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PROBLEMS AND SOLUTIONS

PROBLEMS

02.5.1. *A Mixingale Inequality Using an Exponential Moment*, proposed by Robert M. De Jong. The mixingale property can be used to prove laws of large numbers and central limit theorems. The inequality that follows shows that in the case where $E\exp(|X_t|) < \infty$ and the X_t sequence is strong mixing and not bounded it is possible to come up with an inequality that is just a little sharper than the well-known mixingale inequality. This is because in case $E\exp(|X_t|) < \infty$ and $|X_t|$ is not uniformly bounded, the convergence rate to zero as $m \to \infty$ in equation (2) is better than what could be obtained if the existence of some arbitrarily high moment is assumed in the inequality of equation (1).

Strong (or α-) mixing sequences X_t are defined as satisfying

$$\sup_{k\geq 0} \sup_{A\in\mathcal{G}_1^k,\, B\in\mathcal{G}_{k+m}^{\infty}} |P(A\cap B) - P(A)P(B)| = \alpha(m),$$

where $\mathcal{G}_a^b = \sigma(\{X_a,\ldots,X_b\})$ and where $\alpha(m)\to 0$ as $m\to\infty$. It follows from this definition that measurable functions of strong mixing processes are again strong mixing, and it is well known that strong mixing sequences satisfy the mixingale property

$$\|E(X_t|\mathcal{G}_{-\infty}^{t-m}) - EX_t\|_p \leq 6\alpha(m)^{1/p-1/r}\|X_t\|_r \qquad \textbf{(1)}$$

for $1\leq p\leq r$ and $r\in[1,\infty]$, where $\|X_t\|_p = (E|X_t|^p)^{1/p}$ (see, e.g., Davidson, 1994, Theorem 14.2). Show that for all $p > 1$,

$$\|E(X_t|\mathcal{G}_{-\infty}^{t-m}) - EX_t\|_p \leq C_p\alpha(m)^{1/p}\log(1+\alpha(m)^{-1})E\exp(|X_t|). \qquad \textbf{(2)}$$

REFERENCE

Davidson J. (1994) *Stochastic Limit Theory: An Introduction for Econometricians*. (Advanced Texts in Econometrics). Oxford, Oxford University Press.

02.5.2. *Durbin–Watson Statistic and Random Individual Effects*, proposed by Stanislav Anatolyev. Consider the standard one-way error component model with random effects (Baltagi, 2001):

$$y_{it} = x_{it}'\beta + \mu_i + v_{it}, \qquad i=1,\ldots,n, \qquad t=1,\ldots,T, \qquad \textbf{(3)}$$

where β is $k\times 1$, μ_i are random individual effects, $\mu_i \sim IID(0,\sigma_\mu^2)$, v_{it} are idiosyncratic shocks, $v_{it}\sim IID(0,\sigma_v^2)$, and μ_i and v_{it} are independent of x_{it} for all i and t and mutually independent. The equations are arranged so that the

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index t is faster than the index i. Consider running ordinary least squares (OLS) on the original regression (3); running OLS on the within regression

$$y_{it} - \bar{y}_{i.} = (x_{it} - \bar{x}_{i.})'\beta + v_{it} - \bar{v}_{i.}, \qquad i = 1,\ldots,n, \qquad t = 1,\ldots,T, \tag{4}$$

where $\bar{z}_{i.} = T^{-1}\sum_{t=1}^{T} z_{it}$ for $z = y, x, v$; running OLS on the between regression

$$\bar{y}_{i.} = \bar{x}_{i.}'\beta + \mu_i + \bar{v}_{i.}, \qquad i = 1,\ldots,n, \qquad t = 1,\ldots,T, \tag{5}$$

with T replications of the equation for each individual i; and running OLS on the generalized least squares transformed regression

$$y_{it} - \hat{\theta}\bar{y}_{i.} = (x_{it}' - \hat{\theta}\bar{x}_{i.})'\beta + (1 - \hat{\theta})\mu_i + v_{it} - \hat{\theta}\bar{v}_{i.}, \qquad i = 1,\ldots,n, \quad t = 1,\ldots,T, \tag{6}$$

where $\hat{\theta}$ is a consistent (as $n \to \infty$ and T stays fixed) estimate of $\theta = 1 - \sigma_v \big/ \sqrt{\sigma_v^2 + T\sigma_\mu^2}$. When each OLS estimate is obtained using a typical regression package, the Durbin–Watson statistic is provided among the regression output. Derive the probability limits of the four Durbin–Watson statistics, as $n \to \infty$ and T stays fixed. Using the obtained result, propose an asymptotic test for individual effects based on the Durbin–Watson statistic.

REFERENCE

Baltagi, B.H. (2001) *Econometric Analysis of Panel Data*. New York: Wiley.

SOLUTIONS

01.5.1. *Some Properties of a Generalized Two-Error Components Matrix*—Solution 1, proposed, by Franc J.G.M. Klaassen and Jan R. Magnus.

(a) Denote the submatrix $\alpha_1 J_1 + \beta_1(I_{T_1} - J_1)$ by Ω_{11} and define Ω_{22} accordingly. Assume first that Ω_{11} is nonsingular. Then, using a standard result on the determinant of a partitioned matrix (Magnus and Neudecker, 1999, p. 25),

$$|\Omega| = |\Omega_{11}||\Omega_{22} - \gamma^2 \iota_2 \iota_1' \Omega_{11}^{-1} \iota_1 \iota_2'| = \alpha_1 \beta_1^{T_1 - 1} \left| \Omega_{22} - \frac{\gamma^2 T_1 T_2}{\alpha_1} J_2 \right|$$

$$= \beta_1^{T_1 - 1} \beta_2^{T_2 - 1} \Delta.$$

Next, let $|\Omega_{11}| = 0$. Then, $\Omega_{11} + \epsilon I_{T_1} = (\alpha_1 + \epsilon)J_1 + (\beta_1 + \epsilon)(I_{T_1} - J_1)$ will be nonsingular for all ϵ sufficiently small, and the result follows by the continuity of the determinant, letting $\epsilon \to 0$.

