Ergod. Th. & Dynam. Sys. (1998), **18**, 17–39
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Lalley's theorem on periodic orbits of hyperbolic flows

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(*Received* 18 *March* 1996 *and accepted in revised form* 31 *October* 1996)

Abstract. We give an asymptotic estimate for the number of periodic orbits of an Anosov flow which are subject to multi-dimensional constraints. We also study their spatial distribution. For instance, we describe the distribution of periodic orbits with respect to homology classes, for both homologically full Anosov flows and suspensions of Anosov transformations.

0. *Introduction*

A basic feature of hyperbolic dynamics is global homogeneity. It leads to surprisingly precise asymptotic estimates for global quantities. In this paper, we are interested in the number and the distribution of periodic orbits satisfying various conditions. Margulis [**Ma69**] gave an asymptotic estimate for the number of periodic orbits of length smaller than T as T goes to infinity. His proof is based on the strong mixing property (see [**Le95**, **EM93**] for further use of this approach). For the geodesic flow in constant negative curvature, an estimate of the asymptotic number of closed geodesics in a given homology class was given by Adachi and Sunada [**AS87**], and improved to an asymptotic expansion by Phillips and Sarnak [**PS87**] (see [**Ep87**] for the finite volume case). Equidistribution properties in this set-up are given by Zelditch [**Ze89**]. Their proofs are based on the Selberg trace formula.

Further general results use the Bowen–Ratner symbolic representation of hyperbolic flows. A major achievement is due to Lalley [**La87**, **La89a**]: drawing an analogy between the counting problem and the renewal theorem in probability theory, he described the asymptotic number and the spatial distribution of periodic orbits which are subject to multidimensional constraints. He also introduced the notion of asymptotic directions. This has nice applications to the case of a geodesic flow of a surface of variable negative curvature, the constraints being of homological type [**La89b**]. Along the same lines, Dal'bo and Peigné were able to estimate the number of closed geodesics on manifolds

https://doi.org/10.1017/S0143385798100330 Published online by Cambridge University Press

with cusps [**DP95a**, **DP95b**, **DP96**]. A parallel approach, in a series of papers by Parry and Pollicott [**PP86**, **Po91**], Katsuda and Sunada [**KS90**] and Sharp [**Sh93**], uses precise information on zeta functions and Tauberian theorems to obtain similar asymptotics.

One of the purposes of this paper is to present what we feel is a simple proof of Lalley's results which combines the 'probabilistic' approach and the 'number theoretical' approach. As in Lalley, the counting problem is translated into the study of so-called 'counting measures' which are reminiscent of the renewal measures appearing in probability theory. Then, their asymptotic behavior is shown to be related to the singularities on the critical line of twisted zeta functions. Our approach relies on a Tauberian theorem which seems simpler than Katsuda and Sunada's one and which could be useful to deduce an asymptotic expansion from a better knowledge of the location of other poles of these zeta functions. Another purpose is to improve Lalley's result in the sense that the range of validity of his statement is fully extended. This allows us, for instance, to get new results on the distribution of periodic orbits for suspension flows.

1. *Statement of results*

Let M be a compact Riemannian manifold, (ϕ_t) a topologically mixing Anosov flow on M and for $d \geq 0$ let $F = (F_1, F_2, \ldots, F_d)$ be an $\mathbb{R}^d$-valued Hölder continuous function on M. Denote by $\mathcal{P}$ the set of periodic orbits for (ϕ_t), and for $\wp$ in $\mathcal{P}$, set $l(\wp)$ as the length of $\wp$ and $\int_\wp F$ as the vector $\int_0^{l(\wp)} F(\phi_s(x))\,ds$, where x is any chosen point in $\wp$. We let A be the closed subgroup of $\mathbb{R}^d$ generated by the periods $\int_\wp F$ when $\wp$ varies in $\mathcal{P}$. Say that a set of functions $\{G_1, \ldots, G_k\}$ is cohomologically independent with respect to the flow if no nontrivial linear combination of these functions is of the form $(d/dt)U \circ \phi_{t|t=0}$ for some Hölder continuous function U on M. By extracting from $\{F_1, \ldots, F_d\}$ a cohomologically independent subset, we may—and shall—assume that A is of rank d, i.e. A is isomorphic to $\mathbb{R}^k \times \mathbb{Z}^{d-k}$ for some integer k, $0 \leq k \leq d$. Our main assumption is the following.

Assumption (A). The closed subgroup $\widetilde{A}$ of $\mathbb{R}^{d+1}$ generated by the vectors $(l(\wp), \int_\wp F)$ when $\wp$ varies in $\mathcal{P}$ coincides with $\mathbb{R} \times A$.

Assumption (A) is true for $d = 0$ by the mixing assumption.

In the following, we consider the Euclidean scalar product on $\mathbb{R}^d$ and a $(d-k)$-dimensional fundamental domain E for A. We denote by $|E|$ the volume of E (with the convention that $|E| = 1$ if $k = d$), and by dy the measure on A given by the product of the Euclidean Lebesgue measure with the counting measure.

The basic example to have in mind is the following. Identify the first cohomology group $H^1(M, \mathbb{Z})$ with $\mathbb{Z}^d \times G$, where G is a finite abelian group and d is the first Betti number of M, by means of a basis (over $H^1(M, \mathbb{R})$) of harmonic forms $\{\omega_1, \ldots, \omega_d\}$ taking integral values on closed orbits. Set $F_i(x) = \langle \omega_i | X \rangle_x$, where X is the tangent vector field of the flow. Then, $\int_\wp F = [\wp]_{\mathrm{TF}}$ is the torsion-free component of the homology class of $\wp$ and A is isomorphic to $\mathbb{Z}^d$ [**PP86**]. Moreover, Assumption (A) was shown to be true for non-circular flows by Sharp [**Sh93**]. One could also consider other linear properties of periodic orbits, such as the values of the integrals along the orbit of several positive functions with disjoint supports, or the lengths of homotopic closed

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geodesics for several negatively-curved metrics on the same manifold.

The main object to be studied is the following family of counting measures. Fix ξ in $\mathbb{R}^d$ and a real positive number T, which is going to be large. For any test functions $g_0 : \mathbb{R} \to \mathbb{R}$ and $g : A \to \mathbb{R}$ with compact support, consider the functional N_T^ξ defined by

$$N_T^\xi(g_0 \otimes g) = \sum_{\wp \in \mathcal{P}} g_0(l(\wp) - T) g\left(\int_\wp F - [T\xi] \right),$$

where $[T\xi]$ denotes the unique point in A such that $T\xi \in [T\xi] + E$. Observe that since the number of periodic orbits with bounded length is finite, $N_T^\xi(g_0 \otimes g)$ is finite as soon as g_0 has compact support. In order to see what it means in the case of homological constraints, let g_0 be the characteristic function of the interval $[-\delta, 0]$ and let g be the Dirac function δ_α for some homology class α. Then

$$N_T^\xi(g_0 \otimes g) = \#\{\wp \in \mathcal{P}; T - \delta \le l(\wp) \le T \text{ and } [\wp]_{\mathrm{TF}} = \alpha + [T\xi]\}.$$

When ξ is taken to be 0, this is just the number of closed orbits in a given homology class with length between $T - \delta$ and T, whereas when $\xi \neq 0$ is fixed, this counts the number of closed orbits homologically in a given asymptotic direction. An intermediate counting function can be obtained by considering ξ of the order of $1/\sqrt{T}$ thus obtaining 'large deviations' type results; see [**La89b**].

Observe that, since g_0 and g have compact support, $N_T^\xi(g_0 \otimes g)$ vanishes for large T unless ξ is close to the value of $(1/l(\wp)) \int_\wp F$ for some periodic orbit $\wp$. Hence, we shall only consider $N_T^\xi(g_0 \otimes g)$ when ξ belongs to the compact convex set C of $\mathbb{R}^d$ given by

$$C = \left\{ \int_M F\, dm; m \in \mathcal{M}^\phi \right\},$$

where $\mathcal{M}^\phi$ is the set of ϕ_t-invariant probability measures on M. The following proposition characterizes the interior of C (and answers a question in [**La87**]). For $u = (u_1, \ldots, u_d)$ in $\mathbb{R}^d$, let $P(u)$ be the pressure of $\langle u|F\rangle = \sum u_i F_i$, that is

$$P(u) = \sup_{m \in \mathcal{M}^\phi} h_m + \left\langle u \middle| \int F\, dm \right\rangle,$$

where h_m denotes the metric entropy of the flow with respect to m. Under Assumption (A), the function P is analytic and strictly convex [**Ru78**] and we have the following.

