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A GENERALISATION OF HIGHER-RANK GRAPHS

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Abstract

We introduce 'generalised higher-rank k-graphs' as a class of categories equipped with a notion of size. They extend not only higher-rank k-graphs, but also the Levi categories introduced by the first author as a categorical setting for graphs of groups. We prove that examples of generalised higher-rank k-graphs can be constructed using Zappa–Szép products of groupoids and higher-rank graphs.

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1. Introduction

Higher-rank k-graphs are a class of categories important in the theory of C^*-algebras. They were formally introduced in [3] by reconceptualising the work of [11]. In this paper, we generalise such categories and show that recent work [1] is a special case leading us to an interesting class of left cancellative categories [12, 13]. What connects this paper and [1] are Zappa–Szép products of categories. Specifically, we observed that [1, Lemma 3.4] has a significant intersection with axioms (C1)–(C3) and (SS1)–(SS8) of [6] (also listed in Section 4 below) which describe the Zappa–Szép product of a category by a groupoid. This means that Zappa–Szép products of a category by a groupoid can be viewed as the correct setting for thinking about groupoids acting by partial symmetries on categories. This was exactly the idea that lay behind [5] where self-similar group actions of groups on free monoids are viewed as special cases of Zappa–Szép products of groups on free monoids and are used to construct a class of left cancellative monoids called *left Rees monoids*. This idea is explored in [9] and then generalised to categories in [10]; we shall see below that the categories that arise there, called Levi categories, are examples of generalised higher-rank 1-graphs. Levi categories provide a categorical setting for graphs of groups.

In this section we recall the basic definitions needed to read this paper. Categories C are always small and consist only of arrows. The set of identities of C is denoted by C_o. A subcategory of C is said to be *wide* if it contains all the identities of C.

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For $x \in C$ denote by $\mathbf{d}(x)$ the unique identity such that $x\mathbf{d}(x)$ is defined, and by $\mathbf{r}(x)$ the unique identity such that $\mathbf{r}(x)x$ is defined. We call the identity $\mathbf{d}(x)$ the *domain* of x and the identity $\mathbf{r}(x)$ the *range* of x. The product xy exists if and only if $\mathbf{d}(x) = \mathbf{r}(y)$; we also write $\exists xy$ in this case. An arrow x in a category C is said to be *invertible* if there is an arrow $x^{-1} \in C$ such that $x^{-1}x = \mathbf{d}(x)$ and $xx^{-1} = \mathbf{r}(x)$. We also say that x is an *isomorphism*. A category in which every element is an isomorphism is called a *groupoid*. If C is a category, we shall denote its set of isomorphisms by G. All identities of a category are isomorphisms. We shall say that an isomorphism which is not an identity is *nontrivial*. Clearly, G is a wide subcategory of C. More generally, the category C is *left cancellative* if $ax = ay$ implies that $x = y$, and is *right cancellative* if $xb = yb$ implies that $x = y$. A category which is both left and right cancellative is said to be *cancellative*. If C is a category and $a \in C$, define

$$aC = \{ax \colon x \in C \text{ and } \exists ax\},$$

the *principal right ideal generated by a*, and define

$$Ca = \{xa \colon x \in C \text{ and } \exists xa\},$$

the *principal left ideal generated by a*. The two-sided case CaC has the obvious definition. Observe that $aC \subseteq \mathbf{r}(a)C$. Thus every principal right ideal is contained in a principal right ideal of the form eC where e is an identity. We say that a principal right ideal aC is *maximal* if it is properly contained in $\mathbf{r}(a)C$ and is the maximal principal right ideal in $\mathbf{r}(a)C$. An element $a \in C$ of a category is called an *atom* if a is noninvertible, and if $a = bc$ then either b or c is invertible. A category C is said to be *equidivisible* if $ab = cd$ implies either that there is an arrow u such that $a = cu$ and $d = ub$ or there exists an arrow v such that $c = av$ and $b = vd$; this definition was first stated for monoids [4].