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(b) To find the eigenvalues of Ω, we notice that the matrix $\Omega - \lambda I_T$ has the same structure as Ω but with parameters $\alpha_i - \lambda$ and $\beta_i - \lambda$ instead of α_i and β_i $(i = 1,2)$. Hence,

$$|\Omega - \lambda I_T| = (\beta_1 - \lambda)^{T_1 - 1}(\beta_2 - \lambda)^{T_2 - 1}((\alpha_1 - \lambda)(\alpha_2 - \lambda) - \gamma^2 T_1 T_2).$$

The eigenvalues are therefore β_1 ($T_1 - 1$ times), β_2 ($T_2 - 1$ times), and ξ_1 and ξ_2, where $\xi_1 + \xi_2 = \alpha_1 + \alpha_2$ and $\xi_1 \xi_2 = \Delta$. Because Ω is positive definite if and only if all its eigenvalues are positive, the result follows.

(c) Let S_1 be a $T_1 \times (T_1 - 1)$ matrix such that $S_1' S_1 = I_{T_1 - 1}$ and $S_1' \iota_1 = 0$. Then $S_1 S_1' = I_{T_1} - J_1$. Let S_2 be defined similarly. Let Λ denote the diagonal $T \times T$ matrix of eigenvalues β_1 ($T_1 - 1$ times), β_2 ($T_2 - 1$ times), ξ_1 and ξ_2 and define the $T \times T$ matrix

$$V = \begin{pmatrix} S_1 & 0 & \theta_1 \iota_1 & \omega_1 \iota_1 \\ 0 & S_2 & \theta_2 \iota_2 & \omega_2 \iota_2 \end{pmatrix}.$$

Then one verifies that $\Omega V = V\Lambda$ and $V'V = I_T$ for suitable choices of θ_1, ω_1, θ_2, and ω_2. Hence, $\Omega^p = V\Lambda^p V'$, and this has the same form as Ω.

(d) For $p = -1$, we find

$$\phi_1 = \alpha_2/\Delta, \qquad \phi_2 = \alpha_1/\Delta, \qquad \delta = -\gamma/\Delta,$$

and for $p = -\frac{1}{2}$ we obtain

$$\phi_1 = \frac{\alpha_2 + \Delta^{1/2}}{\Delta^{1/2}\theta}, \qquad \phi_2 = \frac{\alpha_1 + \Delta^{1/2}}{\Delta^{1/2}\theta}, \qquad \delta = -\frac{\gamma}{\Delta^{1/2}\theta},$$

where $\theta = \sqrt{\alpha_1 + \alpha_2 + 2\Delta^{1/2}}$.

(e) Finally, part (e) follows from (c) by direct calculation.

REFERENCE

Magnus, J.R. & H. Neudecker (1999) *Matrix Differential Calculus with Applications in Statistics and Econometrics*, rev. ed. New York: Wiley.

01.5.1 *Some Properties of a Generalized Two-Error Components Matrix*—Solution 2, proposed by Tom Wansbeek. The notation ι, J, Δ, Ω is used according to the formulation of the problem. Moreover, let $j_1 \equiv \iota_1 / \sqrt{T_1}$ (so $j_1 j_1' = J_1$), G_1 is such that $G_1 \perp \iota_1, G_1' G_1 = I_{T_1}, G_1 G_1' = I_{T_1} - J_1$ (of rank $T_1 - 1$), $\gamma_* \equiv \gamma\sqrt{T_1 T_2}$ (so $\Delta = \alpha_1 \alpha_2 - \gamma_*^2$),

$$L \equiv \begin{bmatrix} j_1 & 0 & G_1 & 0 \\ 0 & j_2 & 0 & G_2 \end{bmatrix}, \qquad A \equiv \begin{bmatrix} \alpha_1 & \gamma_* \\ \gamma_* & \alpha_2 \end{bmatrix}, \qquad B \equiv \begin{bmatrix} \beta_1 I_{T_1 - 1} & 0 \\ 0 & \beta_2 I_{T_2 - 1} \end{bmatrix},$$

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with j_2 and G_2 analogous. Note that L is orthonormal so $|L| = 1$, $|A| = \Delta$, and

$$\Omega = \begin{bmatrix} j_1 & 0 \\ 0 & j_2 \end{bmatrix} \begin{bmatrix} \alpha_1 & \gamma_* \\ \gamma_* & \alpha_2 \end{bmatrix} \begin{bmatrix} j_1 & 0 \\ 0 & j_2 \end{bmatrix}' + \begin{bmatrix} \beta_1(I_{T_1} - J_1) & 0 \\ 0 & \beta_2(I_{T_2} - J_2) \end{bmatrix}$$
$$= L \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} L'. \tag{1}$$

This implies

(a) $|\Omega| = |A||B| = \Delta\beta_1^{T_1-1}\beta_2^{T_2-1}$.

(b) $\Omega > 0 \Leftrightarrow$ all eigenvalues of Ω *positive* $\Leftrightarrow$ all eigenvalues of A *and* B *positive* $\Leftrightarrow$ $\alpha_1 > 0, \alpha_2 > 0$, $|A| = \Delta > 0, \beta_1 > 0, \beta_1 > 0$.

(c) From (1): $\Omega^p = L \begin{bmatrix} A^p & 0 \\ 0 & B^p \end{bmatrix} L'$ with $B^p = \begin{bmatrix} \beta_1^p I_{T_1-1} & 0 \\ 0 & \beta_2^p I_{T_2-1} \end{bmatrix}$.

(d) For $p = -1$

$$A^{-1} = \frac{1}{\Delta} \begin{bmatrix} \alpha_2 & -\gamma_* \\ -\gamma_* & \alpha_1 \end{bmatrix},$$

so then $\phi_1 = \alpha_2/\Delta, \phi_2 = \alpha_1/\Delta, \delta = -\gamma_*/\Delta$. For $p = -\frac{1}{2}$, the expressions for ϕ_1, ϕ_2, and δ correspond with the elements (1,1), (2,2), and (1,2), respectively, of the symmetric square root of A^{-1}. These expressions are tedious and not insightful. For the purpose of transformation from a generalized least squares to an ordinary least squares model, however, $\Omega^{-1/2}$ need not be symmetric, allowing for simple expressions. This can be based on $A^{-1} = CC'$, with

$$C \equiv \begin{bmatrix} \sqrt{\alpha_2/\Delta} & 0 \\ -\gamma_*/\sqrt{\alpha_2\Delta} & 1/\sqrt{\alpha_2} \end{bmatrix}.$$

Then $\Omega^{-1/2}$ has upper-right block equal to zero and $\phi_1 = \sqrt{\alpha_2/\Delta}$, $\phi_2 = 1/\sqrt{\alpha_2}$, and $\delta = -\gamma_*/\sqrt{\alpha_2\Delta}$.

