if every point $x \in M$ admits a neighbourhood U and a differentiable function $u : U \to \mathbb{R}$ such that $du = \theta|_U$.

We note that the conditon "locally exact" is strictly stronger than "weakly closed", since it excludes points of nondifferentiability of the local primitives u. This fact is decisive in Proposition 3·2 and Corollary 3·3 below. On the other hand, calibrations are always locally exact since, in addition to being weakly closed, they are assumed to be continuous.

Obviously, if θ is locally exact then the pullback of θ to the universal or to the abelian covering space of M is exact.

If θ is locally exact and of constant comass we choose a covering space $p : \bar{M} \to M$ such that $p^*\theta$ is exact, i.e.

$$p^*\theta = du$$

for some differentiable function $u : \bar{M} \to \mathbb{R}$. We let $\bar{g} = p^*g$ denote the lifted metric and $\bar{g}^*$ the dual scalar product on $T^*\bar{M}$. The Hamiltonian $\bar{H} : T^*\bar{M} \to \mathbb{R}$ induced by $\bar{g}$ is defined by

$$\bar{H}(\pi) := \frac{1}{2}\bar{g}^*(\pi, \pi)$$

for all $\pi \in T^*\bar{M}$. If θ is of constant comass λ and $du = p^*\theta$, then $u : \bar{M} \to \mathbb{R}$ solves the Hamilton–Jacobi equation

$$\bar{H}(du) = \frac{1}{2}\lambda.$$

In this situation we have the following additional regularity, cf. [**18**, theorem 1 and the footnote on p. 520] and [**5**, theorem 3·1].

PROPOSITION 3·2. *Let θ be a locally exact one-form of constant norm on M. Then θ is Lipschitz with Lipschitz constant that only depends on the Riemannian metric.*

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Proposition 3·2 has the following consequence:

COROLLARY 3·3. *Let $\theta : M \to S(T^*M)$ be a locally exact one-form of constant norm one. Then θ is a Lipschitz cc-calibration, and the set of curves calibrated by θ forms an oriented, Lipschitz foliaton of M by $\mathbb{R}$-homologically minimising geodesics.*

Proof. By Proposition 3·2, θ is Lipschitz. Thus the oriented line field calibrated by θ is Lipschitz. Hence it induces a one-dimensional, oriented Lipschitz foliation of M. The leaves of this foliation are $\mathbb{R}$-homologically minimising, cf. Section 2A. In particular, they are geodesic.

If $p : \bar{M} \to M$ is a covering such that $p^*\theta = du$ for some $C^{1,1}$-function $u : \bar{M} \to \mathbb{R}$, then the lifts $\bar{c}$ of the geodesics calibrated by θ are shortest connections between level hypersurfaces of u: If $\bar{c}$ is defined on the interval $[s, t] \subseteq \mathbb{R}$ and $u_0 = u(\bar{c}(s))$, $u_1 = u(\bar{c}(t))$, then $L(\bar{c}|[s, t]) = d(u^{-1}(u_0), u^{-1}(u_1))$. Here the distance $d(u^{-1}(u_0), u^{-1}(u_1))$ between the level hypersurfaces $u^{-1}(u_0)$ and $u^{-1}(u_1)$ is defined as the infimum of the lengths of curves connecting $u^{-1}(u_0)$ and $u^{-1}(u_1)$.

We will now discuss the question how rare cc-calibrations of degree one on a Riemannian manifold M are. Let $\mathcal{G}$ denote the set of Riemannian metrics on M that admit a cc-calibration of degree one. First note that $\mathcal{G} \neq \varnothing$ implies that the Euler characteristic of M is zero and that $H^1_{dR}(M) \neq \{0\}$. So many closed manifolds do not admit any cc-calibration of degree one. The Arzela–Ascoli theorem and Proposition 3·2 imply that $\mathcal{G}$ is closed in the C^0-topology. It is easy to see from Corollary 3·3 that $\mathcal{G}$ does not contain a nonempty open set with respect to the C^1-topology, cf. [**1**, section 9]. On the other hand, KAM theory implies that, on a torus $M = T^m$, the set $\mathcal{G}$ contains a nonempty open set in the C^k-topology for sufficiently large k.

We will now show that every cc-calibration of degree one has minimal comass norm in its cohomology class. Note that every locally exact one-form θ determines a cohomology class $[\theta] \in H^1(M, \mathbb{R})$.