PROPOSITION 1.1. *The map $u \in \mathbb{R}^d \mapsto \nabla P(u)$ is a diffeomorphism between $\mathbb{R}^d$ and the interior $\mathring{C}$ of C.*

The proof of Proposition 1.1 is given in §3, together with additional information on the convex set C.

Thus, for any vector ξ in the interior of C, there exists a unique $u = u^\xi$ in $\mathbb{R}^d$ such that $\xi = \nabla P(u)$. By the thermodynamical formalism, the Gibbs state m^ξ of the function $\langle u^\xi|F\rangle$ satisfies $\xi = \nabla P(u) = \int_M F\, dm^\xi$. Thus m^ξ belongs to the weakly compact

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subset $\mathcal{K}^\xi$ of $\mathcal{M}^\phi$ given by

$$\mathcal{K}^\xi = \left\{ m \in \mathcal{M}^\phi ; \int F\, dm = \xi \right\}$$

and m^ξ is the unique measure maximizing entropy on $\mathcal{K}^\xi$. We let $H(\xi)$ be the entropy of m^ξ. We shall see in §3 that H is concave and, more precisely, that $H''(\xi) = -(P''(u^\xi))^{-1}$ is negative definite at all points ξ in $\mathring{C}$. With the above notations, we can now state the following.

THEOREM 1.2. *Under Assumption (A), for any continuous functions g_0 and g with compact support,*

$$N_T^\xi(g_0 \otimes g) \sim \frac{c(\xi) b(T,\xi)}{(2\pi)^{d/2}} \frac{e^{TH(\xi)}}{T^{1+d/2}} \int_{\mathbb{R}} e^{P(u^\xi)x} g_0(x)\, dx \int_A e^{-\langle u^\xi | y\rangle} g(y)\, dy$$

as T tends to $+\infty$. The convergence is uniform for ξ in a compact subset of the interior of C. The constant $c(\xi)$ is equal to $|E|\sqrt{|\det H''(\xi)|}$ and the function $b(T,\xi)$ is a bounded oscillating term given by $b(T,\xi) = \exp\langle u^\xi | T\xi - [T\xi]\rangle$.

Clearly, Theorem 1.2 holds true when $g_0 \otimes g$ is the characteristic function of a cube or any other Riemann integrable functions g_0 and g. This theorem is due to Lalley [**La87**] when ξ belongs to an open subset of C, and in the case of the geodesic flow on a surface with negative curvature, for ξ in a neighborhood of the origin [**La89b**]. Note that the problem of estimating N_T^ξ when ξ belongs to the boundary of C remains open.

As a consequence of Theorem 1.2 we can state a spatial distribution result. For $\wp$ in $\mathcal{P}$, write $\epsilon_\wp$ for the probability measure on M defined by

$$\epsilon_\wp(f) = \frac{1}{l(\wp)} \int_\wp f.$$

Then for ξ in the interior of C, the above measure m^ξ is approximated by the averages of the measures $\epsilon_\wp$ with the same behavior.

THEOREM 1.3. *Let B be a cube in A and set for $\xi \in \mathring{C}$, $T > 0$, $\delta > 0$*

$$\mathcal{P}_T^\xi(\delta, B) = \left\{ \wp \in \mathcal{P};\ T - \delta \le l(\wp) \le T \text{ and } \int_\wp F \in B + [T\xi] \right\}.$$

Then,

$$\lim_{T\to+\infty} \frac{1}{\#\mathcal{P}_T^\xi(\delta, B)} \sum_{\wp \in \mathcal{P}_T^\xi(\delta,B)} \epsilon_\wp = m^\xi.$$

This theorem is due to Bowen [**Bo72**] in the constraint-free case, and to Lalley [**La87**] for some values of ξ. Our proof of Theorem 1.3, to be given in §2.4, uses a large deviation result by Kifer [**Ki94**]. Indeed, using this result, together with Theorem 1.2, one can see that the proportion of periodic orbits $\wp \in \mathcal{P}_T^\xi(\delta, B)$ such that $\epsilon_\wp$ is not in a given neighborhood of m^ξ tends to 0 exponentially fast. Hence, most of the orbits in $\mathcal{P}_T^\xi(\delta, B)$ are distributed according to m^ξ.

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Let us detail some consequences of Theorem 1.2 in the basic example described above, i.e. when the constraints are of homological type. One gets

$$\begin{aligned}\#\{\wp \in \mathcal{P};\, T-\delta \le l(\wp) \le T \text{ and } [\wp]_{\mathrm{TF}} = \alpha + [T\xi]\} \\ \sim \frac{c(\xi)b(T,\xi)}{(2\pi)^{d/2}} \frac{e^{TH(\xi)}}{T^{1+d/2}} \left(\frac{1-e^{-P(u^\xi)\delta}}{P(u^\xi)} \right) e^{-\langle u^\xi | \alpha \rangle} \quad \text{if } P(u^\xi) \neq 0 \\ \sim \frac{c(\xi)b(T,\xi)}{(2\pi)^{d/2}} \frac{e^{TH(\xi)}}{T^{1+d/2}} \delta e^{-\langle u^\xi | \alpha \rangle} \quad \text{if } P(u^\xi) = 0\end{aligned}$$

for any ξ in the interior of C. There is an interesting dichotomy depending on whether 0 is in the interior of C or not.

Assume first that $0 \in \mathring{C}$. By Sharp [**Sh93**] this is equivalent to the existence of a fully supported measure m such that $\int F\, dm = 0$ or, equivalently, such that the winding cycle Φ_m associated to m and defined by

$$\Phi_m : H^1(M, \mathbb{R}) \ni \omega \to \int_M \langle \omega | X \rangle \, dm$$

vanishes identically. By Proposition 1.1, this measure m can be taken as the Gibbs state m^0 of $\langle u^0 | F \rangle$ for some u^0 in $\mathbb{R}^d$. Observe that $P(u^0) = H(0)$ is positive and that P, being minimal at $u = u^0$, does not vanish. Take $\xi = 0$ in the above formula. By a standard summation technique, we recover the following.

THEOREM 1.4. ([**Sh93**]) *Suppose there exists a fully supported measure for which the associated winding cycle vanishes. Then, for any homology class α,*

$$\pi(T,\alpha) = \#\{\wp \in \mathcal{P};\, l(\wp) \le T;\, [\wp]_{\mathrm{TF}} = \alpha\} \sim \frac{c(0)}{(2\pi)^{d/2}} \frac{e^{TH(0)}}{H(0)T^{1+d/2}} e^{-\langle u^0 | \alpha \rangle}$$

as T goes to infinity, where $c(0) = |E| \sqrt{|\det H''(0)|}$.

It follows in this case, as already noticed by Sharp, that the flow is homologically full, i.e. in each homology class there exists at least one, and in fact infinitely many, periodic orbits. Let us recall that when the winding cycle Φ_{top} associated with the measure of maximal entropy vanishes, e.g. for geodesic flows on manifolds with negative curvature, then $u^0 = 0$ and $H(0) = h_{\mathrm{top}}$ is the topological entropy of the flow. In this set-up Theorem 1.4, which was obtained by Katsuda and Sunada [**KS90**], implies that $\pi(T,\alpha)/\pi(T,\beta) \to 1$ when T is large, i.e. the orbits are equally distributed among homology classes. On the contrary, when Φ_{top} does not vanish, the exponent of growth $H(0)$ is strictly smaller that h_{top} and $\pi(T,\alpha)/\pi(T,\beta)$ tends to $\exp\langle \beta - \alpha | u^0 \rangle$: the orbits are no longer equally distributed. From Theorem 1.2 we can also estimate the number of periodic orbits in any direction, in fact we shall show (see Theorem 3.5) the following.

THEOREM 1.5. *Suppose there exists a fully supported measure for which the associated winding cycle vanishes. Then for $\xi \in \mathring{C}$,*

$$\#\{\wp \in \mathcal{P};\, l(\wp) \le T;\, [\wp]_{\mathrm{TF}} = [T\xi]\} \sim \frac{c(\xi)b(T,\xi)}{(2\pi)^{d/2}} \frac{e^{TH(\xi)}}{P(u^\xi)T^{1+d/2}}.$$

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Now consider the case when 0 does not belong to the interior of C. We cannot say much about the flow if 0 belongs to the boundary of C, but when 0 is not in C, the flow (ϕ_t) is known to be circular, i.e. a suspension of an Anosov transformation [**Sc57**, **Fr82**]. In this last case, there are at most finitely many periodic orbits in any fixed homology class α. Call a half-line in $\mathbb{R}^d \simeq H_1(M, \mathbb{R})$ a ϕ-homological direction if there are infinitely many periodic orbits whose homology class is at a bounded distance from that half-line. As a further consequence of Theorem 1.2, we can now identify the set of ϕ-homological directions and describe how the number of periodic orbits grows along these directions.