(e) Follows directly from $J_1 x_1 = \iota_1 \bar{x}_1$ and $\iota_1 \iota_2' x_2 = \iota_1 T_2 \bar{x}_2$.

01.5.2. *The Characteristic Function from a Family of Truncated Normal Distributions*—Solution,[1] proposed by Karim Abadir and Tassos Magdalinos. The posers of the problem have raised an important question. We can offer a solution to a more general problem, which covers characteristic functions of continuous distributions. We then use these results to obtain formulae for the moments, whose existence is guaranteed because of double truncation. We give alternative formulae for the characteristic function (and also for the moments) because some are easier to apply than others in different circumstances. We illustrate some of the general formulae with two common distributions.

Define a continuous scalar variate x with characteristic function $\varphi_x(\tau)$, density function $f_x(u)$, and distribution function $F_x(u) \equiv \Pr(x \leq u)$. The variate x is doubly truncated to $y \in (a, b)$, where $b > a$, and its characteristic function is given by the integral

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$$\varphi_y(\tau) := \frac{1}{F_x(b) - F_x(a)} \int_a^b f_x(u) \mathrm{e}^{\mathrm{i}u\tau} \mathrm{d}u. \tag{1}$$

Consider the function $g(c, \tau) := \int_{-\infty}^{c} f_x(u) \mathrm{e}^{\mathrm{i}u\tau} \mathrm{d}u$, which is bounded because

$$\left| \int_{-\infty}^{c} f_x(u) \mathrm{e}^{\mathrm{i}u\tau} \mathrm{d}u \right| \leq \int_{-\infty}^{c} f_x(u) \mathrm{d}u \leq 1$$

for $\tau \in \mathbb{R}$ and can be used to write

$$\varphi_y(\tau) := \frac{g(b, \tau) - g(a, \tau)}{F_x(b) - F_x(a)}.$$

Later on, we will make some simplifying assumptions about the derivatives of $f_x(u)$, denoted by $f_x^{(n)}(u)$ for $n \in \mathbb{N}$. Define $\mathrm{D}_u := \frac{\mathrm{d}}{\mathrm{d}u}$ as the derivative operator. We will show that the following proposition applies.

PROPOSITION 1. *The characteristic function of y is*

$$\begin{aligned}
\varphi_y(\tau) &= \frac{1}{F_x(b) - F_x(a)} \sum_{j=0}^{\infty} \frac{(b-a)^{j+1}}{(j+1)!} \mathrm{D}_u^j (\mathrm{e}^{\mathrm{i}u\tau} f_x(u))|_{u=a} \\
&= \frac{\mathrm{e}^{\mathrm{i}a\tau}}{F_x(b) - F_x(a)} \sum_{j=0}^{\infty} \frac{(b-a)^{j+1}}{(j+1)!} \sum_{k=0}^{j} \binom{j}{k} (\mathrm{i}\tau)^k f_x^{(j-k)}(a) \\
&= \frac{(b-a)\mathrm{e}^{\mathrm{i}a\tau}}{F_x(b) - F_x(a)} \sum_{j=0}^{\infty} \frac{(b-a)^j f_x^{(j)}(a)}{(j+1)!} {}_1F_1(j+1; j+2; \mathrm{i}(b-a)\tau) \\
&= \frac{(b-a)\mathrm{e}^{\mathrm{i}b\tau}}{F_x(b) - F_x(a)} \sum_{j=0}^{\infty} \frac{(b-a)^j f_x^{(j)}(a)}{(j+1)!} {}_1F_1(1; j+2; \mathrm{i}(a-b)\tau),
\end{aligned}$$

where the confluent hypergeometric function, ${}_1F_1$, simplifies to the incomplete gamma function (see Abadir, 1999), implying

$$\mathrm{E}((y-a)^n) = \frac{(b-a)^{n+1}}{F_x(b) - F_x(a)} \sum_{j=0}^{\infty} \frac{(b-a)^j f_x^{(j)}(a)}{j!(j+n+1)},$$

$$\mathrm{E}((y-b)^n) = \frac{-(a-b)^{n+1} n!}{F_x(b) - F_x(a)} \sum_{j=0}^{\infty} \frac{(b-a)^j f_x^{(j)}(a)}{(j+n+1)!},$$

$$\begin{aligned}
\mathrm{E}(y^n) &= \frac{b-a}{F_x(b) - F_x(a)} \sum_{j=0}^{\infty} \frac{(b-a)^j f_x^{(j)}(a)}{j!} \sum_{k=0}^{n} \binom{n}{k} \frac{a^{n-k}(b-a)^k}{j+k+1} \\
&= \frac{n!(b-a)}{F_x(b) - F_x(a)} \sum_{j=0}^{\infty} (b-a)^j f_x^{(j)}(a) \sum_{k=0}^{n} \frac{b^{n-k}(a-b)^k}{(n-k)!(j+k+1)!}
\end{aligned}$$

as alternative representations of the moments. ■

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To derive the first expression for $\varphi_y(\tau)$, expand $g(b,\tau)$ in the neighborhood of point a as

$$\begin{aligned}\varphi_y(\tau) &= \frac{1}{F_x(b)-F_x(a)}\left(\sum_{j=0}^{\infty}\frac{(b-a)^j}{j!}\,\mathrm{D}_u^j\left(\int_{-\infty}^{b}\mathrm{e}^{\mathrm{i}u\tau}f_x(u)\mathrm{d}u\right)\Bigg|_{u=a}-g(a,\tau)\right)\\ &= \frac{1}{F_x(b)-F_x(a)}\sum_{j=1}^{\infty}\frac{(b-a)^j}{j!}\,\mathrm{D}_u^j\left(\int_{-\infty}^{b}\mathrm{e}^{\mathrm{i}u\tau}f_x(u)\mathrm{d}u\right)\Bigg|_{u=a}\\ &= \frac{1}{F_x(b)-F_x(a)}\sum_{j=0}^{\infty}\frac{(b-a)^{j+1}}{(j+1)!}\,\mathrm{D}_u^j(\mathrm{e}^{\mathrm{i}u\tau}f_x(u))\Bigg|_{u=a}, \qquad (2)\end{aligned}$$

where we have used a change of index in the last step. More generally,

$$\int_a^b h(u)\mathrm{d}u=\int_{-\infty}^{b}h(u)\mathrm{d}u-\int_{-\infty}^{a}h(u)\mathrm{d}u=\sum_{j=0}^{\infty}\frac{(b-a)^{j+1}}{(j+1)!}\,h^{(j)}(a) \qquad (3)$$

for an arbitrary integrable function $h(\cdot)$, also differentiable at point a. If the density is not differentiable at $u=a$, then the same method can be used to expand *both* $g(b,\tau)$ and $g(a,\tau)$ around a common point where the derivatives would be well behaved.