PROPOSITION 3·4. *Let θ be a cc-calibration of degree one on M. Then the cohomology class $[\theta] \in H^1(M, \mathbb{R})$ of θ has comass norm one, i.e.*

$$\|[\theta]\|^* = 1 (= \|\theta\|^*_\infty).$$

Proof. We will find a closed one-current T calibrated by θ so that our claim will follow from Proposition 2·2. By Corollary 3·3 the set $A_\theta \subseteq S(TM)$ of unit vectors calibrated by θ is closed and invariant under the geodesic flow. Since $S(TM)$ is compact there exists a Borel probability measure μ on $S(TM)$ that is invariant under the geodesic flow and such that $\mathrm{spt}(\mu) \subseteq A_\theta$. The measure μ defines a one-current T_μ by setting

$$T_\mu(\omega) = \int_{S(TM)} \omega d\mu$$

for all one-forms ω. The invariance of μ implies that T_μ is closed. Since μ is a probability measure we have

$$T_\mu(\omega) \leqslant \|\omega\|^*_\infty$$

for all one-forms ω. On the other hand, we have $T_\mu(\theta) = \|\theta\|^*_\infty = 1$, since $\mathrm{spt}(\mu) \subseteq A_\theta$ and $\theta|_{A_\theta} = 1$. This implies that T_μ has mass $\mathbf{M}(T_\mu) = 1$, and that θ calibrates T_μ.

Now our claim follows from Proposition 2·2.

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Remark 3·5. Since T_μ is calibrated by θ, we see that T_μ has minimal mass in its homology class. This shows that μ is a minimal invariant measure as defined by J. Mather [**14**].

4. *Calibrations of constant comass that do not calibrate any submanifold*

In this section, we show that for $k \geqslant 2$ there exist smooth cc-calibrations of degree k not calibrating any k-dimensional submanifold. In particular, there is no analogue of Corollary 3·3 in dimensions $k \geqslant 2$. Here we rely on [**11**, proposition 2·7]. In this proposition, the authors assume that there is a closed k-form θ that is positive on the leaves of an oriented k-dimensional foliation $\mathcal{F}$ of M, and prove that there exists a Riemannian metric g on M such that, with respect to g, θ is a calibration calibrating the leaves of $\mathcal{F}$. Actually, g can be so chosen, that θ calibrates precisely the k-planes tangent to $\mathcal{F}$, see [**11**, theorem 3·1']. For our purpose it is important to note that the proof of [**11**, proposition 2], does not use the integrability of the distribution tangent to $\mathcal{F}$, but holds for any oriented k-plane field on which θ is positive. Thus we have:

PROPOSITION 4·1. *Let $\mathcal{D}$ be a k-dimensional, oriented distribution on M, and let θ be a smooth, closed k-form that is positive on every k-plane of $\mathcal{D}$. Then there exists a Riemannian metric g on M such that, with respect to g, θ is a calibration that calibrates precisely the k-planes of $\mathcal{D}$.*

Note that the calibration θ from Proposition 4·1 is obviously a cc-calibration since, at any point $x \in M$, it calibrates the k-plane $\mathcal{D}_x$ in TM_x.

Using Proposition 4·1 we can easily find examples of smooth cc-calibrations of any degree $k \geqslant 2$ that do not calibrate any k-dimensional submanifold.

Example 4·2. We set $M = T^m = \mathbb{R}^m/\mathbb{Z}^m$ and $\theta = dx_1 \wedge \cdots \wedge dx_k$ with $2 \leqslant k < m$, where $(x_1, \ldots, x_m)$ denote the standard coordinates on $\mathbb{R}^m$. We will construct a k-dimensional, oriented distribution $\mathcal{D}$ such that θ is positive on $\mathcal{D}$ and such that $\mathcal{D}$ does not have any integral manifold. Then Proposition 4·1 provides a Riemannian metric g on T^m such that θ is a calibration not calibrating any submanifold. We choose a C^∞-function $\tilde{f} : T^{k-1} \to \mathbb{R}$ with isolated critical points and define $f : T^m \to \mathbb{R}$ by $f(x_1, \ldots, x_m) = \tilde{f}(x_1, \ldots, x_{k-1})$. Starting from the standard vector fields $E_1, \ldots, E_m$ on T^m, we consider the vector fields $X_1 = E_1, \ldots, X_{k-1} = E_{k-1}, X_k = E_k + fE_m$ and the distribution $\mathcal{D}$ on T^m that is spanned and oriented by $(X_1, \ldots, X_k)$.

For $1 \leqslant i \leqslant k-1$ we have

$$[X_i, X_k] = [E_i, E_k + fE_m] = D_i f \cdot E_m.$$

Note that $\theta(X_1, \ldots, X_k) = 1$, and that

$$[X_i, X_k]|_x \notin \mathcal{D}_x \text{ if } D_i f(x) \neq 0.$$

By construction of f we have $D_1 f(x) = 0, \ldots, D_{k-1} f(x) = 0$ if and only if $x = (\bar{x}, x_k, \ldots, x_m)$ where $\bar{x} \in T^{k-1}$ is one of the finitely many critical points of $\tilde{f}$. So, $\mathcal{D}$ satisfies the infinitesimal integrability conditions $[X_i, X_k] \in \mathcal{D}$ for $1 \leqslant i \leqslant k-1$ only on the union of the finitely many subtori $\{\bar{x}\} \times T^{m-k+1}$, where $D\tilde{f}(\bar{x}) = 0$. Since $k \geqslant 2$ the tangent spaces to these subtori do not contain the corresponding fibers of $\mathcal{D}$. This implies that $\mathcal{D}$ does not admit any integral manifold. Using Proposition 4·1 we choose a Riemannian metric g on T^m such that θ is a calibration that calibrates precisely the fibers of $\mathcal{D}$.

