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Rigidity and non-recurrence along sequences

V. BERGELSON†, A. DEL JUNCO‡, M. LEMAŃCZYK§ and J. ROSENBLATT¶

† *Department of Mathematics, Ohio State University, Columbus, OH 43210, USA*
(e-mail: vitaly@math.ohio-state.edu)

‡ *Department of Mathematics, University of Toronto, Toronto, M5S 3G3, Canada*
(e-mail: deljunco@math.toronto.edu)

§ *Faculty of Math and Computer Science, Nicolaus Copernicus University, Toruń, Poland*
(e-mail: mlem@mat.uni.torun.pl)

¶ *Department of Mathematics, University of Illinois at Urbana-Champaign, Urbana, IL 61801, USA*
(e-mail: rosnbltt@illinois.edu)

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Abstract. We study two properties of a finite measure-preserving dynamical system and a given sequence (n_m) of positive integers, namely rigidity and non-recurrence. Our goal is to find conditions on the sequence which ensure that it is, or is not, a rigid sequence or a non-recurrent sequence for some weakly mixing system or more generally for some ergodic system. The main focus is on weakly mixing systems. For example, we show that for any integer $a \geq 2$ the sequence $n_m = a^m$ is a sequence of rigidity for some weakly mixing system. We show the same for the sequence of denominators of the convergents in the continued fraction expansion of any irrational α. We also consider the stronger property of IP-rigidity. We show that if (n_m) grows fast enough then there is a weakly mixing system which is IP-rigid along (n_m) and non-recurrent along $(n_m + 1)$.

1. *Introduction*

Let $(X, \mathcal{B}, \mu, T)$ be a dynamical system: that is, we have a non-atomic probability space $(X, \mathcal{B}, \mu)$ and an *invertible measure-preserving transformation* (map for short) T of $(X, \mathcal{B}, \mu)$. Unless otherwise specified we will always assume that T is ergodic. We consider only non-atomic Lebesgue spaces $(X, \mathcal{B}, \mu)$, that is, ones which are measure-theoretically isomorphic to the unit interval with Lebesgue measure. Unless otherwise specified we will use the symbol T (and sometimes S) generically throughout with the understanding that it always refers to a measure-preserving map of a Lebesgue probability space. Often we will refer to T simply as a map, without specifying $(X, \mathcal{B}, \mu)$. For a given increasing sequence (n_m) of positive integers we study two properties of the dynamical system $(X, \mathcal{B}, \mu, T)$ along (n_m), namely

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rigidity and non-recurrence. Rigidity means that $\mu(T^{n_m}A \triangle A) \to 0$ for all $A \in \mathcal{B}$, while non-recurrence means that there is an $A \in \mathcal{B}$ such that $\mu(A \cap T^{n_m}A) = 0$ for all m. Our main interest is in finding classes of sequences of whole numbers (n_m) which are rigid, or non-recurrent, for some weakly mixing dynamical system. In the process of doing this, we will see that there are parallels and distinctions between the class of rigid sequences and the class of non-recurrent sequences, both for ergodic transformations in general and for weakly mixing transformations in particular. In several instances we also consider systems with discrete spectrum. An important question left open by our work is the following: given any sequence rigid for some T with discrete spectrum, must it be rigid for some weakly mixing T? Recently, Adams [**4**] has answered this question affirmatively. But we still do not know the answer to the converse question.

First, in §2, we briefly review some background material about rigidity and weak mixing. We also consider the more restrictive property of IP-rigidity. (We defer the definition for now; see the second paragraph following Remark 2.2.) We will see that both rigidity and IP-rigidity can be viewed as spectral properties. We recall the construction of stationary Gaussian measures (Gaussians for short) from a finite (symmetric) measure on the circle as well as the construction of Poisson suspensions of infinite measure-preserving systems. We observe that, via the use of Gaussians, a sequence is rigid for some weakly mixing T if and only if it is rigid for some weakly mixing unitary operator U on a (complex) Hilbert space $\mathcal{H}$, in the sense that $\|U^{n_m}x - x\| \to 0$ for all $x \in \mathcal{H}$. (Weak mixing for a unitary operator simply means continuous spectrum, that is, no eigenvectors.) For completeness we include a short proof of the known result that the class of maps which are rigid along some subsequence of any prescribed infinite subset of $\mathbb{N}$ is a dense G_δ subset of the group of all measure-preserving maps.

We then proceed to consider various obstructions to rigidity. These are mainly of an algebraic nature, for example no polynomial sequence is rigid. Also, the sequence of primes is not rigid for any ergodic T. We also show that any sequence which is a 'good' sequence for L_2-convergence of ergodic averages (see §2 for the definition) cannot be a rigidity sequence.

Next, in §3 we give some sufficient conditions for a sequence to be rigid for some weakly mixing T and also simply for some ergodic T. The section is divided into subsections according to the methods used. Each method gives several results. Interestingly, in some cases we can obtain the same result by very different methods. Some of the tools used are harmonic analysis on the circle, rank-one constructions (preserving both finite and infinite measures), cocycle constructions, stationary Gaussian measures, Poisson suspensions and number theory. Here are some samples of what we prove.

(a) If $\lim_{m\to\infty}(n_{m+1}/n_m) = \infty$, then (n_m) is a rigidity sequence for some weakly mixing transformation T. We show this in two ways. The first (Proposition 3.6) is by taking a closed perfect subset E of the circle on which $z^{n_m} \to 1$ for all z, choosing any continuous measure on E and then passing to the corresponding Gaussian. The second (Proposition 3.15) is by a cutting and stacking construction which yields an ergodic infinite measure-preserving rank-one transformation S, rigid along (n_m), and then taking its Poisson suspension (or one can also use the Gaussian corresponding to any spectral

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measure of S). We also show (Proposition 3.16) that the stronger growth condition $\sum_{m\geq 1}(n_m/n_{m+1}) < \infty$ implies that (n_m) is IP-rigid for some weakly mixing rank-one T.

(b) If $n_{m+1}/n_m \geq 2$ is an integer for all m then (n_m) is rigid for some weakly mixing system. We have several proofs: firstly (Proposition 3.10) by direct construction of a measure on the circle for which z^{n_m} converges to 1 in measure (and passing to the associated Gaussian), and secondly (Proposition 3.15) again by construction of an infinite measure-preserving rank-one T. See also (e) below. As a special case, if a is an integer greater than 1 then (a^m) is rigid for a weakly mixing T. However, $(a^m + p(m) : m \geq 1)$, where $p \in \mathbb{Q}[x]$ is any non-zero polynomial taking integer values on $\mathbb{Z}$, is not a rigidity sequence for any ergodic transformation. See Remark 2.16(b).

(c) In §3.4, we make some remarks about the rate of growth of rigid sequences. Aaronson [**1**, Theorem 4] has proved roughly that there are sequences which are rigid for some weakly mixing map and which grow as slowly as one likes, subject only to the necessary condition that the density is zero. His proof is rather intricate. We give short easy proofs of the same result for systems with discrete spectrum and of a weaker result in the case of weak mixing.

(d) In §3.5, we show that there is no 'universal' weakly mixing rigid transformation in the sense that, given any weakly mixing rigid T, there is a weakly mixing rigid S such that $T \times S$ is not rigid. In fact we show something stronger, namely that S can be chosen so that $T \times S$ has absolutely continuous spectrum in the orthocomplement of the rigid subspaces corresponding to T and S.

(e) In §3.6.2, we show how cocycle constructions can be used to construct rigidity sequences for weakly mixing transformations. One particular result is Corollary 3.34: if $(p_n/q_n : n \geq 1)$ are the convergents associated with the continued fraction expansion of an irrational number, then (q_n) is rigid for a weakly mixing transformation. In particular, for example, the Fibonacci sequence is rigid for a weakly mixing transformation. We also give a third proof of the fact that if $n_{m+1}/n_m \geq 2$ is an integer for all m then (n_m) is rigid for some weakly mixing T.

In §4, we consider non-recurrence. We know that for weakly mixing transformations the sequences exhibiting non-recurrence must have density zero, while for translations they can have density arbitrarily close to 1. Also, even for ergodic transformations, sequences exhibiting non-recurrence cannot have certain types of algebraic structure; for example, as is well known, (n^2) is a sequence of recurrence. It is known (see Furstenberg [**18**]) that any lacunary sequence is non-recurrent for some circle rotation. We conjecture that any lacunary sequence is also non-recurrent for some weakly mixing transformation. The following are our main results on non-recurrence.

(a) If T is rigid along (n_m) then there is a subsequence (l_m) of (n_m) such that we have IP-rigidity along (l_m) and non-recurrence along $(l_m + 1)$. More strongly, given any $K > 0$, (l_m) can be chosen so that there is a single set C which is non-recurrent along $(l_m + k)$ for all m and $0 < |k| \leq K$. See Proposition 4.5 and Remark 4.7.

(b) If T is any rank-one transformation with tower heights h_m and with bounded spacers then the sequence $(h_m - 1)$ is non-recurrent for T. In particular, there are many sequences (n_m) which are non-recurrent for a weakly mixing map and for which n_{m+1}/n_m is bounded.

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(c) If $\sum_{m=1}^{\infty}(n_m/n_{m+1}) < \infty$ then there is a weakly mixing rank-one T such that for any $K > 0$ there is a set C such that $\mu(T^{n_m+k}C \cap C) = 0$ for all m and $0 < |k| \le K$, and at the same time T is IP-rigid along (n_m). In particular, focusing on the non-recurrence, if $\sum_{m=1}^{\infty}(n_m/n_{m+1}) < \infty$ then there is a weakly mixing T which is not recurrent along (n_m). See Proposition 4.12 and Remark 4.13.

When this paper was in the final draft, we learned of the work of Eisner and Grivaux [**12**]. Our papers are largely complementary, although we do cover some of the same basic issues. We will refer to their work several times in the appropriate places. There is also a recent paper by Aaronson *et al* [**2**] on IP-rigidity which (among other things) improves on one of our results (see Proposition 3.16). Also, when this paper was being prepared for proofreading and publication, we learned of the work of Adams [**4**]; he has shown in a cutting and stacking construction that if (n_m) is rigid for an ergodic map, then it is also rigid for a weakly mixing map.

2. *Preliminaries and some obstructions to rigidity*

2.1. *Preliminaries.* If a and b are elements of some metric space (X, d) then we will often use the notation $a \overset{\epsilon}{\sim} b$ to mean $d(a, b) < \epsilon$. In particular, for subsets A, B of a finite measure space $(X, \mathcal{B}, \mu)$ we use the (pseudo)metric $d(A, B) = \mu(A \triangle B)$.

Given a dynamical system $(X, \mathcal{B}, \mu, T)$, we have the associated unitary operator U_T (the so-called Koopman operator) on $L_{2,0}(\mu)$ defined by $U_T f = f \circ T$. Here $L_{2,0}(\mu)$ denotes the space of complex-valued square-integrable functions of mean 0. When the meaning is clear we will often denote $U_T f$ by Tf. Given any unitary operator U on a Hilbert space $\mathcal{H}$ and any $x \in \mathcal{H}$, we have the spectral measure ν_x^U, a measure on the circle $\mathbb{T}$ which is uniquely determined by the requirement that its Fourier transform is the sequence $(\langle U^n x, x\rangle)$, that is, $\int_{\mathbb{T}} z^n \, d\nu = \langle U^n x, x\rangle$. We use the notation ν_f^T if $U = U_T$ and $x = f \in L_{2,0}(\mu)$. We assume that the reader is familiar with the basic notions in the spectral theory of unitary operators as found in, for example, an appendix in Parry [**42**]. We denote the maximal spectral type of U_T or U by ν^T or ν^U. Systems $(X, \mathcal{B}, \mu, T)$ and $(Y, \mathcal{C}, \nu, S)$ are said to be spectrally disjoint when the measures ν^T and ν^S are mutually singular. $(X, \mathcal{B}, \mu, T)$ and $(Y, \mathcal{C}, \nu, S)$ are said to be disjoint (in the sense of Hahn and Parry [**20**]) if the only invariant measure on $\mathcal{B} \otimes \mathcal{C}$ with marginals μ and ν is $\mu \times \nu$. It is well known that spectral disjointness implies disjointness. See, for example, [**20**, Theorem 2.1]. Actually, spectral disjointness is strictly stronger than disjointness. For example, let T be a K-automorphism and S a zero-entropy transformation with a countable Lebesgue spectrum. Then T and S have the same spectrum, but are disjoint.

We use the group $\mathcal{G} = \mathrm{Aut}(X, \mathcal{B}, \mu)$ of all invertible measure-preserving transformations S on $(X, \mathcal{B}, \mu)$. As usual, we identify S with S' whenever the two agree off a null set. As a subgroup of the group of all unitary operators on $L_2(\mu)$, $\mathcal{G}$ inherits the topology of strong operator convergence. That is, a sequence (S_n) in $\mathcal{G}$ converges to $S \in \mathcal{G}$ if $\|f \circ S_n - f \circ S\|_2 \to 0$ as $n \to \infty$ for all $f \in L_2(X, \mu)$. Equivalently, $S_n A \to A$ for all $A \in \mathcal{B}$. It is well known that $\mathcal{G}$ is a Polish space (we will encounter a complete metric for it later). By a *generic property* of T we mean a property that holds on at least a dense G_δ subset of $\mathcal{G}$, and any set containing a dense G_δ set is called a *generic set*.

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We start by recalling the notion of weak mixing for a system $(X, \mathcal{B}, \mu, T)$ with the following standard result which summarizes some of the many possible characterizations of weak mixing. Recall that a sequence (n_m) is called mixing for T if $\mu(T^{n_m}A \cap B) \to \mu(A)\mu(B)$ as $m \to \infty$, for all $A, B \in \mathcal{B}$. Recall also that the density of a subset $E \subset \mathbb{N}$ is defined as $D(E) = \lim_{n\to\infty}(1/n)|\{1, \ldots, n\} \cap E|$ (where $|\cdot|$ denotes cardinality), if the limit exists. In any case the upper and lower densities $\overline{D}(E)$ and $\underline{D}(E)$ are defined by replacing lim with lim sup and lim inf. When considering sequences (n_m) of positive integers we will always mean a strictly increasing sequence so it can be identified with the set $E = \{n_m : m \in \mathbb{N}\}$ of its values, and hence we can speak of the density of a sequence.

PROPOSITION 2.1. *For any T the following conditions are equivalent.*

(1) *$T \times T$ is ergodic.*
(2) *U_T has no eigenfunctions orthogonal to the constant functions.*
(3) *ν_f^T is a continuous measure for each $f \in L_{2,0}(\mu)$.*
(4) *ν^T is continuous.*
(5) *For each $A, B \in \mathcal{B}$ we have $N^{-1}\sum_{n=1}^{N} |\mu(T^nA \cap B) - \mu(A)\mu(B)| \to 0$.*
(6) *For each $A, B \in \mathcal{B}$ we have $\mu(T^nA \cap B) \to \mu(A)\mu(B)$ along some subsequence of density one (which may depend on A, B).*
(7) *T has a mixing sequence of density one.*
(8) *T has a mixing sequence.*

Any one of these conditions may be taken as the definition of T weak mixing. As is well known, weak mixing is a generic property (see, for example, Halmos [**21**]).

Remark 2.2. Each of the conditions in Proposition 2.1 has an analogue for an abstract unitary operator and they are also equivalent in this case.

We now turn to rigidity. We say that a strictly increasing sequence (n_m) of positive integers is rigid for T if $T^{n_m} \to \mathrm{Id}$ in the group $\mathcal{G}$. Explicitly, $T^{n_m}A \to A$ for all $A \in \mathcal{B}$. We say that T is rigid if it has some rigidity sequence.

We will also be interested in the stronger property of IP-rigidity (see, for example, Furstenberg and Weiss [**19**]). The IP set generated by a sequence (n_m) consists of all finite sums of elements with distinct indices in the sequence and is denoted

$$\Sigma = \mathrm{FS}(n_m) = \{\sigma = n_{m_1} + \cdots + n_{m_k} : k \in \mathbb{N}, m_1 < \cdots < m_k\}.$$

Since we consider only strictly increasing sequences (n_m) we may write $E = \{n_m : m \geq 1\} \subset \mathbb{N}$ and

$$\Sigma = \mathrm{FS}(E) = \{\sigma = a_1 + \cdots + a_k : k \in \mathbb{N} : a_1 < \cdots < a_k \in E\}.$$

Given $\sigma = a_1 + \cdots + a_k \in \Sigma$ with $a_1 < \cdots < a_k$, we say that $\sigma \to \infty$ (IP) if a_1 tends to ∞. We say that T is IP-rigid along (n_m) if $T^\sigma \to \mathrm{Id}$ in the strong operator topology as $\sigma \to \infty$ (IP).