The set of natural numbers, along with 0, is denoted by $\mathbb{N}$. We denote by $\mathbb{N}^k$ the monoid of all k-tuples of elements of $\mathbb{N}$, and we denote the elements of $\mathbb{N}^k$ by bold lower-case letters such as $\mathbf{m}$. The additive identity of $\mathbb{N}^k$ is denoted by $\mathbf{0}$. There is a partial order $\leq$ defined on $\mathbb{N}^k$ by $\mathbf{m} \leq \mathbf{n}$ if and only if $\mathbf{m} + \mathbf{p} = \mathbf{n}$ for some $\mathbf{p} \in \mathbb{N}^k$. The element of $\mathbb{N}^k$ which is 0 everywhere except at position i where it takes the value 1 is denoted by $\mathbf{e}_i$.

A category C is called a *higher-rank* k-graph if there is a functor $\delta \colon C \to \mathbb{N}^k$ satisfying the *unique factorisation property* (UFP) as follows: if $\delta(a) = \mathbf{m} + \mathbf{n}$, then there exist unique elements b, c such that $a = bc$, $\delta(b) = \mathbf{m}$ and $\delta(c) = \mathbf{n}$. It follows that C is a cancellative category with no nonidentity invertible elements and $\delta^{-1}(\mathbf{0})$ consists only of identities [3]. Higher-rank k-graphs can be used to construct groups that include and therefore generalise the classical Thompson groups [7, 8]. In the theory of C^*-algebras, the category C is usually assumed to be countable, but we do not need this assumption here.

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2. Categories with size functors

Let C be a small category. By a *size functor* on C, we mean a functor $\lambda\colon C \to \mathbb{N}^k$ such that $\lambda^{-1}(\mathbf{0}) = G$ consists of all and only the invertible elements of C. Thus G is the subgroupoid of C consisting of all the invertible elements. In this section we describe some elementary results about categories equipped with size functors.

LEMMA 2.1. *Let C be a category equipped with a size functor λ with groupoid of invertible elements G.*

(1) *$aC = bC$ if and only if $aG = bG$.*
(2) *$Ca = Cb$ if and only if $Ga = Gb$.*
(3) *$CaC = CbC$ if and only if $GaG = GbG$.*

PROOF. We prove (1) since the proof of (2) follows by symmetry. The proof of (3) is then analogous. Only one direction needs to be proved. Suppose that $aC = bC$. Then $a = bx$ and $b = ay$ for some $x, y \in C$. Then $a = bx = ayx$. Thus $\lambda(a) = \lambda(a) + \lambda(y) + \lambda(x)$. It follows that $\mathbf{0} = \lambda(x) + \lambda(y)$. Thus $\lambda(x) = \lambda(y) = \mathbf{0}$. We deduce that both x and y are invertible. The result follows. □

We adopt semigroup notation and write $a \,\mathcal{R}\, b$ when $aC = bC$ and $a \,\mathcal{L}\, b$ when $Ca = Cb$. We also write $a \,\mathcal{J}\, b$ when $CaC = CbC$.

LEMMA 2.2. *Let C be a category equipped with a size functor λ with groupoid of invertible elements G. The element a is invertible if and only if $aC = eC$ for some identity e.*

PROOF. If a is invertible, then $aC = aa^{-1}C$. Conversely, suppose that $aC = eC$. Then by Lemma 2.1, $a = eg = g$ where g is invertible. □

We now relate atoms to principal right ideals.

LEMMA 2.3. *Let C be a category equipped with a size functor λ with groupoid of invertible elements G. Then $a \in C$ is an atom if and only if aC is maximal.*

PROOF. Suppose that a is an atom. If $aC \subseteq bC$ then $a = bc$ for some $c \in C$. If b is invertible then $bC = bb^{-1}C$. If c is invertible then $aC = bC$. It follows that aC is maximal. Conversely, suppose that aC is maximal and that $a = bc$. Then $aC \subseteq bC$. Thus either $aC = bC$ or $bC = eC$ for some identity e. If the latter then, by Lemma 2.2, b is invertible. If the former then $a = bg$ for some invertible element g. It follows that $bc = bg$. Thus $\lambda(c) = \mathbf{0}$ and so c is invertible. □

The fact that the usual order on $\mathbb{N}^k$ has the property that there are only a finite number of elements below each element is used below.