THEOREM 1.6. *Assume that the flow is circular and satisfies Assumption (A). Then the set of ϕ-homological directions is a convex cone whose interior $\mathcal{CH}$ coincides with the cone generated by $\mathring{C}$ and the set $D = \{\xi \in \mathring{C}; P(u^\xi) = 0\}$ is a basis for $\mathcal{CH}$. Moreover, uniformly in ξ varying in a compact subset of D,*

$$\#\{\wp \in \mathcal{P} : [\wp]_{\mathrm{TF}} = [T\xi]\} \sim |E| \frac{b(T,\xi)}{(2\pi)^{(d-1)/2}} \sqrt{\frac{|\det H''(\xi)|}{|H''(\xi)(\xi,\xi)|}} \frac{e^{TH(\xi)}}{T^{1+(d-1)/2}}.$$

Theorem 1.6 is a particular case of a general statement for flows with constraints which are not necessarily of homological type (see Theorem 3.2). It follows from Theorem 1.2 by summing the contributions corresponding to different lengths of orbits, and relies on a precise description of the foliated structure of the interior of C when it does not contain 0.

2. *The proofs of Theorems 1.2. and 1.3*

The proof of Theorem 1.2 is divided into three steps. In the first step, after a slight modification of the counting measure, we express it in terms of the logarithm of twisted zeta functions. In the second step, we deduce its asymptotic behavior from the knowledge of the singularities of these zeta functions. In the third step, we take care of the uniformity in ξ.

2.1. *Counting measures and zeta functions.*

Let g_0 and g be two functions with compact support defined on $\mathbb{R}$ and A respectively, T a positive real number and ξ a vector in $\mathbb{R}^d$ such that $T\xi$ belongs to A. Observe that for any real number σ and any vector u in $\mathbb{R}^d$,

$$\begin{aligned} & N_T^\xi(e^{-\sigma\cdot} g_0 \otimes e^{\langle u|\cdot\rangle} g) \\ &= \sum_{\wp\in\mathcal{P}} e^{-\sigma(l(\wp)-T)} g_0(l(\wp)-T) e^{\langle u|\int_\wp F - T\xi\rangle} g\left(\int_\wp F - T\xi\right) \\ &= e^{T(\sigma-\langle u|\xi\rangle)} \sum_{\wp\in\mathcal{P}} e^{-\sigma l(\wp)+\langle u|\int_\wp F\rangle} g_0(l(\wp)-T) g\left(\int_\wp F - T\xi\right) \\ &= e^{T(\sigma-\langle u|\xi\rangle)} M_T^{(\sigma,u,\xi)}(g_0 \otimes g), \end{aligned}$$

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where the modified counting measure $M_T^{(\sigma,u,\xi)}$ is defined by

$$M_T^{(\sigma,u,\xi)} = \sum_{\wp\in\mathcal{P}} e^{-\sigma l(\wp)+\langle u|\int_\wp F\rangle}\delta_{(l(\wp)-T)}\otimes\delta_{(\int_\wp F-T\xi)}.$$

We first need to recall some results on the Poincaré series:

$$Z(\sigma,u)=\sum_{\wp\in\mathcal{P}} e^{-\sigma l(\wp)+\langle u|\int_\wp F\rangle}\quad\text{and}\quad Z'(\sigma,u)=\sum_{\wp\in\mathcal{P}'} e^{-\sigma l(\wp)+\langle u|\int_\wp F\rangle},$$

where $\mathcal{P}'$ denotes the set of prime periodic orbits. Both Z and Z' converge when $\sigma > P(u)$ by the definition of the pressure $P(u)$ of the function $\langle u|F\rangle$.

We want to extend these functions to the complex domain and then denote by $(s,z)=(\sigma+it,u+iv)$ a generic point in the space $H(\tilde{A})$ of complex homomorphisms from $\tilde{A}=\mathbb{R}\times A$ to $\mathbb{C}$. Thus $s\in\mathbb{C}$, $u\in\mathbb{R}^d$ and v belongs to A^*, the dual group of A, which is $\mathbb{R}^d$ when $A=\mathbb{R}^d$, which is a d-dimensional torus when $A\simeq\mathbb{Z}^d$, or a mixture of both in the general case. We shall normalize the Haar measure dv on A^* in such a way that in a neighborhood of the trivial character, it coincides with the Euclidean measure on $\mathbb{R}^d$.

Since the series

$$Z(s,z)=\sum_{\wp\in\mathcal{P}} e^{-sl(\wp)+\langle z|\int_\wp F\rangle},\quad Z'(s,z)=\sum_{\wp\in\mathcal{P}'} e^{-sl(\wp)+\langle z|\int_\wp F\rangle}$$

are absolutely convergent on the domain

$$\Omega=\{(s,z)\in H(\tilde{A});\,\mathrm{Re}\,s > P(\mathrm{Re}\,z)\}$$

the functions Z and Z' are analytic on Ω. There is a strong relationship between these functions and the logarithm of the twisted zeta function ζ:

$$\zeta(s,z)=\prod_{\wp\in\mathcal{P}'}(1-e^{-sl(\wp)+\langle z|\int_\wp F\rangle})^{-1}$$

since, at least formally,

$$\log\zeta(s,z)=\sum_{k\geq 1}\frac{1}{k}Z'(ks,kz)\quad\text{whereas}\quad Z(s,z)=\sum_{k\geq 1}Z'(ks,kz).$$

Now, $Z'(ks,kz)$ is finite when $\mathrm{Re}\,s > P(k\,\mathrm{Re}\,z)/k$ and the function $f\colon t\to P(tu)/t$ is decreasing since $f'(t)=-(1/t^2)H(\nabla P(tu))$ is negative. Therefore, it is likely that $\sum_{k\geq 2}Z'(ks,kz)$ converges and is analytic on a domain containing $\overline{\Omega}$ in its interior, so that the singular behavior on the boundary of Ω of $\log\zeta$ and Z should be the same. This is expressed in the next proposition, in a way which is technical but suited to our purposes.

PROPOSITION 2.1. *Fix N any positive integer and endow the space $\mathcal{C}^N$ of functions of class C^N on $\mathbb{R}\times A^*$ with the topology of uniform convergence of derivatives on compact sets. For any compact set K_0 in $\mathbb{R}^d$, there exist:*

(i) *a neighborhood $U=U_1\times U_2$ of the origin in $\mathbb{R}\times A^*$ (as small as desired);*

(ii) *a cut-off function $\rho\in\mathcal{C}^N$ which is 0 outside U and 1 at the origin;*

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(iii) *a continuous map* $u \to A_u$ *from* K_0 *to* $\mathcal{C}^N$ *such that, when* u *belongs to* K_0,

$$\lim_{\sigma \to P(u)} Z(\sigma + it, u + iv) = -\rho(t, v) \log(P(u) + it - P(u + iv)) + A_u(t, v),$$

where $P(u + iv)$ *is an analytic extension of* P *on the neighborhood* $K_0 \times U_2$ *of* K_0 *and* log *is the principal branch of the complex logarithm. In particular, the function* $(t, v) \to \lim_{\sigma \to P(u)} Z(\sigma + it, u + iv)$ *is locally integrable on* $\mathbb{R} \times A^*$.

Moreover, the convergence above is dominated on compact sets, i.e. for any K *compact in* $\mathbb{R} \times A^*$, *there exist constants* C *and* D *such that, for any* $u \in K_0$ *and any real number* $\sigma > P(u)$,

$$\begin{aligned} |Z(\sigma + it, u + iv)| &\leq -C \log |P(u) + it - P(u + iv)| \quad \text{if } (t, v) \in U, \\ |Z(\sigma + it, u + iv)| &\leq D \quad \text{if } (t, v) \in K \backslash U. \end{aligned}$$

The proof of this proposition is mostly based on the book by Parry and Pollicott [**PP**] and is given in the appendix. Let us just check here that $P(u)+it-P(u+iv)$ belongs to the domain $\mathbb{C}-]-\infty, 0]$ of the logarithm when (t, v) is small and different from $(0, 0)$. We have

$$|P(u + iv) - P(u) - i\langle \nabla P(u) | v \rangle + \tfrac{1}{2} P''(u)(v, v)| = o(|v|^2).$$

Thus, when u varies in the compact set K_0, one can choose U_2 small enough so that $\mathrm{Re}(P(u) + it - P(u + iv))$ is strictly positive if $v \in U_2$, $v \neq 0$. When $v = 0$ (this is the case when $d = 0$), $\log(it)$ is defined for any non-zero real number t by the formula $\log(it) = \log|t| + (\pi/2)\,\mathrm{sgn}(t)$.