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Since $\mathcal{D}$ does not admit any integral manifolds we conclude that θ does not calibrate any submanifold.

5. *Calibrations of codimension one and integrability of the calibrated distribution*

In this section we study how the minimality of the comass norm of a cc-calibration θ of codimension one is related to the integrability of the hyperplane field calibrated by θ. Relying on results from [**2**] and [**4**] we prove that integral hypersurfaces exist if θ has minimal comass norm in its cohomology class. An example will show that, even in this case, the calibrated hyperplane field can be non-integrable on a dense and open set. Finally, we present an example of a smooth cc-calibration such that the calibrated hyperplane field gives rise to a foliation by minimal hypersurfaces that does not carry any invariant measure. In this case, in spite of the integrability of the calibrated hyperplane field, the comass norm of the calibration is not minimal in its cohomology class.

We recall that, according to our Definition 2·1, calibrations need only be continuous and weakly closed.

THEOREM 5·1. *Let M be a closed, oriented Riemannian manifold,* $\dim M = m$, *and let θ be a calibration on M of degree $m-1$. Then the comass norm $\|\theta\|^*_\infty = 1$ of θ is minimal in the cohomology class of θ if and only if there exists an oriented, measured lamination on M whose leaves are hypersurfaces calibrated by θ.*

Proof. We first assume that θ has minimal comass in its cohomology class. Then Proposition 2·2 implies that there exists a closed $(m-1)$-current $T \neq 0$ calibrated by θ. Hence T has minimal mass in its homology class. If μ_T is the variation measure of T and $\vec{T}$ is its unit $(m-1)$-vectorfield, then we have

$$\langle \theta(x), \vec{T}(x) \rangle = 1 \tag{5·1}$$

for μ_T-almost every $x \in M$, see (2·9). On the other hand, according to [**2**, theorem 1 or remark 2·10], the current T can be represented by an oriented, measured lamination $(\mathcal{L}, \mu)$. Fact 2·8 implies that, for μ_T-almost every $x \in M$, $\vec{T}(x)$ determines the oriented tangent hyperplane of the leaf of $\mathcal{L}$ through x. Hence equation (5·1) implies that the leaves of $\mathcal{L}$ are calibrated by θ.

Conversely, suppose there exists an oriented, measured lamination on M whose leaves are hypersurfaces calibrated by θ. This lamination defines a closed $(m-1)$-current $T \neq 0$, cf. [**19**] or Section $2D_2$, and T is calibrated by θ. Now Proposition 2·2 implies that θ has minimal comass norm in its cohomology class.

In Section 6 we will prove a version of Theorem 5·1 that is valid for calibrations θ of arbitrary degree k, under the assumptions that θ is simple and of class C^1, see Theorem 6·1. Since simpleness of θ is automatically satisfied for $k = m-1$, Theorem 5·1 is more general than Theorem 6·1 only in relaxing the C^1-condition on θ.

Remark 5·2. In the codimension one case there is the following version of the Lipschitz regularity result that holds in the one-dimensional case, see Proposition 3·2. Let θ be a calibration of degree $m-1$ on an m-dimensional Riemannian mannifold. Suppose θ calibrates the leaves of an oriented lamination $\mathcal{L}$. Then $\theta|_{\mathrm{spt}(\mathcal{L})}$ is locally Lipschitz with local Lipschitz constants that only depend on the Riemannian metric. This is proved for the case of foliatons in [**21**, theorem 1·1 and remark 1·10], and the proof carries over to the case of laminations.

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Remark 5·3. Combining Example 4·2 and Theorem 5·1 we easily obtain examples of cc-calibrations θ of codimension one that do not calibrate any non-zero closed current of finite mass. By Theorem 5·1, the comass norm of such θ is larger than the comass norm of its cohomology class. In Section 6 we will find such examples also for other codimensions, see Example 6·7.

Next we present an example showing that under the assumptions of Theorem 5·1 it can happen that the calibrated hyperplane field is non-integrable outside the support of a closed current calibrated by θ.