Given any increasing sequence (n_m) of integers, we consider the family

$$\mathcal{A} = \mathcal{A}(n_m) = \{A \in \mathcal{B} : T^{n_m}A \to A\}.$$

By a factor algebra of T we mean any sub σ-algebra of $\mathcal{B}$ which is T-invariant ($T\mathcal{B} = \mathcal{B}$). The following result is well known and elementary (see, for example, Walters [**54**]).

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LEMMA 2.3. *$\mathcal{A}(n_m) \subset \mathcal{B}$ is a factor algebra of T. $\mathcal{A}(n_m)$ is the maximal σ-algebra $\mathcal{A} \subset \mathcal{B}$ such that*

$$T^{n_m}|_{\mathcal{A}} \to \mathrm{Id}|_{\mathcal{A}} \quad \text{as } m \to \infty.$$

Moreover,

$$L_2(X, \mathcal{A}(n_m), \mu) = \{f \in L_2(X, \mathcal{B}, \mu) : \|f \circ T^{n_m} - f\|_2 \to 0\}.$$

Remark 2.4. We may also define $\mathcal{A}'(n_m) = \{A \in \mathcal{B} : T^{\sigma} A \to A \text{ as } \sigma \to \infty \text{ (IP)}\}$ and obtain an IP-version of Lemma 2.3.

Evidently, rigidity of T is a spectral property of T; that is, it can be defined in terms of U_T only. For a given sequence (n_m), transformation T and function $f \in L_2(\mu)$, we say that (n_m) is rigid for T and f if $f \circ T^{n_m} \to f$ in L_2-norm. By Lemma 2.3 this is equivalent to the fact that $T|_{\mathcal{A}}$ is rigid where $\mathcal{A}$ is the factor algebra generated by f. (By this we mean the smallest factor algebra for which f is measurable.) Let $\nu = \nu_f^T$ and recall that $\hat{\nu}(n_m) = \langle f \circ T^{n_m}, f \rangle$. The following result is elementary.

PROPOSITION 2.5. *Fix the map T and a sequence (n_m). For $f \in L_2(\mu)$ the following are equivalent:*

(1) $\|T^{n_m} f - f\|_2 \to 0$;
(2) $T^{n_m} f \to f$ *in the weak topology of* $L_2(\mu)$;
(3) $\widehat{\nu_f^T}(n_m) \to \|f\|_2^2$;
(4) $z^{n_m} \to 1$ *in* $L_2(\mathbb{T}, \nu_f^T)$;
(5) $z^{n_m} \to 1$ *in measure with respect to* ν_f^T.

Recall that for any T we have an $F \in L_{2,0}(\mu)$ such that $\nu_F^T = \nu^T$, the maximal spectral type of T, and that for any $f \in L^2(\mu)$ we have $\nu_f^T \ll \nu_F^T$. Using this fact and (5) in Proposition 2.5 we obtain the following well-known observation.

COROLLARY 2.6. *T is rigid if and only if for each function $f \in L_2(\mu)$ there exists (n_m) (which may depend on f) such that $f \circ T^{n_m} \to f$ in $L_2(\mu)$.*

There is an analogous observation for rigidity in terms of sets which is due to Thouvenot. We include a short proof.

PROPOSITION 2.7. *T is rigid if and only if for each $A \in \mathcal{B}$ there is a sequence (n_m), depending on A, such that $T^{n_m} A \to A$.*

Proof. We only need to prove the converse. It is easy to see that the hypothesis implies that the entropy is $h(P, T) = 0$ for any two-set partition $P = \{A, X \setminus A\}$. It follows that $h(P, T) = 0$ for every finite partition P, so the entropy of T is 0. Then by Krieger's generator theorem [**31**] we can conclude that T has a two-set generator $\{A, X \setminus A\}$. As there is an (n_m) which is rigid for T and A it follows by Lemma 2.3 that (n_m) is rigid for T. □

It is a standard fact that the set of rigid S is a generic subset of $\mathcal{G}$. This follows, for example, from [**27**, Theorems 1.1 and 3.3]. In fact, the following stronger result is also known. We include a short proof for completeness as we do not know of a reference.

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PROPOSITION 2.8. *Given an increasing sequence (n_m) of natural numbers, let $\mathcal{R}$ be the set consisting of all $S \in \mathcal{G}$ such that $S^{l_m} \to \mathrm{Id}$ for some subsequence (l_m) of (n_m). Then $\mathcal{R}$ is a generic subset of $\mathcal{G}$.*

Proof. Let $\{P_n, n = 0, 1, \ldots\}$ be an increasing sequence of finite partitions of X which generates $\mathcal{B}$. For $S, T \in \mathcal{G}$ define

$$d_n(S, T) = \sum_{A \in P_n} \mu(SA \triangle TA) + \mu(S^{-1}A \triangle T^{-1}A).$$

Then it is easy to verify that $d = \sum_{n \geq 1} 2^{-n} d_n$ is a complete metric on $\mathcal{G}$ compatible with the topology of $\mathcal{G}$ and

$$d(S, T) \leq 4\mu\{x : Sx \neq Tx\}.$$

Now for each $N > 0$ let

$$\mathcal{U}_N = \{S \in \mathcal{G} : d(S^{n_m}, \mathrm{Id}) < N^{-1}, \text{ for some } n_m > N\}.$$

$\mathcal{U}_N$ is open in $\mathcal{G}$ and it is easy to see that $\mathcal{R} = \bigcap_N \mathcal{U}_N$.

So, in order to complete the proof, it remains only to show that for each N the closure is $\overline{\mathcal{U}_N} = \mathcal{G}$. In fact, if we denote by $\mathcal{E}$ the set of ergodic S then $\overline{\mathcal{E}} = \mathcal{G}$ so it suffices to show that $\overline{\mathcal{U}_N} \supset \mathcal{E}$. Given any $S \in \mathcal{E}$, $N > 0$ and $\epsilon > 0$, choose $n_m > N$ and then by Rokhlin's lemma find B such that $B, SB, \ldots, S^{n_m - 1}B$ are pairwise disjoint and such that $E = X \setminus \bigcup_{0 \leq i \leq n_m - 2} S^i B$ satisfies $\mu(E) < \epsilon$. Define $S' \in \mathcal{G}$ by

$$S'x = \begin{cases} Sx & \text{if } x \in \displaystyle\bigcup_{0 \leq i \leq n_m - 2} S^i B, \\ S^{1-n_m}x & \text{if } x \in S^{n_m - 1}B, \\ x & \text{otherwise.} \end{cases}$$

Then we have $(S')^{n_m} = \mathrm{Id}$, so $S' \in \mathcal{U}_N$ (since $n_m > N$) and $\{x : Sx \neq S'x\} \subset E$ so $d(S, S') < 4\mu\{x : Sx \neq S'x\} < 4\epsilon$. This establishes the required density. □

COROLLARY 2.9. *Given a rigid T, the set of S such that $T \times S$ is rigid is generic.*

Proof. Suppose that (n_m) is rigid for T. Then the generic S is rigid along some subsequence (k_m) (depending on S) of (n_m). Thus S and T are both rigid along k_m and it follows easily that $T \times S$ is rigid along (k_m). □

The antithesis of weak mixing is discrete spectrum: T has discrete spectrum if and only if its maximal spectral type is a purely atomic measure, that is, $L_2(\mu)$ has an orthonormal basis consisting of eigenvectors for U_T. Under the standing assumption that T is ergodic this is equivalent to requiring that T be isomorphic to an isometry of a compact metric space X such that at least one orbit is dense (and hence all orbits are dense), together with its unique invariant probability measure μ. In this situation, X can be given a group structure in a natural way making it into a monothetic compact topological group G, μ is seen to be the Haar measure on G and T is translation by an element of G. Henceforth we will refer to such T as ergodic isometries, and indeed the fact that they are ergodic isometries is the only thing we will use in this paper. Ergodic isometries are easily seen to be rigid. In §3 we will construct rigid sequences for them having special properties.

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Next we briefly recall the construction of stationary Gaussian measures on $\mathbb{R}^{\mathbb{Z}}$ (Gaussian processes in probabilistic language), denoted *GMC*. We refer to [**10**] for a more detailed treatment. See also de la Rue [**46**]. We start with a finite symmetric Borel measure ν on $\mathbb{T}$, that is, $\nu(A) = \nu(\overline{A})$ for all $A \subset \mathbb{T}$. (The assumption that ν is symmetric is classical and simply ensures that its Fourier coefficients are real and the corresponding process will be real-valued. It is also possible to start with an arbitrary finite measure and obtain a complex-valued Gaussian process.) On $X = \mathbb{R}^{\mathbb{Z}}$ we define the coordinate functions $X_n(x) = x(n)$, $x \in X$. Then there is a unique stationary (shift-invariant) probability measure μ on X such that for any finite subset $I \subset \mathbb{Z}$ the random variables $\{X_n : n \in I\}$ are jointly Gaussian and $\langle X_0, X_n \rangle = \widehat{\nu}(n)$ for all n. The system $(X, \mathcal{B}, \mu, S)$, where $\mathcal{B}$ is the Borel σ-algebra of X and S is the shift map, is referred to as the Gaussian shift associated to ν. Of course, $\nu = \nu^T_{X_0}$. When we wish to emphasize the role of ν we will write $S = G_\nu$. We will need the fact (see, for example, [**10**]) that the maximal spectral type σ of G_ν has components ν^{*n}, $n \in \mathbb{N}$, so if ν is a probability measure, σ is represented by $\sum_{n \geq 1} 2^{-n} \nu^{*n}$. Here ν^{*n} denotes the nth convolution power.

Using Gaussian shifts we can see that the question of rigidity of a sequence for some weakly mixing T is the same as the question of rigidity for some weakly mixing unitary U. In this result, we say (n_m) is rigid for a probability measure ν on the circle if $\widehat{\nu}(n_m) \to 1$ as $m \to \infty$.

PROPOSITION 2.10. *Given a sequence $\{n_m\}$ the following are equivalent.*

(1) (n_m) *rigid for some weakly mixing unitary operator.*
(2) (n_m) *is rigid for a weakly mixing dynamical system.*
(3) (n_m) *is rigid for a continuous measure on the circle.*

Proof. The equivalence of (1) and (3) is contained in Proposition 2.5, and (2) implies (1) is immediate, so we need only show that (3) implies (2). If (n_m) is rigid for a continuous probability measure ν on $\mathbb{T}$, by symmetrization we may assume without loss of generality that ν is symmetric.

Now let $(\mathbb{R}^{\mathbb{Z}}, \mathcal{B}, \mu, T)$ be the Gaussian shift corresponding to ν, which is weakly mixing. Then the fact that $\int_{\mathbb{T}} z^{n_m} \, d\nu \to 1$ as $m \to \infty$ means that $X_{n_m} = X_0 \circ T^{n_m} \to X_0$ in L_2 norm, by Proposition 2.5. Now let $\mathcal{A} = \{B \in \mathcal{B} : T^{n_m} B \to B\}$. By Lemma 2.3 the factor algebra $\mathcal{C}$ generated by X_0 is contained in $\mathcal{A}$. But $\mathcal{C} = \mathcal{B}$ since $\mathcal{C}$, being invariant, contains all the X_i. This means that (n_m) is rigid for T. □

Remark 2.11. In view of Remark 2.4 we also see that (n_m) is IP-rigid for a weakly mixing T if and only if it is IP-rigid for some weakly mixing U.

We will also have occasion to use another probabilistic construction, namely the Poisson suspension. See Cornfeld *et al* [**10**], Kingman [**28**], and Neretin [**41**] for information about Poisson suspensions. Briefly, here is the idea. Let T be a transformation of a standard Lebesgue space $(X, \mathcal{B}, \mu)$, where μ is an infinite σ-finite measure. We define a probability space $(\widetilde{X}, \widetilde{\mathcal{B}}, \widetilde{\mu})$. The points of the configuration space $\widetilde{X}$ are infinite countable subsets $\widetilde{x} = \{x_n : n \geq 1\}$ of X. Given a set $A \in \mathcal{B}$ of finite measure, we define $N_A : \widetilde{X} \to \mathbb{N} \cup \{\infty\}$ by setting

$$N_A(\widetilde{x}) = \#\{n \in \mathbb{N} : x_n \in A\}.$$

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Then $\widetilde{\mathcal{B}}$ is defined as the smallest σ-algebra of subsets of $\widetilde{X}$ making all variables N_A, $\mu(A) < +\infty$, measurable. The measure $\widetilde{\mu}$ is the unique probability measure (see [**28**] for details) such that:

- the variables N_A, $\mu(A) < \infty$, satisfy the Poisson law with parameter $\mu(A)$;
- for each family $A_1, \ldots, A_k$ of pairwise disjoint subsets of X of finite measure the corresponding variables $N_{A_1}, \ldots, N_{A_k}$ are (jointly) independent.

Then we define $\widetilde{T}$ on $\widetilde{X}$ by setting

$$\widetilde{T}(\{x_n\}) = (\{Tx_n\})$$

and the system $(\widetilde{X}, \widetilde{\mathcal{B}}, \widetilde{\mu}, \widetilde{T})$ is called the Poisson suspension of T. Then $\widetilde{T}$ is ergodic if and only if T has no non-trivial invariant sets of finite measure. In this case, $\widetilde{T}$ turns out to be weakly mixing and, moreover, the maximal spectral type of $\widetilde{T}$ has components $(\sigma)^{*n}$, $n \in \mathbb{N}$, where σ is the maximal spectral type of T.

2.2. *Obstructions to rigidity.* Proposition 2.1 implies that if (n_m) is rigid for a weakly mixing map then (n_m) has density zero. However, more is true for ergodic T in general.

PROPOSITION 2.12.

(a) *Suppose that T is ergodic. If (n_m) is a rigidity sequence for T then $n_{m+1} - n_m \to \infty$ (and hence (n_m) has density zero).*

(b) *Suppose that T is totally ergodic, that is, every non-zero power of T is ergodic. If $\|f_0 \circ T^{n_m} - f_0\|_2 \to 0$ as $m \to \infty$ for some non-constant $f_0 \in L_2(X, \mu)$, then $n_{m+1} - n_m \to \infty$.*

Proof. In both (a) and (b) if the conclusion fails then there is a value $d \geq 1$ such that $n_{m+1} - n_m = d$ infinitely often. In case (a) the rigidity implies that $T^d = \mathrm{Id}$, which is impossible for an ergodic T on a non-atomic measure space. In case (b) it follows that $T^d f_0 = f_0$, contradicting total ergodicity. □

Remark 2.13. An analogous result is true for unitary operators U. In case (a) ergodicity of T must be replaced by the requirement that the maximal spectral type of U is not concentrated on any finite subgroup of the roots of unity. In case (b) the assumption on T should be replaced by the assumption that the maximal spectral type is not concentrated on the group of all roots of unity.

Part (a) of Proposition 2.12 is a special case of the following result.

PROPOSITION 2.14. *Suppose that $F(x_1, \ldots, x_K) = \sum_{k=1}^{K} c_k x_k$, where $c_1, \ldots, c_K \in \mathbb{Z}\backslash\{0\}$. Suppose that $(n_m) \subset \mathbb{N}$ is an increasing sequence. Assume that there is a non-zero $d \in \mathbb{Z}$, such that for any $M \geq 1$, there are $m_k \geq M$ for all $k = 1, \ldots, K$, such that $F(n_{m_1}, \ldots, n_{m_K}) = d$. Then (n_m) is not a rigidity sequence for any ergodic transformation T.*

Proof. As in the proof of Proposition 2.12, if (n_m) is a rigidity sequence then the hypothesis implies that $T^d = \mathrm{Id}$, which is impossible. □

Remark 2.15. As in Proposition 2.12, if we assume that T is totally ergodic then we conclude that we cannot have $f \circ T^{n_m} \to f$ unless f is constant.

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Remark 2.16.

(a) If $p \in \mathbb{Q}[x]$ is any non-zero polynomial taking integer values on $\mathbb{Z}$ then the sequence $(p(m))$ cannot be rigid for any ergodic transformation. The easiest way to see this is to note that if $(p(m))$ is rigid then the difference sequence $q(m) = p(m+1) - p(m)$ is a polynomial of degree one less than p which is also rigid. Thus by induction a constant sequence would be rigid, a contradiction. For example, if $p(n) = n^2$, then $q(n) = p(n+1) - p(n) = 2n + 1$ and $q(n+1) - q(n) = 2$. So $p(n+2) - 2p(n+1) + p(n) = q(n+1) - q(n) = 2$ so we may take $F(x, y, z) = x - 2y + z$ in Proposition 2.14.

Actually, more is true. No sequence which contains the terms n^2, $(n+1)^2$, $(n+2)^2$ for infinitely many n is a sequence of rigidity. Of course, a similar statement is true for an arbitrary polynomial.