Due to the lack of control of the behavior of $Z(\sigma + it, u + iv)$ when t or v is large, we now introduce the class $\mathcal{H}^+$ of those functions $h : \mathbb{R} \times A \to \mathbb{C}$ such that:
(i) h is real nonnegative everywhere on $\mathbb{R} \times A$; and
(ii) the Fourier transform $\hat{h}$ of h belongs to $C^N(\mathbb{R} \times A^*)$ and has compact support.
The class $\mathcal{H}$ is the linear span of $\{e^{i\langle (t,v)|\cdot\rangle} h;\ h \in \mathcal{H}^+, (t, v) \in \mathbb{R} \times A^*\}$.

Now fix a function h in $\mathcal{H}$. Using the Fourier inversion formula, we get that when $\sigma > P(u)$,

$$\begin{aligned} &M_T^{(\sigma,u,\xi)}(h) \\ &\quad = \sum_{\wp \in \mathcal{P}} e^{-\sigma l(\wp) + \langle u | \int_\wp F \rangle} |E| \int_{\mathbb{R} \times A^*} e^{-it(l(\wp) - T) + i\langle v | \int_\wp F - T\xi \rangle} \hat{h}(t, -v) \frac{dt\, dv}{(2\pi)^{d+1}} \\ &\quad = |E| \int_{\mathbb{R} \times A^*} e^{iT(t - \langle v | \xi \rangle)} Z(\sigma + it, u + iv) \hat{h}(t, -v) \frac{dt\, dv}{(2\pi)^{d+1}}, \end{aligned}$$

where the second equality is obtained by applying Fubini's theorem to the convergent series Z on the compact support K of $\hat{h}$. For this compact set K, we can now use Proposition 2.1 and get, by the Lebesgue dominated convergence theorem, that

$$\lim_{\sigma \to P(u)} M_T^{(\sigma,u,\xi)}(h) = |E| \int_{\mathbb{R} \times A^*} e^{iT(t - \langle v | \xi \rangle)} \lim_{\sigma \to P(u)} Z(\sigma + it, u + iv) \hat{h}(t, -v) \frac{dt\, dv}{(2\pi)^{d+1}}. \quad (1)$$

On the other hand, when h belongs to $\mathcal{H}^+$, $M_T^{(\sigma,u,\xi)}(h)$ is a sum of positive terms, each one increasing as σ decreases to $P(u)$. Thus, the monotone convergence theorem implies

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that

$$\lim_{\sigma \to P(u)} M_T^{(\sigma,u,\xi)}(h) = M_T^{(P(u),u,\xi)}(h). \tag{2}$$

Comparing (1) and (2), we see that for any function h in the class $\mathcal{H}^+$, $M_T^{(P(u),u,\xi)}(h)$ is *finite*. Again using the Lebesgue dominated convergence theorem, we get that formula (2) is also valid for any function $h \in \mathcal{H}$. This gives the main formula:

$$M_T^{(P(u),u,\xi)}(h) = |E| \int_{\mathbb{R}\times A^*} e^{iT(t-\langle v|\xi\rangle)} \lim_{\sigma \to P(u)} Z(\sigma + it, u + iv)\hat{h}(t,-v)\frac{dt\,dv}{(2\pi)^{d+1}}. \tag{3}$$

Hence, $M_T^{(P(u),u,\xi)}(h)$ can be expressed as the Fourier transform of a function having a singularity at the origin and which is regular elsewhere. In the next section we are going to study the asymptotic behavior of such integrals.

2.2. *Fourier analysis.* We now fix the compact set K_1 of possible values of ξ contained in the interior of the convex set C. Since the distance between $T\xi$ and $[T\xi]$ is uniformly bounded, one can find T_0 large enough so that the set $\{[T\xi]/T; \xi \in K_1, T \geq T_0\}$ is also contained in a compact set K_2 included in the interior of C. Using Proposition 1.1, let $K_0 = (\nabla P)^{-1}(K_2)$ be the set of vectors u in $\mathbb{R}^d$ such that $\nabla P(u)$ belongs to K_2. To stress the dependence on T of the variables we also set $\xi^T = [T\xi]/T$, $u^T = (\nabla P)^{-1}(\xi^T)$ for $\xi \in K_1$, $T \geq T_0$, and we introduce the counting measure

$$M_T^{\xi} = M_T^{(P(u^T),u^T,\xi^T)}.$$

The aim of this section is to prove the following.

PROPOSITION 2.2. *Let h be a function in the class $\mathcal{H}$. Then, uniformly with respect to ξ in K_1, one has*

$$\lim_{T\to+\infty} T^{1+d/2} M_T^{\xi}(h) = \frac{c(\xi)}{(2\pi)^{d/2}} \int_{\mathbb{R}\times A} h(x,y)\,dx\,dy,$$

where $c(\xi) = |E|/\sqrt{\det P''(u^{\xi})}$.

Proof. Applying formula (3) and Proposition 2.1 of §2.1 leads to

$$\begin{aligned} M_T^{\xi}(h) &= |E| \int_{\mathbb{R}\times A^*} e^{iT(t-\langle v|\xi^T\rangle)} A_{u^T}(t,v)\hat{h}(t,-v)\frac{dt\,dv}{(2\pi)^{d+1}} \\ &\quad -|E| \int_{\mathbb{R}\times A^*} e^{iT(t-\langle v|\xi^T\rangle)} \rho(t,v)\log(P(u^T)+it-P(u^T+iv))\hat{h}(t,-v)\frac{dt\,dv}{(2\pi)^{d+1}}. \end{aligned}$$

We first look at the first term in the above sum. Since $A_{u^T}\hat{h}$ is of class C^N with compact support, it follows from standard Fourier analysis that

$$\text{if } T \geq T_0, \quad T^N|\text{first term}| \leq c\|A_{u^T}\hat{h}\|_{C^N}$$

for some absolute constant c. Since u^T varies in the compact set K_0 and $u \to A_u$ is continuous, we see that taking $N > 1 + d/2$ ensures that the first term decreases faster than $T^{-1-d/2}$ uniformly in ξ in K_1.

The behavior of the second term is now given by the following.

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LEMMA 2.3. *Let Φ a function on $\mathbb{R}^{d+1}$ of class C^N, $N > 1 + d/2$, whose support is contained in a sufficiently small neighborhood U' of $(0,0)$ and consider the functional S_T^u defined by*

$$S_T^u(\Phi) = -\int_{\mathbb{R}^{d+1}} e^{iT(t-\nabla P(u).v)} \log\left(P(u) + it - P(u+iv)\right) \Phi(t,v)\, dt\, dv.$$

Then, uniformly in $u \in K_0$,

$$\lim_{T\to+\infty} T^{d/2+1} S_T^u(\Phi) = \frac{(2\pi)^{1+d/2}}{\sqrt{\det P''(u)}} \Phi(0,0).$$

When $d = 0$, the corresponding statement holds with the convention that $\det P''(u) = 1$.

Let us see how Lemma 2.3 implies Proposition 2.2. Let U be the neighborhood of $(0,0)$ in $\mathbb{R} \times A^*$ given by Proposition 2.1. If U is chosen small enough, one can identify U with a neighborhood U' of the origin in $\mathbb{R}^{d+1}$. Applying Lemma 2.3 to the function $\rho\hat{h}$ and noticing that $(\rho\hat{h})(0,0) = \hat{h}(0,0) = \int h(x,y)\, dx\, dy$ gives the result (and the right amount of 2π's).

Proof of Lemma 2.3. Notice first that when there are no constraints on the orbits, that is when $d = 0$, $S_T^u(\Phi)$ reduces to $S_T(\Phi) = \int_{\mathbb{R}} e^{iTt} \log it\ \Phi(t)\, dt$ and that the statement of the lemma is a simple exercise in distribution theory. In dimension $d \geq 1$, since we did not find a reference for this statement we sketch a proof for the sake of completeness. Observe first that $S_T^u(\Phi)$ can also be written as

$$S_T^u(\Phi) = -\int_{\mathbb{R}^{d+1}} e^{iTt} \log(it + P(u) - P(u+iv) + i\nabla P(u)\cdot v)\Phi(t + \nabla P(u)\cdot v, v)\, dt\, dv.$$

Set $R(u,v) = P(u) - P(u+iv) + i\nabla P(u)\cdot v = \frac{1}{2}P''(u)(v,v) + o(|v|^2)$. From the fact that $\operatorname{Re} R(u,v) \geq c|v|^2$ and $|\operatorname{Im} R(u,v)| \leq |v|^3$, one can see that the function $1/(it + R(u,v))$ is integrable at the origin in $\mathbb{R}^{d+1}$. Then using Proposition (2.17) of [**Ba88**], one can integrate by parts with respect to the variable t to get

$$S_T^u(\Phi) = \frac{1}{T}\int_{\mathbb{R}^{d+1}} e^{iTt} \frac{1}{it + R(u,v)} \Phi(t + \nabla P(u)\cdot v, v)\, dt\, dv - \frac{1}{T} S_T^u\left(\frac{\partial\Phi}{\partial t}\right).$$