Example 5·4. We consider the 3-torus $T^3 = \mathbb{R}^3/\mathbb{Z}^3$ with the standard coordinates (x_1, x_2, x_3) mod $\mathbb{Z}^3$. First, we will construct a 2-dimensional, oriented distribution $\mathcal{D}$ on T^3 such that the 2-torus $x_3 \equiv 0$ mod $\mathbb{Z}$ is an integral manifold of $\mathcal{D}$ that contains all integral manifolds of $\mathcal{D}$. $\mathcal{D}$ will be generated by vector fields

$$X = E_1 + fE_3 \text{ and } Y = E_2,$$

where E_1, E_2, E_3 denote the standard vector fields in the directions of the coordinate lines. We choose the orientation of $\mathcal{D}$ so that (X, Y) is positively oriented. We assume that $f \in C^\infty(T^3, \mathbb{R})$ satisfies the following properties:

$$f \text{ vanishes precisely on the 2-torus } x_3 \equiv 0 \mod \mathbb{Z}; \tag{5·2}$$

$$\frac{\partial f}{\partial x_2}(x_1, x_2, x_3) = 0 \text{ precisely if } x_3 \equiv 0 \mod \mathbb{Z}, \text{ or } x_2 \equiv 0 \mod \mathbb{Z}, \text{ or } x_2 \equiv \frac{1}{2} \mod \mathbb{Z}. \tag{5·3}$$

More specifically, we can take

$$f(x_1, x_2, x_3) = (2 + \cos(2\pi x_2))(1 - \cos(2\pi x_3)).$$

Note that $[X, Y] = (\partial f/\partial x_2)E_3$. So, by property (5·3), the integrability condition $[X, Y]|_x \in \mathcal{D}_x$ is satisfied only on the three 2-tori defined by the equations $x_3 \equiv 0$ mod $\mathbb{Z}$, $x_2 \equiv 0$ mod $\mathbb{Z}$, $x_2 \equiv 1/2$ mod $\mathbb{Z}$, respectively. By property (5·2), the 2-torus $x_3 \equiv 0$ mod $\mathbb{Z}$ is an integral manifold of $\mathcal{D}$, while the 2-tori $x_2 \equiv 0$ mod $\mathbb{Z}$ and $x_2 \equiv \frac{1}{2}$ mod $\mathbb{Z}$ are transversal to $\mathcal{D}$ at all points with $x_3 \not\equiv 0$ mod $\mathbb{Z}$. Hence every integral manifold of $\mathcal{D}$ is a nonempty open subset of the 2-torus $x_3 \equiv 0$ mod $\mathbb{Z}$. Now we consider the closed 2-form $\theta = dx_1 \wedge dx_2$ on T^3. Since $\theta(X, Y) = 1$ we can use Proposition 4·1 to obtain a Riemannian metric g on T^3 such that, with respect to g, θ is a calibration that calibrates precisely the distribution $\mathcal{D}$. Hence the integral manifold $x_3 \equiv 0$ mod $\mathbb{Z}$ of $\mathcal{D}$ is an $\mathbb{R}$-homologically minimising, closed 2-current calibrated by θ. In particular, by Theorem 5·1, θ realises the minimum of the comass norm in its cohomology class. On the other hand, the plane field $\mathcal{D}$ calibrated by θ does not have any integral manifolds outside the 2-torus $x_3 \equiv 0$ mod $\mathbb{Z}$. So, the cc-calibration θ calibrates precisely the non-empty open subsets of the 2-torus $x_3 \equiv 0$ mod $\mathbb{Z}$.

Finally, we present an example of a smooth cc-calibration θ of degree two on a closed, Riemannian 3-manifold such that the plane field calibrated by θ is everywhere integrable, but such that θ does not realise the infimum of the comass norm in its cohomology class.

Example 5·5. Let M be a closed manifold that admits a codimension one Anosov flow, i.e. the unstable foliation $\mathcal{F}$ of the flow has codimension one. Moreover, we assume that $\mathcal{F}$ is orientable. It is well known that the leaves of $\mathcal{F}$ have exponential growth, see [**16**,

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example (1) on p. 451]. In particular, $\mathcal{F}$ does not have a compact leaf. In this case, [**11**, corollary 7·4] implies that $\mathcal{F}$ can be calibrated, i.e. there exist a Riemannian metric on M and a smooth calibration θ of codimension one that calibrates all the leaves of $\mathcal{F}$. Hence the hyperplane field calibrated by θ is integrable. On the other hand, [**17**, corollary 6·5] implies that $\mathcal{F}$ does not admit an invariant measure. So, by Theorem 5·1, θ does not realise the infimum of the comass norm in its cohomology class. We note that all these objects can be explicitly constructed starting with the standard Anosov automorphism $A : T^2 \to T^2$, where $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ and A is induced by the matrix

$$\tilde{A} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \in SL_2(\mathbb{Z}).$$

Then we can take M as the mapping cylinder of A, and the Anosov flow on M is the suspension of A. A metric and a calibration as above can easily be found in coordinates adapted to the eigenspaces of $\tilde{A}$.

