The Journal of Symbolic Logic
Volume 86, Number 1, March 2021

THE RELATION BETWEEN TWO DIMINISHED CHOICE PRINCIPLES

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Abstract. For every $n \in \omega \setminus \{0,1\}$ we introduce the following weak choice principle:

$\mathrm{nC}^-_{<\aleph_0}$: *For every infinite family $\mathcal{F}$ of finite sets of size at least n there is an infinite subfamily $\mathcal{G} \subseteq \mathcal{F}$ with a selection function $f : \mathcal{G} \to [\bigcup \mathcal{G}]^n$ such that $f(F) \in [F]^n$ for all $F \in \mathcal{G}$.*

Moreover, we consider the following choice principle:

KWF^- : *For every infinite family $\mathcal{F}$ of finite sets of size at least 2 there is an infinite subfamily $\mathcal{G} \subseteq \mathcal{F}$ with a Kinna–Wagner selection function. That is, there is a function $g : \mathcal{G} \to \mathcal{P}(\bigcup \mathcal{G})$ with $\emptyset \neq f(F) \subsetneq F$ for every $F \in \mathcal{G}$.*

We will discuss the relations between these two choice principles and their relations to other well-known weak choice principles. Moreover, we will discuss what happens when we replace $\mathcal{F}$ by a linearly ordered or a well-ordered family.

§1. Introduction and definitions. In the present work we investigate the relations between the following weak choice principles:

1.1. Known choice principles. In [1] and [2], De La Cruz and Di Prisco introduced several diminished versions of weak choice principles and studied the relations between them. Among them are:

- ACF^-: For every infinite family $\mathcal{F}$ of non-empty finite sets there is an infinite subfamily $\mathcal{G} \subseteq \mathcal{F}$ with a choice function.

For $n \in \omega \setminus \{0,1\}$ we define

- C^-_n: Every infinite family $\mathcal{F}$ of n-element sets has an infinite subfamily $\mathcal{G} \subseteq \mathcal{F}$ with a choice function.

In [9], Keremedis and Wajch introduced the following weak choice principle:

- KWF^-: For every infinite family $\mathcal{F}$ of finite sets of size at least 2 there is an infinite subfamily $\mathcal{G} \subseteq \mathcal{F}$ with a Kinna–Wagner selection function. That is, there is a function $g : \mathcal{G} \to \mathcal{P}(\bigcup \mathcal{G})$ with $\emptyset \neq f(F) \subsetneq F$ for every $F \in \mathcal{G}$.

1.2. New choice principle. For every set S let $[S]^n := \{T \subseteq S \mid |T| = n\}$ and $\mathrm{fin}(S) := \{T \subseteq S \mid |T| \text{ is finite}\}$. For all $n \in \omega \setminus \{0,1\}$ define

Received July 19, 2020.
2020 *Mathematics Subject Classification*. 03E25, 03E35.
Key words and phrases. weak forms of the axiom of choice, consistency results, Fraenkel–Mostowski permutation models of ZFA+ ¬AC, Pincus' transfer theorems, partial choice for infinite families of n-element sets, partial choice for infinite families of finite sets.

2021, Association for Symbolic Logic
0022-4812/21/8601-0019
DOI:10.1017/jsl.2021.20

https://doi.org/10.1017/jsl.2021.20 Published online by Cambridge University Press
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- $\mathrm{nC}^-_{<\aleph_0}$: For every infinite family $\mathcal{F}$ of finite sets of at least n there is an infinite subfamily $\mathcal{G} \subseteq \mathcal{F}$ with a selection function $f : \mathcal{G} \to [\bigcup \mathcal{G}]^n$ such that $f(F) \in [F]^n$ for all $F \in \mathcal{G}$.

Moreover, we will mention the following choice principles:

1.3. Other choice principles.

- $\mathrm{AC}(\mathrm{LO}, \mathrm{LO})$: Every linearly orderable family of non-empty linearly orderable sets has a choice function.
- $\mathrm{UT}(\mathrm{WO}, \mathrm{fin}, \mathrm{WO})$: The union of a well-orderable family of finite sets is well-orderable.

For all $n \in \omega \setminus \{0,1\}$ define

- C_n: Every family of n-element sets has a choice function.
- KW^-_n: For every infinite family of n-element sets there is an infinite subfamily $\mathcal{G} \subseteq \mathcal{F}$ with a Kinna–Wagner selection function. That is, there is a function $f : \mathcal{G} \to [\bigcup \mathcal{G}]^{<n}$ such that $\emptyset \neq f(F) \subsetneq F$ for all $F \in \mathcal{G}$ with $|F| \geq 2$.

We will add the prefix "LO" in order to indicate that we replace the family $\mathcal{F}$ by a linearly orderable family. And we use the prefix "WO" to indicate that we replace the family $\mathcal{F}$ by a well-orderable family. So for example for $n \in \omega \setminus \{0,1\}$

- $\mathrm{nLOC}^-_{<\aleph_0}$: For every infinite, linearly orderable family $\mathcal{F}$ of finite sets of size at least n there is an infinite subfamily $\mathcal{G} \subseteq \mathcal{F}$ with a selection function $f : \mathcal{G} \to [\bigcup \mathcal{G}]^n$ such that $f(F) \in [F]^n$ for all $F \in \mathcal{G}$.
- WOACF^-: For every infinite, well-orderable family $\mathcal{F}$ of non-empty finite sets there is an infinite subset $\mathcal{G} \subseteq \mathcal{F}$ with a choice function.

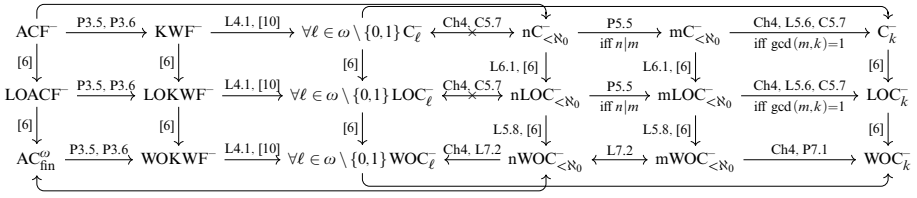
Note that WOACF^- is equivalent to the following well-known choice principle:

- $\mathrm{AC}^\omega_{\mathrm{fin}}$: Every countably infinite family $\mathcal{F}$ of non-empty finite sets has an infinite subfamily $\mathcal{G} \subseteq \mathcal{F}$ with a choice function.

Analogously, WOKWF^- is equivalent to

- Every countably infinite family $\mathcal{F}$ of non-empty finite sets has an infinite subfamily $\mathcal{G} \subseteq \mathcal{F}$ with a Kinna–Wagner selection function.

The diagram below indicates the relations between our weak forms of the Axiom of Choice. An arrow indicates that the implication holds in ZF. The labels of the implication arrows indicate where the proof can be found. For example the label Ch4, L5.6, C5.7 means Chapter 4, Lemma 5.6, and Corollary 5.7. The implications $\mathrm{LOACF}^- \Rightarrow \mathrm{nLOC}^-_{<\aleph_0}$ and $\forall l \in \omega \setminus \{0,1\}\, \mathrm{LOC}^-_n \Rightarrow \mathrm{LOC}^-_k$ also hold, however, to make the diagram easier to read, we omitted these arrows.



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In [6] Halbeisen and Tachtsis found permutation models

$$\mathcal{N}_n \models \mathrm{AC}(\mathrm{LO},\mathrm{LO}) \wedge \neg \mathrm{C}_n^-$$

and

$$\mathcal{V}_n \models \mathrm{UT}(\mathrm{WO},\mathrm{fin},\mathrm{WO}) \wedge \neg \mathrm{LOC}_n^-$$

for all $n \in \omega \setminus \{0,1\}$. (The definitions of these two permutation models can be found in Sections 6 and 5.) These results prove that the implications

$$\mathrm{ACF}^- \Rightarrow \mathrm{LOACF}^- \Rightarrow \mathrm{AC}^\omega_{\mathrm{fin}};$$

$$\mathrm{KWF}^- \Rightarrow \mathrm{LOKWF}^- \Rightarrow \mathrm{WOKWF}^-;$$

$$\forall \ell \in \omega \setminus \{0\}\, \mathrm{C}_\ell^- \Rightarrow \forall l \in \omega \setminus \{0\}\, \mathrm{LOC}_\ell^- \Rightarrow \forall \ell \in \omega \setminus \{0\}\, \mathrm{WOC}_\ell^- \text{ and}$$

$$\mathrm{C}_n^- \Rightarrow \mathrm{LOC}_n^- \Rightarrow \mathrm{WOC}_n^- \text{ for all } m \in \omega \setminus \{0,1\}$$

are all strict.