We now use the relation $1/z = \displaystyle\int_0^{+\infty} e^{-zT'} dT'$ which holds for any complex number z with $\operatorname{Re} z > 0$. Since $\operatorname{Re} R(u,v) > 0$ when $v \neq 0$, we can write

$$\begin{aligned}\Delta_T^u(\Phi) &:= S_T^u(\Phi) + \frac{1}{T} S_T^u\left(\frac{\partial\Phi}{\partial t}\right)\\ &= \frac{1}{T}\int_{\mathbb{R}^{d+1}}\int_0^{+\infty} e^{it(T-T')} e^{-T'R(u,v)} \Phi(t + \nabla P(u).v, v)\, dt\, dv\, dT'.\end{aligned}$$

Assume that $\Phi(t,v) = \Phi_1(t)\Phi_2(v)$. Integrating with respect to t leads to

$$\begin{aligned}\Delta_T^u(\Phi) &= \frac{1}{T}\int_0^{+\infty} \hat{\Phi}_1(T-T')\left(\int_{\mathbb{R}^d} e^{i(T-T')\nabla P(u).v} e^{-T'R(u,v)} \Phi_2(v)\, dv\right) dT'\\ &= \frac{1}{T}\int_0^{+\infty} \hat{\Phi}_1(T-T') A_T^u(T')\, dT'.\end{aligned}$$

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We shall now split the integral into two terms according to whether $T' \leq T/2$ or $T' \geq T/2$. On the region $T' \leq T/2$, one has $T - T' \geq T/2$ and since Φ_1 is of class C^N with compact support, $|\hat{\Phi}_1(T - T')| \leq c(T - T')^N \leq c'T^N$ for some constant c and $c' = 2^N c$. On the other hand, $|A_T^u(T')|$ is less than $c'' = \int_{\mathbb{R}^d} \Phi_2$. Therefore

$$\frac{1}{T}\left| \int_0^{T/2} \hat{\Phi}_1(T - T') A_T^u(T')\, dT' \right| \leq \frac{c'c''}{T^{1+N}} \int_0^{T/2} dT' \leq \frac{c'''}{T^N}$$

and this decreases faster that $T^{-d/2-1}$ when $N > d/2 + 1$ uniformly in u. Let us now consider the region $T' \geq T/2$. We have

$$\begin{aligned}
&\frac{1}{T} \int_{T/2}^{+\infty} \hat{\Phi}_1(T - T') A_T^u(T')\, dT' \\
&= \frac{1}{T} \int_{T/2}^{+\infty} \frac{\hat{\Phi}_1(T - T')}{(T')^{d/2}} \left(\int_{\mathbb{R}^d} e^{i(T-T')\nabla P(u)\cdot v/\sqrt{T'}} e^{-T' R(u, v/\sqrt{T'})} \Phi_2(v/\sqrt{T'})\, dv \right) dT' \\
&= \frac{1}{T} \int_{-\infty}^{T/2} \frac{\hat{\Phi}_1(T'')}{(T - T'')^{d/2}} \\
&\quad \times \left(\int_{\mathbb{R}^d} e^{i(T''\nabla P(u)\cdot v)/\sqrt{T-T''}} e^{-(T-T'')R(u, v/\sqrt{T-T''})} \Phi_2\left(\frac{v}{\sqrt{T - T''}} \right) dv \right) dT''.
\end{aligned}$$

When T goes to infinity $T^{d/2}/(T - T'')^{d/2}$ converges to 1 , $T''\nabla P(u) \cdot v/\sqrt{T - T''}$ tends to 0, $(T - T'')R(u, v/\sqrt{T - T''})$ tends to $\frac{1}{2}P''(u)(v, v)$ and $\Phi_2(v/\sqrt{T - T''})$ tends to $\Phi_2(0)$. Using the fact that $T - T'' \geq T/2$ it is easy to see that the convergence is dominated. Finally, we get that

$$\begin{aligned}
T^{d/2} \int_{T/2}^{+\infty} \hat{\Phi}_1(T - T') A_T^u(T')\, dT' &\to \int_{\mathbb{R}} (\hat{\Phi}_1(t)\, dt) \left(\int_{\mathbb{R}^d} e^{-\frac{1}{2}P''(u)(v,v)}\, dv \right) (\Phi_2(0)) \\
&= \frac{2\pi(\sqrt{2\pi})^d}{\sqrt{\det P''(u)}} \Phi(0, 0).
\end{aligned}$$

A little more care in this last step shows that the convergence is uniform in u. Thus

$$T^{1+d/2} \Delta_T^u(\Phi) \to \frac{(2\pi)^{1+d/2}}{\sqrt{\det P''(u)}} \Phi(0, 0)$$

if Φ is of class C^N. Repeating the same arguments, one can see that

$$|\Delta_T^u(\Phi)| \leq \frac{M\|\Phi\|_{C^K}}{T^{\inf(K, d/2+1)}} \quad \text{if } \Phi \text{ is of class } C^K$$

for some absolute constant M only depending on U. Finally, we have

$$S_T^u(\Phi) = \Delta_T^u(\Phi) - \frac{1}{T}\Delta_T^u\left(\frac{\partial \Phi}{\partial t}\right) + \cdots + \frac{(-1)^{N-1}}{T^{N-1}} \Delta_T^u\left(\frac{\partial^{N-1}\Phi}{\partial t^{N-1}}\right) + \frac{(-1)^N}{T^N} S_T^u\left(\frac{\partial^N \Phi}{\partial t^N}\right).$$

All the additional terms decrease faster than $T^{d/2+1}$ and this finishes the (sketch of) proof of Lemma 2.3 in the case of product functions. This immediately implies the lemma for any function Φ of class C^N supported in U'. □

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2.3. *From the class $\mathcal{H}$ to the class of functions with compact support.* Let ν_T^ξ be the renormalised measure

$$\nu_T^\xi = \frac{(2\pi)^{d/2}}{c(\xi)} T^{d/2+1} M_T^\xi.$$

What we have shown so far is that for any function h in the class $\mathcal{H}$, $\nu_T^\xi(h)$ converges uniformly to $\lambda(h)$, where λ is the product of the Lebesgue measure on $\mathbb{R}$ with the measure dy on A. For fixed ξ, it is a standard fact in measure theory that this implies the convergence of $\nu_T^\xi(G)$ to $\lambda(G)$ if G belongs to the space $C_c(\mathbb{R} \times A)$ of continuous functions on $\mathbb{R} \times A$ with compact support (see [**Bre**]). It is not obvious to us that the uniformity in ξ goes through (or that it follows from [**Sto**], as claimed in [**La87**, Theorem F]) and we shall prove the following.

LEMMA 2.4. *Let ν_T^ξ be a family of positive measures on $\tilde{A}$. If, for any function h in the class $\mathcal{H}$, $\nu_T^\xi(h)$ is finite and converges to $\lambda(h)$ uniformly in ξ as T goes to infinity, then the same holds for any continuous function G with compact support.*

We only prove Lemma 2.4 when $\tilde{A} = \mathbb{R}^{d+1}$ since the general case can be treated with obvious modifications. Fix a function G with compact support K. Consider the set of functions $\{h_a, a \in \mathbb{R}^+\}$ in $\mathcal{H}^+$ defined by

$$h_a(x_0, \ldots, x_d) = \prod_{i=0}^{d} \left(\frac{\sin a x_i}{a x_i} \right)^{2N}.$$

Choose a small enough so that h_a does not vanish on K and introduce the probability measure defined for any bounded measurable function F by

$$\mu_T^\xi(F) = \frac{\nu_T^\xi(F h_a)}{\nu_T^\xi(h_a)}.$$

Now let $F = G/h_a$. It suffices to prove that $\mu_T^\xi(F)$ converges uniformly to $\lambda(Fh_a)/\lambda(h_a)$. Fix $\epsilon > 0$ and choose:

– b small enough that $(1 - h_b) \leq \epsilon/\|F\|_\infty$ on K;
– a compact cube K' containing K such that $h_b \leq \epsilon/\|F\|_\infty$ outside of K';
– a trigonometrical polynomial P such that $\|P - F\|_\infty \leq \epsilon$ on K'.

We can write

$$\mu_T^\xi(F) = \mu_T^\xi(P h_b) + \mu_T^\xi((F - P)h_b) + \mu_T^\xi(F(1 - h_b)).$$

The first term converges uniformly to $\lambda(P h_b h_a)/\lambda(h_a)$ since $P h_b h_a$ belongs to $\mathcal{H}$. Observe that $(F - P)h_b$ is less than ϵ on K' by the choice of P and less that 2ϵ outside of K' since P is uniformly bounded by $2\|F\|_\infty$. As $|F(1 - h_b)|$ is less than ϵ everywhere, the error term is less than 3ϵ uniformly in ξ. For the same reasons, the limit is 3ϵ-close to $\lambda(Fh_a)/\lambda(h_a)$. This proves the lemma. □

We can now finish the proof of Theorem 1.2.