To derive the second expression for $\varphi_y(\tau)$, the formula for the derivatives of products of functions (say, h_1 and h_2),

$$\mathrm{D}^j(h_1h_2)=\sum_{k=0}^{j}\binom{j}{k}\mathrm{D}^k(h_1)\mathrm{D}^{j-k}(h_2), \qquad (4)$$

allows us to write (2) explicitly as

$$\begin{aligned}\varphi_y(\tau) &= \frac{1}{F_x(b)-F_x(a)}\sum_{j=0}^{\infty}\frac{(b-a)^{j+1}}{(j+1)!}\sum_{k=0}^{j}\binom{j}{k}\mathrm{D}_u^k(\mathrm{e}^{\mathrm{i}u\tau})\mathrm{D}_u^{j-k}(f_x(u))\Bigg|_{u=a}\\ &= \frac{\mathrm{e}^{\mathrm{i}a\tau}}{F_x(b)-F_x(a)}\sum_{j=0}^{\infty}\frac{(b-a)^{j+1}}{(j+1)!}\sum_{k=0}^{j}\binom{j}{k}(\mathrm{i}\tau)^k f_x^{(j-k)}(a). \qquad (5)\end{aligned}$$

The final expressions for $\varphi_y(\tau)$ are obtained as follows. By a standard technique of rewriting absolutely convergent double series (see the discussion following equation (6) for the conditions) with typical term $\vartheta(\cdot)$ as

$$\sum_{j=0}^{\infty}\sum_{k=0}^{j}\vartheta(j,k)=\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\vartheta(j+k,k),$$

we have the alternative formulation

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$$\varphi_y(\tau) = \frac{(b-a)\mathrm{e}^{\mathrm{i}a\tau}}{F_x(b)-F_x(a)} \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{(b-a)^{j+k}(i\tau)^k}{j!k!(j+k+1)} f_x^{(j)}(a)$$

$$= \frac{(b-a)\mathrm{e}^{\mathrm{i}a\tau}}{F_x(b)-F_x(a)} \sum_{j=0}^{\infty} \frac{(b-a)^j f_x^{(j)}(a)}{(j+1)!} {}_1F_1(j+1; j+2; \mathrm{i}(b-a)\tau)$$

$$= \frac{(b-a)\mathrm{e}^{\mathrm{i}b\tau}}{F_x(b)-F_x(a)} \sum_{j=0}^{\infty} \frac{(b-a)^j f_x^{(j)}(a)}{(j+1)!} {}_1F_1(1; j+2; \mathrm{i}(a-b)\tau). \quad \textbf{(6)}$$

The ratio test for convergence can be applied to yield the sufficient condition for absolute convergence that, as $n \to \infty$, we have $f_x^{(n+1)}(u)/f_x^{(n)}(u) = o(n)$. A weaker condition that would involve also $|b-a|$ can be obtained if required. Even if the derivatives increase too fast with j, the formal series can be converted into convergent ones often easily (see, e.g., Abadir, 1999). In general, one only needs the necessary assumption that $\varphi_y(\tau)$ is not lacunary in some neighborhood including $\tau = 0$. This condition is satisfied in our problem when we recall (1), because of the double truncation (implying the existence of moments) and the integrability of the function $f_x(u)$ (which is a density).

Formula (6) is particularly useful. The last two expressions there imply, respectively, that

$$\begin{aligned}
\mathrm{E}((y-a)^n) &= \mathrm{D}_{\mathrm{i}\tau}^n(\mathrm{E}(\mathrm{e}^{\mathrm{i}(y-a)\tau}))|_{\tau=0} \\
&= \mathrm{D}_{\mathrm{i}\tau}^n(\varphi_y(\tau)\mathrm{e}^{-\mathrm{i}a\tau})|_{\tau=0} \\
&= \frac{b-a}{F_x(b)-F_x(a)} \sum_{j=0}^{\infty} \frac{(b-a)^j f_x^{(j)}(a)}{(j+1)!} \\
&\quad \times \mathrm{D}_{\mathrm{i}\tau}^n({}_1F_1(j+1; j+2; \mathrm{i}(b-a)\tau))|_{\tau=0} \\
&= \frac{(b-a)^{n+1}}{F_x(b)-F_x(a)} \sum_{j=0}^{\infty} \frac{(b-a)^j f_x^{(j)}(a)}{j!(j+n+1)} \qquad \textbf{(7)}
\end{aligned}$$

or

$$\begin{aligned}
\mathrm{E}((y-b)^n) &= \mathrm{D}_{\mathrm{i}\tau}^n(\mathrm{E}(\mathrm{e}^{\mathrm{i}(y-b)\tau}))|_{\tau=0} \\
&= \mathrm{D}_{\mathrm{i}\tau}^n(\varphi_y(\tau)\mathrm{e}^{-\mathrm{i}b\tau})|_{\tau=0} \\
&= \frac{b-a}{F_x(b)-F_x(a)} \sum_{j=0}^{\infty} \frac{(b-a)^j f_x^{(j)}(a)}{(j+1)!} \\
&\quad \times \mathrm{D}_{\mathrm{i}\tau}^n({}_1F_1(1; j+2; \mathrm{i}(a-b)\tau))|_{\tau=0} \\
&= \frac{-(a-b)^{n+1}n!}{F_x(b)-F_x(a)} \sum_{j=0}^{\infty} \frac{(b-a)^j f_x^{(j)}(a)}{(j+n+1)!}. \qquad \textbf{(8)}
\end{aligned}$$