LEMMA 2.4. *Let C be a category equipped with a size functor λ with groupoid of invertible elements G. Each noninvertible element can be written as finite product of atoms.*

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PROOF. Let a be a noninvertible element. If a is an atom then we are done. If not, then we can write $a = bc$ where neither b nor c is invertible. Observe that $\lambda(a) = \lambda(b) + \lambda(c)$ and, by assumption, $\lambda(b), \lambda(c) < \lambda(a)$. If b and c are atoms then we are done, otherwise the procedure can be repeated. □

The groupoid G acts on both sides of the set of atoms.

LEMMA 2.5. *Let C be a category equipped with a size functor λ with groupoid of invertible elements G.*

(1) *If g is invertible, a is an atom and $\exists ga$ then ga is an atom.*
(2) *If g is invertible, a is an atom and $\exists ag$ then ag is an atom.*
(3) *Let X be a transversal of the set of generators of the maximal principal right ideals. Let g be invertible and let a be an atom such that $\exists ga$. Then $ga = xh$ for some $x \in X$ and invertible h.*

PROOF. (1) First, ga cannot be invertible because this would imply that a was invertible. Suppose that $ga = bc$. Then $a = (g^{-1}b)c$. Thus $g^{-1}b$ is invertible, in which case b is invertible or c is invertible. It follows that ga is an atom. The proof of (2) is similar to the proof of (1).

(3) We proved in (1) that ga is an atom. Thus $gx \mathcal{R} x$ for some $x \in X$. It follows that $gx = xh$ for some $h \in G$ by Lemma 2.1. □

We now come to our first main result.

PROPOSITION 2.6. *Let C be a category equipped with a size functor λ with groupoid of invertible elements G. Let X be a transversal of the set of generators of the maximal principal right ideals. Then $C = \langle X \rangle G$ where $\langle X \rangle$ is the subcategory generated by the elements X.*

PROOF. Let $a \in C$ be any noninvertible element. By Lemma 2.4, $a = a_1 \dots a_n$ where each a_i is an atom. Let $a_1 \mathcal{R} x_1$ where $x_1 \in X$ by Lemma 2.5. Then $a_1 = x_1 h_1$ for some $h_1 \in G$. Thus $a = x_1(h_1 a_2)a_3 \dots a_n$. We now apply Lemma 2.5 to the atom $h_1 a_2$. Repeating this, we deduce that $a = x_1 \dots x_n h$ where $x_i \in X$ and $h \in G$. □

3. Generalised higher-rank k-graphs

Proposition 2.6 gives us some insight into the structure of categories equipped with size functors, but to obtain more detailed results, we shall need to assume more than the mere existence of a size functor. The next definition is the main one of this paper. It extends the definition of a higher-rank k-graph in a categorically natural way by incorporating isomorphisms.

DEFINITION 3.1. Let C be a category equipped with a size functor λ to $\mathbb{N}^k$. We say that λ satisfies the *weak factorisation property* (WFP) if the following statement holds. let $\lambda(a) = \mathbf{m} + \mathbf{n}$. Then there exist $a_1, a_2 \in C$ where $a = a_1 a_2$, $\lambda(a_1) = \mathbf{m}$ and $\lambda(a_2) = \mathbf{n}$,

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such that if $a = b_1b_2$, $\lambda(b_1) = \mathbf{m}$ and $\lambda(b_2) = \mathbf{n}$, then $b_1 = a_1g$ and $b_2 = g^{-1}a_2$ for some invertible element g. We call such a category a *generalised higher-rank k-graph*.