(b) Here is another example of the same idea. Suppose that $a \geq 2$ is any integer, p is a non-zero polynomial as in (a), $n_m = a^m + p(m)$, and suppose that n_m were rigid for some ergodic T. Then $l_m = n_{m+1} - an_m = p(m+1) - ap(m)$ would also be rigid for T, contradicting (a), since $p(m+1) - ap(m)$ is a polynomial (of the same degree as $p(m)$). Later we will see that (a^m) itself *is* a rigid sequence for both a weakly mixing transformation (Proposition 3.10) and an ergodic isometry.

Proposition 2.14 can in turn be used to prove the following result. Here it is more convenient to speak of a subset of $\mathbb{N}$ rather than a sequence.

PROPOSITION 2.17. *Suppose that the set A is an infinite subset of $\mathbb{N} \cup \{0\}$ such that $0 \in A$ and for some integers $c_1, \ldots, c_L$ the set of sums*

$$c_1 A + \cdots + c_L A = \left\{ \sum_{l=1}^{L} c_l a_l : a_1, \ldots, a_L \in A \right\}$$

has positive upper density. Then A cannot be rigid for any ergodic transformation.

Remark 2.18. The requirement that $0 \in A$ is purely a notational convenience, so that $c_1 A + \cdots + c_L A$ includes sums of fewer than L elements. Of course, adjoining 0 to a subset of $\mathbb{N}$ does not change its rigidity character.

Proof. Suppose that the condition on the set of sums holds and L is the smallest possible value for which this holds for some $c_1, \ldots, c_L$. Let $B = c_1 A + \cdots + c_L A$ and assume that it has upper density $D > 0$. Let $A_N = A \cap [N, \infty)$ and $B_N = c_1 A_N + \cdots + c_L A_N$. B_N can be obtained from B by deleting a finite set and a finite number of translates of sets of the form $c_{i_1} A + \cdots + c_{i_{L'}} A$ with $L' < L$. Since these sets are all assumed to be of upper density zero, B_N also has upper density D for every N. Thus for each N there is at least one pair $\sigma_1, \sigma_2 \in B_N$ such that $0 < \sigma_2 - \sigma_1 < D^{-1} + 1$. (In fact, there must be infinitely many such pairs.) It follows that there is some d, $0 < d < D^{-1} + 1$, such that for infinitely many N there are $\sigma_1, \sigma_2 \in B_N$ such that $\sigma_2 - \sigma_1 = d$. Hence if we set $F(x_1, \ldots, x_{2L}) = \sum_{k=1}^{L} c_k x_k - \sum_{k=1}^{L} c_k x_{k+L}$ we see that F and A satisfy the hypothesis of Proposition 2.14. □

Remark 2.19. As an example, Proposition 2.17 applies to the set P of prime numbers, because there is a $K > 0$ such that every integer greater than 1 is the sum of at most K

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primes, a result of Schnirelman (see Ramaré [**44**] for a stronger result). This means that the sum of K copies of $P \cup \{0\}$ has density one. Similarly, for any $c \in \mathbb{Z}$ it is easy to see that the sum of K copies of $(P + c) \cup \{0\}$ has bounded gaps and hence positive upper density, so $P + c$ is not a rigid sequence.

Another example of non-rigid sequences is the class of so-called norm-good sequences. Recall that a sequence (n_m) is *norm-good* if for all measure-preserving transformations T, the averages $(1/M)\sum_{m=1}^{M} f \circ T^{n_m}$ converge in L_2-norm for all $f \in L_2(X, \mu)$. See Rosenblatt [**47**] for some background about norm-good averaging methods. In particular, Theorem 1 there shows that (n_m) is norm-good if and only if for each $\gamma \in \mathbb{T}$, the limit $\lim_{M\to\infty}(1/M)\sum_{m=1}^{M} \gamma^{n_m}$ exists. (This is also true if the sequence is just norm-good for all ergodic transformations.) Thus the following result applies to norm-good sequences.

PROPOSITION 2.20. *Suppose that (n_m) is an increasing sequence such that $\lim_{M\to\infty}(1/M)\sum_{m=1}^{M} \gamma^{n_m}$ exists for all but countably many values $\gamma \in \mathbb{T}$. Then (n_m) cannot be a rigidity sequence for an ergodic transformation.*

Proof. Let $h(\gamma)$ denote $\lim_{M\to\infty}(1/M)\sum_{m=1}^{M} \gamma^{n_m}$ if it exists. We know that $h(\gamma) = 0$ on a dense set because $(\gamma^{n_m} : m \geq 1)$ is uniformly distributed almost everywhere (a.e.) with respect to Lebesgue measure; see Kuipers and Niederreiter [**32**]. Suppose that (n_m) is rigid for an ergodic transformation T and let $\nu = \nu_T$ denote the maximal spectral type, normalized to be a probability measure. We recall that ergodicity of T easily implies that the support of ν is $\mathbb{T}$. The rigidity assumption tells us that $\int_{\mathbb{T}} \gamma^{n_m}\, d\nu(\gamma) \to 1$. If γ belongs to the point spectrum of T, that is, $\nu(\gamma) > 0$, then we have $\gamma^{n_m} \to 1$, so $h(\gamma) = 1$. Thus the countably many values where h may not exist all have ν-measure 0, so h exists ν-a.e. It follows that $\int_{\mathbb{T}} h(\gamma)\, d\nu(\gamma) = \lim_{M\to\infty}(1/M)\sum_{m=1}^{M} \int_{\mathbb{T}} \gamma^{n_m}\, d\nu(\gamma) = 1$. But then since $|h| \leq 1$ a.e. with respect to ν, which is a probability measure, it must be that $h(\gamma) = 1$ a.e. with respect to ν. Since ν has full support we conclude that $h = 1$ on a dense set in $\mathbb{T}$. Apply Proposition 27 in Rosenblatt [**47**] to the functions $\phi_M(\gamma) = (1/M)\sum_{m=1}^{M} \gamma^{n_m}$. It follows that h fails to exist on a dense G_δ set, contradicting our assumption that it exists except for a countable set. □

Remark 2.21. We do not know if the class of sequences satisfying the hypotheses in Proposition 2.20 is strictly larger than the class of norm-good sequences.

Proposition 2.20 gives a new proof that a non-constant polynomial sequence (n_m) is not rigid, since $(n_m\alpha)$ is uniformly distributed for all irrational α, that is, for the function h in the proof above, we have $h(\gamma) = 0$ whenever γ is not a root of unity. Of course, the argument we have already seen using successive differences is more elementary. Another interesting source of examples is provided by tempered functions as in Bergelson and Håland [**8**, Lemma 5.12]. See Kuipers and Niederreiter [**32**] for these examples and general information on uniform distribution of sequences. Proposition 2.20 also gives another proof of the fact that $P + c$, where $P = (p_m)$ again denotes the primes, is not rigid for any ergodic map, since $((p_m + c)\alpha)$ is uniformly distributed modulo 1 for any irrational α [**32**]. Indeed, if f is any non-constant polynomial taking integer values on $\mathbb{Z}$, then $(f(p_m))$ cannot be rigid either since $(f(p_m)\alpha)$ is uniformly distributed by a result of Rhin [**45**].

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3. *Examples of sequences which are rigid for some weakly mixing map*

We give a number of different approaches here for constructing weakly mixing transformations that have a specific type of sequence as a rigidity sequence. These methods are sometimes overlapping, but the different approaches give us insights into the issues nonetheless. Also, there are a variety of number-theoretic and harmonic analysis connections with some of these methods; these are also explored in this section.

3.1. *Existence of sets supporting rigid measures.* The most obvious way to obtain a continuous measure on $\mathbb{T}$ which is rigid for a sequence (n_m) is to ensure that it is supported on a set $K \subset \mathbb{T}$ such that $\gamma^{n_m} \to 1$ for all $\gamma \in K$. Accordingly, given (n_m), consider the group

$$\mathcal{R} = \mathcal{R}(n_m) = \Big\{\gamma \in \mathbb{T} : \lim_{m\to\infty} \gamma^{n_m} = 1\Big\},$$

which is a Borel subset of $\mathbb{T}$. Thus, if $\mathcal{R}$ is uncountable, a result of Sierpiński [**53**, p. 228] guarantees that $\mathcal{R}$ contains a closed perfect subset of $\mathbb{T}$ and hence contains a Cantor set, that is, a homeomorphic copy of $\{0, 1\}^{\mathbb{N}}$. We remark that in all the situations we consider we will always have available a direct construction of a closed perfect set K on which $z^{n_m} \to 1$ pointwise so we actually have no need of Sierpiński's result. (Note: for convenience we use the term 'uncountable' for any set which has the cardinality of $\mathbb{R}$.)

It is easy to see that a Cantor set supports a continuum of pairwise mutually singular continuous probability measures ν (for example, take all product measures of the form $(p, 1-p)^{\mathbb{N}}$ on $\{0, 1\}^{\mathbb{N}}$). For any such ν we evidently have $z^{n_m} \to 1$ ν-a.e. so this gives us uncountably many pairwise spectrally disjoint weakly mixing unitary operators for which (n_m) is rigid. Moreover, for any such ν the Gaussian shift G_ν is rigid along (n_m) so (n_m) is rigid for at least one weakly mixing map T. In fact, if we take some extra care, we can find a continuum of such Gaussian shifts which are pairwise spectrally disjoint, and hence disjoint as dynamical systems, in the sense of Furstenberg [**17**].

Remark 3.1. For the proof of the following proposition we will need the notion of an independent subset K of T. The meaning of this is that for every finite subset $\{z_1, \ldots, z_m\}$ of K and $n_1, \ldots, n_m \in \mathbb{Z}$ we have $z_1^{n_1} \cdots z_m^{n_m} = 1$ if and only if $n_l = 0$ for all l, in other words $z_1, \ldots, z_m$ are free (Abelian) generators for the group they generate. A Kronecker subset K of $\mathbb{T}$ is one such that any continuous $\mathbb{T}$-valued function φ on K is a uniform limit of characters of $\mathbb{T}$, that is, there is a sequence (k_m) such that $z^{k_m} \to \varphi$ uniformly on K. Any Kronecker set is independent and it is not hard to construct a closed perfect Kronecker subset of $\mathbb{T}$; see, for example, Rudin [**49**]. In fact, the same argument shows that any closed perfect subset of $\mathbb{T}$ contains a closed perfect Kronecker set, which is *a fortiori* independent. See also [**50**].

PROPOSITION 3.2. *Suppose that $\mathcal{R} = \mathcal{R}(n_m)$ is uncountable. Then there is a continuum $\{K_t, t \in [0, 1]\}$ of pairwise disjoint symmetric closed perfect subsets K_t of $\mathcal{R}$ such that for any choice of continuous symmetric probability measures ν_t, $t \in [0, 1]$, with ν_t supported on K_t, the corresponding Gaussian maps G_{ν_t} are pairwise spectrally disjoint.*

Proof. Given any subset $A \subset \mathbb{T}$ and $n > 0$, we denote $A^n = \{g_1 g_2 \cdots g_n : g_i \in A\}$. Note that A^n is compact whenever A is compact. By Remark 3.1 we can find an independent

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Cantor set $K \subset \mathcal{R}$. It is elementary that K is a disjoint union of Cantor subsets L_t, $t \in [0, 1]$. (In the proof of this fact the subsets are naturally indexed by $\{0, 1\}^{\mathbb{N}}$ rather than [0,1].) We let $K_t = K_t \cup (K_t)^{-1}$ be the symmetrization of L_t. The independence of K implies that the K_t, $t \in [0, 1]$, are pairwise disjoint. Now suppose that, for each t, ν_t is a continuous symmetric probability measure supported on K_t. The maximal spectral type σ_t of the Gaussian map G_{ν_t} is $\sum_{n\geq 1} 2^{-n} \nu_t^{*n}$, which is supported on the F_σ-set $\widehat{K}_t = \bigcup_{n\geq 1} K_t^n$. Now for $t \neq t'$ the independence of K easily implies that $\widehat{K}_t \cap \widehat{K}_{t'} = \{1\}$. Since each σ_t is continuous, so $\sigma_t\{1\} = 0$, it follows that σ_t and $\sigma_{t'}$ are supported on disjoint sets, as desired. □

Thus at least in this situation there are uncountably many different maps with the same rigidity sequence.

Question 3.3. In general, if (n_m) is a rigid sequence for some weakly mixing map T, is it a rigid sequence for uncountably many different weakly mixing T? Here 'different' could mean, for example, spectrally disjoint, disjoint or simply non-isomorphic, in decreasing order of strength.

Remark 3.4. Similarly, given (n_m) and the IP set Σ it generates, we can define

$$\mathcal{R}' = \{z : z^\sigma \to 1 \text{ as } \sigma \to \infty \text{ (IP)}\}.$$

If $\mathcal{R}'$ is uncountable, then, just as in Proposition 3.2, we again obtain a continuum of pairwise spectrally disjoint Gaussian shifts having (n_m) as an IP-rigidity sequence.

If an ergodic T has discrete spectrum and $T^{n_m} \to \mathrm{Id}$ then $z^{n_m} \to 1$ for each z belonging to the countably infinite point spectrum of T. Thus, if $\mathcal{R}(n_m)$ is finite, then no ergodic transformation with discrete spectrum has this sequence as a rigidity sequence. Moreover, if $\mathcal{R}(n_m) = \{1\}$ and (n_m) is rigid for an ergodic T then the point spectrum of T is reduced to $\{1\}$ so T is necessarily weakly mixing.

Question 3.5. Is there a sequence (n_m) such that $\mathcal{R}(n_m)$ is finite but (n_m) is rigid for some weakly mixing map?

A possible candidate is given by $n_m = 2^m + 3^m$, for which $\mathcal{R}(n_m) = \{1\}$. To see this, note that $n_{m+1} - 2n_m = 3^m$ and $3n_m - n_{m+1} = 2^m$. So any $\gamma \in \mathcal{R}(n_m)$ must have both γ^{3^m} and γ^{2^m} tending to 1 as $m \to \infty$. It follows that γ is both a dyadic and a triadic root of unity and hence $\gamma = 1$. We know that (2^m) and (3^m) are rigid sequences for some weakly mixing map T (see Proposition 3.10 below) but we do not know if this is the case for $(2^m + 3^m)$, and, as we have observed, in this case ergodicity of T would already force weak mixing.

Transferring from the circle to [0, 1) (but retaining the topology of the circle) the uncountability of $\mathcal{R}(n_m)$ is equivalent to the statement that

$$\lim_{m\to\infty} \|n_m x\| = 0, \tag{3.1}$$

holds for uncountably many $x \in [0, 1)$. Here $\|t\| = d(t, \mathbb{Z})$ for $t \in \mathbb{R}$. The following is a result of Eggleston [**13**], which is not hard to prove.

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PROPOSITION 3.6. *If* $\lim_{m\to\infty}(n_{m+1}/n_m) = \infty$ *then equation (3.1) holds on a closed perfect subset of* [0, 1). *Hence by Proposition 3.2 there is a continuum of pairwise spectrally disjoint weakly mixing T for which* (n_m) *is rigid.*

Remark 3.7. In the case where n_{m+1}/n_m is always an integer, Eisner and Grivaux [**12**] have shown that the hypothesis in Proposition 3.6 can be weakened to $\limsup(n_{m+1}/n_m) = \infty$. On the other hand, Eggleston also shows in [**13**] that if n_{m+1}/n_m is bounded then (3.1) holds on at most a countable set.

For IP-rigidity we need the following lemma.

LEMMA 3.8. *Given a sequence* (n_m), *let* Σ *denote the* IP *set it generates and* σ *a typical element of* Σ. *For* $x \in [0, 1)$ *the following are equivalent.*

(1) $\|\sigma x\| \to 0$ *as* $\sigma \to \infty$ *(IP).*

(2) $\sum_m \|n_m x\| < \infty$.

Proof. First suppose that $\sum_m \|n_m x\| < \infty$. Given $\epsilon > 0$, choose M such that $\sum_{m\geq M} \|n_m x\| < \epsilon$. Then for any $m_1, m_2, \ldots, m_k > M$,

$$\|(n_{m_1} + \cdots + n_{m_k})x\| \leq \|n_{m_1 x}\| + \cdots + \|n_{m_k} x\| < \epsilon.$$

This shows that $\|\sigma x\| \to 0$ as $\sigma \to \infty$ (IP).

Now suppose that $\|\sigma x\| \to 0$ as $\sigma \to \infty$ (IP) and consider first the case where x is irrational. Choose M such that for any distinct $m_1, m_2, \ldots, m_k > M$ we have $\|(n_{m_1} + \cdots + n_{m_k})x\| \leq \frac{1}{100}$. In particular, $\|n_m x\| < \frac{1}{100}$ for $m > M$. For $t \in \mathbb{R}$ let $\{t\}$ denote the fractional part of t, so $\|t\|$ is either $\{t\}$ or $1 - \{t\}$, as $\{t\} \leq \frac{1}{2}$ or $\{t\} > \frac{1}{2}$.