All independence results in this paper are obtained by using permutation models. See for example [3] for basics about them. These permutation models give us models of ZFA. Note that all weak choice principles we will consider in this paper are injectively boundable (see, e.g., [12] or [7, Note 203] for the definitions). For example, KWF^- is injectively boundable because

$\mathrm{KWF}^- \iff \forall x(|x|_- \leq \omega \rightarrow$ every infinite family of finite sets of size at least 2 whose union is equal to x has a Kinna–Wagner selection function.)

So by [12], all results can be transferred into ZF.

§2. The basic form of the permutation models. In this section we introduce the basic form all permutation models in this paper will have (Figure 1).

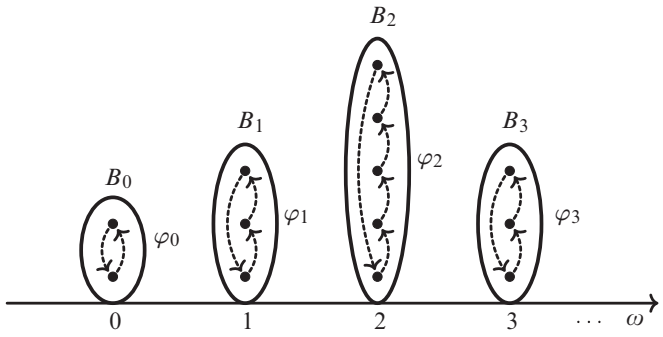


FIGURE 1. A permutation model with $\ell_0 = 2, \ell_1 = 3, \ell_2 = 5, \ell_3 = 3$, and $\mathbb{B} = \omega$.

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Let $\mathbb{B} \in \{\omega, \mathbb{Q}\}$ and let $\mathbb{L} := \langle \ell_i \mid i \in \mathbb{B} \rangle$ be a sequence of natural numbers. For all $n \in \mathbb{B}$ define a set

$$B_n := \{b_0^n, b_1^n, \ldots, b_{\ell_n - 1}^n\}.$$

We call these sets blocks and assume that they are pairwise disjoint. We start with a ground model $\mathcal{M} \models \mathrm{ZFA} + \mathrm{AC}$ with a set of atoms

$$\mathcal{A} := \bigcup_{n \in \mathbb{B}} B_n.$$

For all $n \in \mathbb{B}$ let

$$\varphi_n \colon \mathcal{A} \to \mathcal{A}$$
$$b_i^k \mapsto \begin{cases} b^n_{(i+1 \mod \ell_n)} & \text{if } k = n; \\ b_i^k & \text{otherwise.} \end{cases}$$

In other words, φ_n permutes the block B_n cyclically and fixes all atoms outside of B_n. Let G be a group of permutations of $\mathcal{A}$ and let $\mathcal{I} \subseteq \mathcal{P}(\mathcal{A})$ be a normal ideal. Let the normal filter on G be generated by the subgroups

$$\mathrm{fix}_G(E) := \{\varphi \in G \mid \forall a \in E \ (\varphi(a) = a)\},$$

where $E \in \mathcal{I}$. Let $\mathcal{V}(\mathbb{B}, \mathbb{L}, G, \mathcal{I})$ be the class of all hereditarily symmetric sets in $\mathcal{M}$. Then $\mathcal{V}(\mathbb{B}, \mathbb{L}, G, \mathcal{I})$ is a permutation model.

§3. A new permutation model.

3.1. The model. Let $\mathbb{B} := \omega$ and $\mathbb{L} := \langle p_i \mid i \in \omega \rangle$, where p_i is the ith prime number. Now we define the group G of permutations of $\mathcal{A}$ by requiring

$$\varphi \in G \text{ if and only if } \varphi = \prod_{k \in \omega} \varphi_k^{\alpha_k},$$

with $\alpha_k \in \omega$ and $\alpha_k = 0$ for all but finitely many $k \in \omega$. Note that G is a commutative group. Let

$$\mathcal{I} := \left\{ E \subseteq \mathcal{A} \mid \lim_{n \to \infty} \frac{|\{k \in \omega \mid B_k \cap E \neq \emptyset\} \cap n|}{n} = 0 \right\}.$$

In other words, E is in $\mathcal{I}$ iff the set of all indices of blocks E intersects non-trivially is a set of density zero in ω.

LEMMA 3.1. *$\mathcal{I}$ is a normal ideal.*

PROOF. Let $E, F \in \mathcal{I}$. Then also $E \cup F \in \mathcal{I}$ because

$$\lim_{n \to \infty} \frac{|\{k \in \omega \mid B_k \cap (E \cup F) \neq \emptyset\} \cap n|}{n}$$
$$\leq \lim_{n \to \infty} \frac{|\{k \in \omega \mid B_k \cap E \neq \emptyset\} \cap n|}{n} + \frac{|\{k \in \omega \mid B_k \cap F \neq \emptyset\} \cap n|}{n} = 0.$$

Moreover, for every $\varphi \in G$ and all $E \in \mathcal{I}$ we have that $\varphi(E) \in \mathcal{I}$ because

$$\{k \in \omega \mid B_k \cap E \neq \emptyset\} = \{k \in \omega \mid B_k \cap \varphi(E) \neq \emptyset\}.$$

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Furthermore we have that

- $\emptyset \in \mathcal{I}$,
- if $E \in \mathcal{I}$ and $F \subseteq E$, then $F \in \mathcal{I}$ and
- $\{a\} \in \mathcal{I}$ for all $a \in \mathcal{A}$. ⊣

Define $\mathcal{V}_0 := \mathcal{V}(\mathbb{B}, \mathbb{L}, G, \mathcal{I})$.

3.2. The relation between ACF⁻ and KWF⁻. Since every choice function is also a Kinna–Wagner selection function, ACF⁻ implies KWF⁻ in ZF. In this section we will show that the opposite implication cannot be proven in ZF. Note that this result only holds because to show that KWF⁻ holds for an infinite family $\mathcal{F}$ we only have to find a Kinna–Wagner selection function on an infinite subfamily $\mathcal{G} \subseteq \mathcal{F}$ and not on the whole family $\mathcal{F}$. This is crucial. In fact, ACF and KWF are equivalent in ZF (see [7, Note 70]).

DEFINITION 3.2. A set $E \subseteq \mathcal{A}$ is called closed iff for every $n \in \omega$

$$B_n \cap E \neq \emptyset \Rightarrow B_n \subseteq E.$$

REMARK 3.3. The set of all closed supports can be linearly ordered. The ordering is induced by the ordering on the blocks. The ordering is denoted by "$\preccurlyeq$".

LEMMA 3.4. *Let $x \in \mathcal{V}_0$ and let*

$$\mathcal{E} := \{E \in \mathcal{I} \mid E \text{ is a closed support of } x\}.$$

Then $\bigcap \mathcal{E}$ is a closed support of x. Indeed, $\bigcap \mathcal{E}$ is the minimal closed support of x.