The above definition was motivated by [5, 9, 10] combined with the definition of a higher-rank k-graph. Observe that higher-rank graphs are precisely those generalised higher-rank k-graphs which have only the identities as invertible elements (and are countable). The following result shows that in weak higher-rank graphs, the atoms can be identified by means of the size functor.

LEMMA 3.2. *Let C be a generalised higher-rank k-graph. Then the atoms are precisely the elements $a \in C$ such that $\lambda(a) = \boldsymbol{e}_i$ for some i.*

PROOF. Suppose that $\lambda(a) = \mathbf{e}_i$. We prove that a is an atom. Let $a = bc$. Then $\lambda(a) = \lambda(b) + \lambda(c)$. It follows that either $\lambda(b) = \mathbf{0}$ or $\lambda(c) = \mathbf{0}$, from which we deduce that either b or c is invertible. Now, suppose that $\lambda(a)$ is zero but not equal to any $\mathbf{e}_i$. Then, for some i we can write $\lambda(a) = \mathbf{e}_i + \mathbf{m}$ where $\mathbf{e}_i, \mathbf{m} \neq \mathbf{0}$. Then, by the WFP, $a = bc$ where $\lambda(b) = \mathbf{e}_i$ and $\lambda(c) = \mathbf{m}$. It follows that neither b nor c is invertible and so a cannot be an atom. □

The following is an immediate corollary to Lemma 3.2.

COROLLARY 3.3. *Let C be a generalised higher-rank 1-graph. Then the atoms are precisely those elements $a \in C$ such that $\lambda(a) = 1$.*

The following result provides an alternative characterisation of generalised higher-rank 1-graphs and shows that our generalisation contains other examples apart from higher-rank k-graphs. Recall that a *Levi category* [10] is an equidivisible category equipped with a size functor $\lambda\colon C \to \mathbb{N}$ such that $\lambda^{-1}(1)$ consists of all and only the atoms of C. We introduced Levi categories in [10] to describe graphs of groups.

THEOREM 3.4.

(1) *Every Levi category is a generalised higher-rank 1-graph.*
(2) *Every generalised higher-rank 1-graph is a Levi category.*

PROOF. (1) Let C be a Levi category. Denote its size functor by $\lambda\colon C \to \mathbb{N}$. Let a be a noninvertible element. By Lemma 2.4, we may write $a = a_1 \dots a_m$ where the a_i are atoms. By assumption, $\lambda(a_i) = 1$ and so $\lambda(a) = m$. Let $m = s + t$. Then, using the above representation of a as a product of atoms, we may find elements x and y such that $a = xy$ where $\lambda(x) = s$ and $\lambda(y) = t$. Specifically, in the case where $s, t \neq 0$, define $x = a_1 \dots a_s$ and $y = a_{s+1} \dots a_m$. If $s = 0$ then chose $x = \mathbf{r}(a_1)$ and $y = a$, and if $t = 0$ then choose $x = a$ and $y = \mathbf{d}(a_m)$. Now suppose that $a = wz$ where $\lambda(w) = s$ and $\lambda(z) = t$. Then $a = wz = xy$ where $\lambda(w) = \lambda(x)$ and $\lambda(z) = \lambda(y)$. We use equidivisibility, and without loss of generality, $w = xu$ and $y = uz$ for some $u \in C$. Thus $\lambda(u) = 0$ and so u is invertible. It follows that $w = xu$ and $z = u^{-1}y$.

(2) Let C be a generalised higher-rank 1-graph. Because of Lemma 3.3, it is enough to prove that C is equidivisible. Let $ab = cd$. There are two cases. We prove the first,

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and the second follows by symmetry. Suppose that $\lambda(a) \geq \lambda(c)$. Then $\lambda(c) + m = \lambda(a)$ for some natural number m. By the WFP, we can write $a = xy$ where $\lambda(x) = \lambda(c)$ and $\lambda(y) = m$. Thus $x(yb) = cd$. Observe that $\lambda(x) = \lambda(c)$ and so $\lambda(yb) = \lambda(d)$. By the WFP, there is an invertible element g such that $x = cg$ and $yb = g^{-1}d$. It follows that $a = xy = c(gy)$ and $d = (gy)b$. Put $u = gy$ and we have $a = cu$ and $d = ub$. □

4. The structure of a class of generalised higher-rank k-graphs

In this section we shall show how to construct a class of generalised higher-rank k-graphs. The following result is a step in this direction.