Let

$$E_0 = \{m > M : \{n_m x\} < \tfrac{1}{2}\} \quad \text{and} \quad E_1 = \{m > M : \{n_m x\} > \tfrac{1}{2}\}.$$

Now suppose that $\sum_m \|n_m x\| = \infty$. Then either $\sum_{m\in E_0} \|n_m x\|$ or $\sum_{m\in E_1} \|n_m x\|$ is infinite. Suppose for the sake of argument that we have the first case (the second case is similar). Let $E_0 = \{m_1 < m_2 < \cdots\}$ and let k denote the minimal value such that $\sum_{i=1}^k \|n_{m_i} x\| > \frac{1}{2}$. Since each $\|n_{m_i} x\| = \{n_{m_i} x\}$ and $0 < \{n_{m_i} x\} < \frac{1}{100}$ we see that

$$\frac{1}{2} < \left\{\sum_{i=1}^k n_{m_i} x\right\} = \sum_{i=1}^k \{n_{m_i} x\} = \sum_{i=1}^k \|n_{m_i} x\| < \frac{1}{2} + \frac{1}{100}.$$

Thus $\|\sum_{i=1}^k n_{m_i} x\| > \frac{1}{2} - \frac{1}{100}$ which contradicts our choice of M.

In the case where $x = p/q$ is rational then $\|n_m x\|$ is either zero or at least $1/q$ so a similar but simpler argument shows that both conditions on x are equivalent to the fact that $\|n_m x\| = 0$ eventually. □

Now we quote the following result of Erdős and Taylor [**14**], whose proof is straightforward.

PROPOSITION 3.9. *If* $\sum_m (n_m/n_{m+1}) < \infty$ *then* $\sum_m \|n_m x\| < \infty$ *holds on a closed perfect subset of* [0, 1). *Hence by Lemma 3.8 and Remark 3.4 there is a continuum of Gaussian shifts which are* IP*-rigid along* (n_m) *and pairwise spectrally disjoint.*

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3.2. *Symbolic approach.* In this section we consider the case of sequences (n_m) such that n_m divides n_{m+1} for every m. In this case the problem of constructing a rigid measure can be transferred in a natural way to a sequence space where there is a natural way to construct a measure with the desired rigidity. See also Propositions 3.15 and 3.38 for other approaches to the following result.

PROPOSITION 3.10. *Given an increasing sequence (n_m) such that n_{m+1}/n_m is always an integer strictly greater than* 1*, there is a weakly mixing T which is rigid along (n_m).*

Proof. Transferring the problem to the unit interval, by Proposition 2.10 it suffices to construct a continuous probability measure ν on $[0, 1]$ such that for all $\epsilon > 0$,

$$\nu\{t \in [0, 1] : 0 \le \{n_m t\} < \epsilon\} \to 1. \tag{3.2}$$

Let $a_1 = n_1$ and $a_m = n_{m+1}/n_m$ for $m > 1$, $\mathbf{a} = (a_m)$, $A_m = \{0, 1, \ldots, a_m - 1\}$ and for each interval $I \subset \mathbb{N}$ (finite or infinite) denote by $A(I)$ the space of functions (sequences) $x : I \to \mathbb{N}$ such that $x(m) \in A_m$ for each $m \in I$. For each $M \ge 1$ define $\varphi_M : A(\mathbb{N}) \to [0, 1]$ by

$$\varphi_M(x) = \sum_{m \ge M} \frac{x(m)}{n_m},$$

which is continuous and injective off a countable subset of $A(\mathbb{N})$. In other words, $\varphi_1(x)$ is the number whose $\mathbf{a}$-adic expansion is x. Note that we have the relation $\{n_m \varphi_1(x)\} = \varphi_m(x)$.

Now define a probability measure π on $X = A(\mathbb{N})$ as follows. For each integer $k \ge 0$ let $N_k = k(k+1)/2$ so that $N_{k+1} - N_k = k + 1$ for $k \ge 1$. For each $k \ge 1$ let $I_k = \{N_k, N_k + 1, \ldots, N_{k+1} - 1\}$. Let π_k denote the probability measure on $A(I_k)$ which gives mass $1 - 1/(k+1)$ to the constant sequence $\overline{0} \in A(I_k)$ and is equidistributed over the remaining points of $A(I_k)$. Then define π on $A(\mathbb{N})$ as the product of the π_k. π is continuous since the mass of the constant sequence $\overline{0} \in A(\mathbb{N})$ is $\prod_{k \ge 1}(1 - 1/(k+1)) = 0$ and clearly all other points of $A^{\mathbb{N}}$ have smaller mass. We define $\nu = \pi \circ \varphi_1^{-1}$ and we claim that (3.2) holds.

To see this, first note that if $x \in A(\mathbb{N})$ and $x_m, x_{m+1}, \ldots, x_{m+l-1}$ are all 0 then $\varphi_m x \le 2^{-l}$, since each $a_m \ge 2$. Define D_k to be the set of all $x \in A(\mathbb{N})$ such that $x(m) = 0$ for all $m \in I_k \cup I_{k+1}$ and note that for each k we have $\mu(D_k) = (1 - 1/k)(1 - 1/(k+1))$. For $m \in \mathbb{N}$ let $k(m) = k$ if $m \in I_k$. Observe that if $x \in D_{k(m)}$ then at least the first $k(m) + 2$ entries of $(x_m, x_{m+1}, \ldots)$ are 0 (at least one in $I_{k(m)}$ and then exactly $k(m) + 1$ in $I_{k(m)+1}$). Thus

$$\{n_m \varphi_1(x)\} = \varphi_m(x) < 2^{-k(m)-2} < 2^{-k(m)}.$$

This shows that for a fixed l, once $k(m) > l$ then

$$\varphi^{-1}\{t \in [0, 1] : \{n_m t\} < 2^{-l}\} \supset D_{k(m)},$$

so

$$\nu(\varphi^{-1}\{t \in [0, 1] : \{n_m t\} < 2^{-l}\}) > \mu(D_{k(m)}) \to 1 \quad \text{as } m \to \infty. \qquad \square$$

COROLLARY 3.11. *Given any integer $a \ge 2$, the sequence $(a^m : m \ge 1)$ is rigid for some weakly mixing dynamical system.*

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Remark 3.12.

(a) It is interesting to note that, for example, (2^m) is a rigid sequence for both a weakly mixing map and an ergodic isometry (the binary odometer) while $(2^m + 1)$ is not rigid for any ergodic map (Remark 2.16(b)).

(b) Eisner and Grivaux [**12**] give an inductive construction of a sequence (n_m) satisfying

$$\lim_{m\to\infty} \frac{n_{m+1}}{n_m} = 1$$

that is rigid for some weakly mixing dynamical system. A famous concrete example of a sequence with this 'anti-lacunarity' property is $\{2^k 3^l : k, l \geq 1\}$. Is this sequence rigid for some weakly mixing T? Note: results of Ajtai *et al* [**5**] show that for any sequence (ϵ_m) decreasing to 0, no matter how slowly, there exists an increasing sequence (n_m) such that $n_{m+1}/n_m \geq 1 + \epsilon_m$ and n_m is not rigid for any ergodic system. Indeed, they give examples where $\lim_{M\to\infty}(1/M)\sum_{m=1}^{M} \exp(2\pi i n_m x) = 0$ for all $x \in (0, 1)$, and hence for any positive Borel measure $\nu \neq \delta_0$, $\lim_{M\to\infty}(1/M)\sum_{m=1}^{M} \widehat{\nu}(n_m) = 0$.

Now we turn to the question of IP-rigidity for sequences as in Proposition 3.10. Retaining the notation in the proof of that Proposition, if T denotes the **a**-odometer then it is easy to see that (n_m) is IP-rigid for T. (In terms of the rank-one constructions described in §3.3, T is the map obtained by starting with a tower of height a_1 and cutting the mth tower into a_{m+1} slices and stacking with no spacers.) Then T is, of course, an ergodic isometry. If (n_m) goes to infinity sufficiently quickly then Proposition 3.9 shows that we can also have IP-rigidity for a weakly mixing T. However, if n_{m+1}/n_m is bounded the following result shows that we never have IP-rigidity for a weakly mixing T.

PROPOSITION 3.13. *Suppose that $(n_m)_{m\geq 1}$ is an increasing sequence of positive integers and that n_{m+1}/n_m is always an integer greater than* 1 *and less than C for some constant C. Then the sequence (n_m) is not a sequence of* IP*-rigidity for any weakly mixing map.*

Proof. Every positive integer k has a unique **a**-adic expansion $k = \sum_{l\geq 0} x_l n_l$, where the x_l are integers such that $0 \leq x_l < a_l$ and all but finitely many are zero. We write $x_l =: x_l(k)$ to indicate the dependence on k. Now suppose that T is weakly mixing and IP-rigid along (n_m). The meaning of this in terms of **a**-adic expansions is that for all $\epsilon > 0$ there is an M such that for any k satisfying

$$x_l(k) = 0 \quad \text{for } 0 \leq l \leq M \quad \text{and} \quad x_l(k) \in \{0, 1\} \quad \text{for } l > M \tag{3.3}$$

we have $T^k \overset{\epsilon}{\sim} \mathrm{Id}$, the approximation being with respect to a metric compatible with the strong topology. Now it is clear that any integer k with $x_l(k) = 0$ for $0 \leq l \leq M$, that is, any integer k divisible by n_M, can be expressed as the sum of at most C integers satisfying (3.3). It follows that for all k divisible by n_M we have $T^k \overset{C\epsilon}{\sim} \mathrm{Id}$. In other words,

$$T^{n_M r} \overset{C\epsilon}{\sim} \mathrm{Id} \quad \text{for all } r \geq 1. \tag{3.4}$$

Fixing any $f \in L_{2,0}(\mu)$ of norm 1, if $C\epsilon$ is sufficiently small (3.4) implies that $\|T^{rn_M} f - f\|_2 < \frac{1}{4}$, whence $|\langle T^{rn_M} f, f\rangle| > \frac{3}{4}$. This contradicts the fact that, since T^{n_M} is weakly mixing, there is a set of r of density one along which $\langle T^{rn_M} f, f\rangle \to 0$. □

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Question. If $\rho_m = n_{m+1}/n_m$ is bounded, but not always an integer, is it possible for (n_m) to be a sequence of IP-rigidity for some weakly mixing transformation? If not, then how fast does (ρ_m) have to grow for (n_m) to be a sequence of IP-rigidity for some weakly mixing transformation? In other words, can we improve on Proposition 3.9?

3.3. *Rank-one constructions.* In this section, given an increasing sequence (n_m) such that either $n_{m+1}/n_m \to \infty$ or n_{m+1}/n_m is an integer greater than 1 for each m, we will explicitly construct an infinite measure-preserving rank-one map T such that (n_m) is rigid for T. Here, as in the finite measure-preserving case, rigid means that $T^{n_m} \to \mathrm{Id}$ in the strong operator topology which is equivalent to $\mu(T^{n_m} E \triangle E) \to 0$ for every set of finite measure. Then, as observed earlier in §2.1, either the Poisson suspension or the Gaussian shift corresponding to any spectral measure for T is weakly mixing and rigid along (n_m).

If $\sum_{m\geq 1}(n_m/n_{m+1}) < \infty$, we also give a direct construction of a finite measure-preserving rank-one T which is IP-rigid along (n_m). While Proposition 3.9 already provides examples of weakly mixing T IP-rigid along (n_m), the rank-one construction is more concrete and is of independent interest. Moreover, this construction will be used again in the proof of Proposition 4.12.

We assume that the reader has some familiarity with rank-one constructions, but the following is a quick refresher. We will not discuss rank-one maps in the abstract but simply give a recipe for their construction. For more details about the theory of rank-one maps, see [**40**] or [**16**].

Any rank-one T may be realized concretely as a so-called cutting and stacking construction yielding a measure-preserving map of a finite or infinite interval $X \subset \mathbb{R}$. We now proceed to describe the process of cutting and stacking. By a *tower* τ we mean a finite ordered collection $\{I_0, I_1, \ldots, I_{h-1}\}$ of pairwise disjoint finite intervals in $\mathbb{R}$ of equal length whose union is an interval. Then τ defines a map $T = T_\tau$ by decreeing that, for $l < h-1$, T maps I_l to I_{l+1} by translation. The intervals I_l are called the *levels* of τ, h is the *height* of τ, and the *width* of τ is the common length of the levels. We also denote by $|\tau|$ the union of the levels of τ. We envision the levels $I_0, I_1, \ldots, I_{h-1}$ stacked above each other in order, with I_0 at the bottom, I_1 directly above it, and so on up to I_{h-1} at the top. Thus T maps each level (except for the top) to the level above it by moving each point to the point which sits directly above it in the tower. T is (not yet) defined on the top level.

If $\tau = \{I_i : 0 \leq i < h\}$ and $\tau' = \{I_i : h \leq i < h + h'\}$ are two towers of equal width such that $|\tau| \cap |\tau'| = \varphi$ and $|\tau| \cup |\tau'|$ is again an interval then the *concatenation* $\tau * \tau'$ is defined as the tower $\{I_i : 0 \leq i < h'\}$. Note that $T_{\tau*\tau'}$ is compatible with T_τ and $T_{\tau'}$ but is also defined on I_{h-1}, where T_τ was not. This concept extends in an obvious way to more than two towers. By a *refinement* of τ we mean a tower τ' obtained from τ as follows. We choose an integer $c \geq 2$ and slice τ into c subtowers $\tau^1, \ldots, \tau^c$ of height h and of equal width, by dividing the base I_0 into subintervals $I_0^1, \ldots, I_0^c$ of equal width w and taking τ^i to be the subtower of height h over I_0^i. We also choose an integer $s \geq 0$ and take s disjoint intervals $J_1, \ldots, J_s$ of width w which are contained in $\mathbb{R}\backslash|\tau|$ and such that $|\tau| \cup \bigcup_{i=1}^{j} J_i$ is again an interval. We will refer to the J_i as *spacers*. Now we define τ' to

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be the concatenation

$$\sigma_0 * \tau_1 * \sigma_1 * \tau_2 * \cdots * \sigma_{c-1} * \tau_c * \sigma_c,$$

where the σ_i are disjoint towers whose levels come from the spacers and together exhaust the spacers. We will refer to the τ_i as copies of τ in τ'.

Now we define T by starting with a tower τ_0 of height 1 consisting of a single level X_0 and then for $n > 0$ inductively defining τ_{n+1} to be any refinement of τ_n. Letting $X_n = |\tau_n|$ and $X = \bigcup_{n\geq 1} X_n$, then X is an interval and T is defined a.e. on X by defining T to be T_{τ_n} on X'_n, where X'_n is X_n with the top level removed. It is easy to see that T is an automorphism of a co-null subset of X. Note that X may be finite or infinite depending on how much spacer material is added at each stage of the construction. A key point is that if we are given any subset E of X having finite measure and $\epsilon > 0$ then, for all sufficiently large n, E can be approximated within ϵ by a union of levels of τ_n. A standard approximation argument shows that T is always ergodic so if X has infinite measure it follows that T is weakly mixing. We remark that, up to isomorphism, T is completely determined by the description we have given, that is, it does not depend on the exact intervals chosen nor on what their ordering as subsets of $\mathbb{R}$ is.

We let $S_n = X_{n+1} \backslash X_n$ and $\epsilon_n = \mu(S_n)/\mu(X_{n+1})$, the fraction of the levels of τ_{n+1} which are not contained in a level of τ_n; that is, they are spacers added at stage $n+1$ of the construction. We observe that X is infinite precisely when $\sum_{n=1}^{\infty} \epsilon_n = \infty$.

Now we establish a general scheme for showing that a sequence (n_m) is rigid for the rank-one T just constructed. We will say that a time $l > 0$ is ϵ-rigid for a set E if $T^l E \overset{\epsilon\mu(E)}{\sim} E$. For a fixed τ_M we will say that a time $l > 0$ is ϵ-rigid for τ_M if l is ϵ-rigid for each level of τ_M, or equivalently if l is ϵ-rigid for the base of τ_M. Note that if this is the case then l is ϵ-rigid for any union of levels of τ_M. Consequently, if l is ϵ-rigid for τ_M then it is also ϵ-rigid for any $\tau_{M'}$, $M' \leq N$.

LEMMA 3.14. *Suppose that we have a sequence $\epsilon_M \to 0$ and that for every m there is an $M(m)$ such that $(M(m))$ increases to infinity and (n_m) is $\epsilon_{M(m)}$-rigid for $\tau_{M(m)}$. Then (n_m) is rigid for T. If we assume in addition that $\sum_{M\geq 1} \epsilon_M < \infty$ then (n_m) is* IP*-rigid for T.*

Proof. Suppose first that $\epsilon_M \to 0$. For any set E which is a union of levels of some τ_M once $M(m) \geq M$ we will know that (n_m) is $\epsilon_{M(m)}$-rigid for τ_M so we see that (n_m) is rigid for E. Since the class of such sets E is dense in the class of sets having finite measure, it follows that (n_m) is rigid for T.