PROOF. Note that $\bigcap \mathcal{E}$ is a closed set. Now we will show that $\bigcap \mathcal{E}$ is a support of x. Let $\varphi \in \mathrm{fix}_G(\bigcap \mathcal{E})$. Without loss of generality we can assume that $\varphi = \varphi_{n_0}^{\alpha_{n_0}} \circ \varphi_{n_1}^{\alpha_{n_1}} \circ \cdots \circ \varphi_{n_{\ell-1}}^{\alpha_{n_{\ell-1}}}$ with $\varphi_{n_i} \in \mathrm{fix}_G(\bigcap \mathcal{E})$ for all $i < \ell$, $\alpha_{n_i} \in \{1, 2, \ldots, p_{n_i} - 1\}$ for all $i < \ell$ and pairwise different $n_0, n_1, \ldots, n_{\ell-1} \in \omega$. For every $i < \ell$ there is an $E \in \mathcal{E}$ such that

$$\varphi_{n_i} \in \mathrm{fix}_G(E).$$

Since E is a support of x we have that $\varphi_{n_i} \in \mathrm{sym}_G(x)$, i.e., $\varphi_{n_i}(x) = x$. This proves that $\varphi \in \mathrm{sym}_G(x)$. Therefore, $\bigcap \mathcal{E}$ is a support of x. Moreover, $\bigcap \mathcal{E}$ is the minimal support of x by definition of $\mathcal{E}$. ⊣

PROPOSITION 3.5. *We have that $\mathcal{V}_0 \models \mathrm{KWF}^-$.*

PROOF. Let $\mathcal{F} \in \mathcal{V}_0$ be an infinite family of pairwise disjoint finite sets of size at least 2 with minimal closed support $E_{\mathcal{F}}$. For every $x \in \bigcup \mathcal{F}$ the orbit of x is defined by

$$\mathrm{orb}(x) := \{\varphi(x) \mid \varphi \in \mathrm{fix}_G(E_{\mathcal{F}})\} \subseteq \bigcup \mathcal{F}.$$

Note that for all $x, x' \in \bigcup \mathcal{F}$ we have that

$$\mathrm{orb}(x) = \mathrm{orb}(x') \text{ or } \mathrm{orb}(x) \cap \mathrm{orb}(x') = \emptyset.$$

Since $E_{\mathcal{F}}$ is a support of every $\mathrm{orb}(x)$, the set of all orbits

$$\mathcal{O} := \left\{\mathrm{orb}(x) \mid x \in \bigcup \mathcal{F}\right\}$$

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can be well-ordered in $\mathcal{V}_0$. Let $Y \in \mathcal{F}$. Our goal is to choose a proper subset of Y. Let $\mathrm{orb}(x)$ be smallest possible with respect to the well-ordering on $\mathcal{O}$ such that $Y \cap \mathrm{orb}(x) \neq \emptyset$. If $Y \cap \mathrm{orb}(x) \neq Y$ we are done. So assume that $Y \cap \mathrm{orb}(x) = Y$. Moreover, we can assume that each $y \in Y$ has the same minimal closed support E_Y because otherwise we could choose the elements in Y with the "$\preccurlyeq$"-smallest closed support. Let $\Lambda_Y \subseteq \omega$ with

$$E_Y \setminus E_{\mathcal{F}} = \bigcup_{k \in \Lambda_Y} B_k.$$

If there is an infinite subfamily $\mathcal{F}' \subseteq \mathcal{F}$ with $E_Y \setminus E_{\mathcal{F}} = \emptyset$ for all $Y \in \mathcal{F}'$, the set $\bigcup \mathcal{F}'$ is well-orderable and we are done. So without loss of generality we can assume that $E_Y \setminus E_{\mathcal{F}} \neq \emptyset$ for all $Y \in \mathcal{F}$. For every $Y \in \mathcal{F}$ define

$$\tilde{G}_Y := \left\{ \prod_{k \in N} \varphi_k^{\alpha_k} \in G \mid N \in \mathrm{fin}(\Lambda_Y) \text{ and } \alpha_k \in \{0, 1, \ldots, p_k - 1\} \text{ for all } k \in N \right\}.$$

In other words, $\tilde{G}_Y$ is the subgroup of G which permutes only the atoms in $\bigcup_{k \in \Lambda_Y} B_k = E_Y \setminus E_{\mathcal{F}}$. For all $\varphi \in \tilde{G}_Y$ and all $k \in \Lambda_Y$ let

$$\varphi|_k := \alpha_k.$$

The elements in $\tilde{G}_Y$ can be ordered lexicographically (induced by the ordering on the blocks and the exponents of the φ_k, $k \in \Lambda_Y$). The ordering on $\tilde{G}_Y$ is denoted by $\leq_{\tilde{G}_Y}$. For all $y, y' \in Y$ and all $k \in \Lambda_Y$ let

$$\mathrm{dist}_k(\langle y, y' \rangle) := \varphi|_k,$$

where φ is the $\leq_{\tilde{G}_Y}$-smallest function in $\tilde{G}_Y$ with $\varphi(y) = y'$. For every $Y \in \mathcal{F}$ let

$$k_Y := \min\{k \in \Lambda_Y \mid \exists y, y' \in Y \ (\mathrm{dist}_k(\langle y, y' \rangle) \neq 0)\}.$$

Let $A := \bigcup \{B_{k_Y} \mid Y \in \mathcal{F}\}$. If A is a finite set, let $B := A$ and $\mathcal{G} := \mathcal{F}$. Otherwise there is an infinite subset $\mathcal{H} \subseteq \mathcal{F}$ such that $B := \bigcup \{B_{k_Y} \mid Y \in \mathcal{H}\} \in \mathcal{I}$ and we define $\mathcal{G} := \{Y \in \mathcal{F} \mid B_{k_Y} \subseteq B\}$. Note that $\mathcal{G} \in \mathcal{V}_0$ with support $E_{\mathcal{F}} \cup B$. For every $Y \in \mathcal{G}$ define

$$[Y] := \{\varphi(Y) \mid \varphi \in \mathrm{fix}_G(E_{\mathcal{F}} \cup B)\} \subseteq \mathcal{G}$$

and choose (in the ground model) a $Y_0 \in [Y]$ and a $y_0 \in Y_0$. Let

$$Z_{Y_0} := \{y \in Y_0 \mid \mathrm{dist}_{k_{Y_0}}(\langle y_0, y \rangle) = 0\} \subsetneq Y_0,$$

and for every $\varphi \in \mathrm{fix}_G(E_{\mathcal{F}} \cup B)$ let

$$Z_{\varphi(Y)} := \varphi(Z_Y) \subsetneq \varphi(Y).$$

Our goal is to show that

$$g : \mathcal{G} \to \mathcal{P}\left(\bigcup \mathcal{G}\right)$$
$$Y \mapsto Z_Y$$

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is a well-defined function in $\mathcal{V}_0$ with support $E_{\mathcal{F}} \cup B$. So we have to prove the following claim:

CLAIM. *Assume that there are* $\varphi,\varphi' \in \mathrm{fix}_G(E_{\mathcal{F}} \cup B)$ *with* $\varphi(Y_0) = \varphi'(Y_0)$. *Then we have that* $\varphi(Z_{Y_0}) = \varphi'(Z_{Y_0})$.