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As for $\mathrm{E}(y^n)$, it can be obtained recursively by expanding the binomials $(y-a)^n$ or $(y-b)^n$; or directly from

$$\begin{aligned}\mathrm{E}(y^n) &= \frac{\varphi_y^{(n)}(0)}{\mathrm{i}^n} \\ &= \frac{b-a}{F_x(b)-F_x(a)} \sum_{j=0}^{\infty} \frac{(b-a)^j f_x^{(j)}(a)}{(j+1)!} \\ &\quad \times \mathrm{D}_{\mathrm{i}\tau}^n (\mathrm{e}^{\mathrm{i}a\tau}\, {}_1F_1(j+1; j+2; \mathrm{i}(b-a)\tau))|_{\tau=0} \\ &= \frac{b-a}{F_x(b)-F_x(a)} \sum_{j=0}^{\infty} \frac{(b-a)^j f_x^{(j)}(a)}{j!} \sum_{k=0}^{n} \binom{n}{k} \frac{a^{n-k}(b-a)^k}{j+k+1} \end{aligned} \tag{9}$$

or

$$\begin{aligned}\mathrm{E}(y^n) &= \frac{\varphi_y^{(n)}(0)}{\mathrm{i}^n} \\ &= \frac{b-a}{F_x(b)-F_x(a)} \sum_{j=0}^{\infty} \frac{(b-a)^j f_x^{(j)}(a)}{(j+1)!} \\ &\quad \times \mathrm{D}_{\mathrm{i}\tau}^n (\mathrm{e}^{\mathrm{i}b\tau}\, {}_1F_1(1; j+2; \mathrm{i}(a-b)\tau))|_{\tau=0} \\ &= \frac{n!(b-a)}{F_x(b)-F_x(a)} \sum_{j=0}^{\infty} (b-a)^j f_x^{(j)}(a) \sum_{k=0}^{n} \frac{b^{n-k}(a-b)^k}{(n-k)!(j+k+1)!}. \end{aligned} \tag{10}$$

We now illustrate some of these formulae.

Example 1.

Let $x \sim \mathrm{N}(\mu, \sigma^2)$ with density $f_x(a) = \sigma^{-1}\phi((a-\mu)/\sigma)$ and distribution $F_x(a) = \Phi((a-\mu)/\sigma)$. Then, for for the first expression of $\varphi_y(\tau)$ in the proposition, we work out

$$\begin{aligned}\mathrm{D}_a^j(\mathrm{e}^{\mathrm{i}a\tau} f_x(a)) &= \mathrm{D}_a^j \left(\frac{\mathrm{e}^{\mathrm{i}a\tau}}{\sigma\sqrt{2\pi}}\, \mathrm{e}^{-\frac{1}{2}\left(\frac{a-\mu}{\sigma}\right)^2} \right) \\ &= \frac{\mathrm{e}^{\mathrm{i}\mu\tau - \frac{1}{2}\sigma^2\tau^2}}{\sigma} \mathrm{D}_a^j \left(\phi\left(\frac{a-\mu-\mathrm{i}\tau\sigma^2}{\sigma} \right) \right) \\ &= (-1)^j \frac{\mathrm{e}^{\mathrm{i}\mu\tau - \frac{1}{2}\sigma^2\tau^2}}{\sigma^{j+1}\sqrt{2\pi}}\, D_j^- \left(\frac{a-\mu-\mathrm{i}\tau\sigma^2}{\sigma} \right), \end{aligned}$$

where $D_j^-(\cdot)$ is a modified parabolic cylinder function that is related to Hermite polynomials when $j = 0,1,\ldots$, and the standard normal cumulative distribution function (c.d.f.) when $j = -1$ (see Abadir, 1999). By a change of

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index, followed by using an addition theorem for parabolic cylinder functions (see Prudnikov, Brychkov, and Marichev, 1986, p. 651), we have

$$\begin{aligned}
\varphi_y(\tau) &= \frac{\mathrm{e}^{\mathrm{i}\mu\tau-\frac{1}{2}\sigma^2\tau^2}}{\left[\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)\right]\sqrt{2\pi}} \\
&\quad\times\left(D_{-1}^{-}\left(\frac{a-\mu-\mathrm{i}\tau\sigma^2}{\sigma}\right)-\sum_{j=0}^{\infty}\frac{\left(\frac{a-b}{\sigma}\right)^j}{j!}D_{j-1}^{-}\left(\frac{a-\mu-\mathrm{i}\tau\sigma^2}{\sigma}\right)\right) \\
&= \frac{\mathrm{e}^{\mathrm{i}\mu\tau-\frac{1}{2}\sigma^2\tau^2}}{\left[\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)\right]\sqrt{2\pi}} \\
&\quad\times\left(D_{-1}^{-}\left(\frac{a-\mu-\mathrm{i}\tau\sigma^2}{\sigma}\right)-D_{-1}^{-}\left(\frac{b-\mu-\mathrm{i}\tau\sigma^2}{\sigma}\right)\right) \\
&\equiv \frac{\varphi_x(\tau)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)} \\
&\quad\times\left(\Phi\left(-\frac{a-\mu-\mathrm{i}\tau\sigma^2}{\sigma}\right)-\Phi\left(-\frac{b-\mu-\mathrm{i}\tau\sigma^2}{\sigma}\right)\right).
\end{aligned}$$

The derivation of the moment of the truncated variate, y, is straightforward. The formula for differentiating products (4) gives us

$$\begin{aligned}
\varphi_y^{(n)}(\tau) &= \frac{1}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\sum_{k=0}^{n}\binom{n}{k}\varphi_x^{(k)}(\tau) \\
&\quad\times \mathrm{D}^{n-k}\left(\Phi\left(-\frac{a-\mu-\mathrm{i}\tau\sigma^2}{\sigma}\right)-\Phi\left(-\frac{b-\mu-\mathrm{i}\tau\sigma^2}{\sigma}\right)\right) \\
&= \frac{1}{\sqrt{2\pi}\left[\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)\right]}\sum_{k=0}^{n}\binom{n}{k}\varphi_x^{(k)}(\tau) \\
&\quad\times(\mathrm{i}\sigma)^{n-k}\left(D_{n-k-1}^{-}\left(\frac{a-\mu-\mathrm{i}\tau\sigma^2}{\sigma}\right)-D_{n-k-1}^{-}\left(\frac{b-\mu-\mathrm{i}\tau\sigma^2}{\sigma}\right)\right)
\end{aligned}$$

by Abadir (1999, p. 307). Exceptionally in the case of the normal, this additional formula relates *directly* the moments of y (from $\varphi_y^{(n)}(0)$) to those of x (from $\varphi_x^{(n)}(0)$). For example,