6. *Simple Calibrations and Integrability of the Calibrated Distribution*

In this section we prove the following partial generalisation of Theorem 5·1 to calibrations of arbitrary degree. Recall that a k-form ω is called simple if, for all $x \in M$, the k-covector ω_x can be represented as the wedge product of k elements of TM_x^*.

THEOREM 6·1. *Let M be a closed, oriented Riemannian manifold, and let θ be a calibration on M that is simple and of class C^1. Then the comass norm $\|\theta\|_\infty^* = 1$ of θ is minimal in the cohomology class of θ if and only if there exists an oriented, measured lamination on M whose leaves are calibrated by θ.*

The proof of Theorem 6·1 is given after the proof of Theorem 6·8 below.

By the following remarks we intend to put Theorem 6·1 into perspective.

According to Proposition 2·2 a calibration θ of degree k calibrates a non-zero closed current of finite mass if (and only if) θ realises the infimum of the comass norm in its cohomology class. If $k = 1$ or $k = \dim M - 1$ this implies the existence of submanifolds calibrated by θ, see Corollary 3·3 and Theorem 5·1. If $1 < k < \dim M - 1$ the existence of a non-zero closed current calibrated by θ does not imply that there exist submanifolds calibrated by θ. Indeed, examples by M. Zworski [**24**, proposition 3] and by M. Gromov [**10**, $4{\cdot}A_2$] show that in general closed k-currents of finite mass with $1 < k < \dim M - 1$ will not be decomposable into integral k-currents, not even if they are calibrated by a smooth calibration. In particular, the example in [**10**, $4{\cdot}A_2$] shows that the existence of a non-zero closed k-current calibrated by θ does not imply that θ calibrates any submanifold. On the positive side, [**24**, proposition 1] shows that such a decomposition is locally possible if the closed current T is smooth and simple, i.e. if μ_T is a smooth volume element and if $\vec{T}$ is a smooth and simple k-vectorfield.

So, Theorem 6·1, see also Theorem 6·8 below, generalises [**24**, proposition 1], by showing that smoothness of μ_T need not be assumed. It would be interesting to know if the assumption that θ be C^1 can be relaxed to Lipschitz continuity, or even to continuity, of θ. On the other hand, the example in [**24**] shows that simpleness of θ alone is not sufficient. The main difficulty in the proof of Theorem 6·1 results from the fact that the standard smoothing methods either destroy the simpleness of $\vec{T}$ or the closedness of T.

Note that many of the results in this section are local, and so we do not assume that the manifold M is closed.

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It is helpful to introduce the following definitions:

Definition 6·2. Let $\mathcal{D}$ be an oriented distribution on M. A current T of finite mass is a *distribution current* for $\mathcal{D}$ if, for μ_T-almost all $x \in M$, $\vec{T}(x)$ is simple and determines the fiber $\mathcal{D}_x$ of $\mathcal{D}$.

Distribution currents are special cases of the structure currents considered by D. Sullivan in [**22**].

Definition 6·3. A C^1-distribution $\mathcal{D}$ is infinitesimally integrable at x, if $[X, Y]|_x \in \mathcal{D}_x$ holds for all C^1-sections X, Y of $\mathcal{D}$.

The following proposition generalises [**24**, lemma A].

PROPOSITION 6·4. *Let $\mathcal{D}$ be an oriented C^1-distribution. If T is a closed distribution current for $\mathcal{D}$, then $\mathcal{D}$ is infinitesimally integrable on spt(T).*

Proof. Let $x \in \text{spt}(T)$ and choose a C^1-frame $X_1, \ldots, X_m$ in a neighbourhood U of x such that

$$\vec{T}|_U = X_1 \wedge \cdots \wedge X_k.$$

Let $\pi_1, \ldots, \pi_m$ denote the coframe dual to $X_1, \ldots, X_m$. We argue by contradiction and assume that $\mathcal{D}$ is not infinitesimally integrable at x. Then there exist indices $1 \leqslant i < j \leqslant k$ such that $[X_i, X_j]|_x \notin \mathcal{D}_x$, and, hence, $\alpha \in \{k+1, \ldots, m\}$ such that $\pi_\alpha([X_i, X_j])|_x \neq 0$. We consider the $(k-1)$-form

$$\omega = \pi_\alpha \wedge \pi_1 \wedge \cdots \wedge \widehat{\pi_i} \wedge \cdots \wedge \widehat{\pi_j} \wedge \cdots \wedge \pi_k.$$

Note that ω is defined on U and of class C^1. We compute

$$\begin{aligned}\langle d\omega, \vec{T}\rangle &= d\omega(X_1 \wedge \cdots \wedge X_k)\\ &= \sum_{i=1}^{k} (-1)^{i-1} X_i(\omega(X_1, \ldots, \widehat{X_i}, \ldots, X_k))\\ &\quad + \sum_{1 \leqslant r < s \leqslant k} (-1)^{r+s} \omega([X_r, X_s], X_1, \ldots, \widehat{X_r}, \ldots, \widehat{X_s}, \ldots, X_k).\end{aligned}$$