PROOF OF THE CLAIM. Assume that $\varphi,\varphi' \in \mathrm{fix}_G(E_{\mathcal{F}} \cup B)$ with

$$\varphi(Y_0) = \varphi'(Y_0),$$

but $\varphi(Z_{Y_0}) \neq \varphi'(Z_{Y_0})$. So there is a $y_0 \in Z_{Y_0}$ with $\varphi'(y_0) \notin \varphi(Z_{Y_0})$. Therefore,

$$y_1 := \varphi^{-1}\varphi'(y_0) \notin Z_{Y_0}$$

and $\varphi^{-1}\varphi'|_{k_{Y_0}} = 0$ because φ and φ' are both in $\mathrm{fix}_G(E_{\mathcal{F}} \cup B)$. Since $y_1 \notin Z_{Y_0}$, the $\leq_{\tilde{G}_{Y_0}}$-smallest function $\psi \in \tilde{G}_{Y_0}$ with

$$\psi(y_0) = y_1$$

satisfies $\psi|_{k_{Y_0}} \neq 0$. Define $\gamma := \psi^{-1}\varphi^{-1}\varphi'$ and let

$$L := \{\ell \in \omega \setminus \{k_{Y_0}\} \mid \gamma|_\ell \neq 0\} \in \mathrm{fin}(\omega).$$

By the Chinese Remainder Theorem there is an $\alpha \in \omega$ such that

$$\begin{aligned} &\alpha \equiv 0 \mod p_\ell \text{ for all } \ell \in L \text{ and} \\ &\alpha \equiv 1 \mod p_{k_{Y_0}}. \end{aligned}$$

So

$$y_0 = \gamma(y_0) = \gamma^\alpha(y_0) = \varphi_{k_{Y_0}}^{\psi|_{k_{Y_0}}}(y_0) \;\Rightarrow\; \varphi_{k_{Y_0}}(y_0) = y_0.$$

So $E_{Y_0} \setminus B_{k_{Y_0}}$ is also a support of y_0. Indeed, if $\eta \in \mathrm{fix}_G(E_{Y_0} \setminus B_{k_{Y_0}})$ we have that

$$\eta(y_0) = \varphi_{k_{Y_0}}^{\eta|_{k_{Y_0}}}(y_0) = y_0.$$

This is a contradiction to the fact that E_{Y_0} is the minimal closed support of y_0. $\dashv_{\text{Claim}}$

This claim shows that the Kinna–Wagner selection function g is in the model $\mathcal{V}_0$ and therefore KWF^- holds. $\dashv$

PROPOSITION 3.6. *We have that*

$$\mathcal{V}_0 \models \neg \mathrm{AC}^\omega_{\mathrm{fin}}.$$

PROOF. For every $n \in \omega$ let

$$F_n := \Big\{E \subseteq \bigcup_{k \leq n} B_k \mid \forall k \leq n\ (|E \cap B_k| = 1)\Big\}$$

and

$$\mathcal{F} := \{F_n \mid n \in \omega\}.$$

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This is a well-orderable family of finite sets in $\mathcal{V}_0$ with support $\emptyset$. Assume towards a contradiction that there is an infinite subfamily $\mathcal{G} \subseteq \mathcal{F}$ with a choice function

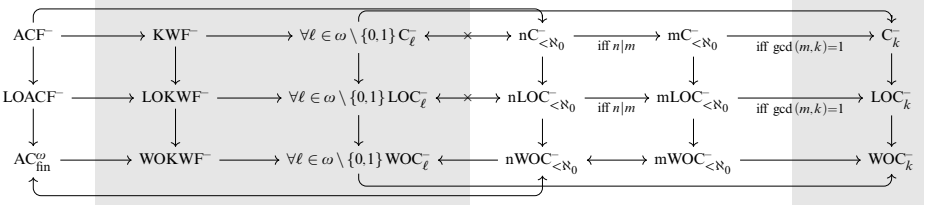
$$g : \mathcal{G} \to \bigcup \mathcal{G}.$$

Let $E_g \in \mathcal{I}$ be a support of g. Let $k_0 \in \omega$ with $B_{k_0} \cap E_g = \emptyset$. Since $\mathcal{G}$ is an infinite set there is an $n_0 > k_0$ with $F_{n_0} \in \mathcal{G}$. But then we have that

$$\varphi_{k_0}(g(F_{n_0})) \neq g(F_{n_0}) = g(\varphi_{k_0}(F_{n_0})).$$

So E_g is not a support of g and therefore, $g \notin \mathcal{V}_0$. ⊣

3.3. Summary. In the diagram below, all weak choice principles with a gray background hold in $\mathcal{V}_0$ and all weak choice principles with a white background do not hold in $\mathcal{V}_0$.



§4. Second permutation model.

4.1. The model. Let $\mathbb{B} := \omega$ and $\mathbb{L} := \langle p_i \mid i \in \omega \rangle$, where p_i is the ith prime number. Now we define the group G of permutations of $\mathcal{A}$ by requiring

$$\varphi \in G \text{ if and only if } \varphi = \prod_{k \in \omega} \varphi_k^{\alpha_k}$$

with $\alpha_k \in \omega$ and $\alpha_k = 0$ for all but finitely many $k \in \omega$. Let $\mathcal{I} := \mathrm{fin}(\mathcal{A})$. Define

$$\mathcal{V}_{\mathrm{fin}} := \mathcal{V}(\mathbb{B}, \mathbb{L}, G, \mathcal{I}).$$

4.2. The relation between $\forall \ell \in \omega\, \mathrm{C}^-_\ell$ and KWF⁻. By applying KWF⁻ several times to a family of n-element sets we can prove that KWF⁻ implies $\forall \ell \in \omega \setminus \{0\}\ \mathrm{C}^-_\ell$. Moreover, Lévy proved in [10] that

$$\mathcal{V}_{\mathrm{fin}} \models \forall \ell \in \omega \setminus \{0\}\, \mathrm{C}_\ell\,.$$

So it remains to prove that $\mathcal{V}_{\mathrm{fin}} \models \neg \mathrm{KWF}^-$.

LEMMA 4.1. *We have that* $\mathcal{V}_{\mathrm{fin}} \models \neg \mathrm{WOKWF}^-$.

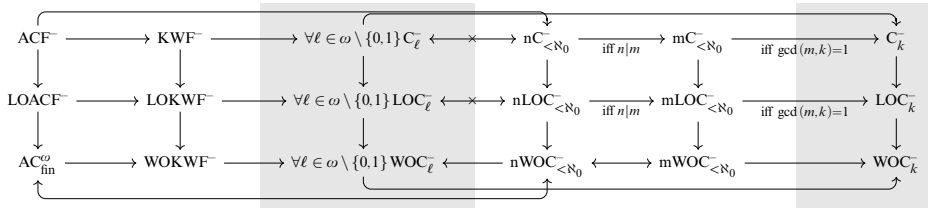
PROOF. Let $\mathcal{F} := \{B_n \mid n \in \omega\}$ be the set of all blocks. Assume towards a contradiction that there is an infinite subset $\mathcal{G} \subseteq \mathcal{F}$ with a Kinna–Wagner selection function $g : \mathcal{G} \to \mathcal{P}(\bigcup \mathcal{G})$. Let $E_g \in \mathrm{fin}(\mathcal{A})$ be a support of g. Let $n_0 \in \omega$ such that $B_{n_0} \in \mathcal{G}$ and $E_g \cap B_{n_0} \neq \emptyset$. Then we have that

$$\varphi_{n_0}(g(B_{n_0})) \neq g(B_{n_0}) = g(\varphi_{n_0}(B_{n_0})).$$

So E_g is not a support of g. This is a contradiction and therefore, $\mathcal{V}_{\mathrm{fin}} \models \neg \mathrm{WOKWF}^-$. ⊣

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4.3. Summary. In the diagram below, all weak choice principles with a gray background hold in $\mathcal{V}_{\text{fin}}$ and all weak choice principles with a white background do not hold in $\mathcal{V}_{\text{fin}}$. Note that $\mathcal{V}_{\text{fin}} \models \neg \text{nWOC}^-_{<\aleph_0}$ because $\text{AC}^\omega_{\text{fin}}$ holds iff $\text{nWOC}^-_{<\aleph_0}$ holds.



§5. Third permutation model.