Let $G = g_0 \otimes g$ be continuous with compact support K on $\mathbb{R} \times A$. Using the relation $P(u^T) - \langle u^T | \xi^T \rangle = H(\xi^T)$, we find that

$$T^{d/2+1} e^{T H(\xi^T)} N_T^\xi(g_0 \otimes g) = \frac{c(\xi)}{(2\pi)^{d/2}} \nu_T^\xi(e^{P(u^T)\cdot} g_0 \otimes e^{\langle u^T | \cdot \rangle} g).$$

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Notice that u^T tends to u^ξ as T goes to infinity. From the fact that the family $\{e^{P(u)\cdot} g_0 \otimes e^{\langle u|\cdot\rangle} g; u \in K_0\}$ is equicontinuous on the support of G, it follows that

$$T^{d/2+1} e^{TH(\xi^T)} N_T^\xi(g_0 \otimes g) \to \frac{c(\xi)}{(2\pi)^{d/2}} \lambda(e^{P(u^\xi)\cdot} g_0 \otimes e^{\langle u^\xi|\cdot\rangle} g)$$

as T tends to $+\infty$, uniformly in ξ in the compact set K_1, for any compact set K_1 included in the interior of C. It only remains to notice that

$$e^{TH(\xi^T)} = e^{TH(\xi)} e^{T(H(\xi^T)-H(\xi))} \sim e^{TH(\xi)} e^{\langle \nabla H(\xi)|[T\xi]-T\xi\rangle}$$

to get the statement of Theorem 1.2.

Remark. There are cases ($k = d$, or $k < d$ and $\xi = 0$) where ξ^T, u^T do not depend on T. Then the analysis in §§2.1 and 2.2 can be easily improved to get an asymptotic expansion of $N_T^\xi(h)e^{-TH(\xi)}$ up to any order in T^{-1}, for functions h in the class $\mathcal{H}$.

2.4. *Proof of Theorem 1.3.* Let $\mathcal{P}_T^\xi(\delta, B) = \{\wp \in \mathcal{P}; T - \delta \le l(\wp) \le T \text{ and } \int_\wp F \in B + [T\xi]\}$. We want to estimate the number of orbits in $\mathcal{P}_T^\xi(\delta, B)$ such that $\epsilon_\wp$ is not in a given neighborhood of m^ξ. Let us first recall the following large deviation estimate [**Ki94**].

THEOREM. (Kifer) *If $\mathcal{K}$ is a closed subset of $\mathcal{M}^\phi$, then*

$$\limsup_{T\to+\infty} \frac{1}{T} \log \#\{\wp \in \mathcal{P}; T - \delta \le l(\wp) \le T \text{ and } \epsilon_\wp \in \mathcal{K}\} \le \sup_{m\in\mathcal{K}} h_m.$$

Let U be a neighborhood of m^ξ. Since m^ξ is the unique invariant measure maximizing entropy on $\mathcal{K}^\xi = \{m \in \mathcal{M}^\phi; \int F\,dm = \xi\}$, we can find $\delta_0 > 0$ such that $h_m \le H(\xi) - \delta_0$ if $m \in \mathcal{K}^\xi \cap U^c$. For $\varepsilon > 0$, consider the closed ε-neighborhood of $\mathcal{K}^\xi$ given by

$$\mathcal{K}^\varepsilon = \left\{m \in \mathcal{M}^\phi; \left| \int F\,dm - \xi \right| \le \varepsilon \right\}.$$

Again using the upper semi-continuity of the entropy, one can find ε small enough so that

$$h_m \le H(\xi) - \delta_0/2 \quad \text{if } m \in \mathcal{K}^\varepsilon \cap U^c.$$

Now, for T large enough, all periodic orbits in $\mathcal{P}_T^\xi(\delta, B)$ satisfying $\epsilon_\wp \notin U$ are such that $\epsilon_\wp$ belongs to $\mathcal{K}^\varepsilon \cap U^c$. Applying Kifer's result to the closed set $\mathcal{K}^\varepsilon \cap U^c$, we see that the number of such orbits is eventually smaller than $e^{T(H(\xi)-\delta_0/4)}$. On the other hand, by Theorem 1.2, the number of orbits in $\mathcal{P}_T^\xi(\delta, B)$ is eventually larger than $e^{T(H(\xi)-\delta_0/6)}$. It follows that most of the orbits in $\mathcal{P}_T^\xi(\delta, B)$ are in U. This immediately implies Theorem 1.3.

3. *The convex set C and the proofs of Theorems 1.5 and 1.6*

In this section, we use the notations introduced in §1. Let us first give the proof of Proposition 1.1. Set

$$\begin{aligned} H(\xi) &= \max\left\{h_m; m \in \mathcal{M}^\phi, \int F\,dm = \xi\right\} \quad \text{if } \xi \in C \\ &= -\infty \quad \text{otherwise.} \end{aligned}$$

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Observe that by definition $P(u) = \max\{H(\xi) + \langle u|\xi\rangle; \xi \in \mathbb{R}^d\}$. Now, the function H is concave and upper semi-continuous since the map $m \to h_m$ is itself upper semi-continuous for Anosov flows. It follows that $-H$ is the Legendre transform of P. According to Theorem 26.5 of [**Roc**], it suffices to check that P is of Legendre type which, in this case, only means that the function P is differentiable and that $P''(u)$ is positive definite for all u. It follows from [**BR75**, Proposition 3b], and from [**Ru78**], that the function P is analytic. The value of $P''(u)(u', u')$ is computed in [**KS90**] as the asymptotic variance of the function $\langle u'|F\rangle$ under the equilibrium state of the function $\langle u|F\rangle$. Assumption (A) in particular implies that the set $\{1, F_1, \ldots, F_d\}$ is cohomologically independent, hence [**Ra73**, Theorem 3.1] implies that $P''(u)$ is positive definite. This proves Proposition 1.1.

Furthermore, the function H is differentiable on the interior of C and the inverse map of ∇P is given by $-\nabla H$. Thus, for $u^\xi = -\nabla H(\xi)$, one has

$$P(u^\xi) = H(\xi) + \langle u^\xi|\xi\rangle, \quad \nabla P(u^\xi) = \xi \quad \text{and} \quad P''(u^\xi)H''(\xi) = -\operatorname{Id}.$$

Set m^ξ for the Gibbs state of the function $\langle u^\xi|F\rangle$. By thermodynamical formalism, one knows that

$$P(u^\xi) = h_{m^\xi} + \Big\langle u^\xi \Big| \int F\, dm^\xi \Big\rangle, \quad \nabla P(u^\xi) = \int F\, dm^\xi$$
$$\text{and} \quad h_m + \Big\langle u^\xi \Big| \int F\, dm \Big\rangle < P(u^\xi) \quad \text{if } m \neq m^\xi.$$

Hence, for $\xi \in \mathring{C}$, $H(\xi) = h_{m^\xi}$ and for any invariant measure m with $\int F\, dm = \xi$, $h_m \leq H(\xi)$ with equality if and only if $m = m^\xi$.

We shall now study in more detail the structure of C when 0 does not belong to it.

PROPOSITION 3.1. *Assume $0 \notin C$. Write D for the set of ξ in $\mathring{C}$ such that $P(u^\xi) = 0$ and $\mathcal{D}$ for the cone generated by D. Then $\mathcal{D}$ coincides with the convex cone $\mathcal{C}$ generated by $\mathring{C}$.*

Proof. First, D is not empty. In fact, since C is disjoint from 0, there is v in $\mathbb{R}^d$ such that $\langle v|\xi\rangle \geq 1$ for any ξ in C and therefore for any $u \in \mathbb{R}^d$, $t \to P(u + tv)$ takes all values in $\mathbb{R}$. By Proposition 1.1, $\xi \to P(u^\xi)$ takes all values as well.