Using the fact that $\pi_\alpha(X_i) = 0$ for $1 \leqslant i \leqslant k$, we see that the first summand vanishes, while the second reduces to $(-1)^{i+j}\pi_\alpha([X_i, X_j])$. Hence

$$\langle d\omega, \vec{T}\rangle = (-1)^{i+j}\pi_\alpha([X_i, X_j]).$$

Since $\pi_\alpha([X_i, X_j])|_x \neq 0$ we can find $f \in C^1(M, \mathbb{R})$ with compact support in U such that

$$f(x)\pi_\alpha([X_i, X_j])|_x > 0,$$

while $f \cdot \pi_\alpha([X_i, X_j]) \geqslant 0$ everywhere on U. Now $f\omega$ can be considered a $(k-1)$-form on M with support in U. Since T is closed we have

$$\begin{aligned} 0 &= T(d(f\omega)) = \textstyle\int_M \langle d(f\omega), \vec{T}\rangle d\mu_T\\ &= \textstyle\int_M \langle df \wedge \omega, \vec{T}\rangle d\mu_T + \int_M f\langle d\omega, \vec{T}\rangle d\mu_T.\end{aligned}$$

Now, the first summand is zero since $\omega|\mathcal{D}^{k-1} = 0$, while the preceding inequalities together with $x \in \text{spt}(T) = \text{spt}(\mu_T)$ imply that

$$\int_M f\langle d\omega, \vec{T}\rangle d\mu_T = (-1)^{i+j} \int_M f\pi_\alpha([X_i, X_j])d\mu_T \neq 0.$$

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This contradicts $T(d(f\omega)) = 0$, and proves our claim.

COROLLARY 6·5. *Let $\mathcal{D}$ be an oriented, k-dimensional C^1-distribution on a manifold M, and assume that the set of points at which $\mathcal{D}$ is infinitesimally integrable has k-dimensional Hausdorff measure zero. Then there is no non-zero closed distribution current for $\mathcal{D}$.*

Remark 6·6. It suffices to assume that the k-dimensional integral geometric measure is zero.

Proof. Since a closed current of finite mass is a flat current, [**6**, 4·1·20] implies that the support of a non-zero closed distribution current would have positive k-dimensional integral geometric measure, in contradiction to Proposition 6·4 and our hypothesis. Note that the k-dimensional Hausdorff measure dominates the k-dimensional integral geometric measure, cf. [**6**, 2·10·6].

Using Proposition 4·1 and Corollary 6·5 we can easily find examples of cc-calibrations on compact manifolds that do not calibrate any closed current of finite mass. In particular, according to Proposition 2·2, such cc-calibrations do not realise the infimum of the comass norm in their cohomology classes. Here, we give examples for the dimensions $2 \leqslant k \leqslant \dim M - 2$. For the case $k = \dim M - 1$ see Remark 5·4. For $k = 1$ such examples do not exist by Proposition 3·4.

Example 6·7. On the torus $T^m = \mathbb{R}^m/\mathbb{Z}^m$ with the standard coordinates $(x_1, \ldots, x_m)$ mod $\mathbb{Z}^m$ we consider the closed k-form

$$\theta = dx_1 \wedge \cdots \wedge dx_k$$

for $2 \leqslant k \leqslant m - 2$. We choose functions $f, g \in C^\infty(T^m, \mathbb{R})$, such that

$$\left(\frac{\partial f}{\partial x_1}(x), \frac{\partial g}{\partial x_1}(x)\right) \neq (0, 0)$$

for all $x \in T^m$, e.g. $f(x) = \cos(2\pi x_1)$, $g(x) = \sin(2\pi x_1)$. Denoting the standard vector fields on T^m by $E_1, \ldots, E_m$, we define vector fields $X_1, \ldots, X_k$ by

$$X_1 = E_1, \ldots, X_{k-1} = E_{k-1}, X_k = E_k + fE_{m-1} + gE_m.$$

Since

$$[X_1, X_k] = \frac{\partial f}{\partial x_1} E_{m-1} + \frac{\partial g}{\partial x_1} E_m$$

and $k < m - 1$, we conclude that the oriented distribution $\mathcal{D}$ generated by $(X_1, \ldots, X_k)$ is nowhere infinitesimally integrable. Since θ is positive on $\mathcal{D}$, we can use Proposition 4·1 to obtain a Riemannian metric on T^m such that θ is a cc-calibration that calibrates precisely the distribution $\mathcal{D}$. Then θ does not calibrate any non-zero closed current of finite mass since such a current would be a closed distribution current for $\mathcal{D}$. This would contradict Proposition 6·4 and the fact that $\mathcal{D}$ is nowhere infinitesimally integrable.