LEMMA 4.1. *Let C be a generalised higher-rank k-graph with groupoid of invertible elements G. Choose and fix a transversal X of the maximal right principal ideals. Let $\mathcal{X} = \langle X \rangle$. Then C is left cancellative if and only if $ug = u$ implies $g = \mathbf{d}(a)$ where $u \in \mathcal{X}$ and $g \in G$.*

PROOF. Only one direction need be proved. We prove first that if $a \in C$ and $g \in G$ then $a = ag$ implies that $g = \mathbf{d}(a)$. Let $a \in C$ be noninvertible. Then $a = uh$ where $u \in \mathcal{X}$ and $h \in G$ by Proposition 2.6. Thus $ag = uhg = uh$. It follows that $u = uhgh^{-1}$. By assumption, $\mathbf{d}(u) = hgh^{-1}$ from which we deduce that $g = \mathbf{d}(a)$. Now let $ab = ac$. Then there is an invertible element g such that $a = ag$ and $b = g^{-1}c$. By assumption, $g = \mathbf{d}(a)$ and so $b = c$, as claimed. □

We shall study weak higher-rank k-graphs that satisfy a strengthened form of the condition stated in Lemma 4.1. Let C be a generalised higher-rank k-graph. Choose and fix a transversal X of the maximal right principal ideals. Let $\mathcal{X} = \langle X \rangle$. We say that C satisfies the $\mathcal{R}$-condition if $u, v \in \mathcal{X}$ and $u = vg$ in C implies that $u = v$ and g is an identity. By Lemma 4.1, the category C is left cancellative. We can now refine Proposition 2.6.

THEOREM 4.2. *Let C be a generalised higher-rank k-graph satisfying the $\mathcal{R}$-condition, let X be a transversal of the set of generators of the maximal principal right ideals, let $\mathcal{X}$ be the wide subcategory of C generated by X and let G be the groupoid of invertible elements of C. Then $\mathcal{X}$ is a higher-rank k-graph, and every element of C has a unique representation as an element of $\mathcal{X}G$.*

PROOF. Observe that apart from the identities the subcategory $\mathcal{X}$ does not contain any invertible elements because each nonidentity element of $\mathcal{X}$ is a product of atoms and if a is an atom then $\lambda(a) \neq \mathbf{0}$. It follows that $\mathcal{X}$ is a subcategory with only the identities as invertible elements. Let $a \in \mathcal{X}$. Define $d(a) = \lambda(a)$. This is clearly a functor. It remains to show that we have the UFP. Let $a \in \mathcal{X}$ be such that $d(a) = \mathbf{m} + \mathbf{n}$. Then, by the WFP, there are elements $b, c \in C$ such that $a = bc$ where $d(b) = \mathbf{m}$ and $d(c) = \mathbf{n}$. We may write $b = ug$ for some $u \in \mathcal{X}$ and $g \in G$ and we may write $c = vh$ in a similar way. Thus $a = ugvh$. Observe that $d(b) = d(u)$ and $d(c) = d(v)$. We may write $gv = wk$. Observe that $d(w) = d(v)$. Thus $a = uwkg$ where $a, u, w \in \mathcal{X}$ and $k, g \in G$. We now invoke the $\mathcal{R}$-condition, and deduce that $a = uw$ where $d(u) = \mathbf{m}$ and $d(w) = \mathbf{n}$. We now prove