Let us now prove that the cone $\mathcal{D}$ is convex. Let $\xi_1 = \nabla P(u_1)$ and $\xi_2 = \nabla P(u_2)$ be two points in D, thus $P(u_1) = P(u_2) = 0$. For any $\alpha \in]0, 1[$, consider the linear function ψ defined by $\psi(u) = \langle \alpha\xi_1 + (1-\alpha)\xi_2|u\rangle$. Since P is strictly convex, we know that

$$\langle \nabla P(u')|u - u'\rangle \leq P(u) - P(u') \text{ with equality iff } u = u'.$$

Therefore, for any u in the closed convex set $C_0 = \{u; P(u) \leq 0\}$, we have $\langle \nabla P(u_i)|u\rangle \leq \langle \nabla P(u_i)|u_i\rangle$, $i = 1, 2$, and ψ is bounded from above on C_0. Recall that the condition $0 \notin C$ ensures that C_0 is unbounded. We claim that $\psi(u) \to -\infty$ if $u \in C_0$ and $\|u\| \to +\infty$. Indeed, if not true, then there exists a sequence (w_n) going to infinity in C_0 and $A > -\infty$ such that $\langle \nabla P(u_1)|w_n\rangle \geq A$ for all n. Set $v_n = (w_n - u_1)/|w_n - u_1|$ and $t_n = |w_n - u_1|$ so that $w_n = u_1 + t_n v_n$. We may assume that v_n converges to some

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vector v. Since $u_1 + tv_n \in C_0$ for all $t \le t_n$, $u_1 + tv \in C_0$ for any positive number t. In particular, $\langle \nabla P(u_1)|v\rangle$ is less than $P(u_1+v) - P(u_1)$ by the strict convexity of P. Thus $\langle \nabla P(u_1)|v\rangle < 0$. On the other hand, dividing the inequality $\langle \nabla P(u_1)|u_1 + t_n v_n\rangle \ge A$ by t_n and letting t_n go to $+\infty$ leads to $\langle \nabla P(u_1)|v\rangle \ge 0$, a contradiction. This proves our claim and shows that ψ attains its maximum value at some point u_α in C_0. Observe first that $P(u_\alpha) = 0$ since when $P(u) < 0$, one can increase $\psi(u)$ by going in the direction of $\alpha\xi_1 + (1-\alpha)\xi_2$ and still staying in C_0. Therefore, $\xi_\alpha = \nabla P(u_\alpha)$ belongs to D. Now let v be any vector such that $\langle \nabla P(u_\alpha)|v\rangle < 0$. Then $t \to P(u_\alpha + tv)$ is decreasing at $t = 0$ and for small $t > 0$, $u_\alpha + tv \in C_0$. Hence, $\psi(u_\alpha + tv) \le \psi(u_\alpha)$, i.e. $\langle \alpha\xi_1 + (1-\alpha)\xi_2|v\rangle \le 0$. This shows that $\alpha\xi_1 + (1-\alpha)\xi_2$ is colinear to ξ_α, and that $\mathcal{D}$ is convex.

Obviously, $\mathcal{D} \subset \mathcal{C}$. Suppose now that $\mathcal{C}$ is not included in $\mathcal{D}$. Since both cones are open and convex, this means that there exist $v \in \mathbb{R}^d$ and ξ_0 in C such that $\langle v|\xi\rangle < 0$ for all $\xi \in D$ but $\langle v|\xi_0\rangle > 0$. Fix ξ in D and consider the function $\phi : t \to P(u^\xi + tv)$. We have $\phi(0) = 0$, $\phi'(0) < 0$ and $\phi(t) \to +\infty$ as $t \to +\infty$ since $\phi(t) \ge \langle u^\xi + tv|\xi_0\rangle$. Hence, there exists $t_0 > 0$ such that $\phi(t_0) = 0$ and therefore $\xi' = \nabla P(u^\xi + t_0 v)$ belongs to D. Then

$$0 = P(u^\xi) - P(u^\xi + t_0 v) > \langle \xi'| - t_0 v\rangle,$$

a contradiction. This ends the proof of Proposition 3.1. □

Remark. For each $\xi \in \mathring{C}$, there exists numbers $0 < t_-(\xi) < 1 < t_+(\xi)$ such that $t\xi \in \mathring{C}$ iff $t \in]t_-(\xi), t_+(\xi)[$. With the same proof, Proposition 3.1 holds true with D replaced by $D_c = \{\xi \in \mathring{C};\ P(u^\xi) = c\}$ for any value of c. Thus D_c is also a basis for the cone $\mathcal{C}$. Consequently, the increasing function $t \to P(u^{t\xi})$ takes all values from $-\infty$ to $+\infty$ as t varies in the interval $]t_-(\xi), t_+(\xi)[$.

Theorem 1.6 of §1 is now a particular case of the following.

THEOREM 3.2. *Assume that $d > 0$, Assumption (A) holds, and that 0 does not belong to the convex set C. Let $D = \{\xi \in \mathring{C};\ P(u^\xi) = 0\}$ be a basis for the cone generated by $\mathring{C}$. For $\xi \in D$, any function g with compact support in A, and T positive, set*

$$\eta_{T,\xi}(g) = \sum_{\wp \in \mathcal{P}} g\left(\int_\wp F - [T\xi]\right).$$

Then, $\eta_{T,\xi}(g)$ is finite and we have, as T goes to infinity,

$$\eta_{T,\xi}(g) \sim \frac{c'(\xi) b(T,\xi)}{(2\pi)^{(d-1)/2}} \frac{e^{TH(\xi)}}{T^{1+(d-1)/2}} \int_A e^{-\langle u^\xi|y\rangle} g(y)\, dy$$

uniformly for ξ in any compact subset of D. Here,

$$c'(\xi) = |E| \sqrt{\left|\frac{\det H''(\xi)}{H''(\xi)(\xi,\xi)}\right|} \quad \text{and} \quad b(T,\xi) = \exp\langle u^\xi | T\xi - [T\xi]\rangle.$$

In particular, for any half-line $\dot{\xi} = \{t\xi;\ t > 0\}$ with $\xi \in D$, there exist infinitely many periodic orbits $\wp$ such that $\int_\wp F$ lies at a bounded distance from the half-line $\dot{\xi}$.

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Remark. From Theorem 3.2 one can deduce the behavior of $\eta_{T,\zeta}(g)$ for any vector $\zeta \in \mathring{C}$ by writing $\zeta = \xi/t$ for some $\xi \in D$ and $t > 0$. Then $\eta_{T,\zeta}(g) = \eta_{Tt,\xi}(g)$. In fact it follows from the proof of Theorem 3.2 that D gives the right reference point in the constraint direction $\dot{\xi}$, in the sense that when ξ belongs to D and T is large, there are many orbits with $\int_\wp F$ close to $T\xi$ and most of them are of length actually close to T.

Proof of Theorem 3.2. We may assume g is non-negative and we let K be a compact subset of D.

(a) The main contribution. Fix ε small enough so that the set $\{\xi/t; \xi \in K, 1-3\varepsilon \le t \le 1+3\varepsilon\}$ is contained in $\mathring{C}$ and set

$$\eta'_{T,\xi}(g) = \sum_{\wp\in\mathcal{P}, |l(\wp)-T|\le 2\varepsilon T} g\Big(\int_\wp F - [T\xi]\Big).$$

LEMMA 3.3. *For T large enough,*

$$\eta'_{T,\xi}(g) \sim_\varepsilon \frac{c'(\xi)b(T,\xi)}{(2\pi)^{(d-1)/2}} \frac{e^{TH(\xi)}}{T^{1+(d-1)/2}} \int_A e^{-\langle u^\xi | y\rangle} g(y)\, dy$$

uniformly for ξ in K,where by $f \sim_\varepsilon g$, we mean $(1+O(\varepsilon))^{-1} \le f/g \le 1+O(\varepsilon)$.

Proof. Fix $\tau > 0$ and let g_0 be a non-negative continuous function supported in $[-\tau,\tau]$ such that $\int_{\mathbb{R}} g_0(s)\, ds = 1$. By Theorem 1.2, uniformly in (ξ,t) with $\xi \in K$ and $|t-1| \le 3\varepsilon$, $\sum_{\wp\in\mathcal{P}} g_0(l(\wp)-Tt) g(\int_\wp F - [T\xi]) = N_{Tt}^{\xi/t}(g_0 \otimes g)$ is equivalent as T goes to infinity to

$$\frac{c(\xi/t)b(Tt,\xi/t)}{(2\pi)^{d/2}} \frac{e^{TtH(\xi/t)}}{(tT)^{1+d/2}} \int_{\mathbb{R}} e^{P(u^{\xi/t})x} g_0(x)\, dx \int_A e^{-\langle u^{\xi/t}|y\rangle} g(y)\, dy.$$

We write this last expression as

$$\frac{b(T,\xi)}{(2\pi)^{d/2}T^{1+d/2}} g_\xi(t) \frac{b(Tt,\xi/t)}{b(T,\xi)} e^{Th_\xi(t)},$$

where we set

$$g_\xi(t) = \frac{c(\xi/t)}{t^{1+d/2}} \int_{\mathbb{R}} e^{P(u^{\xi/t})x} g_0(x)\, dx \int_A e^{-\langle u^{\xi/t}|y\rangle} g(y)\, dy \quad \text{and} \quad h_\xi(t) = tH(\xi/t).$$