Now suppose that $\sum_{M\geq 1} \epsilon_M < \infty$. As we have already observed, we know that if E is a union of levels of some τ_M then there is an m_0 such that

$$\mu(T^{n_m} E \triangle E) < \epsilon_{M(m)} \mu(E) \text{ once } m \geq m_0. \tag{3.5}$$

Now suppose that $(\delta_m) \in \{0, 1\}^{\mathbb{N}}$ is a sequence which is 0 off the interval

$$\{m_0, m_0 + 1, \ldots, m_0 + k\},$$

where $k \geq 0$ is arbitrary, and set $n = \sum_{m=m_0}^{m_0+k} \delta_m n_m$. Applying the estimate (3.5) repeatedly, we see that

$$\mu(T^n E \Delta E) < \left(\sum_{m=m_0}^{m_0+k} \delta_m \epsilon_m\right) \mu(E) < \left(\sum_{m\geq m_0} \epsilon_m\right)(\mu(E)).$$

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Since the tail of the sum goes to 0 this shows that E is IP-rigid along (n_m). As before, we can now extend this to all $E \in \mathcal{B}$, so (n_m) is IP-rigid for T. □

PROPOSITION 3.15. *Suppose that either $n_{m+1}/n_m \to \infty$ as $m \to \infty$ or $n_{m+1}/n_m \geq 2$ is an integer for all m. Then in each case there exists an infinite measure-preserving rank-one transformation T such that T is rigid along (n_m). Consequently, via the GMC, (n_m) is rigid for some weakly mixing finite measure-preserving T'.*

Proof. We will use the notation established in our description of rank-one constructions and in Lemma 3.14 throughout the following proof. Suppose first that $n_{m+1}/n_m \to \infty$ as $m \to \infty$. We introduce the notation $h_m := n_m$ as, in this case, these will be the heights of the rank-one towers we construct. Write $h_{m+1} = q_m h_m + r_m$, $0 \leq r_m < h_m$. Define $p_m < q_m$ to be the least integer $l \geq 0$ such that $(r_m + lh_m)/h_{m+1} > 1/m$ (p_m may be zero) and let $\epsilon_m = (r_m + p_m h_m)/h_{m+1}$. Thus we have $\sum_{m=1}^{\infty} \epsilon_m = \infty$. Moreover, $0 < \epsilon_m - 1/m < h_m/h_{m+1} \to 0$ so $\epsilon_m \to 0$.

Construct T as follows. Start with a tower τ_1 of height h_1 and suppose that the towers $\tau_1, \ldots, \tau_m$ have been constructed. Form τ_{m+1} by slicing τ_m into $s_m := q_m - p_m$ columns, stacking these directly above each other and then following them by $h_{m+1} - s_m h_m = r_m + p_m h_m$ spacers to create a tower τ_{m+1} of height h_{m+1}. Thus $\mu(S_m)/\mu(X_{m+1}) = \epsilon_m$, and since $\sum_{m=1}^{\infty} \epsilon_m = \infty$ we see that the measure of the space we have constructed is infinite.

We now check that T is rigid along (h_m). If E is a level of τ_m and $E_1, \ldots, E_{s_m}$ are its pieces in τ_{m+1} then $T^{h_m} E_i = E_{i+1}$, except for $i = s_m$. It follows that $\mu(T^{h_m} E \backslash E) < (1/s_m)\mu(E)$ so h_m is $(2/s_m)$-rigid for τ_m. Since $s_m \to \infty$ the rigidity follows by Lemma 3.14.

Now suppose that n_{m+1}/n_m is an integer greater than 1 for all m. For simplicity we will consider only the case $n_m = 2^m$. The general case would be notationally more complicated but essentially the same. Let $h_M = 2^{M^2}$ so

$$q_M := \frac{h_{M+1}}{h_M} = 2^{2M+1} \to \infty.$$

Let $p_M \geq 0$ be the least integer r so that $r/q_M \geq 1/M$ and let $\epsilon_M = p_M/q_M$. As before, we have $\sum_{M=1}^{\infty} \epsilon_M = \infty$ and $\epsilon_M \to 0$.

We construct the towers τ_M for T as before, by concatenating $s_M := q_M - p_M$ copies of τ_M and adding $p_M h_M$ spacers to get the tower τ_{M+1} of height h_{M+1}. As before, the space on which T acts has infinite measure and we need only check the rigidity of the sequence $(n_m) = (2^m)$.

For $m > 0$ let $M = M(m) > 0$ be the integer such that $M^2 \leq m < (M+1)^2$. We want to see that $n_m = 2^m$ is approximately rigid for τ_M. So suppose that E is a level of τ_M. E is a union of levels $E_1, \ldots, E_l$, $l = s_M s_{M+1}$, of τ_{M+2}. These occur with period h_M in τ_{M+2}, except for gaps occupied by the spacers in S_M and S_{M+1}. Since there is a block of spacers at the top of τ_{M+2} of length at least $h_{M+1} > 2^m$ we see that for each i, $1 \leq i \leq l$, $T^{2^m} E_i$ is still a level of τ_{M+2}. Thus, since h_M divides 2^m, if $T^{2^m} E_i$ is not contained in E it must lie in $S_M \cup S_{M+1}$. It follows that

$$\mu(T^{2^m} E \backslash E) \leq \mu(E)\left(\frac{p_M}{q_M} + \frac{p_{M+1}}{q_{M+1}}\right) = \mu(E)(\epsilon_M + \epsilon_{M+1}),$$

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so 2^m is $2(\epsilon_M + \epsilon_{M+1})$-rigid for τ_M. Since $M = M(m) \to \infty$ as $m \to \infty$ and $\epsilon_M \to 0$ as $M \to \infty$ it follows by Lemma 3.14 that (2^m) is rigid for T. □

PROPOSITION 3.16. *Suppose that* $\sum_{m\geq 1}(n_m/n_{m+1}) < \infty$. *Then there is a finite measure-preserving, weakly mixing, rank-one transformation* T *that is* IP*-rigid along* (n_m).

Proof. By omitting the first few terms of (n_m) we may assume without loss of generality that $n_{m+1} > 3n_m$. Let $n_{m+1} = q_m n_m + r_m$, where $n_m + 1 \leq r_m \leq 2n_m$. We construct the rank-one towers τ_m of height $h_m := n_m$ for T as follows. Start with a tower τ_1 of height h_1. Once τ_m has been defined with height h_m we take τ_{m+1} to be the concatenation of $[q_m/2]$ copies of τ_m, a single spacer, $q_m - [q_m/2]$ copies of τ_m and then $r_m - 1$ spacers, so τ_{m+1} has height h_{m+1}. The resulting T is finite measure-preserving because

$$\sum_{m\geq 1} \frac{r_m}{n_{m+1}} < \sum_{m\geq 1} \frac{2n_m}{n_{m+1}} < \infty$$

by our hypothesis.

Now we claim that

$$n_m \text{ is } \epsilon_m\text{-rigid for } \tau_m, \quad \text{where } \epsilon_m = \frac{4n_m}{n_{m+1}}. \tag{3.6}$$

If E is a level of τ_m and F is one of its pieces in τ_{m+1} first note that $T^{n_m}F$ is still a level of τ_{m+1}, since we have at least n_m spacers at the end of τ_{m+1}. So, from the structure of τ_{m+1} it is clear that $T^{n_m}F \subset E$ unless F is in one of the two copies of τ_m preceding a block of spacers in τ_{m+1}. It follows that $\mu(T^{n_m}E\backslash E) \leq 2n_m/n_{m+1}$ which implies (3.6). The IP-rigidity follows by Lemma 3.14.

Now we give a classical argument that shows that T is weakly mixing, due to the presence of the solitary spacer. Suppose that $f \in L_2(X, \mu)$, $|\lambda| = 1$ and $Tf = \lambda f$. Then $|f|$ is invariant and since T is ergodic we may assume that $|f| = 1$. Given $\epsilon > 0$ we can find m and f' which is a linear combination of the characteristic functions of the levels of τ_m such that $\|f - f'\|_2 < \epsilon$. In addition, we may assume that $|f'| = 1$ on X_m.

Note that for any k,

$$T^k f' \overset{\epsilon}{\sim} T^k f = \lambda^k f \overset{\epsilon}{\sim} \lambda^k f',$$

so

$$\|T^k f' - \lambda^k f'\|_2 < 2\epsilon, \text{ that is, } \int_X |T^k f' - \lambda^k f'|^2 \, d\mu < 4\epsilon^2.$$

Let E_1 denote the union of the first $l := [q_m/4]h_m$ levels of τ_{m+1} and E_2 the union of the l levels before the first spacer in τ_{m+1}. We observe that $(T^l f')|_{E_1} = f'|_{E_1}$ so

$$\begin{aligned} 4\epsilon^2 > \int_X |T^l f' - \lambda^l f'|^2 \, d\mu &> \int_{E_1} |T^l f' - \lambda^l f'|^2 \, d\mu \\ &= \int_{E_1} |f' - \lambda^l f'|^2 \, d\mu < 4\epsilon^2 = |1 - \lambda^l|^2 \int_{E_1} |f'|^2 \, d\mu \\ &= |1 - \lambda^l|^2 \mu(E_1). \end{aligned}$$

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If m is sufficiently large then $\mu(E_1) > \frac{1}{5}$ so we conclude that $|1 - \lambda^l| < \sqrt{20\epsilon} =: \epsilon'$. Now similar considerations with E_2 and $l+1$ replacing E_1 and l show that $|1 - \lambda^{l+1}| < \epsilon'$. It follows that $2\epsilon' > |\lambda^l - \lambda^{l+1}| = |1 - \lambda|$. Since $\epsilon' > 0$ is arbitrary we conclude that $\lambda = 1$. □

Remark 3.17. Aaronson *et al* [**2**, Proposition 1.3] have recently proved a generalization of Proposition 3.16 in which the summability is required only over an infinite set of indices n, and for the remaining n one has that b_n divides b_{n+1}. Their proof proceeds via a similar rank-one construction.

3.4. *Rates of growth.* Aaronson [**1**, Theorem 4] has proved what seems to be the best possible theorem about how slowly a rigid sequence for a weakly mixing map can grow. His result can be restated in the following form.

PROPOSITION 3.18. (Aaronson) *Given any sequence* (a_m) *of density* 0*, there is a sequence* (b_m) *such that* $b_m < a_m$ *for all* m *and* b_m *is rigid for some weakly mixing map.*

This is surprising in view of the fact that all of the concrete examples we have in this paper of rigid sequences for weak mixing have exponential growth. In particular, for example, Aaronson's result implies that there is a sequence (n_m) which is rigid for weak mixing such that $n_m < m \log m$. The proof of Aaronson's theorem is not easy so we make some remarks in this section about some weaker statements which can be proved quite easily. Specifically, we prove a result analogous to Aaronson's for ergodic isometries and also a weaker version of the weak mixing result which gives, rather than $b_m < a_m$ for all m as in Aaronson's result, just $b_m < a_m$ infinitely often.

PROPOSITION 3.19. *If* T *is an ergodic isometry and* (g_m) *is a sequence such that* $g_m \to \infty$ *then there is a rigid sequence* n_m *for* T *such that* $n_{m+1} - n_m < g_m$ *for all* $m \geq 1$.

Proof. We review some well-known facts about ergodic isometries. Fixing a point x_0 with dense orbit, since T is an isometry for each fixed k we have that $d(x, T^k x)$ is constant on the orbit of x_0. Since the orbit is dense it follows that $d(x, T^k x)$ is constant for all $x \in X$. We also see that the orbit of any point is dense, that is, T is minimal. Thus if U is any open set then there is a $d > 0$ such that $\bigcup_{k=0}^{d-1} T^k U = X$, that is, every orbit interval of length k contains at least one point in U. Using the uniform metric on the space of self-maps of X,

$$d_u(S_1, S_2) = \sup_x d(S_1 x, S_2 x),$$

it follows from the above remarks that, given any $\epsilon > 0$, there is a $d > 0$ such that any integer interval of length d contains at least one k such that $d(T^k, \mathrm{Id}) < \epsilon$. Finally, we observe that (n_m) is rigid for T if and only if $d_u(T^{n_m}, \mathrm{Id}) \to 0$.

Take any sequence $\epsilon_i \to 0$ and for each i find d_i such that any integer interval of length d_i contains at least one k such that $d_u(T^k, \mathrm{Id}) < \epsilon_i$. Now for each i find m_i such that $g_m > d_i$ for all $m > m_i$. We may assume without loss of generality that (m_i) is strictly increasing. Now we construct the sequence (n_m) inductively as follows. Start by defining $n_1 = 1$. Suppose that $n_1, n_2, \ldots, n_m$ have been chosen. Now, if $2 \leq m < m_1$ simply define $n_{m+1} = n_m + g_m - 1$. If $m_i \leq m < m_{i+1}$ then by our choice of m_i we may choose n_{m+1}

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so that $n_m < n_{m+1} \le n_m + d_i$ and $d(T^{n_{m+1}}, \mathrm{Id}) < \epsilon_i$. Thus $n_{m+1} - n_m \le d_i < g_m$ since $m > m_i$. Moreover, we have $d_u(T^{n_m}, \mathrm{Id}) \to 0$ so (n_m) is rigid for T. □

Passing to the weakly mixing case, we can show that one can have rigidity sequences whose density is infinitely often greater than a prescribed sequence going to 0. It is easy to see that this implies, as we claimed earlier, that given any sequence (a_m) of density 0 there is a sequence (b_m) such that $b_m < a_m$ for infinitely many m and (b_m) is rigid for weak mixing.

PROPOSITION 3.20. *Given any sequence $(d_N : N \ge 1)$ such that $d_N \to 0$ as $N \to \infty$, there exist a weakly mixing transformation and a rigidity sequence $\mathbf{n} = (n_m)$ for T such that $D(N, \mathbf{n}) > d_N$ for infinitely many N.*

Proof. We will informally describe how to inductively construct a rank-one map on a probability space with the desired property. So, suppose that, at the nth stage of the construction, we have arrived at a tower τ_n of height h_n. Then choose $N_n > h_n$ large enough so that $d_{N_n} < 1/2h_n$. A further condition will be imposed on N_n later. Define the tower τ_{n+1} at the next stage to be the concatenation of N_n^2 copies of τ_n followed by a single spacer and then another N_n^2 copies of τ_n. Doing this for all n defines the rank-one map T acting on the space X, which is a finite interval endowed with its normalized Lebesgue measure μ, but we are still free to impose additional growth conditions on (N_n). In particular, once τ_n has been defined we can ensure that $\mu(|\tau_n|)$ is as close to 1 as we please by choosing N_i, $i \ge n$ sufficiently large. Evidently, X has finite measure and T will be weakly mixing due to the single spacer in the middle of τ_{n+1}, as in the proof of Proposition 3.16.

Now consider $E_n = h_n\mathbb{N} \cap \{1, \ldots, N_n\}$. Note that E_n is non-empty since $N_n > h_n$. Given any sequence $\epsilon_n \to 0$, we can ensure that all $l \in E_n$ are ϵ_n-rigid for τ_n simply by making N_n sufficiently large. Thus $E = \bigcup_n E_n$ is a rigid sequence for T. Finally, note that since E_n is non-empty $D(N_n, E_n) > 1/2h_n > d_{N_n}$, so certainly $D(N_n, E) > d_{N_n}$. □

3.5. *Disjoint rigidity.* In this section, we adapt an argument of LaFontaine [**34**], which relies heavily on work of Salem [**51, 52**], to show that if T is a weakly mixing rigid map then there is a weakly mixing rigid map S such that $T \times S$ is not rigid. In fact we show something stronger, namely Proposition 3.23 below. Actually rigidity of T is not needed for Proposition 3.23 but if the only conclusion desired is non-rigidity of $T \times S$ then the only interesting case is that of a rigid T, since otherwise $T \times S$ is trivially non-rigid.

We identify the circle with $\mathbb{T} = \mathbb{R}/\mathbb{Z} \simeq [0, 1)$. A finite measure ω on $[0, 1)$ may be considered as a measure on $\mathbb{T}$ or as a measure on $\mathbb{R}$ which is supported on $[0, 1)$. When we want to consider it as a measure on $\mathbb{T}$ we write $\omega_{\mathbb{T}}$, otherwise ω is to be considered as a measure on $\mathbb{R}$. For example, if γ is another measure on $[0, 1)$, $\omega_{\mathbb{T}} * \gamma_{\mathbb{T}}$ means convolution in $\mathbb{T}$, whereas $\omega * \gamma$ means convolution in $\mathbb{R}$ (which may no longer be supported on $[0, 1)$).