Next we prove that a closed k-current T of finite mass can be represented by an oriented, measured lamination if its k-vectorfield $\vec{T}$ is the restriction to $\mathrm{spt}(T)$ of a simple k-vectorfield of class C^1. The proof relies on Proposition 2·7, Proposition 6·4 and on [**11**, lemma 1.14 and corollary 1.20]. We assume that M is closed only to be consistent with the notation introduced in Section 2. The result is local and holds on arbitrary manifolds for "closed distribution currents of locally finite mass".

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THEOREM 6·8. *Let $\mathcal{D}$ be an oriented C^1-distribution on a closed, oriented manifold, and let T be a closed distribution current for $\mathcal{D}$. Then T can be represented by an oriented, measured lamination whose leaves are integral manifolds of $\mathcal{D}$.*

Proof. Suppose X is a C^1-vectorfield tangent to $\mathcal{D}$, and let φ_t denote the flow of X. Then the closed distribution current T is invariant under φ_t, i.e.

$$(\varphi_t)_\# T = T.$$

This follows from [**11**, corollary 1·20] . Note that the integrability of $\mathcal{D}$ that is assumed in [**11**, corollary 1·20], is not used in the proof. In particular, we conclude that $\mathrm{spt}(T)$ is invariant under φ_t. Hence condition (b) in Proposition 2·7 is satisfied for $S = \mathrm{spt}(T)$. Proposition 6·4 implies that condition (a) in Proposition 2·7 is satisfied on $S = \mathrm{spt}(T)$ as well. So, from Proposition 2·7, we obtain an oriented C^1-lamination structure $\mathcal{L}$ on $\mathrm{spt}(T)$ the leaves of which are integral manifolds of $\mathcal{D}$ with the orientation induced from $\mathcal{D}$. Finally, we will construct an invariant measure μ for $\mathcal{L}$ so that $T = T(\mathcal{L}, \mu)$. According to the remark following Proposition 2·7, there is an oriented lamination atlas $\mathcal{A}$ of $\mathcal{L}$, the charts of which are restrictions to $S^\varphi = S \bigcap U^\varphi$ of C^1-charts $\varphi : U^\varphi \to D^k \times \mathbb{R}^{m-k}$ of M. For technical reasons, we will identify D^k with $\mathbb{R}^k$ via an orientation preserving diffeomorphism and assume that $\varphi : U^\varphi \to \mathbb{R}^k \times \mathbb{R}^{m-k}$ for $\varphi \in \mathcal{A}$. We choose a Riemannian metric on M, and obtain a unit k-vectorfield $\vec{T}$ and a Radon measure μ_T corresponding to T. For every $\varphi \in \mathcal{A}$ we have the Radon measure $\mu_T^\varphi = \varphi_\#(\mu_T)$ on $\mathbb{R}^k \times \mathbb{R}^{m-k}$ with support $\mathbb{R}^k \times T^\varphi$. If ω is a k-form on $\mathbb{R}^m$ with compact support and of class C^1, and $\varphi \in \mathcal{A}$, then

$$T(\varphi^*\omega) = \int_{\mathbb{R}^m} \langle \omega, \varphi_* \vec{T} \rangle d\mu_T^\varphi = \int_{\mathbb{R}^m} \langle \omega, \partial_1 \wedge \ldots \wedge \partial_k \rangle g^\varphi d\mu_T^\varphi, \tag{6·1}$$

where $g^\varphi = |\partial_1^\varphi \wedge \ldots \wedge \partial_k^\varphi|^{-1} \circ \varphi^{-1}$. Since T is closed we have $T(\varphi^* d\pi) = 0$ for every compactly supported $(k-1)$-form of class C^1 on $\mathbb{R}^m$. Using (6·1) we see that this implies

$$\int_{\mathbb{R}^m} \partial_i f \cdot g^\varphi d\mu_T^\varphi = 0 \tag{6·2}$$

for every $f \in C^1(\mathbb{R}^m, \mathbb{R})$ with compact support and every $i \in \{1, \ldots, k\}$.

According to [**11**, lemma 1·14] (see also Lemma 6·9 below), we obtain a Radon measure $\tilde{\mu}^\varphi$ on $\mathbb{R}^{m-k}$, so that

$$g^\varphi d\mu_T^\varphi = \mathcal{L}^k \otimes d\tilde{\mu}^\varphi, \tag{6·3}$$

where $\mathcal{L}^k$ denotes k-dimensional Lebesgue measure. Note that $\mathrm{spt}(\mu_T^\varphi) = \mathbb{R}^k \times T^\varphi$ and, hence, $\mathrm{spt}(\tilde{\mu}^\varphi) = T^\varphi$. The restriction μ^φ of $\tilde{\mu}^\varphi$ to T^φ will turn out to be the local representative of our invariant measure μ with $T = T(\mathcal{L}, \mu)$. We first prove a local version of this equality $T = T(\mathcal{L}, \mu)$. Let ω be a compactly supported k-form of class C^1 on $\mathbb{R}^m$. Using (6·1)-(6·3) we obtain