5.1. The models. Let $n \in \omega$ and let $\mathbb{B} := \mathbb{Q}$. For all $i \in \mathbb{Q}$ let $n_i := n$ and define $\mathbb{L} := \langle n_i \mid i \in \mathbb{Q} \rangle$. For all order preserving bijections $\psi : \mathbb{Q} \to \mathbb{Q}$ we define

$$\begin{aligned} \rho_\psi : \mathcal{A} &\to \mathcal{A} \\ b^k_j &\mapsto b^{\psi(k)}_j. \end{aligned}$$

Now we define the group G by requiring

$$\alpha \in G \iff \alpha = \rho_\psi \varphi,$$

for an order preserving bijection $\psi : \mathbb{Q} \to \mathbb{Q}$ and $\varphi = \prod_{i \in \mathbb{Q}} \varphi_i^{k_i}$ with $k_i \in \omega$ and $k_i = 0$ for all but finitely many $i \in \mathbb{Q}$. Let $\mathcal{I} := \text{fin}(\mathcal{A})$. Define

$$\mathcal{V}_n := \mathcal{V}(\mathbb{B}, \mathbb{L}, G, \mathcal{I}).$$

5.2. The relation between $\text{nC}^-_{<\aleph_0}$ and $\text{mC}^-_{<\aleph_0}$.

REMARK 5.1. Let $\alpha \in G$, let $i_0 \in \mathbb{Q}$, and let $B_{j_0} := \alpha(B_{i_0})$. Then we have that

$$\alpha \varphi_{i_0} = \varphi_{j_0} \alpha.$$

LEMMA 5.2. *Let $k \in \omega \setminus \{0\}$ and let p be a prime number. Then we have that*

$$\mathcal{V}_{p^k} \models \{\text{p}^\ell \mid 0 \leq \ell \leq \text{k}\}\text{C}^-_{<\aleph_0},$$

where $\{\text{p}^\ell \mid 0 \leq \ell \leq \text{k}\}\text{C}^-_{<\aleph_0}$:= *For every infinite family $\mathcal{F}$ of non-empty finite sets there is an infinite subset $\mathcal{G} \subseteq \mathcal{F}$ with a selection function $f : \mathcal{G} \to \mathcal{P}(\bigcup \mathcal{G})$ such that $|f(F)| \in \{p^\ell \mid 0 \leq \ell \leq k\}$ for all $F \in \mathcal{G}$.*

PROOF. In the first part of this proof we will follow [6, Proof of Fact 4]. Let $\mathcal{F} \in \mathcal{V}_{p^k}$ be an infinite family of pairwise disjoint non-empty finite sets. Let $E_\mathcal{F} = \bigcup\{B_i \mid i \in I\} \in \mathcal{I}$ be a support of $\mathcal{F}$. If $\bigcup \mathcal{F}$ is well-orderable in $\mathcal{V}_{p^k}$ we are done. So assume that $\bigcup \mathcal{F}$ is non-well-orderable. Therefore, there exists a $z \in \bigcup \mathcal{F}$ which is not supported by $E_\mathcal{F}$. Let $E_z = \bigcup\{B_i \mid i \in I'\}$ be a support of z. We assume that $E_\mathcal{F} \subsetneq E_z$ and that $\{B_i \mid i \in I'\}$ has the fewest possible blocks B_j outside of $E_\mathcal{F}$. In other words, $\bigcup\{B_i \mid i \in I' \setminus \{j\}\}$ is not a support of z for all $j \in I' \setminus I$. As in [6, Proof of Fact 4] we can show that there is a linearly orderable partition $\mathcal{Y}$ of an infinite subset $y \subseteq \bigcup \mathcal{F}$.

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Let "$\preccurlyeq$" be the linear order on $\mathcal{Y}$. Moreover, $\mathcal{Y} = \{U_\alpha \mid \alpha \in \mathrm{fix}_G(E_z \setminus B_{i_0})\}$, where $i_0 := \max(I' \setminus I)$ and $U_\alpha \subseteq \{\alpha\varphi_{i_0}^j(z) \mid j < p^k\}$ for all $\alpha \in \mathrm{fix}_G(E_z \setminus B_{i_0})$. Note that by [6, Proof of Fact 4] for all $\alpha_0, \alpha_1 \in \mathrm{fix}_G(E_z \setminus B_{i_0})$ with $U_{\alpha_0} = U_{\alpha_1}$ we have that $\alpha_0(B_{i_0}) = \alpha_1(B_{i_0})$ and $|U_\alpha|$ divides p^k.

So far we followed [6, Proof of Fact 4]. Now we have to continue differently. Let

$$\mathcal{G} := \{F \in \mathcal{F} \mid y \cap F \neq \emptyset\}.$$

Let $F \in \mathcal{G}$ and let $U_\alpha \in \mathcal{Y}$ be the "$\preccurlyeq$"-least element with $F \cap U_\alpha \neq \emptyset$. Let $F_{\mathcal{Y}} := F \cap U_\alpha$. For all $y_0, y_1 \in F_{\mathcal{Y}}$ define the distance from y_0 to y_1

$$\mathrm{dist}(\langle y_0, y_1\rangle) := \min\{\ell \in \omega \mid \alpha\varphi_{i_0}^\ell\alpha^{-1}(y_0) = y_1\} > 0.$$

In other words, $\mathrm{dist}(\langle y_0, y_1\rangle)$ is the least natural number ℓ such that if $y_0 = \alpha\varphi_{i_0}^m(z)$, then $y_1 = \alpha\varphi_{i_0}^{m+\ell}(z)$.

Claim. *The distance* $\mathrm{dist}(\langle y_0, y_1\rangle)$ *is well-defined.*

Proof of the Claim. Assume towards a contradiction that there are α_0, α_1 with $U_{\alpha_0} = U_{\alpha_1}$ and $y_0, y_1 \in U_{\alpha_0} = U_{\alpha_1}$ such that

$$\mathrm{dist}_{\alpha_0}(\langle y_0, y_1\rangle) \neq \mathrm{dist}_{\alpha_1}(\langle y_0, y_1\rangle),$$

where

$$\mathrm{dist}_{\alpha_k}(\langle y_0, y_1\rangle) := \min\{\ell \in \omega \mid \alpha_k\varphi_{i_0}^\ell\alpha_k^{-1}(y_0) = y_1\}$$

for $k \in \{0,1\}$. Let $d_0 := \mathrm{dist}_{\alpha_0}(\langle y_0, y_1\rangle)$ and $d_1 := \mathrm{dist}_{\alpha_1}(\langle y_0, y_1\rangle)$. Let $\ell_0, \ell_1 \in \omega$ with $y_0 = \alpha_0\varphi_{i_0}^{\ell_0}(z) = \alpha_1\varphi_{i_0}^{\ell_1}(z)$. Without loss of generality we assume that $\ell_0 < \ell_1$. By replacing z by $\varphi_{i_0}^{\ell_0}(z)$ if necessary, we can assume that

$$y_0 = \alpha_0(z) = \alpha_1\varphi_{i_0}^\ell(z) \tag{1}$$

for an $\ell \in \omega$. Moreover we have that

$$y_1 = \alpha_0\varphi_{i_0}^{d_0}(z) = \alpha_1\varphi_{i_0}^{\ell+d_1}(z). \tag{2}$$

We assume that $d_1 > d_0$. The case when $d_0 > d_1$ can be treated similarly. By (1) we have that

$$\alpha_1^{-1}\alpha_0(z) = \varphi_{i_0}^\ell(z). \tag{3}$$

As mentioned above we have that $\alpha_0(B_{i_0}) = \alpha_1(B_{i_0}) =: B_{j_0}$ because $U_{\alpha_0} = U_{\alpha_1}$. So by (2) and Remark 5.1 we have that

$$\varphi_{j_0}^{d_0}\alpha_0(z) = \alpha_0\varphi_{i_0}^{d_0}(z) = \alpha_1\varphi_{i_0}^{\ell+d_1}(z) = \varphi_{j_0}^{d_1}\alpha_1\varphi_{i_0}^\ell(z) \Rightarrow \alpha_0(z) = \varphi_{j_0}^{d_1-d_0}\alpha_1\varphi_{i_0}^\ell(z).$$

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With (3) we get that

$$\alpha_0(z) = \varphi_{j_0}^{d_1-d_0}\alpha_1\alpha_1^{-1}\alpha_0(z) = \varphi_{j_0}^{d_1-d_0}\alpha_0(z) = \alpha_0\varphi_{i_0}^{d_1-d_0}(z).$$

So $z = \varphi_{i_0}^{d_1-d_0}(z)$. This is a contradiction to the minimality of d_1. $\dashv_{\text{Claim}}$