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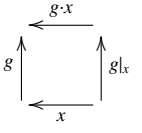
that u and w are unique with these properties. Suppose that $a = u_1w_1$ where $u_1, w_1 \in \mathcal{X}$ and $d(u_1) = \mathbf{m}$ and $d(w_1) = \mathbf{n}$. Then by the WFP, there is an isomorphism l such that $u_1 = ul$ and $w_1 = l^{-1}w$. By the $\mathcal{R}$-condition, we deduce that $u_1 = u$ and so $w_1 = w$. Thus $\mathcal{X}$ is a higher-rank k-graph. It only remains to show that every element has a unique factorisation. Let $ug = vh$ where $u, v \in \mathcal{X}$ and $g, h \in G$. Then $u = v(hg^{-1})$. By the $\mathcal{R}$-condition, $hg^{-1} = 1$ and $u = v$; thus $g = h$ also. □

Our goal now is to show how to construct categories of the above type from higher-rank k-graphs and groupoids. We shall use *Zappa–Szép products* [2] or, equivalently, what are termed *self-similar groupoid actions* in [1]. The motivation for doing this comes from [5] and the sequence of papers that followed [6, 9, 10].

Let $\mathcal{X}$ be a category and let G be a groupoid such that $G_o = \mathcal{X}_o$. We suppose that there is a function $G \times \mathcal{X} \to \mathcal{X}$ given by $(g, x) \mapsto g \cdot x$, whenever $\mathbf{d}(g) = \mathbf{r}(x)$, and a function $G \times \mathcal{X} \to G$ given by $(g, x) \mapsto g|_x$, whenever $\mathbf{d}(g) = \mathbf{r}(x)$, such that the axioms (C1), (C2), (C3) and (SS1)–(SS8) listed below all hold.

(C1) $\mathbf{r}(g \cdot x) = \mathbf{r}(g)$.
(C2) $\mathbf{d}(g \cdot x) = \mathbf{r}(g|_x)$.
(C3) $\mathbf{d}(x) = \mathbf{d}(g|_x)$.

This information is summarised by the following diagram:



(SS1) $\mathbf{r}(x) \cdot x = x$.
(SS2) $(gh) \cdot x = g \cdot (h \cdot x)$.
(SS3) $g \cdot \mathbf{d}(g) = \mathbf{r}(g)$.
(SS4) $\mathbf{r}(x)|_x = \mathbf{d}(x)$.
(SS5) $g|_{\mathbf{d}(g)} = g$.
(SS6) $g|_{xy} = (g|_x)|_y$.
(SS7) $(gh)|_x = g|_{h \cdot x}h|_x$.
(SS8) $g \cdot (xy) = (g \cdot x)(g_x \cdot y)$.

We call this a *Zappa–Szép action* of G on $\mathcal{X}$ or a *self-similar action of G on $\mathcal{X}$*.

REMARK 4.3. Zappa–Szép products are two-sided semidirect products. There is a left action of G on $\mathcal{X}$ denoted by $(g, x) \mapsto g \cdot x$ and there is a right action of $\mathcal{X}$ on G denoted by $(g, x) \mapsto g|_{\mathbf{d}(g)}$. The axioms (SS1), (SS2), (SS5) and (SS6) tell us that we are dealing with actions. Axioms (SS3) and (SS4) tell us how identities act and axioms (SS7) and (SS8) tell us how the category structures on G and $\mathcal{X}$ interact with the actions.

We can use self-similar actions to build categories, as we now show.

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PROPOSITION 4.4. (1) *Let there be a Zappa–Szép action of the groupoid G on the category $\mathcal{X}$. Define $\mathcal{X} \bowtie G$ to be the set of all ordered pairs (x, g) where $\boldsymbol{d}(x) = \boldsymbol{r}(g)$. Define*

$$\boldsymbol{d}(x, g) = (\boldsymbol{d}(g), \boldsymbol{d}(g)) \quad \text{and} \quad \boldsymbol{r}(x, g) = (\boldsymbol{r}(x), \boldsymbol{r}(x)).$$