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$$\begin{aligned}
\mathrm{E}(y) &= \frac{\varphi_y'(0)}{\mathrm{i}} \\
&= \frac{\sigma\left(D_0^-\left(\frac{a-\mu}{\sigma}\right) - D_0^-\left(\frac{b-\mu}{\sigma}\right)\right) + \mu\left(D_{-1}^-\left(\frac{a-\mu}{\sigma}\right) - D_{-1}^-\left(\frac{b-\mu}{\sigma}\right)\right)}{\sqrt{2\pi}\left[\Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)\right]} \\
&\equiv \sigma \frac{\phi\left(\frac{a-\mu}{\sigma}\right) - \phi\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)} + \mu \frac{\Phi\left(-\frac{a-\mu}{\sigma}\right) - \Phi\left(-\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)} \\
&= \mu + \sigma \frac{\phi\left(\frac{a-\mu}{\sigma}\right) - \phi\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)}
\end{aligned}$$

by $\Phi(-z) = 1 - \Phi(z)$. The latter equation for $\mathrm{E}(y)$ can be obtained by using any of the expectations in Proposition 1 and then collecting sums. Alternatively, for the simple $\mathrm{E}(y)$, we could integrate $\int_a^b u f_x(u)\mathrm{d}u$ by parts. Both alternatives will be illustrated in the next example. ■

The previous example is not the ideal illustration! As always, a general formula comes at the expense of simplicity of some of its special cases. For the case of normals, the integral in (1) could be worked out directly by completing the square in the exponential. The following example is, however, less easily worked out from the integral in (1) but is straightforward to obtain from our formula.

Example 2.

Let $x \sim \mathrm{F}(m, n)$, with $x, m, n \in \mathbb{R}_+$, density

$$f_x(a) = \frac{\frac{m}{n}\left(\frac{m}{n}a\right)^{\frac{m}{2}-1}}{B\left(\frac{m}{2}, \frac{n}{2}\right)\left(1 + \frac{m}{n}a\right)^{\frac{m+n}{2}}},$$

and distribution (derived in Abadir, 1999, p. 301)

$$F_x(a) = \frac{2af_x(a)}{m}\,{}_2F_1\left(1, \frac{m+n}{2}; \frac{m}{2}+1; \frac{am}{am+n}\right),$$

where, if required, the hypergeometric function can be directly expanded in a series that is convergent for all finite a. The variate x has moments of order

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less than $n/2$ only, but the doubly truncated version has moments of all order. We now only need to calculate the derivatives of $f_x(a)$ and we have all the ingredients of our explicit formula for the characteristic function of the truncated $\mathrm{F}(m,n)$. Such a distribution can be used to represent variance ratios that are restricted to lie within a specified interval. One could derive the corresponding results for a truncated non-central $\mathrm{F}(m,n,\delta)$ similarly.

The density of $\mathrm{F}(m,n)$ can be written as

$$f_x(a) = \frac{m\left(-1+1+\frac{m}{n}a\right)^{\frac{m}{2}-1}}{nB\left(\frac{m}{2},\frac{n}{2}\right)\left(1+\frac{m}{n}a\right)^{\frac{m+n}{2}}}$$

$$= \frac{m}{nB\left(\frac{m}{2},\frac{n}{2}\right)}\sum_{\ell=0}^{\infty}\binom{\frac{m}{2}-1}{\ell}\frac{(-1)^\ell}{\left(1+\frac{m}{n}a\right)^{\frac{n}{2}+\ell+1}},$$

giving

$$f_x^{(j)}(a) = \frac{-\left(-\frac{m}{n}\right)^{j+1}}{B\left(\frac{m}{2},\frac{n}{2}\right)}\sum_{\ell=0}^{\infty}\binom{\frac{m}{2}-1}{\ell}\frac{\Gamma\left(j+\frac{n}{2}+\ell+1\right)(-1)^\ell}{\Gamma\left(\frac{n}{2}+\ell+1\right)\left(1+\frac{m}{n}a\right)^{j+\frac{n}{2}+\ell+1}}$$

$$= \frac{\left(\frac{m}{n}\right)^{j+1}j!}{B\left(\frac{m}{2},\frac{n}{2}\right)\left(1+\frac{m}{n}a\right)^{j+\frac{n}{2}+1}}\binom{-\frac{n}{2}-1}{j}$$

$$\times\, {}_2F_1\left(1-\frac{m}{2}, j+\frac{n}{2}+1; \frac{n}{2}+1; \frac{1}{1+\frac{m}{n}a}\right)$$

$$= \frac{f_x(a)j!}{a^j}\binom{-\frac{n}{2}-1}{j}\,{}_2F_1\left(-j, \frac{m+n}{2}; \frac{n}{2}+1; \frac{1}{1+\frac{m}{n}a}\right), \qquad \textbf{(11)}$$

the latter function being a finite series of $j+1$ terms (to see why, recall (4) and consider $f_x(a)$ as a product of two elementary functions). Substituting into the last expression of $\varphi_y(\tau)$ in Proposition 1, we have

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$$\varphi_y(\tau) = \frac{(b-a)f_x(a)\mathrm{e}^{\mathrm{i}b\tau}}{F_x(b)-F_x(a)} \sum_{j=0}^{\infty} \binom{-\frac{n}{2}-1}{j} \frac{\left(\frac{b}{a}-1\right)^j}{j+1}$$

$$\times {}_2F_1\left(-j, \frac{m+n}{2}; \frac{n}{2}+1; \frac{1}{1+\frac{m}{n}a}\right) {}_1F_1(1; j+2; \mathrm{i}(a-b)\tau).$$

The ${}_2F_1$ function is of exponential order in $-j$ when $m/2 \in \mathbb{N}$, and it is of order $j^{-(m+n)/2}$ otherwise. In the latter case, convergence depends on the magnitude of b/a. Nevertheless, the formula can be used for obtaining general results that do not depend on b/a, by analytic continuation, as we now show.