$$\begin{aligned} T(\varphi^*\omega) = \textstyle\int_{\mathbb{R}^m} \langle \omega, \partial_1, \wedge \ldots \wedge \partial_k \rangle g^\varphi d\mu_T^\varphi &= \textstyle\int_{\mathbb{R}^{m-k}} (\int_{\mathbb{R}^k \times \{y\}} \omega) d\tilde{\mu}^\varphi(y) \\ &= \textstyle\int_{T^\varphi} (\int_{\mathbb{R}^k \times \{y\}} \omega) d\mu^\varphi(y). \end{aligned}$$

This corresponds to equation (2·14). On the one hand, it proves the invariance property (2·13) for the collection of local measures μ^φ, $\varphi \in \mathcal{A}$, and, on the other hand, it implies that indeed $T = T(\mathcal{L}, \mu)$.

For completeness, we state and prove the fact used to prove (6·3) above.

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LEMMA 6·9. *Let μ be a Radon measure on $\mathbb{R}^m$ such that*

$$\int_{\mathbb{R}^m} \partial_i f \, d\mu = 0$$

holds for every $i \in \{1, \dots, k\}$ and every $f \in C_0^1(\mathbb{R}^m, \mathbb{R})$. Then there exists a unique Radon measure ν on $\mathbb{R}^{m-k}$ such that $\mu = \mathcal{L}^k \otimes \nu$.

Proof. For $v \in \mathbb{R}^k \times \{0\} \subseteq \mathbb{R}^m$ let $T_v : \mathbb{R}^m \to \mathbb{R}^m$ denote the translation

$$T_v(x) = x + v.$$

For $f \in C_0^1(\mathbb{R}^m, \mathbb{R})$, we define $g : \mathbb{R}^k \times \{0\} \to \mathbb{R}$ by

$$g(v) = \int_{\mathbb{R}^m} f \circ T_v \, d\mu.$$

Then g is of class C^1 and, by assumption, $(\partial_i g)(v) = \int_{\mathbb{R}^m} \partial_i (f \circ T_v) d\mu = 0$ for $i \in \{1, \dots, k\}$. Since $C_0^1(\mathbb{R}^m, \mathbb{R})$ is dense in $L^1(\mu)$, we conclude that $(T_v)_{\#}\mu = \mu$ for all $v \in \mathbb{R}^k \times \{0\}$. For every Borel set $B \subseteq \mathbb{R}^{m-k}$ we define a Borel measure μ_B on $\mathbb{R}^k$ by setting

$$\mu_B(C) := \mu(C \times B)$$

for every Borel set $C \subseteq \mathbb{R}^k$. Now $(T_v)_{\#}\mu = \mu$ for all $v \in \mathbb{R}^k \times \{0\}$ implies that μ_B is a translation invariant Radon measure on $\mathbb{R}^k$, hence a multiple of Lebesgue measure $\mathcal{L}^k$, say

$$\mu_B = r_B \mathcal{L}^k$$

for some number $r_B \geqslant 0$. If we define

$$\nu(B) = r_B,$$

then we have $\mu(C \times B) = \mathcal{L}^k(C)\nu(B)$ for all Borel sets $C \subseteq \mathbb{R}^k$, $B \subseteq \mathbb{R}^{m-k}$, and ν is easily seen to be a Radon measure. This implies $\mu = \mathcal{L}^k \otimes \nu$. Uniqueness of ν is obvious.

Proof of Theorem 6·1. Using Theorem 6·8 we can argue as in the proof of Theorem 5·1. If the comass norm of θ is minimal in the cohomology class of θ, then by Proposition 2·2 there exists a closed current $T \neq 0$ calibrated by θ, i.e. $\langle \theta(x), \vec{T}(x) \rangle = 1$ for μ_T-almost all $x \in \operatorname{spt}(T)$. The natural isomorphism between T^*M and TM induced by the Riemannian metric associates to θ a simple k-vectorfield ξ of class C^1. On the open set $\{x \in M | \theta(x) \neq 0\}$ the k-vectorfield ξ determines an oriented C^1-distribution $\mathcal{D}$. Moreover, the equality $\langle \theta(x), \vec{T}(x) \rangle = 1$ is easily seen to imply that $\vec{T}(x) = \xi(x)$. Hence T is a closed distribution current for $\mathcal{D}$. Hence, by Theorem 6·8, cf. also the sentence preceding Theorem 6·8, T can be represented as an oriented, measured lamination the leaves of which are calibrated by θ. For the converse note that if $(\mathcal{L}, \mu)$ is an oriented, measured lamination whose leaves are calibrated by θ then $T = T(\mathcal{L}, \mu)$ is a non-zero closed current calibrated by θ.

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