Define a product by

$$(x, g)(y, h) = (x(g \cdot y), g|_y h)$$

when $\boldsymbol{d}(x, g) = \boldsymbol{r}(y, h)$. Then $\mathcal{X} \bowtie G$ is a category with set of identities (e, e) where $e \in G_o = \mathcal{X}_o$. Define

$$\iota_{\mathcal{X}} \colon \mathcal{X} \to \mathcal{X} \bowtie G; \quad x \mapsto (x, \boldsymbol{d}(x)).$$

This is an injective functor. Define

$$\iota_G \colon G \to \mathcal{X} \bowtie G; \quad g \mapsto (\boldsymbol{r}(g), g).$$

This is an injective functor. Observe that if $(x, g) \in \mathcal{X} \bowtie G$ then $(x, g) = \iota_{\mathcal{X}}(x)\iota_G(g)$.

(2) *Let C be a category with groupoid of isomorphisms G and wide subcategory $\mathcal{X}$ such that $C = \mathcal{X}G$ uniquely. Then C is isomorphic to $\mathcal{X} \bowtie G$.*

PROOF. (1) The proofs are special cases of general arguments given in [2]. We describe the salient points. Define $\mathbf{d}(x, g) = (\mathbf{d}(g), \mathbf{d}(g))$ and $\mathbf{r}(x, g) = (\mathbf{r}(x), \mathbf{r}(x))$. Observe that the product $(x, g)(y, h)$ is defined if and only if $\mathbf{d}(g) = \mathbf{r}(y)$. It is routine now to check that $\mathcal{X} \bowtie G$ is a category with set of identities (e, e) where $e \in G_o = \mathcal{X}_o$. The product $(x, \mathbf{d}(x))(y, \mathbf{d}(y))$ exists if and only if $\exists xy$ and the product is equal to $(x(\mathbf{d}(x) \cdot y), \mathbf{d}(x)|_y \mathbf{d}(y))$ which simplifies to $(xy, \mathbf{d}(y))$ using axioms (SS1) and (SS4). The product $(\mathbf{r}(g), g)(\mathbf{r}(h), h)$ exists if and only if $\exists gh$ and the product is equal to $(\mathbf{r}(g)(g \cdot \mathbf{r}(h), g|_{\mathbf{r}(h)} h)$ which simplifies to $(\mathbf{r}(g), gh)$ using axioms (SS3) and (SS5).

(2) Let $g \in G$ and $x \in \mathcal{X}$ be such that gx is defined. Then by assumption, there are unique elements $g \cdot x \in \mathcal{X}$ and $g|_x \in G$ such that $gx = (g \cdot x)g|_x$. Checking domains and ranges leads to the verification of axioms (C1), (C2) and (C3) for a Zappa–Szép product. The verification of axioms (SS1)–(SS8) follows from using the following equalities: $x = \mathbf{r}(x)x$; $(gh)x = g(hx)$; $g = g(g^{-1}g)$; $g(xy) = (gx)y$. It is now routine to check that C is isomorphic to $\mathcal{X} \bowtie G$. □

We now add extra structure to the categories that form the basis of our Zappa–Szép product.

LEMMA 4.5. *Let there be a Zappa–Szép action of the groupoid G on the category $\mathcal{X}$. If $\mathcal{X}$ is left cancellative so too is $\mathcal{X} \bowtie G$*

PROOF. Suppose that $(x, g)(y, h) = (x, y)(z, k)$. Then $x(g \cdot y) = x(g \cdot z)$ and $g|_y h = g|_z k$. By left cancellation $g \cdot y = g \cdot z$. Thus $y = z$ and so $h = k$. □

The following consequence of the Zappa–Szép axioms will be needed below.