Let $F \in \mathcal{G}$. If $|F_{\mathcal{Y}}| \in \{p^\ell \mid 0 \leq \ell \leq k\}$ we are done. Otherwise let

$$d_F := \min\{\text{dist}(\langle y_0, y_1\rangle) \mid y_0, y_1 \in F_{\mathcal{Y}}\}$$

and define

$$F_{\mathcal{Y},1} := \{y \in F_{\mathcal{Y}} \mid \exists y_1 \in F_{\mathcal{Y}}\ (\text{dist}(\langle y, y_1\rangle) = d_F)\}.$$

Note that $F_{\mathcal{Y},1} \subsetneq F_{\mathcal{Y}}$ because $|F_{\mathcal{Y}}| \neq p^\ell$ for each $0 \leq \ell \leq k$. If $F_{\mathcal{Y},1} \in \{p^\ell \mid 0 \leq \ell \leq k\}$ we are done. Otherwise we repeat this procedure with $F_{\mathcal{Y},1}$ instead of $F_{\mathcal{Y}}$. After at most p^k iterations we have chosen a subset $H \subseteq F$ from every $F \in \mathcal{G}$ with

$$|H| \in \{p^\ell \mid 0 \leq \ell \leq k\}. \qquad \dashv$$

PROPOSITION 5.3. *Let p be a prime number and let $k \in \omega \setminus \{0\}$. Then we have that*

$$\mathcal{V}_{p^k} \models \forall \ell \in \omega \setminus \{0\}\, \ell \mathrm{p}^{\mathrm{k}}\mathrm{C}^{-}_{<\aleph_0}.$$

PROOF. Since $\mathrm{p}^{\mathrm{k}}\mathrm{C}^{-}_{<\aleph_0} \Rightarrow \ell\mathrm{p}^{\mathrm{k}}\mathrm{C}^{-}_{<\aleph_0}$ for all $\ell \in \omega \setminus \{0\}$ it suffices to show that

$$\mathcal{V}_{p^k} \models \mathrm{p}^{\mathrm{k}}\mathrm{C}^{-}_{<\aleph_0}.$$

Let $\mathcal{F} \in \mathcal{V}_{p^k}$ be an infinite family of finite sets of size at least p^k. By Lemma 5.2 we have that

$$\mathcal{V}_{p^k} \models \{\mathrm{p}^\ell \mid 0 \leq \ell \leq \mathrm{k}\}\mathrm{C}^{-}_{<\aleph_0}.$$

We iteratively apply $\{\mathrm{p}^\ell \mid 0 \leq \ell \leq \mathrm{k}\}\mathrm{C}^{-}_{<\aleph_0}$ to the set $\mathcal{F}$ until we find an infinite subset $\mathcal{G} \subseteq \mathcal{F}$ such that for each $F \in \mathcal{G}$ there are an $n_F \in \omega$ and pairwise disjoint sets

$$F_0, F_1, \ldots, F_{n_F-1} \in \mathcal{P}(F)$$

with $\left|\bigcup_{i<n_F} F_i\right| \geq p^k$ and $|F_i| \in \{p^\ell \mid 0 \leq \ell \leq k\}$ for each $i < n_F$. Choose the least subset $\alpha \subseteq n_F$ with respect to the lexicographical ordering on $\mathcal{P}(\{0,1,\ldots,n_F-1\})$ such that

$$\left|\bigcup_{\beta\in\alpha} F_\beta\right| = p^k.$$

So we can indeed choose exactly p^k elements from each $F \in \mathcal{G}$. $\dashv$

LEMMA 5.4. *Let $m \in \omega \setminus \{0,1\}$, let p be a prime number, and let $k \in \omega \setminus \{0\}$ with $p^k \nmid m$. Then we have that*

$$\mathcal{V}_{p^k} \models \neg\mathrm{mLOC}^{-}_{<\aleph_0}.$$

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PROOF. Let $\ell \in \omega \setminus \{0\}$ be minimal such that $p^{\ell k} > m$. Let

$$\mathcal{F} := \left\{ \prod_{t=0}^{\ell-1} B_{i_t} \mid i_0, i_1, \ldots, i_{\ell-1} \in \mathbb{Q} \text{ are pairwise different} \right\} \in \mathcal{V}_{p^k}.$$

This is an infinite, linearly orderable family of finite sets. Assume towards a contradiction that there is an infinite subset $\mathcal{G} \subseteq \mathcal{F}$ with a selection function

$$f : \mathcal{G} \to \left[\bigcup \mathcal{G}\right]^m$$

in $\mathcal{V}_{p^k}$ such that $f(F) \in [F]^m$ for every $F \in \mathcal{G}$. Let $E_f \in \mathrm{fin}(\mathcal{A})$ be a support of f. There is a set $F_0 = \prod_{t=0}^{\ell-1} B_{i_t} \in \mathcal{G}$ such that $B_{i_{t_0}} \cap E_f = \emptyset$ for a $t_0 \in \{0, \ldots, \ell - 1\}$. Let

$$F_0^x := \{z \in f(F_0) \mid z(i_{t_0}) = x\}$$

for all $x \in B_{i_{t_0}}$. Since $p^k \nmid m$ there are $x_0, x_1 \in B_{i_{t_0}}$ with $|F_0^{x_0}| \neq |F_0^{x_1}|$. Therefore we have that

$$f(\varphi_{i_{t_0}}(F_0)) = f(F_0) \neq \varphi_{i_{t_0}}(f(F_0)).$$

So $E_{\mathcal{F}}$ is not a support of f. This is a contradiction. ⊣

PROPOSITION 5.5. *Let $m, n \in \omega \setminus \{0,1\}$. Then we have that*

$$\mathrm{nC}^-_{<\aleph_0} \Rightarrow \mathrm{mC}^-_{<\aleph_0} \quad \textit{and} \quad \mathrm{nLOC}^-_{<\aleph_0} \Rightarrow \mathrm{mLOC}^-_{<\aleph_0}$$

if and only if $n \mid m$.

PROOF. First of all assume that $n|m$. Let $\mathcal{F}$ be a family of sets of size at least m. We apply $\mathrm{nC}^-_{<\aleph_0}$ exactly $\frac{m}{n}$ times in order to find an infinite subset $\mathcal{G} \subseteq \mathcal{F}$ with a selection function

$$f : \mathcal{G} \to \left[\bigcup \mathcal{G}\right]^m.$$

Now assume that $n \nmid m$. Then there is a prime number p and a $k \in \omega \setminus \{0\}$ such that $p^k | n$ and $p^k \nmid m$. Then by Proposition 5.3 and Lemma 5.4

$$\mathcal{V}_{p^k} \models \mathrm{nC}^-_{<\aleph_0} \wedge \neg\, \mathrm{mC}^-_{<\aleph_0}.$$

Analogously we can prove that

$$\mathrm{nLOC}^-_{<\aleph_0} \Rightarrow \mathrm{mLOC}^-_{<\aleph_0}$$

if and only if $n \mid m$. ⊣

5.3. The relation between $\mathrm{nC}^-_{<\aleph_0}$ and C^-_m.

LEMMA 5.6. *Let $n, m \in \omega \setminus \{0,1\}$ be relatively prime. Then we have that*

$$\mathrm{nC}^-_{<\aleph_0} \Rightarrow \mathrm{C}^-_m \quad \textit{and} \quad \mathrm{nLOC}^-_{<\aleph_0} \Rightarrow \mathrm{LOC}^-_m$$

in ZF.