We will be working with finite (Borel) measures ω on $\mathbb{R}$. We denote the Fourier transform of ω by

$$\widehat{\omega}(t) = \int_{\mathbb{R}} \exp(-2\pi i t x)\, d\omega(x).$$

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We will denote the fractional part of $x \in \mathbb{R}$ by $\theta(x) = \{x\}$ and the integer part by $[x]$. Given ω, the measure $\Omega = \omega \circ \theta^{-1}$ is the projection of ω onto $[0, 1)$, explicitly for $E \subset [0, 1)$,

$$\Omega(E) = \sum_{n\in\mathbb{Z}} \omega(E + n).$$

Note that $\widehat{\Omega}(n) = \widehat{\omega}(n)$ for each $n \in \mathbb{Z}$. We also remark that Ω is absolutely continuous if and only if ω is absolutely continuous. By m we will denote Lebesgue measure on $\mathbb{R}$.

LEMMA 3.21. *If μ and ν are finite measures on $[0, 1)$ then $\mu * \nu$ is absolutely continuous if and only if $\mu_{\mathbb{T}} * \nu_{\mathbb{T}}$ is absolutely continuous.*

Proof. This follows immediately by observing that $\mu_{\mathbb{T}} * \nu_{\mathbb{T}} = ((\mu * \nu) \circ \theta^{-1})_{\mathbb{T}}$. □

Let us say that a finite measure ω on $\mathbb{R}$ is rigid if $\lim_{m\to\infty} \widehat{\omega}(x_m) = 1$ for some sequence (x_m) of reals converging to ∞.

LEMMA 3.22. *Suppose that ω is a rigid measure on $\mathbb{R}$. Then there is an increasing sequence (n_m) of positive integers going to ∞ such that $\lim_{m\to\infty} \widehat{\omega}(n_m) = 1$.*

Proof. By passing to a subsequence we may assume that $x_m - x_{m-1} \to \infty$. The condition $\lim_{m\to\infty} \widehat{\omega}(x_m) = 1$ is equivalent to convergence of the functions $f_{x_m}(t) := e^{2\pi i x_m t}$ to 1 in measure (with respect to ω). By passing to a subsequence we may assume that the fractional parts $\{x_m\}$ converge to a limit s_0. If follows that $e^{2\pi i[x_m]t} \to e^{-2\pi i s_0 t}$ in measure. But then $e^{2\pi i([x_m]-[x_{m-1}])t} \to 1$ in measure so, setting $n_m = [x_m] - [x_{m-1}]$, we have $\widehat{\omega}(n_m) \to 1$. Moreover, $n_m \to \infty$ because $x_m - x_{m-1} \to \infty$. □

PROPOSITION 3.23. *Assume that $(X, \mathcal{B}_X, p_X, T)$ is a weakly mixing dynamical system. Then there is a weakly mixing, rigid dynamical system $(Y, \mathcal{B}_Y, p_Y, S)$ such that the maximal spectral type of $U_{T\times S}$ in the orthocomplement of the subspace $F := (L_2(p_X) \otimes 1_Y) \oplus (1_X \otimes L_2(p_Y))$ in $L_2(p_X \times p_Y)$ is absolutely continuous. In particular, $T \times S$ is not rigid.*

Proof. Let μ' denote the maximal spectral type of U_T, a continuous measure on $\mathbb{T}$. We will construct a continuous rigid measure γ' on $\mathbb{T}$ such that $\mu' * \gamma'$ is absolutely continuous. Having done so, we can then take $S = G_{\gamma'}$. Since the maximal spectral type of S consists of components γ'^{*k}, $k \geq 1$, the maximal spectral type of $T \times S$ in the orthocomplement of F has components $\mu' * \gamma'^{*k} = (\mu' * \gamma') * \gamma'^{*(k-1)}$ which will be absolutely continuous, since $\mu' * \gamma'$ is.

At this point we will start viewing μ' as a measure, call it μ, on $[0, 1)$. We now refer to the article [**34**] of LaFontaine. The argument in the proof of Theorem II in that paper, beginning at equation (1) (which is equation (3.7) below), shows that there is a continuous probability measure ν supported on a certain Cantor set $F \subset [0, 1]$ such that, for m-almost all $\lambda \in [0, 1)$,

$$\sum_{n\in\mathbb{Z}} |\widehat{\mu}(\lambda n)\widehat{\nu}(n)|^2 < \infty. \tag{3.7}$$

In [**34**] it is shown that F is a Dirichlet subset of $\mathbb{R}$, that is, $\liminf_{x\to\infty} \|e^{ixt} - 1\|_{C(F)} = 0$. This certainly implies that ν is a rigid measure on $\mathbb{R}$.

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Now fix any $\lambda \in [0, 1)$ satisfying (3.7), let μ_λ denote the dilation of μ by λ, ($\mu_\lambda(E) = \mu(\lambda^{-1}E)$) and note that μ_λ is supported on $[0, \lambda] \subset [0, 1)$ and $\widehat{\mu_\lambda}(t) = \widehat{\mu}(\lambda t)$. Thus equation (3.7) shows that $(\mu_\lambda)_{\mathbb{T}} * \nu_{\mathbb{T}}$, and hence also $\mu_\lambda * \nu$, is absolutely continuous. It follows that $(\mu_\lambda * \nu)_{1/\lambda} = \mu * \nu_{1/\lambda}$ is also absolutely continuous. Note that the rigidity of ν implies the rigidity of $\nu_{1/\lambda}$. Now let $\gamma = (\nu_{1/\lambda} \circ \theta^{-1})$ and $\gamma' = \gamma_{\mathbb{T}}$. Since $\mu * \nu_{1/\lambda}$ is absolutely continuous we see that $(\mu * \nu_{1/\lambda}) \circ \theta^{-1} = \mu * \gamma$, and hence also $\mu_{\mathbb{T}} * \gamma_{\mathbb{T}} = \mu' * \gamma'$, is absolutely continuous. Finally, to see that γ' is a rigid circle measure, note that $\widehat{\gamma'}(n) = \widehat{\nu_{1/\lambda}}(n)$. Since $\nu_{1/\lambda}$ is rigid, the rigidity of γ' follows by Lemma 3.22. □

Remark 3.24. By the proof of the proposition above we cannot decide whether or not there is a Lebesgue component in the spectrum of $T \times S$. Some stronger results in this direction were obtained by V. Ryzhikov in 2009 (private communication). In particular, Ryzhikov proved that there are two rigid rank-one automorphisms T and S such that $T \times S$ has a Lebesgue component (T can be taken from a generic set of automorphisms). Moreover, the result about the existence of a Lebsgue component can be achieved in the class of rigid Gaussian automorphisms having simple spectrum.

COROLLARY 3.25. *The transformations T and S in Proposition 3.23 must be spectrally disjoint and hence disjoint.*

Proof. Suppose that (n_m) is rigid for S. If we do not have spectral disjointness then there are non-zero $f \in L_{2,0}(p_X)$ and $g \in L_{2,0}(p_Y)$ whose spectral measures are identical. Since $S^{n_m} g \to g$ it follows that $T^{n_m} f \to f$ which implies that $(S \times T)^{n_m}(f \otimes g) \to f \otimes g$, contradicting the fact that the spectral type of $f \otimes g$ is absolutely continuous. □

COROLLARY 3.26. *Assume that $(X, \mathcal{B}, \mu, T)$ is an ergodic dynamical system. Then T has discrete spectrum if and only if for each weakly mixing rigid system $(Y, \mathcal{C}, \nu, S)$ the Cartesian product system $T \times S$ remains rigid.*

Proof. Assume that T is an ergodic isometry and let S be weakly mixing, say $S^{n_m} \to \mathrm{Id}$. By compactness and passing to a subsequence if necessary, we may assume that $d_u(T^{n_m}, R) \to 0$ for some R in the isometry group of X. Then $T^{n_{m+1}-n_m} \to \mathrm{Id}$ and also $S^{n_{m+1}-n_m} \to \mathrm{Id}$ so $(n_{m+1} - n_m)$ is a rigidity sequence for $T \times S$. To prove the converse, suppose that T does not have discrete spectrum, but $T \times S$ is rigid for each S which is rigid and weakly mixing. Then there is some continuous μ' with $\mu' \ll \nu^T$. The first paragraph in the proof of Proposition 3.23 shows that there is a rigid weakly mixing S such that $\mu' * \nu^S$ is absolutely continuous. Since $\mu' * \nu^S \ll \nu^T * \nu^S$ it follows that $T \times S$ cannot be rigid, a contradiction. □

3.6. *Cocycle methods.*

3.6.1. *Tools.* We will now describe tools using cocycles over rotations to produce weakly mixing transformations with a prescribed sequence as a rigidity sequence. The main application we obtain is that the sequence of denominators in the continued fraction expansion of any irrational α is a rigidity sequence for some weakly mixing transformation. In particular, for example, the Fibonacci sequence is such a sequence. We also use the methods of this section to give another proof of Proposition 3.10.

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We will start with a transformation T having discrete spectrum and its sequence of rigidity. (In fact, for applications, we will consider one-dimensional rotations by irrational α and the sequence given by the denominators in the continued fraction expansion of α.) Then we will consider cocycles over T with values in locally compact Abelian groups. We will then pass to the associated unitary operators (weighted operators) and we will try to 'lift' some rigidity sequences for the rotation to the weighted operator. Once such an operator has continuous spectrum we apply the Gaussian measure construction which preserves rigidity. Another option to obtain 'good' weakly mixing transformations will be to pass to Poisson suspensions (in case we extend by a locally compact and not compact group)—which in a sense will be even easier as ergodicity of Poisson suspension is closely related to the fact that the cocycles are not coboundaries. See §2.1 and also [**10, 11, 48**], for details concerning ergodic properties of Poisson suspensions. Note that at times in this section $\mathbb{T}$ denotes the unit circle in $\mathbb{C}$ with multiplicative notation, and at times it means $[0, 1)$ with addition modulo one. The reader will be able to distinguish which model for the circle is being used by the context of the discussion.

3.6.2. *Compact group extensions and weighted operators.* Assume that T is an ergodic transformation acting on a standard Lebesgue space $(X, \mathcal{B}, p)$. Let G be a compact metric Abelian group with normalized Haar measure λ_G. We consider the probability measure space $(G, \mathcal{B}_G, \lambda_G)$ where $\mathcal{B}_G$ denotes the σ-algebra of λ_G-measurable sets. A measurable map $\varphi : X \to G$ generates a *cocycle* $\varphi^{(\cdot)}(\cdot)$ which is given by $\varphi^{(n)}(\cdot) : X \to G$, $n \in \mathbb{Z}$, by the formula, for $x \in X$,

$$\varphi^{(n)}(x) = \begin{cases} \varphi(x) + \varphi(Tx) + \cdots + \varphi(T^{n-1}x) & \text{if } n > 0, \\ 0 & \text{if } n = 0, \\ -(\varphi(T^{-n}x) + \cdots + \varphi(T^{-1}x)) & \text{if } n < 0. \end{cases}$$

Using T and φ, we define a *compact group extension* T_φ of T which acts on the space $(X \times G, \mathcal{B} \otimes \mathcal{B}_G, p \otimes \lambda_G)$ by the formula

$$T_\varphi(x, g) = (Tx, \varphi(x) + g) \quad \text{for } (x, g) \in X \times G.$$

Notice that for each $n \in \mathbb{Z}$ and $(x, g) \in X \times G$,

$$(T_\varphi)^n(x, g) = (T^n x, \varphi^{(n)}(x) + g). \tag{3.8}$$

The natural decomposition of $L_2(G, \lambda_G)$ using the character group $\widehat{G}$ yields the decomposition

$$L_2(X \times G, p \otimes \lambda_G) = \bigoplus_{\chi \in \widehat{G}} L_2(X, p) \otimes (\mathbb{C}\chi). \tag{3.9}$$

Here $\mathbb{C}\chi$ is the one-dimensional subspace spanned by the character χ. To understand ergodic and other mixing properties of T_φ we need to study the associated Koopman operator U_{T_φ},

$$U_{T_\varphi} F = F \circ T_\varphi \quad \text{for } F \in L_2(X \times G, p \otimes \lambda_G).$$

As the (closed) subspaces $L_2(X, p) \otimes (\mathbb{C}\chi)$ in (3.9) are U_{T_φ}-invariant, we can examine those mixing properties separately on all such subspaces (notice that for $\chi = 1$ we consider

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the original Koopman operator U_T). It is well known and not hard to see that the map $f \otimes \chi \mapsto f$ provides a spectral equivalence of $U_{T_\varphi}|_{L_2(X,p)\otimes(\mathbb{C}\chi)}$ and the operator $V^T_{\chi\circ\varphi}$ acting on $L_2(X, \mathcal{B}, p)$ by the formula

$$V^T_{\chi\circ\varphi} f = \chi \circ \varphi \cdot f \circ T \quad \text{for } f \in L_2(X, p).$$

Each such operator is an example of a *weighted operator* V^T_ξ over T, where $\xi : X \to \mathbb{T}$ is a measurable function with values in the (multiplicative) circle $\mathbb{T}$ and $V^T_\xi f = \xi \cdot f \circ T$.

Assume now that T is an ergodic transformation with discrete spectrum, that is, without loss of generality, we can assume that X is a compact monothetic metric group with $p = \lambda_X$ Haar measure on X, and $Tx = x + x_0$, where $x_0 \in X$ and $\{nx_0 : n \in \mathbb{Z}\}$ is dense in X. Assume that $\xi : X \to \mathbb{T}$ is measurable. By Helson's analysis [**22**] (see also [**25**], for example), we have the following.

THEOREM 3.27. [**22**] *For T as above, the maximal spectral type of V^T_ξ is either discrete or continuous. If it is continuous then either it is singular or it is Lebesgue.*

We will use only the first part of this theorem. An important practical point that comes from this theorem is that once we find a function $f \in L_2(X, \mathcal{B}, p)$ such that the spectral measure $\sigma_f = \sigma_f^{V^T_\xi}$ is continuous, then V^T_ξ has purely continuous spectrum. Consider $f = 1$. Using (3.8), for each $n \in \mathbb{Z}$, we obtain

$$\langle (V^T_\xi)^n 1, 1\rangle = \int_X \xi^{(n)}(x)\, dp(x).$$

It follows that if there is a subsequence $(n_m)_{m\geq 1}$ such that

$$\int_X \xi^{(n_m)}(x)\, dp(x) \to 0, \tag{3.10}$$

then V^T_ξ has continuous spectrum. It is also nice to note in passing that we have a slightly stronger result:

$$\int_X \xi^{(n_m)}(x)\, dp(x) \to 0 \;\Rightarrow\; (V^T_\xi)^{n_m} \to 0 \text{ weakly},$$

which follows directly from (3.10); indeed,

$$\int_X \xi^{(n_m)}(x) f(T^{n_m}x)\overline{f(x)}\, dp(x) \to 0$$

whenever f is a character of X.

It is well known and easy to check that if (n_m) is a rigidity sequence for T, then

$$\xi^{(n_m)} \to 1 \text{ in measure} \Rightarrow\; (V^T_\xi)^{n_m} \to \mathrm{Id} \text{ strongly}.$$

3.6.3. $\mathbb{R}$*-extensions, weighted operators and Poisson suspensions.* We will now consider the case $G = \mathbb{R}$. We assume that $f : X \to \mathbb{R}$ is a cocycle for T acting ergodically on a standard probability Borel space $(X, \mathcal{B}, p)$. We consider T_f,

$$T_f(x, r) = (Tx,\, f(x) + r),$$

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acting on $(X \times \mathbb{R}, p \otimes \lambda_{\mathbb{R}})$. Note that we are now on a standard Lebesgue space with a σ-finite (and not finite) measure. In particular, constants are not integrable.

To study T_f spectrally we will write it slightly differently,

$$T_f = T_{f,\tau},$$

where τ is the natural action of $\mathbb{R}$ on itself by translations: $\tau_t(r) = r + t$ and

$$T_{f,\tau}(x, r) = (Tx, \tau_{f(x)}(r)).$$

This transformation is a special case of a so-called *Rokhlin extension* (of T; see, for example, [**35**]), and the spectral analysis below is similar to the one in [**35**]. So let us just imagine a slightly more general situation,

$$T_{f,\mathcal{S}}(x, y) = (Tx, S_{f(x)}(y)),$$

where $\mathcal{S} = (S_t)$ is a flow acting on $(Y, \mathcal{C}, \nu)$ (ν can be finite or infinite). We will denote the spectral measure of $a \in L_2(Y, \mathcal{C}, \nu)$ (for the Koopman representation $t \mapsto U_{S_t}$ on $L_2(Y, \mathcal{C}, \nu)$) by $\sigma_{a,\mathcal{S}}$.