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LEMMA 4.6. *Let there be a Zappa–Szép action of the groupoid G on the category $\mathcal{X}$. Suppose that $\boldsymbol{r}(g) = \boldsymbol{r}(x)$. Then*

$$g^{-1}|_x = (g|_{g^{-1}\cdot x})^{-1}.$$

PROOF. We calculate from the axioms for a Zappa–Szép product

$$g|_{g^{-1}\cdot x} g^{-1}|_x = \mathbf{r}(g)|_x$$

by axiom (SS7). Thus by axiom (SS4),

$$g|_{g^{-1}\cdot x} g^{-1}|_x = \mathbf{d}(x).$$

The result now follows. □

We now refine Proposition 4.4 by incorporating Lemma 4.6 and applying it to the case where $\mathcal{X}$ is a higher-rank k-graph. Denote by $\delta\colon \mathcal{X} \to \mathbb{N}^k$ the usual size functor. We say that our Zappa–Szép action is *size-preserving* if

$$\delta(g \cdot x) = \delta(x).$$

PROPOSITION 4.7. *Let there be a size-preserving Zappa–Szép action of the groupoid G on the higher-rank k-graph $\mathcal{X}$. Define $\lambda\colon \mathcal{X} \bowtie G \to \mathbb{N}^k$ by $\lambda(x, g) = \delta(x)$. Then this is a well-defined functor and $\mathcal{X} \bowtie G$ is a weak higher-rank k-graph satisfying the $\mathcal{R}$-condition.*

PROOF. It is clear that λ is a size functor. Let (x, g) be an element such that $\lambda(x, g) = \mathbf{m} + \mathbf{n}$. Then by definition $\delta(x) = \mathbf{m} + \mathbf{n}$. But $\mathcal{X}$ is a higher-rank k-graph. Thus there are unique elements u and v such that $x = uv$ where $\delta(u) = \mathbf{m}$ and $\delta(v) = \mathbf{n}$. Observe that $(x, g) = (u, \mathbf{d}(u))(v, g)$ where $\lambda(u, \mathbf{d}(u)) = \mathbf{m}$ and $\lambda(v, g) = \mathbf{n}$. Now suppose that $(x, g) = (w, h)(z, k)$ where $\lambda(w, h) = \mathbf{m}$ and $\lambda(z, k) = \mathbf{n}$. Thus $x = w(h \cdot z)$ and $g = h|_z k$. By uniqueness, $w = u$ and $v = h \cdot z$. Observe that $(w, h) = (u, \mathbf{d}(u))(\mathbf{r}(h), h)$ and $(z, k) = (\mathbf{r}(h), h)^{-1}(v, g)$ where we use the fact that

$$(h|_z)^{-1} = h^{-1}|_v,$$

a consequence of Lemma 4.6. It is routine to check that the $\mathcal{R}$-condition holds. □

We round off this paper with a theorem which pulls together the results of this section.

THEOREM 4.8. *Let C be a generalised higher-rank k-graph satisfying the $\mathcal{R}$-condition. Denote by $\lambda\colon C \to \mathbb{N}^k$ the size functor. Then C is isomorphic to a Zappa–Szép product of the form $\mathcal{X} \bowtie G$ where $\mathcal{X}$ is a higher-rank k-graph, G is a groupoid and the Zappa–Szép action is size-preserving. The functor λ is determined by the size functor δ defined on $\mathcal{X}$.*

PROOF. Let X be a transversal of the set of generators of the maximal principal right ideals, let $\mathcal{X}$ be the wide subcategory of C generated by X and let G be the groupoid of invertible elements of C. Then there is a self-similar action of G on $\mathcal{X}$. Observe that $\lambda(gx) = \lambda(x) = \lambda(g \cdot x)$. There is therefore a size-preserving Zappa–Szép action by

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Proposition 4.7. We may therefore form the category $\mathcal{X} \bowtie G$. Define a function $\theta\colon C \to \mathcal{X} \bowtie G$ by $\theta(a) = (u, g)$ if $a = ug$ where $u \in \mathcal{X}$ and $g \in G$. Then θ is an isomorphism. Observe that $\lambda(a) = \lambda(ug) = \lambda(u)$. Define $\delta(u) = \lambda(u)$ when $u \in \mathcal{X}$. We are done. □

After we posted this paper on the arXiv, we received emails from Eduard Ortega and Enrique Pardo who are working on a paper entitled 'Zappa–Szép products for partial actions of groupoids on left cancellative small categories'.

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