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PROOF. Let $\mathcal{F}$ be an infinite family of pairwise disjoint m-element sets. By the Chinese Remainder Theorem there is an $r \in \omega$ with

$$r \equiv 0 \mod m \quad \text{and} \quad r \equiv 1 \mod n.$$

For every $F \in \mathcal{F}$ let F_0 be the union of $\frac{r}{m}$ copies of F (i.e., $\bigcup\{F \times \{i\} \mid i < \frac{r}{m}\}$). Let

$$\mathcal{G}_0 := \{F_0 \mid F \in \mathcal{F}\}.$$

Apply $\mathrm{nC}^-_{<\aleph_0}$ to $\mathcal{G}_0$. This gives us an infinite subset $\overline{\mathcal{G}}_1 \subseteq \mathcal{G}_0$ and a selection function

$$g_1 : \overline{\mathcal{G}}_1 \to \left[\bigcup \overline{\mathcal{G}}_1\right]^n.$$

Let

$$\mathcal{G}_1 := \{F \setminus g_1(F) \mid F \in \overline{\mathcal{G}}_1\}.$$

Repeat this process another $\lfloor \frac{r}{n} \rfloor - 1$ times. Then $\mathcal{G}_{\lfloor \frac{r}{n} \rfloor}$ is a family of one-element sets. Define

$$\mathcal{G} := \{F \in \mathcal{F} \mid F \cap \left(\bigcup \mathcal{G}_{\lfloor \frac{r}{n} \rfloor}\right) \neq \emptyset\}.$$

From each $F \in \mathcal{G}$ we can choose the element in $F \cap \bigcup \mathcal{G}_{\lfloor \frac{r}{n} \rfloor}$. This gives us a choice function on $\mathcal{G}$ and therefore, C^-_m holds. Analogously we can show that $\mathrm{nLOC}^-_{<\aleph_0} \Rightarrow \mathrm{LOC}^-_m$. ⊣

COROLLARY 5.7. *Let $n,m \in \omega \setminus \{0,1\}$ with $\gcd(m,n) > 1$. Let p be a prime number which divides both, n and m. Then we have that*

$$\mathcal{V}_p \models \mathrm{nC}^-_{<\aleph_0} \wedge \neg\, \mathrm{LOC}^-_m.$$

PROOF. By [6, Theorem 4] we have that $\mathcal{V}_p \models \neg\, \mathrm{LOC}^-_m$. And by Proposition 5.3 we have that $\mathcal{V}_p \models \mathrm{nC}^-_{<\aleph_0}$. ⊣

5.4. The relation between $\mathrm{nWOC}^-_{<\aleph_0}$ and $\mathrm{nLOC}^-_{<\aleph_0}$.

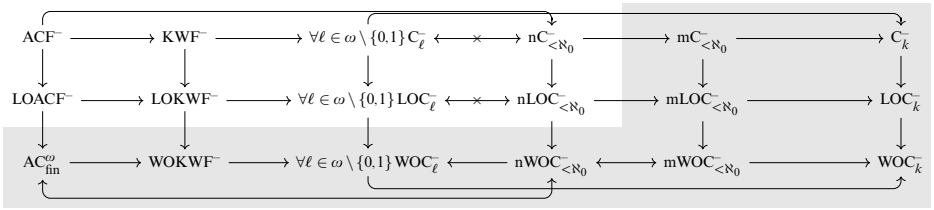
LEMMA 5.8. *Let $n \in \omega \setminus \{0,1\}$ and let $p > n$ be a prime number. Then we have that*

$$\mathcal{V}_p \models \mathrm{nWOC}^-_{<\aleph_0} \wedge \neg\, \mathrm{nLOC}^-_{<\aleph_0}.$$

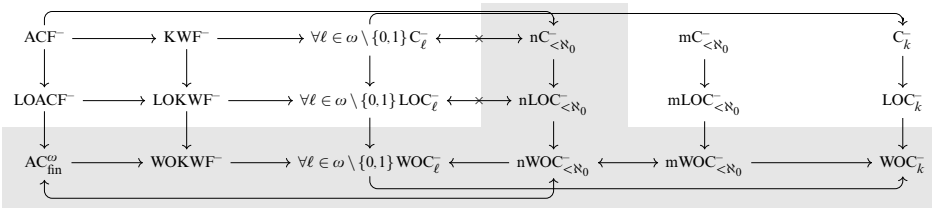
PROOF. This follows from [6, Proof of Theorem 8] and Lemma 5.4. ⊣

5.5. Summary. Assume that $n \mid m$ and that $\gcd(m,k) = 1$. Moreover, assume that there is a prime number p with $p \mid m$ and $p \nmid n$. In the diagram below, all weak choice principles with a gray background hold in $\mathcal{V}_p$ and all weak choice principles with a white background do not hold in $\mathcal{V}_p$.

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Assume that $\gcd(m,k) > 1$ and that $n \nmid m$. Furthermore we assume that there is a prime number p with $p \mid n$, $p \nmid m$, and $p \mid k$. In the diagram below, all weak choice principles with a gray background hold in $\mathcal{V}_p$ and all weak choice principles with a white background do not hold in $\mathcal{V}_p$.



§6. Fourth permutation model.

6.1. The models. Let $n \in \omega$ and let $\mathbb{B} := \omega$. For all $i \in \omega$ let $n_i := n$ and define $\mathbb{L} := \langle n_i \mid i \in \omega \rangle$. For every bijection $\psi : \omega \to \omega$ which moves only finitely many natural numbers we define

$$\begin{aligned} \rho_\psi : \mathcal{A} &\to \mathcal{A} \\ b_i^k &\mapsto b_i^{\psi(k)}. \end{aligned}$$

Now we define the group G of permutations of $\mathcal{A}$ by requiring

$$\alpha \in G \text{ if and only if } \alpha = \rho_\psi \varphi,$$

where $\psi : \omega \to \omega$ is a bijection which moves only finitely many natural numbers and $\varphi = \prod_{i \in \omega} \varphi_i^{k_i}$ with $k_i \in \omega$ and $k_i = 0$ for all but finitely many $i \in \omega$. Let $\mathcal{I} := \mathrm{fin}(\mathcal{A})$. Define

$$\mathcal{N}_n := \mathcal{V}(\mathbb{B}, \mathbb{L}, G, \mathcal{I}).$$

6.2. The relation between $\mathrm{nLOC}^-_{<\aleph_0}$ and $\mathrm{nC}^-_{<\aleph_0}$.

LEMMA 6.1. *Let $n \in \omega \setminus \{0\}$ and let $p > n$ be a prime number. Then we have that*

$$\mathcal{N}_p \models \neg \mathrm{nC}^-_{<\aleph_0} \wedge \mathrm{nLOC}^-_{<\aleph_0}.$$

PROOF. By [6, Proof of Theorem 8] we have that $\mathcal{N}_p \models \mathrm{nLOC}^-_{<\aleph_0}$. So it remains to prove that $\mathcal{N}_p \models \neg \mathrm{nC}^-_{<\aleph_0}$. Let $\mathcal{F} := \{B_i \mid i \in \omega\}$. Assume that there is an infinite

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subset $\mathcal{G} \subseteq \mathcal{F}$ with a selection function

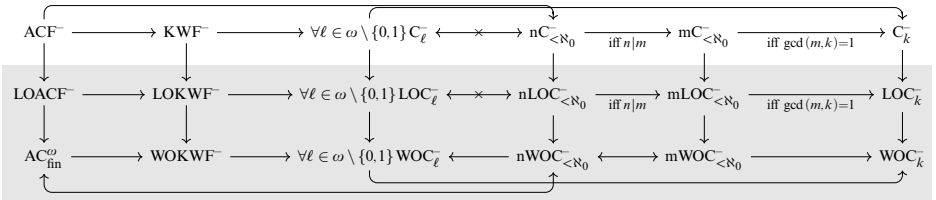
$$f : \mathcal{G} \to \left[\bigcup \mathcal{G}\right]^n.$$

Let $E_f \in \mathrm{fin}(\mathcal{A})$ be a support of f. Since $\mathcal{G}$ is infinite, there is an $i_0 \in \omega$ with $B_{i_0} \in \mathcal{G}$ and $E_f \cap B_{i_0} = \emptyset$. But then we have that

$$\varphi_{i_0}(f(B_{i_0})) \neq f(B_{i_0}) = f(\varphi_{i_0}(B_{i_0})).$$

So E_f is not a support of f. This is a contradiction and therefore, $\mathcal{N}_p \models \neg \mathrm{nC}^-_{<\aleph_0}$. ⊣

6.3. Summary. Let $n,m,k \in \omega \setminus \{0\}$ and let p be a prime number with $p > m$, $p > n$, and $p \mid k$. In the diagram below, all weak choice principles with a gray background hold in $\mathcal{N}_p$ and all weak choice principles with a white background do not hold in $\mathcal{N}_p$.