Substituting (11) into the formula for $\mathrm{E}(\mathrm{y}-\mathrm{a})$ in Proposition 1 gives

$$\mathrm{E}(y-a) = \frac{(b-a)^2 f_x(a)}{F_x(b)-F_x(a)} \sum_{j=0}^{\infty} \binom{-\frac{n}{2}-1}{j} \frac{\left(\frac{b}{a}-1\right)^j}{j+2}$$

$$\times {}_2F_1\left(-j, \frac{m+n}{2}; \frac{n}{2}+1; \frac{1}{1+\frac{m}{n}a}\right),$$

which we can simplify by noting that

$$\mathrm{E}(y-a)\,\frac{F_x(b)-F_x(a)}{a^2 f_x(a)}$$

$$= \sum_{j=0}^{\infty} \binom{-\frac{n}{2}-1}{j} \int_1^{b/a} (z-1)^{j+1} \mathrm{d}z\, {}_2F_1\left(-j, \frac{m+n}{2}; \frac{n}{2}+1; \frac{1}{1+\frac{m}{n}a}\right)$$

$$= \int_1^{b/a} (z-1) z^{\frac{m}{2}-1} \left(\frac{1+\frac{m}{n}a}{1+\frac{m}{n}az}\right)^{\frac{m+n}{2}} \mathrm{d}z$$

by an addition theorem for ${}_2F_1$ (e.g., see Prudnikov, Brychkov, and Marichev 1990, p. 413, #3). The integral can be split into

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$$\mathrm{E}(y-a)\,\frac{F_x(b)-F_x(a)}{a^2 f_x(a)}$$

$$= \int_0^1 (1-z) z^{\frac{m}{2}-1} \left(\frac{1+\frac{m}{n}a}{1+\frac{m}{n}az} \right)^{\frac{m+n}{2}} \mathrm{d}z$$

$$- \int_0^{b/a} (1-z) z^{\frac{m}{2}-1} \left(\frac{1+\frac{m}{n}a}{1+\frac{m}{n}az} \right)^{\frac{m+n}{2}} \mathrm{d}z$$

$$= \int_0^1 (1-z) z^{\frac{m}{2}-1} \left(\frac{1+\frac{m}{n}a}{1+\frac{m}{n}az} \right)^{\frac{m+n}{2}} \mathrm{d}z$$

$$- \int_0^{b/a} \left(\frac{b}{a} - z \right) z^{\frac{m}{2}-1} \left(\frac{1+\frac{m}{n}a}{1+\frac{m}{n}az} \right)^{\frac{m+n}{2}} \mathrm{d}z$$

$$- \left(1 - \frac{b}{a} \right) \int_0^{b/a} z^{\frac{m}{2}-1} \left(\frac{1+\frac{m}{n}a}{1+\frac{m}{n}az} \right)^{\frac{m+n}{2}} \mathrm{d}z$$

where the last term is equal to $-(a-b)F_x(b)/[a^2 f_x(a)]$ by the definition of the distribution function. The remaining two integrals are of the same form, namely, the integral representation of ${}_2F_1$, so we have

$$\mathrm{E}(y-a)[F_x(b)-F_x(a)] = \frac{a^2 f_x(a)}{\frac{m}{2}\left(1+\frac{m}{2}\right)} {}_2F_1\left(2, \frac{m+n}{2}; \frac{m}{2}+2; \frac{am}{am+n}\right)$$

$$- \frac{b^2 f_x(b)}{\frac{m}{2}\left(1+\frac{m}{2}\right)} {}_2F_1\left(2, \frac{m+n}{2}; \frac{m}{2}+2; \frac{bm}{bm+n}\right)$$

$$- (a-b)F_x(b),$$

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where the definitions of $f_x(a)$ and $f_x(b)$ have been used. The density and distributions are given explicitly previously, which completes the derivation of $\mathrm{E}(y)$. Notice how the parameters of these ${}_2F_1$ compare to those in the definition of the distribution function $F_x(\cdot)$, indicating that they are integrals of the c.d.f. (or double integrals of the probability density function). This brings us to the next and final point.

The formula for $\mathrm{E}(y-b)$ in Proposition 1 gives

$$\begin{aligned}
\mathrm{E}(y-b) &= -\frac{1}{F_x(b)-F_x(a)}\sum_{j=0}^{\infty}\frac{(b-a)^{j+2}f_x^{(j)}(a)}{(j+2)!} \\
&= \frac{(b-a)F_x(a)}{F_x(b)-F_x(a)} - \frac{1}{F_x(b)-F_x(a)}\sum_{j=-1}^{\infty}\frac{(b-a)^{j+2}f_x^{(j)}(a)}{(j+2)!} \\
&= \frac{(b-a)F_x(a)}{F_x(b)-F_x(a)} - \frac{1}{F_x(b)-F_x(a)}\sum_{j=0}^{\infty}\frac{(b-a)^{j+1}F_x^{(j)}(a)}{(j+1)!} \\
&= \frac{(b-a)F_x(a)}{F_x(b)-F_x(a)} - \frac{1}{F_x(b)-F_x(a)}\int_a^b F_x(u)\mathrm{d}u
\end{aligned}$$

by $f_x^{(j-1)}(a) = F_x^{(j)}(a)$ and then applying (3). This is equivalent to the integration by parts of

$$\begin{aligned}
\mathrm{E}(y) &= \frac{1}{F_x(b)-F_x(a)}\int_a^b uf_x(u)\mathrm{d}u \\
&= \frac{[uF_x(u)]_a^b}{F_x(b)-F_x(a)} - \frac{1}{F_x(b)-F_x(a)}\int_a^b F_x(u)\mathrm{d}u \\
&= b + \frac{(b-a)F_x(a)}{F_x(b)-F_x(a)} - \frac{1}{F_x(b)-F_x(a)}\int_a^b F_x(u)\mathrm{d}u,
\end{aligned}$$

and the remaining integral can be worked out in the same way that (11) was obtained. This illustrates how some of our formulae simplify to more familiar procedures with specific densities. ■

NOTE

1. An independent solution was proposed by S. Lawford.

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Prudnikov, A.P., Yu.A. Brychkov, & O.I. Marichev (1990) *Integrals and Series*, vol. 3. New York: Gordon and Breach.