The space $L_2(X \times Y, p \otimes \nu)$ is nothing but a tensor product of two Hilbert subspaces, so to understand spectral measures we only need to study spectral measures for tensors $a \otimes b$ and we have

$$\begin{aligned}
&\int_{X\times Y} (a \otimes b) \circ (T_{f,\mathcal{S}})^n \cdot \overline{a \otimes b}\, dp\, d\nu \\
&\quad = \int_X \int_Y a(T^n x)\overline{a(x)} b(S_{f^{(n)}(x)} y)\overline{b(y)}\, dp(x)\, d\nu(y) \\
&\quad = \int_X a(T^n x)\overline{a(x)} \left(\int_Y e^{2\pi i t f^{(n)}(x)}\, d\sigma_{b,\mathcal{S}}(t) \right) dp(x) \\
&\quad = \int_Y \left(\int_X e^{2\pi i t f^{(n)}(x)} a(T^n x)\overline{a(x)}\, dp(x) \right) d\sigma_{b,\mathcal{S}}(t).
\end{aligned}$$

PROPOSITION 3.28. *If $T^{n_k} \to \mathrm{Id}$ and $f^{(n_k)} \to 0$ in measure then (n_k) is a rigidity sequence for $T_{\varphi,\mathcal{S}}$.*

Proof. Take $a \in L^\infty(X, p)$ and notice that by assumption, for each $t \in \mathbb{R}$,

$$\int_X e^{2\pi i t f^{(n_k)}(x)} a(T^{n_k} x)\overline{a(x)}\, dp(x) \to \int_X |a|^2\, dp.$$

By the Lebesgue dominated convergence theorem,

$$\begin{aligned}
\int_Y \left(\int_X e^{2\pi i t f^{(n_k)}(x)} a(T^{n_k} x)\overline{a(x)}\, dp(x) \right) d\sigma_{b,\mathcal{S}}(t) &\to \int_Y \left(\int_X |a|^2\, dp \right) d\sigma_{b,\mathcal{S}} \\
&= \|a \otimes b\|^2_{L_2(p\otimes\lambda_{\mathbb{R}})}.
\end{aligned}$$

□

We need more information about sequences of the form

$$\int_X e^{2\pi i t f^{(n)}(x)} a(T^n x)\overline{a(x)}\, dp(x), \quad n \in \mathbb{Z}.$$

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In fact, they turn out to be again Fourier coefficients of some spectral measures. Indeed, consider V_t acting on $L_2(X, \mathcal{B}, p)$ by the formula

$$(V_t a)(x) = e^{2\pi i t f(x)} a(Tx).$$

This is nothing but a weighted unitary operator and

$$\langle V_t^n a, a\rangle = \int_X e^{2\pi i t f^{(n)}(x)} a(T^n x)\overline{a(x)}\, dp(x).$$

(Notice that we have returned to the finite measure-preserving case.)

Clearly, for $b \in L_2(\mathbb{R})$ with compact support and $\mathcal{S} = \tau$,

$$\begin{aligned}\widehat{\sigma}_{b,\mathcal{S}}(t) &= \int_{\mathbb{R}} b \circ S_t \cdot \overline{b}\, dr \\ &= \int_{\mathbb{R}} b(r+t)\overline{b(r)}\, dr = (b * \overline{b})(-t).\end{aligned}$$

Hence, the Fourier transform of $\sigma_{b,\mathcal{S}}$ is square summable, and therefore this spectral measure is absolutely continuous. In fact, the maximal spectral type of $\mathcal{S}$ is Lebesgue, and we can see the maximal spectral type of U_{T_f} as an integral (against 'Lebesgue' measure) of the maximal spectral types of the family indexed by $t \in \mathbb{R}$ of weighted operators.

Suppose now that U_{T_f} has an eigenvalue c, $|c| = 1$. Then we cannot have that all spectral measures $\sigma_{a\otimes b,T_f}$ are continuous. In fact, we must have that c appears as an eigenvalue for 'many' V_t (on a set of $t \in \mathbb{R}$ of positive Lebesgue measure), and the following result is well known (and left as an exercise).

LEMMA 3.29. *The scalar c is an eigenvalue of V_t if and only if we can solve the functional equation*

$$e^{2\pi i t f} = \frac{c \cdot \xi \circ T}{\xi}$$

in measurable functions $\xi : X \to \mathbb{T}$.

It follows that having an eigenvalue for U_{T_f} means that we can solve the above multiplicative equation on a set of positive Lebesgue measure of $t \in \mathbb{R}$. We are now in the framework of the classical Helson problem (see, for example, [**39**]) of passing from multiplicative coboundaries to additive coboundaries. Using known results in this area ([**39**], and see also the appendix in [**36**]) we obtain the following proposition (remember that constant functions are not elements of $L_2(X \times \mathbb{R}, p \otimes \lambda_{\mathbb{R}})$).

PROPOSITION 3.30. *If U_{T_f} has an eigenvalue then f is an additive quasi-coboundary, that is, there exist a measurable $g : X \to \mathbb{R}$ and $r \in \mathbb{R}$ such that $f(x) = r + g(x) - g(Tx)$ for p-almost every $x \in X$.*

Therefore, if f is a non-trivial cocycle then automatically U_{T_f} has continuous spectrum and classically the Poisson suspension over T_f is ergodic, hence weakly mixing. Recall that from the spectral point of view the Poisson suspension over $(X \times \mathbb{R}, p \otimes \lambda_{\mathbb{R}}, T_f)$ will be the same as the Gaussian functor over $(L_2(X \times \mathbb{R}, p \otimes \lambda_{\mathbb{R}}), U_{T_f})$. In particular, if $(T_f)^{n_t} \to \mathrm{Id}$ on $L_2(X \times \mathbb{R}, p \otimes \lambda_{\mathbb{R}})$ then (n_t) will be a rigidity sequence for the suspension (in view of Proposition 3.28). From the above discussion, it follows that to

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have a weakly mixing transformation $\widetilde{T}_f$ with a rigidity sequence (N_t), we need: (i) f is not an additive coboundary; (ii) $T^{N_t} \to \mathrm{Id}$; (iii) $f^{(N_t)} \to 0$ in measure. (If f is an additive quasi-coboundary satisfying (ii) and (iii) then r is necessarily zero.)

3.6.4. *Denominators of α and rigidity.* We have already seen that the sequence (2^n) is a rigidity sequence for a weakly mixing transformation. We can construct some other explicit examples of rigidity sequences by using known results from the theory of 'smooth' cocycles over one-dimensional rotations. This will allow us to show that if α is irrational, and (q_n) stands for its sequence of denominators then (q_n) is also a rigidity sequence for a weakly mixing transformation. The most interesting case is of course the bounded partial quotient case (for example, for the golden mean).

So assume that $\varphi : \mathbb{T} \to \mathbb{R}$ is a smooth mean-zero cocycle. We use the term 'smooth' here in a not very precise way; it may refer to a good speed of decay of the Fourier transform of φ.

We recall first that one of consequences of the Denjoy–Koksma inequality for AC_0 (absolutely continuous mean-zero) cocycles is that

$$\varphi^{(q_n)} \to 0 \text{ uniformly}$$

for every irrational rotation by α; see [**23**]. Another type of Denjoy–Koksma inequality has been proved in [**3**] for functions φ whose Fourier transform is of order $\mathrm{O}(1/|n|)$—as its consequence we have that

$$\text{if } \widehat{\varphi}(n) = \mathrm{o}\left(\frac{1}{|n|}\right), \widehat{\varphi}(0) = 0 \text{ then } \varphi^{(q_n)} \to 0 \text{ in measure} \tag{3.11}$$

for every rotation by an irrational α.

We would like also to recall another (unpublished) result by Herman [**24**]. While this may not be available, one can see also Krzyżewski [**30**] for generalizations of Herman's result.

THEOREM 3.31. *Assume that a mean-zero $\varphi : \mathbb{T} \to \mathbb{R}$ is in $L_2(\mathbb{T}, \lambda_{\mathbb{T}})$ and its Fourier transform is concentrated on a lacunary subset of $\mathbb{Z}$. Suppose that*

$$\varphi(x) = g(x) - g(x + \alpha), \quad \lambda_{\mathbb{T}}\text{-a.e.}$$

for some irrational $\alpha \in [0, 1)$. Then $g \in L_2(\mathbb{T}, \lambda_{\mathbb{T}})$.

Fix $\alpha \in [0, 1)$ irrational, and let $\alpha = [0 : a_1, a_2, \ldots]$ stand for the continued fraction expansion of α. Denote by (q_n) the sequence of denominators of α: $q_0 = 1$, $q_1 = a_1$ and $q_{n+1} = a_{n+1}q_n + q_{n-1}$ for $n \geq 2$. Then

$$\frac{q_{n+2}}{q_n} = \frac{a_{n+2}q_{n+1} + q_n}{q_n} \geq a_{n+2} + 1 \geq 2.$$

It follows that

$$(q_{2n}) \text{ is lacunary.} \tag{3.12}$$

Moreover,

$$q_n \|q_n\alpha\| \leq 1 \quad \text{for each } n \geq 1. \tag{3.13}$$

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We define $\varphi : \mathbb{T} \to \mathbb{R}$ by

$$\varphi(x) = \sum_{n=0}^{\infty} c_{q_{2n}} \cos 2\pi i q_{2n} x,$$

where for $n \geq 1$,

$$c_{q_{2n}} = \frac{1}{\sqrt{n}} \|q_{2n}\alpha\|.$$

We then have $\varphi : \mathbb{T} \to \mathbb{R}$, $\widehat{\varphi}(n) = o(1/|n|)$ and $(c_{q_{2n}}) \in l_2$ in view of (3.13). Now, suppose that

$$\varphi(x) = g(x) - g(x + \alpha) \tag{3.14}$$

for a measurable $g : \mathbb{T} \to \mathbb{R}$. In view of Theorem 3.31 and (3.12), $g \in L_2(\mathbb{T}, \lambda_{\mathbb{T}})$. Hence,

$$g(x) = \sum_{k=-\infty}^{\infty} b_k e^{2\pi i k x}.$$

Furthermore, by comparing Fourier coefficients on both sides in (3.14),

$$b_k = 0 \quad \text{if } k \neq q_{2n} \quad \text{and} \quad b_{q_{2n}} = c_{q_{2n}}/(1 - e^{2\pi i q_{2n}\alpha})$$

with $(b_{q_{2n}}) \in l_2$. However,

$$|b_{q_{2n}}| = c_{q_{2n}}/\|q_{2n}\alpha\| = 1/\sqrt{n},$$

which is a contradiction. We have thus proved the following proposition.

PROPOSITION 3.32. *For each irrational $\alpha \in [0, 1)$ there is a mean-zero $\varphi : \mathbb{T} \to \mathbb{R}$ such that $\widehat{\varphi}(n) = o(1/|n|)$ and φ is not an additive coboundary.*

Using (3.11) and Proposition 3.28 we hence obtain the following result.

PROPOSITION 3.33. *For φ satisfying the assertion of Proposition 3.32 the sequence (q_n) of denominators of α is a rigidity sequence of T_φ on $L_2(\mathbb{T} \times \mathbb{R}, \lambda_{\mathbb{T}} \otimes \lambda_{\mathbb{R}})$ and U_{T_φ} has continuous spectrum.*

By using the GMC method or by passing to the relevant Poisson suspension we obtain the following corollary.

COROLLARY 3.34. *For each sequence (q_n) of denominators there exists a weakly mixing transformation R such that $R^{q_n} \to \mathrm{Id}$.*

Remark 3.35.

(i) We would like to emphasize that in general the sequence (q_n) of denominators of α is not lacunary. Indeed, assume that $\alpha = [0 : a_1, a_2, \ldots]$ stands for the continued fraction expansion of α. Suppose that for a subsequence (n_k) we have $a_{n_k+1} = 1$, $a_{n_k} \to \infty$. Then by the recurrence formula $q_{m+1} = a_{m+1}q_m + q_{m-1}$ we obtain that

$$\liminf_{n\to\infty} \frac{q_{n+1}}{q_n} = 1.$$

So the sequences of denominators are another type of non-lacunary sequences which can be realized as rigidity sequences for weakly mixing transformations, besides the ones in §3.4.

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(ii) As the above shows, $\{q_n : n \geq 1\}$ is always a Sidon set (see [**26, 49**] for the definition of a Sidon set); indeed,

$$\{q_n : n \geq 1\} = \{q_{2n} : n \geq 1\} \cup \{q_{2n+1} : n \geq 0\}.$$

So, the set of denominators is the union of two lacunary sets and hence is a Sidon set [**26, 49**].

(iii) The assertion of Theorem 3.31 is true for functions whose Fourier transform is concentrated on a Sidon set and when T is an arbitrary ergodic rotation on a compact metric Abelian group (by the proof of the main result in [**22**] or by [**30**]).

(iv) It follows that to prove Proposition 3.32 we could have used all denominators, with (for example) $a_{q_n} = (1/\sqrt{n})\|q_n\alpha\|$.

(v) Eisner and Grivaux [**12**] obtain some results in the direction of Corollary 3.34, but their examples are restricted to badly approximated irrational numbers. Also, Fayad [**15**] obtained a result as in Corollary 3.34 but his result is restricted to the denominators of Liouville numbers. It should be noted, though, that Fayad also gets uniform rigidity for some weakly mixing automorphism of $\mathbb{T}^2$.

There are also more complicated constructions showing that for each irrational α there is an absolutely continuous mean-zero $\varphi : \mathbb{T} \to \mathbb{R}$ which is not a coboundary—see [**37**]. We can then use such cocycles and (3.11) for another proof of the above corollary.

Remark 3.36. Proposition 2.8 shows that, given any increasing sequence (n_m) of integers, there is a weakly mixing transformation R such that $R^{n_{m_k}} \to \mathrm{Id}$ for some subsequence (n_{m_k}) of (n_m). We can use Corollary 3.34 to give a new proof. In fact this follows directly from Corollary 3.34 and the following well-known result; see, for example, [**33**].

LEMMA 3.37. *Given an increasing sequence (n_m) of natural numbers, consider the set of $\alpha \in [0, 1)$ such that a subsequence of (n_m) is a subsequence of denominators of α. This is a generic subset of* $[0, 1)$.

3.6.5. *Integer lacunarity case.* We will now give an alternative proof of Proposition 3.10 using the cocycle methods that have been developed here. Assume that $(n_m)_{m\geq 0}$ is an increasing sequence of positive integers such that $n_0 = 1$, $n_{m+1}/n_m \in \mathbb{Z}$ with

$$\rho_m := n_{m+1}/n_m \geq 2 \quad \text{for } m \geq 0. \tag{3.15}$$

Notice that in view of (3.15) there exists a constant $C > 0$ such that

$$\frac{1}{n_{m+1}^2} + \frac{1}{n_{m+2}^2} + \cdots \leq \frac{C}{n_m^2} \quad \text{for each } m \geq 0.$$

Let $X = \Pi_{m=1}^{\infty}\{0, 1, \ldots, \rho_m - 1\}$ which is a metrizable compact group when we consider the product topology and the addition is meant coordinatewise with carrying the remainder to the right. On X we consider Haar measure p_X which is the usual product measure of uniform measures. Define $Tx = x + \hat{1}$, where

$$\hat{1} = (1, 0, 0, \ldots).$$

The resulting dynamical system is called the (n_m)-*odometer*.

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For each $t \geq 0$ set

$$D_0^{n_m} = \{x \in X : x_0 = x_1 = \cdots = x_{m-1} = 0\}.$$

Note that $\{D_0^{n_m}, TD_0^{n_m}, \ldots, T^{n_m-1}D_0^{n_m}\}$ is a Rokhlin tower filling the whole space X and

$$\hat{1} \in TD_0^{n_m} \quad \text{for each } m \geq 0. \tag{3.16}$$

The character group $\widehat{X}$ of X is discrete and is isomorphic to the (discrete) group of roots of unity of degree n_m, $m \geq 0$. More precisely, for $m \geq 0$, define the character 1_{n_m} by

$$1_{n_m}(x) = \varepsilon_{n_m}^j := e^{2\pi ij/n_m} \quad \text{for } x \in T^j D_0^{n_m}, j = 0, 1, \ldots, n_m - 1.$$

Then $\widehat{X} = \{1_{n_m}^j : j = 0, 1, \ldots, n_m - 1, m \geq 0\}$.