§7. The relation between $\mathrm{nWOC}^-_{<\aleph_0}$ and WOC^-_m.

PROPOSITION 7.1. *Let $n,m \in \omega \setminus \{0,1\}$. Then we have that*

$$\mathrm{nWOC}^-_{<\aleph_0} \Rightarrow \mathrm{WOC}^-_m.$$

PROOF. It suffices to prove that

$$\mathrm{nWOC}^-_{<\aleph_0} \Rightarrow \mathrm{WOKW}^-_{m_0} \text{ for all } 1 < m_0 \leq m$$

because $\mathrm{WOKW}^-_{m_0}$ for all $1 < m_0 \leq m$ implies WOC^-_m. So let $1 < m_0 \leq m$ and let $\mathcal{F} = \{F_k \mid k \in \omega\}$ be a countable infinite family of m_0-element sets. For every $k \in \omega$ let

$$A_k := F_0 \times F_1 \times \cdots \times F_k$$

and define $\mathcal{B} := \{A_k \mid k \in \omega \wedge |A_k| \geq n\}$. Apply $\mathrm{nWOC}^-_{<\aleph_0}$ to the set $\mathcal{B}$. So there is an infinite subset $\mathcal{C} \subseteq \mathcal{B}$ with a selection function

$$f : \mathcal{C} \to \left[\bigcup \mathcal{C}\right]^n.$$

For every $k \in \omega$ with $A_k \in \mathcal{C}$, all $\ell \leq k$ and all $x \in F_\ell$ define

$$n_{\ell,k}(x) := |\{s \in f(A_k) \mid s(\ell) = x\}|.$$

In other words, $n_{\ell,k}(x)$ is the number of sequences in $f(A_k)$ which have the value x at the ℓth position. For each $k \in \omega$ with $A_k \in \mathcal{C}$ let

$$N_k := \{\ell \leq k \mid \exists x_0, x_1 \in F_\ell \ (n_{\ell,k}(x_0) \neq n_{\ell,k}(x_1))\}.$$

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Case 1: $\bigcup_{A_k\in\mathcal{C}} N_k$ is an infinite set. In this case $\mathcal{G} := \left\{F_\ell \mid \ell \in \bigcup_{A_k\in\mathcal{C}} N_k\right\} \subseteq \mathcal{F}$ is an infinite set. And for each $F_\ell \in \mathcal{G}$ let $k_\ell \in \omega$ be minimal such that $\ell \in N_{k_\ell}$. We define

$$g : \mathcal{G} \to \mathcal{P}\left(\bigcup \mathcal{G}\right)$$
$$F_l \mapsto \{x \in F_\ell \mid n_{\ell,k_\ell}(x) \geq n_{\ell,k_\ell}(y) \text{ for all } y \in F_\ell\} \subsetneq F_\ell.$$

Case 2: $\bigcup_{A_k\in\mathcal{C}} N_k$ is a finite set. In this case let $n_0 > n$ for all $n \in \bigcup_{A_k\in\mathcal{C}} N_k$. Choose an $x_0 \in F_{n_0}$. For every $k \geq n_0$ with $A_k \in \mathcal{C}$ let

$$\overline{f}(A_k) := \{s \in f(A_k) \mid s(n_0) = x_0\}.$$

We redefine

$$n_{l,k}(x) := |\{s \in \overline{f}(A_k) \mid s(l) = x\}|$$

for all $k \geq n_0$ with $A_k \in \mathcal{C}$ and all $l \leq k$. Then we repeat the process above. Note that after finitely many repetitions of this process we are in Case 1. ⊣

PROPOSITION 7.2. *For all* $n \in \omega \setminus \{0,1\}$ *we have that* $\mathrm{nWOC}^-_{<\aleph_0} \Rightarrow \mathrm{AC}^\omega_{\mathrm{fin}}$. *In particular for all* $n,m \in \omega \setminus \{0,1\}$ *we have that* $\mathrm{nWOC}^-_{<\aleph_0} \Rightarrow \mathrm{mWOC}^-_{<\aleph_0}$.

PROOF. Let $m,n \in \omega \setminus \{0,1\}$. It suffices to show that $\mathrm{nWOC}^-_{<\aleph_0} \Rightarrow \mathrm{AC}^\omega_{\mathrm{fin}}$ because $\mathrm{AC}^\omega_{\mathrm{fin}} \Rightarrow \mathrm{mWOC}^-_{<\aleph_0}$. Let $\mathcal{F}$ be an infinite, well-ordered family of finite sets. Apply $\mathrm{nWOC}^-_{<\aleph_0}$ to the set $\mathcal{F}$ if there are infinitely many $F \in \mathcal{F}$ with $|F| \geq n$; otherwise use Proposition 7.1. So there is an infinite subfamily $\mathcal{G} \subseteq \mathcal{F}$ with a selection function

$$g : \mathcal{G} \to \left[\bigcup \mathcal{G}\right]^n.$$

By Proposition 7.1 we have that

$$\mathrm{nWOC}^-_{<\aleph_0} \Rightarrow \mathrm{WOC}^-_n.$$

Apply WOC^-_n to $\mathcal{G}$. So we can indeed choose exactly one element from an infinite subset $\mathcal{H} \subseteq \mathcal{F}$. ⊣

§8. Open questions.

1. As in [1] and [2] we can define restricted versions of $\mathrm{nC}^-_{<\aleph_0}$ and C^-_n for every $n \in \omega \setminus \{0,1\}$:

   $\mathrm{nRC}_{\mathrm{fin}}$: *Given any infinite set* x, *there is an infinite subset* $y \subseteq x$ *and a selection function* f *that chooses an* n*-element subset from every* $z \subseteq y$ *containing at least* n *elements.*

   RC_{n} : *Given any infinite set* x, *there is an infinite subset* $y \subseteq x$ *with a choice function on* $[y]^n$.

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The relationship of these two weak choice principles to $\mathrm{kC}^-_{<\aleph_0}$ and C^-_j has already been studied in [4]. However, the following questions are still open:

- For $n \in \{2,3,4,6\}$ we have that $\mathrm{nRC_{fin}} \Rightarrow \mathrm{nC}^-_{<\aleph_0}$. Are there other $n \in \omega$ for which this implication holds?
- Let $n = \prod_{i=1}^{k} p_i^{m_i}$ be the prime decomposition of the natural number n. Is it true that $\mathrm{nRC_{fin}} \Rightarrow \mathrm{RC_m}$ holds iff $m > n$ and $m \equiv 1 \left(\mathrm{mod}\ \prod_{i=1}^{k} p_i\right)$?

2. In [11], Montenegro proved that $\mathrm{RC_n} \Rightarrow \mathrm{C}^-_n$ for all $n \in \{2,3,4\}$. In [5] we proved that $\mathrm{RC_n} \Rightarrow \mathrm{LOC}^-_n$ for $n \in \{5,7\}$. Moreover, we showed that $\mathrm{RC_n} \Rightarrow \mathrm{WOC}^-_m$ holds for $m,n \in \omega \setminus \{0,1\}$ iff for all prime numbers $p_0, \ldots, p_{r-1}, r \in \omega$, such that there are $a_0, \ldots, a_{r-1} \in \omega$ with

$$n = \sum_{i<r} a_i p_i$$

we can find $b_0, \ldots, b_{r-1} \in \omega$ with

$$m = \sum_{i<r} b_i p_i.$$

The following questions are still open:

- Does $\mathrm{RC_6}$ imply LOC^-_6? Or more generally: For which $n \in \omega \setminus \{0,1\}$ does $\mathrm{RC_n}$ imply LOC^-_n?
- For which $n \in \omega \setminus \{0,1\}$ does $\mathrm{RC_n}$ imply C^-_n?

Acknowledgments. I would like to thank the referee for his or her valuable comments and suggestions which helped to improve the quality of the article. This work was partially supported by SNF grant 200021_178851.

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https://doi.org/10.1017/jsl.2021.20 Published online by Cambridge University Press