01.5.3. *Serial Correlation and Asymptotic Variance*—Solution,[1] proposed by Geert Dhaene. Let $\hat{\beta}$ be the ordinary least squares (OLS) estimator based on (x_t, y_t), $t = 1, \ldots, T$, and let $\bar{x} = T^{-1}\sum_t x_t$ and $z_t = x_t - E(x_t)$. Assume that the process (x_t, y_t) is ergodic and that $\mathrm{Var}(x_t)$ and $\mathrm{Var}(z_t e_t)$ exist. Then

$$T^{1/2}(\hat{\beta} - \beta) = T^{1/2}\frac{\sum_t (x_t - \bar{x})e_t}{\sum_t (x_t - \bar{x})^2} = T^{-1/2}\frac{\sum_t z_t e_t}{\mathrm{Var}(x_t)} + o_p(1)$$

and

$$V_\beta = \frac{\mathrm{Var}(z_t e_t)}{[\mathrm{Var}(x_t)]^2}\left[1 + 2\sum_{i=1}^{\infty}\rho_{ze}(i)\right], \tag{1}$$

where $\rho_{ze}(\cdot)$ is the autocorrelation function of $z_t e_t$. Observe that, in general, $\mathrm{Var}(z_t e_t) \neq \mathrm{Var}(x_t e_t)$ and $\rho_{ze}(\cdot) \neq \rho_{xe}(\cdot)$.

(a) Let e_t be any zero mean, stationary, serially correlated process.

Example 1.

If the process x_t is independent of e_t, then

$$V_\beta = \frac{\mathrm{Var}(e_t)}{\mathrm{Var}(x_t)}\left[1 + 2\sum_{i=1}^{\infty}\rho_x(i)\rho_e(i)\right].$$

Thus, we are looking for cases where $\sum_{i=1}^{\infty}\rho_x(i)\rho_e(i)$ vanishes. This occurs, obviously, when x_t is an independent and identically distributed (i.i.d.) process but also in many cases where x_t is serially correlated. It is, for example, always possible to find MA(p) processes, for any $p \geq 2$, such that $\sum_{i=1}^{\infty}\rho_x(i)\rho_e(i) = 0$. As an example, consider the MA(2) process $x_t = \mu + u_t + au_{t-1} + bu_{t-2}$, where u_t is an i.i.d. process with zero mean. Then,

$$\sum_{i=1}^{\infty}\rho_x(i)\rho_e(i) = (1 + a^2 + b^2)^{-1}[a(1+b)\rho_e(1) + b\rho_e(2)],$$

which vanishes for many choices of (a, b).

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Example 2.

Let $x_t = w_t + u_t e_t$, where w_t is a stationary process, u_t is an i.i.d. process with zero mean and unit variance, and w_t, u_t, and e_t are independent processes. Although not independent, x_t and e_t are uncorrelated. We have

$$\begin{aligned}
\operatorname{Var}(x_t) &= \operatorname{Var}(w_t) + \operatorname{Var}(e_t),\\
\operatorname{Var}(z_t e_t) &= \operatorname{Var}(w_t)\operatorname{Var}(e_t) + E(e_t^4),\\
\operatorname{Cov}(z_t e_t, z_{t-i} e_{t-i}) &= \operatorname{Var}(w_t)\operatorname{Var}(e_t)\rho_w(i)\rho_e(i), \qquad i \geq 1.
\end{aligned}$$

Therefore

$$V_\beta = \frac{\operatorname{Var}(w_t)\operatorname{Var}(e_t) + E(e_t^4) + 2\operatorname{Var}(w_t)\operatorname{Var}(e_t)\sum_{i=1}^{\infty}\rho_w(i)\rho_e(i)}{[\operatorname{Var}(w_t) + \operatorname{Var}(e_t)]^2}$$

and V_β reduces to $\operatorname{Var}(e_t)/\operatorname{Var}(x_t)$ if, and only if,

$$\operatorname{Var}(w_t)\sum_{i=1}^{\infty}\rho_w(i)\rho_e(i) = \frac{[\operatorname{Var}(e_t)]^2 - E(e_t^4)}{2\operatorname{Var}(e_t)}. \tag{2}$$

The right-hand side of (2) is nonpositive. Because e_t is serially correlated, $\rho_e(i) \neq 0$ for some $i \geq 1$. Hence there exist many processes w_t such that $\sum_{i=1}^{\infty}\rho_w(i)\rho_e(i)$ is negative or zero (whichever is required), and (2) holds for a suitable choice of $\operatorname{Var}(w_t)$.

(b) Let e_t be a zero mean, stationary, serially uncorrelated process. To have $V_\beta \neq \operatorname{Var}(z_t e_t)/[\operatorname{Var}(x_t)]^2$ it is necessary and sufficient that $\sum_{i=1}^{\infty}\rho_{ze}(i) \neq 0$. This implies that x_t and e_t must be dependent processes.

Example 3.

Let e_t and u_t be identical i.i.d. processes with zero mean and symmetric distribution. Let d_t be an i.i.d. process with $\Pr[d_t = 0] = \Pr[d_t = 1] = 0.5$ and let s_t be a zero mean, stationary process taking values -1 and 1. Let, moreover, the processes e_t, u_t, s_t, and d_t be independent and let $x_t = s_t(d_t e_t + (1 - d_t)u_t)$. The processes e_t and x_t are identical and uncorrelated. Furthermore $z_t = x_t$, and a straightforward calculation shows that

$$\sum_{i=1}^{\infty}\rho_{ze}(i) = \frac{[\operatorname{Var}(e_t)]^2}{E(e_t^4) + [\operatorname{Var}(e_t)]^2}\sum_{i=1}^{\infty}\rho_s(i).$$

Thus, $V_\beta \neq \operatorname{Var}(z_t e_t)/[\operatorname{Var}(x_t)]^2$ if, and only if, the process s_t satisfies $\sum_{i=1}^{\infty}\rho_s(i) \neq 0$. "Most" serially correlated processes have this property.

(c) Some formulae for the asymptotic variance of $\hat{\beta}$ do not take into account serial correlation, such as $V_\beta = \operatorname{Var}(z_t e_t)/[\operatorname{Var}(x_t)]^2$ or $V_\beta = \operatorname{Var}(e_t)/\operatorname{Var}(x_t)$.

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As the examples in (a) and (b) make clear, absence of serial correlation of e_t is neither necessary nor sufficient for the validity of those expressions. Instead, it is the serial correlation of $z_t e_t$ (or the absence thereof) that matters. If such correlation is present, one should use the general formula (1). In some specific cases $\sum_{i=1}^{\infty} \rho_{ze}(i)$ vanishes, but these cases are to be regarded as artifacts rather than as constituting a general rule. The preceding results suggest that it may be more worthwhile to (pre)test for serial correlation in $z_t e_t$ than in e_t.

NOTE

1. An independent solution was proposed by S. Anatolyev, the poser of the problem.

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