From now on we will consider $f \in L_{2,0}(X, p_X)$ whose Fourier transform is 'concentrated' on $\{1_{n_m} : m \geq 0\}$. We have

$$f(x) = \sum_{m=1}^{\infty} a_{n_m} 1_{n_m}(x), \quad \sum_{m=1}^{\infty} |a_{n_m}|^2 < +\infty. \tag{3.17}$$

(A) Small divisors. Assume that f satisfies (3.17) and suppose that

$$f(x) = g(x) - g(x + \hat{1}) \quad \text{for } p_X\text{-almost every } x \in X. \tag{3.18}$$

Suppose, moreover, that $g \in L_2(X, p_X)$. Hence $g(x) = \sum_{\chi \in \widehat{X}} b_\chi \chi(x)$ and by comparison of Fourier coefficients on both sides in (3.17) we obtain

$$b_\chi = 0 \quad \text{whenever } \chi \neq 1_{n_m} \text{ and } a_{n_m} = b_{n_m}(1 - 1_{n_m}(\hat{1})) \text{ for } m \geq 1.$$

Using (3.16), we obtain that $b_{n_m} = a_{n_m}/(1 - \varepsilon_{n_m})$, so

$$|b_{n_m}|^2 = \frac{|a_{n_m}|^2}{|1 - \varepsilon_{n_m}|^2} = n_m^2 |a_{n_m}|^2 \quad \text{for } m \geq 1.$$

We have proved that

$$\text{(3.18) has an } L_2\text{-solution if and only if } (n_m a_{n_m})_m \in l_2. \tag{3.19}$$

(B) Estimate of L_2-norms for the cocycle. Assume that $n \in \mathbb{N}$. Then

$$\begin{aligned} f^{(n)}(x) &= \sum_{m=1}^{\infty} a_{n_m}(1 + 1_{n_m}(\hat{1}) + \cdots + 1_{n_m}((n-1)\hat{1}))1_{n_m}(x) \\ &= \sum_{m=1}^{\infty} a_{n_m}\left(\sum_{j=0}^{n-1} \varepsilon_{n_m}^j\right)1_{n_m}(x). \end{aligned}$$

Fix $n = n_{m_0}$. Then

$$\begin{aligned} f^{(n_{m_0})}(x) &= \sum_{m=1}^{m_0} a_{n_m}\left(\sum_{j=0}^{n_{m_0}-1} \varepsilon_{n_m}^j\right)1_{n_m}(x) + \sum_{m=m_0+1}^{\infty} a_{n_m}\left(\sum_{j=0}^{n_{m_0}-1} \varepsilon_{n_m}^j\right)1_{n_m}(x) \\ &= \sum_{m=m_0+1}^{\infty} a_{n_m} \frac{1 - \varepsilon_{n_m}^{n_{m_0}}}{1 - \varepsilon_{n_m}} 1_{n_m}(x). \end{aligned}$$

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Since $|1-\varepsilon_{n_m}|=1/n_m$ and $|1-\varepsilon_{n_m}^{n_{m_0}}|\le n_{m_0}/n_t$,

$$\|f^{(n_{m_0})}\|^2_{L_2(X,p_X)}\le n_{m_0}^2\sum_{m=m_0+1}^{\infty}|a_{n_m}|^2. \tag{3.20}$$

(C) Sidon sets and a 'good' function. According to [**49**] (see Example 5.7.6 therein) every infinite subset of a discrete group contains an infinite Sidon set. Hence we can choose a subsequence (n_{m_k}) of (n_m) so that

$$\{1_{n_{m_k}}:k\ge 1\}\text{ is a Sidon subset of }\widehat{X}. \tag{3.21}$$

We set

$$f(x)=\sum_{k=1}^{\infty}\frac{1}{\sqrt{k}n_{m_k}}1_{n_{m_k}}(x).$$

Suppose now that (3.18) has a measurable solution g (we should consider f real-valued, so in fact we should consider $f+\overline{f}$ below). In view of (3.21) and [**30**], $g\in L_2(X,\,p_X)$. But

$$\sum_{m=1}^{\infty}|n_m a_{n_m}|^2=\sum_{k\ge 1}\frac{1}{k},$$

so by (3.19), we cannot obtain an L_2-solution. This means that f is not a measurable coboundary. According to (3.20), the definition of f and (3.15) for each $s\ge 1$,

$$\begin{aligned}\|f^{(n_s)}\|^2_{L_2(X,p_X)}&\le n_s^2\sum_{m=s+1}^{\infty}|a_{n_m}|^2=n_s^2\sum_{k\ge 1:n_{m_k}\ge n_{s+1}}^{\infty}|a_{n_{m_k}}|^2\\&=n_s^2\sum_{k=k_s}^{\infty}|a_{n_{m_k}}|^2\le\frac{n_s^2}{k_s}\sum_{j=s+1}^{\infty}\frac{1}{n_j^2}\le\frac{C}{k_s}\to 0\end{aligned}$$

as clearly $k_s\to\infty$ when $s\to\infty$.

Using our general method, we hence proved the following proposition.

PROPOSITION 3.38. *Assume that (n_m) is an increasing sequence of integers with n_{m+1}/n_m being an integer at least 2. Then there exists a weakly mixing transformation R such that $R^{n_m}\to\mathrm{Id}$.*

4. *Non-recurrence*

A sequence (n_m) is said to be recurrent if for all finite measure-preserving T and all subsets A of positive measure we have $\mu(T^{n_m}A\cap A)>0$ for at least one m. Thus our notion of non-recurrent means simply not recurrent. As with rigidity, it is sometimes more convenient to speak of non-recurrent sets. There is a huge literature on recurrence. We cite just a few survey articles: Bergelson [**6, 7**], Furstenberg [**18**] and Kra [**29**]. Examples of recurrent sequences include difference sets $D=E-E$ where E is any infinite subset of $\mathbb{N}$, polynomial sequences $p(m)$ where $p(0)=0$, IP sets in $\mathbb{N}$ and $P-1$ where P denotes

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the set of primes. (P itself is not recurrent for trivial reasons.) If $\mu(T^{n_m}A \cap A) = 0$ for some non-null A and for all m we will say that (n_m) is non-recurrent for T, or sometimes, non-recurrent for T and A.

Remark 4.1. If a sequence (n_m) is non-recurrent for an arbitrary finite measure-preserving T then it is also non-recurrent for some ergodic S. Indeed, it is easy to see that (n_m) will be non-recurrent for all S in a set of ergodic components of T having positive measure.

Remark 4.2. Suppose that T is totally ergodic. Given a sequence (n_m) and a non-null set A such that $T^{n_m}A \cap A = \varnothing$ for all $m > M$, we can find a non-null $B \subset A$ such that $T^{n_m}B \cap B = \varnothing$ for all m. This follows from repeated applications of the observation that, given any non-null set C and $l > 0$, there is a non-null $D \subset C$ such that $T^l D \cap D = \varnothing$, since T^l is ergodic.

The following is a fundamental observation of Furstenberg [**18**].

PROPOSITION 4.3. *Suppose that for $k = 1, \ldots, K$ the set $E_k \subset \mathbb{N}$ is non-recurrent. Then $E = \bigcup_{k=1}^{K} E_k$ is non-recurrent.*

Proof. If E_k is non-recurrent for T_k and A_k then evidently E is non-recurrent for $T_1 \times \cdots \times T_k$ and $A_1 \times \cdots \times A_k$. □

Remark 4.4. If all the T_k are weakly mixing then so is T. In general, if the T_k are just ergodic T may not be ergodic but Remark 4.1 shows that E is still non-recurrent for some ergodic T.

We have the following observation connecting rigidity and non-recurrence.

PROPOSITION 4.5. *Suppose that T is ergodic and rigid along some sequence (a_m). Then (a_m) has a subsequence (n_m) such that $(n_m + 1)$ is non-recurrent for T.*

Proof. Since T is ergodic we can find (for example, by Rokhlin's lemma) a set A of measure $\frac{1}{3}$ such that $TA \cap A = \varnothing$. Then we can find a subsequence (n_m) of (a_m) such that $T^{n_m}A \overset{\epsilon_m}{\sim} A$, where the sequence (ϵ_m) is chosen so that $\sum_{m\geq 1} \epsilon_m < \frac{1}{10}$. Now let

$$B = A \setminus \bigcup_{m\geq 1} T^{-(n_m+1)}A.$$

Then $T^{n_m+1}B \cap B = \varnothing$ by definition so we need only check that B has positive measure. Note that

$$T^{-(n_m+1)}A \cap A = T^{-1}(T^{-n_m}A \cap TA) \overset{\epsilon_m}{\sim} T^{-1}(A \cap TA) = \varnothing,$$

that is, $\mu(T^{-(n_m+1)}A \cap A) < \epsilon_m$. It follows that

$$\begin{aligned}\mu(B) &\geq \mu(A) - \mu\left(\bigcup_m (A \cap T^{-(n_m+1)}A)\right) > \frac{1}{3} - \sum_m \mu(A \cap T^{-(n_m+1)}A) \\ &> \frac{1}{3} - \frac{1}{10}.\end{aligned}$$

□

COROLLARY 4.6. *Given any sequence (n_m), the set of weakly mixing T which is non-recurrent along some subsequence of (n_m) is generic.*

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Proof. This follows from Proposition 4.5, since by Proposition 2.8 the generic T is weakly mixing and rigid along some subsequence of $(n_m - 1)$. □

Remark 4.7. The argument in Proposition 4.5 can be modified to show that if an infinite $E \subset \mathbb{N}$ is rigid for T and $K > 0$ then there is an infinite $F \subset E$ such that $F + \{k : 0 < |k| \leq K\}$ is non-recurrent for T. We need only start with a set A such that $T^k A$, $|k| \leq K$, are mutually disjoint. Then, choosing (n_m) sufficiently rigid for A, a similar modification of A produces the desired subset B. In the manner of Corollary 4.6 this in turn yields the following result.

COROLLARY 4.8. *Given any infinite $E \subset \mathbb{N}$ and $K > 0$, the set of $T \in \mathcal{G}$, such that there is an infinite subset F of E for which $F + \{k : |k| \leq K\}$ is non-recurrent for T, is a generic subset of $\mathcal{G}$.*

It is not hard to see that any lacunary sequence is non-recurrent for some ergodic dynamical system. Indeed, this happens even with ergodic rotations of $\mathbb{T}$; see Pollington [**43**], de Mathan [**38**], and Furstenberg [**18**, p. 220]. They show that for any lacunary sequence (n_m) there is some $\gamma \in \mathbb{T}$ of infinite order and some $\delta > 0$, such that $|\gamma^{n_m} - 1| \geq \delta$ for all $m \geq 1$. The arguments there also give information about the size of the set of rotations that work for a given lacunary sequence. The constructions in these articles are made more difficult by not knowing the degree of lacunarity. But if one just wants some ergodic dynamical system to exhibit non-recurrence, then the construction is easier. This was observed by Furstenberg [**18**]. His argument gives non-recurrence for a translation of a finite-dimensional torus rather than a circle rotation. In short, it goes as follows.

If (n_k) is lacunary, then we can choose K so that the subsequences $(p_{m,j} : m \geq 1)$ given by $p_{m,j} = n_{j+Km}$, $j = 0, \ldots, K-1$, each have lacunary constant $\inf_{m \geq 1}(p_{m+1,j})/p_{m,j} \geq 5$. Then for each j, a standard argument shows that there is a closed perfect set C_j such that for all $\gamma \in C_j$, $|\gamma^{p_{m,j}} - 1| \geq \frac{1}{100}$ for all $m \geq 1$.

Since the sets C_j are all uncountable we can choose $(\gamma_0, \ldots, \gamma_{K-1})$ with each $\gamma_j \in C_j$ and such that $\gamma_0, \ldots, \gamma_{K-1}$ are independent. Then the translation T of the K-torus $\mathbb{T}^K$ given by $T(\alpha_0, \ldots, \alpha_{K-1}) = (\gamma_0\alpha_0, \ldots, \gamma_{K-1}\alpha_{K-1})$ is ergodic. Moreover, as in Proposition 4.3, if we set $A = \{z \in \mathbb{T}^K : |z_j - 1| < \frac{1}{100}\ \forall j = 0, \ldots, K-1\}$ we have (n_k) non-recurrent for T and A.

This naturally raises the following fundamental unanswered question.

Question. Is it the case that any lacunary sequence is non-recurrent for some weakly mixing system?

Remark 4.9. For rotations of the circle, or more generally any ergodic isometry, it is easy to see that there exist very slowly growing non-recurrent sequences, in fact they can always be chosen with density arbitrarily close to 1.

We will see below (Proposition 4.12) that sequences which grow very fast (in some sense, super-lacunary) are always non-recurrent for some weakly mixing map. However, first we would like to single out a class of examples which gives lacunary sequences with

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bounded ratios n_{m+1}/n_m which are non-recurrent for weak mixing. We say that a rank-one map has bounded spacers if there is a constant C such that the lengths of the spacer blocks inserted at each stage m of the construction are all bounded by C, and, moreover, spacers are inserted only between the copies of the mth tower in the $(m+1)$th tower and not before the first or after the final copy. Note that this implies that the distance between two successive copies of the mth tower τ_m in any $\tau_{m'}$, $m' > m$, is at most C. (Of course, this definition refers not just to the map T but to a specific cutting and stacking construction of T.) See the classic example in Chacón [**9**].

PROPOSITION 4.10. *Suppose that T is a rank-one map with bounded spacer lengths and tower heights h_m. Then $(h_m - 1)$ is a non-recurrent sequence for T.*

Proof. Suppose that the spacer lengths are bounded by C and choose an M such that $h_M > 2C$. Let A denote the top level of the Mth tower τ_M. Then it is easy to see that $T^{h_m-1}A \cap A$ is empty for all $m \geq M$. Roughly speaking, if we take $l \gg m \geq M$ and shift the lth tower by h_m then every *m-block* in the shifted tower either aligns exactly with the following m-block (if it is not followed by any spacers) or it lags behind the following m-block by $l \leq C$ (l being the number of spacers following it). Thus if we shift by $h_m - 1$ no occurrence of A in the shifted tower aligns with any occurrence in the unshifted tower.

Since $T^{h_m-1}A \cap A$ is empty for all $m \geq M$ it follows by Remark 4.2 that the full sequence $(h_m - 1)$ is non-recurrent for T. □

Remark 4.11. Setting $E = \{h_m - 1 : m \geq 1\}$, it is clear that almost the same argument shows that for any $K \geq 1$ the set $E - \{0, 1, \ldots, K\}$ is non-recurrent for T.

The Chacón map T is a well-known example of Proposition 4.10. In this case the $(n+1)$th tower is constructed by concatenating three copies of the nth tower and inserting a single spacer between the second and third copies. The initial tower is taken to have height one, and one sees that $h_n - 1 = \frac{3}{2}(3^n - 1)$ is non-recurrent for this map. T is well known to be weakly mixing (by an argument similar to the weak mixing argument in the proof of Proposition 3.16). In fact, we can vary this construction by choosing not to insert the single spacer at some stages of the construction, and as long as we do insert it infinitely often we still get weak mixing. In this way we get many other examples of sequences (n_m) which are non-recurrent for a weakly mixing map.

We also remark that Chacón's map has no rigid sequences but we do have partial rigidity, specifically it is not hard to see that $T^{h_m} \to \frac{1}{2}(\mathrm{Id} + T^{-1})$. It follows that (h_m) and $(h_m + 1)$ are sequences of recurrence for T, while $(h_m - 1)$ is not. For any $K > 0$ we could modify the construction to ensure that for each $0 \leq k \leq K$ the sequence $(h_m + k)$ is a sequence of recurrence for T.

One can define further variants of Chacón's map where at the mth-stage of the construction we cut in two at each stage of the construction and insert δ_m spacers between the two copies where $\delta_m \in \{0, 1\}$. If $\{m : \delta_m = 1\}$ and $\{m : \delta_m = 0\}$ are both infinite this yields weak mixing. (If, for example, $\delta_m = 1$ for all m then one just gets the binary odometer which is not weak mixing.) For this example we have $\lim_{m\to\infty}(h_{m+1}/h_m) = 2$, obviously the smallest lacunary growth rate which we can get from Proposition 4.10. Of course, by varying the number of cuts at each stage we get many other examples.

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If we assume that (n_m) grows sufficiently fast then we can show that it is non-recurrent for weak mixing. Note that the hypothesis in the following proposition is the same as in Proposition 3.16.

PROPOSITION 4.12. *Suppose that we have a sequence* $E = (n_m)$ *such that* $\sum_{k=1}^{\infty}(n_m/n_{m+1}) < \infty$. *Then there is a rank-one weakly mixing map* T *for which* (n_m) *is* IP*-rigid and* $(n_m + 1)$ *is non-recurrent. In particular, there is also a rank-one weakly mixing map which is not recurrent for* (n_m).

Proof. The transformation T constructed in the proof of Proposition 3.16 will do the job. We already know that (n_m) is IP-rigid. Moreover, referring to that proof, specifically to the estimate (3.6), choose M so that

$$\sum_{m \geq M} \epsilon_m < \frac{1}{10} \tag{4.1}$$

and then let A consist of every other level of τ_M, not including the top level. Then for sufficiently large M we have $\mu(A) > \frac{1}{3}$ so the estimates (3.6) and (4.1) allow us to argue as in the proof of Proposition 4.5 that some subset of A does not recur along the sequence $n_m + 1$.

For the second statement just observe that $(n_m - 1)$ satisfies the hypothesis of the proposition if and only if (n_m) does. □

Remark 4.13. As in Remark 4.7, we could also argue here that for any $K \geq 1$ the set $E + \{k : 0 < |k| \leq K\}$ is non-recurrent for T. Moreover, for any $K > 0$ there is a weakly mixing rank-one T which is not recurrent along $E + \{k : |k| \leq K\}$.

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