Observe that the function h_ξ is defined on the interval $I_\xi =]t_+^{-1}(\xi), t_-^{-1}(\xi)[$ and satisfies $h'_\xi(t) = P(u^{\xi/t})$ and $h''_\xi(t) = H''_{\xi/t}(\xi/t^2, \xi/t) < 0$. Therefore h_ξ is concave, and is maximal at $t=1$. Moreover, uniformly in ξ, as t varies in the interval $[1-3\varepsilon, 1+3\varepsilon]$, the function $t \to g_\xi(t)$ is smooth and the function $b(Tt,\xi/t)/b(T,\xi)$ takes its values in an interval $[(1+O(\varepsilon))^{-1}, 1+O(\varepsilon)]$ for all T. By our choice of g_0, as soon as $T\varepsilon > \tau$, we also have for any periodic orbit $\wp$ that

$$T\int_{1-\varepsilon}^{1+\varepsilon} g_0(l(\wp)-Tt)\, dt \le \mathbf{1}_{[T-2\varepsilon T, T+2\varepsilon T]}(l(\wp)) \le T\int_{1-3\varepsilon}^{1+3\varepsilon} g_0(l(\wp)-Tt)\, dt.$$

It follows that up to $(1+O(\varepsilon))$, we have uniformly in ξ, for T large enough, that

$$\int_{1-\varepsilon}^{1+\varepsilon} g_\xi(t) e^{Th_\xi(t)}\, dt \le \frac{(2\pi T)^{d/2}}{b(T,\xi)} \eta'_{T,\xi}(g) \le \int_{1-3\varepsilon}^{1+3\varepsilon} g_\xi(t) e^{Th_\xi(t)}\, dt.$$

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Since the function h_ξ has a non-degenerate maximum at $t = 1$, we have by Laplace's method that both integrals are equivalent to $(\sqrt{2\pi}\, g_\xi(1)/\sqrt{T|h''_\xi(1)|})e^{TH(\xi)}$ as T goes to infinity. Lemma 3.3 follows. □

(b) Other contributions. We want to show that the contribution of both very long and very short orbits is negligible with respect to the previous main contribution. This is another large deviation estimate. Since we did not find a reference for the exact statement we need, we sketch the (standard) proof in the case of short orbits. Namely, set with the notations of part (a) that

$$\eta''_{T,\xi}(g) = \sum_{\wp\in\mathcal{P},\ l(\wp)\le T-2\varepsilon T} g\left(\int_\wp F - [T\xi]\right).$$

LEMMA 3.4. *There exist constants C_0 and $\delta_0 > 0$ such that for all ξ in K,*

$$\eta''_{T,\xi}(g) \le C_0 e^{T(H(\xi)-\delta_0)}.$$

Proof. We have for some constant C,

$$\eta''_{T,\xi}(g) \le \|g\|_\infty \#\left\{\wp\in\mathcal{P}; l(\wp)\le (1-2\varepsilon)T, \left|\int_\wp F - [T\xi]\right| \le C\right\}.$$

If Lemma 3.4 were not true, we could find $\xi_n \in K$, $T_n \to +\infty$, $\delta_n \to 0$, such that

$$\#\left\{\wp\in\mathcal{P}; l(\wp)\le (1-2\varepsilon)T_n, \left|\int_\wp F - [T_n\xi_n]\right| \le C\right\} \ge e^{T_n(H(\xi_n)-\delta_n)}.$$

We may assume that ξ_n converges to $\xi_\infty \in K$. Fix N large and divide the interval $[0, 1-2\varepsilon]$ into N equal intervals. There is some number l, $0 \le l \le 1-2\varepsilon$, such that we have subsequences $T_n \to +\infty$, $\xi_n \to \xi_\infty$, $\delta_n \to 0$, with the property that

$$\#\mathcal{P}(l, N, T_n, \xi_n) \ge \frac{1}{N} e^{T_n(H(\xi_n)-\delta_n)},$$

where

$$\mathcal{P}(l, N, T_n, \xi_n) = \left\{\wp\in\mathcal{P}; \left|\frac{l(\wp)}{T_n} - l\right| \le \frac{1}{2N}, \left|\int_\wp F - [T_n\xi_n]\right| \le C\right\}.$$

Consider the probability measure on M defined by

$$\mu_n = \frac{1}{\#\mathcal{P}(l, N, T_n, \xi_n)} \sum_{\wp\in\mathcal{P}(l,N,T_n,\xi_n)} \varepsilon_\wp$$

and let μ be a weak limit of (μ_n). Since $\mathcal{P}(l, N, T_n, \xi_n)$ is a $(l+1/2N)T_n$-separated set, we have (see, e.g., [**DGS**, p. 145]) that

$$\begin{aligned} h_\mu &\ge \liminf_n \frac{1}{(l+1/2N)T_n} \log \#\mathcal{P}(l, N, T_n, \xi_n) \\ &\ge \frac{1}{l+1/2N} H(\xi_\infty). \end{aligned}$$

On the other hand, since for any orbit $\wp$ in $\mathcal{P}(l, N, T_n, \xi_n)$, $\varepsilon_\wp(F)$ is close to $T_n/l(\wp)$, we have that $\int F\, d\mu = \lambda^{-1}\xi_\infty$ for some number λ such that $|\lambda - l| \le 1/2N$. In particular,

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$l \geq t_+^{-1}(\xi) - 1/2N$. Finally, we get that for some $l \in [t_+^{-1}(\xi) - 1/2N, 1 - 2\varepsilon]$ and $N \geq (2\varepsilon)^{-1}$,

$$\frac{l - 1/2N}{l + 1/2N} H(\xi_\infty) \leq \left(l - \frac{1}{2N}\right) h_\mu \leq \lambda h_\mu \leq h_{\xi_\infty}(\lambda) \leq h_{\xi_\infty}(1 - \varepsilon).$$

In the third inequality, we used the fact that $H(\xi_\infty/\lambda) = \sup\{h_m, \int F\, dm = \xi_\infty/\lambda\}$, and in the fourth inequality, the fact that h_{ξ_∞} is increasing on $[t_+^{-1}(\xi_\infty), 1]$. Letting N go to infinity, this gives $H(\xi_\infty) = h_{\xi_\infty}(1) \leq h_{\xi_\infty}(1 - \varepsilon)$, a contradiction. This ends the proof of Lemma 3.4. □

Observe that if an orbit $\wp$ satisfies $|\int_\wp F - [T\xi]| \leq C$ and $l(\wp)$ is large enough, then $l(\wp) \leq 2Tt_-^{-1}(\xi)$. In particular, $\eta_{T,\xi}(g)$ is finite and one can control the long orbits exactly in the same way as above. Namely, there exist constants $\delta_1 > 0$ and C_1 such that uniformly on K,

$$\sum_{\wp \in \mathcal{P},\ l(\wp) \geq T + 2\varepsilon T} g\left(\int_\wp F - [T\xi]\right) \leq C_1 e^{T(H(\xi) - \delta_1)}.$$

This ends the proof of Theorem 3.2. □

Theorem 3.2 has a counterpart in the case when 0 lies in the interior of C. Theorem 1.5 in §1 is a particular case of the following.

THEOREM 3.5. *Assume that $d > 0$, Assumption (A) and that $0 \in \mathring{C}$. For $\xi \in \mathring{C}$, any function g with compact support on A and $T > 0$, set*

$$\eta_{T,\xi}(g) = \sum_{\wp \in \mathcal{P},\, l(\wp) \leq T} g\left(\int_\wp F - [T\xi]\right).$$

Then, uniformly for ξ in a compact subset set of $\mathring{C}$, we have

$$\eta_{T,\xi}(g) \sim \frac{c(\xi) b(T,\xi)}{(2\pi)^{d/2} P(u^\xi)} \frac{e^{TH(\xi)}}{T^{1+d/2}} \int_A e^{-\langle u^\xi | y \rangle} g(y)\, dy.$$

(The notation is the same as in Theorem 1.2.)

The proof of Theorem 3.5 is completely analogous to that of Theorem 3.2. Now, the main contribution is given by $\eta'_{T,\xi}(g) = \sum_{\wp \in \mathcal{P},\, (1-2\varepsilon)T \leq l(\wp) \leq T} g(\int_\wp F - [T\xi])$. The computation in the proof of Lemma 3.3 shows that uniformly in ξ, $(2\pi T)^{d/2} \eta'_{T,\xi}$ can be estimated by integrals of the form $\int_{1-\cdot}^1 g_\xi(t) e^{Th_\xi(t)}\, dt$, where $h_\xi(t) = tH(\xi/t)$ is now an increasing function from $[t_+^{-1}(\xi), +\infty[$ into $\mathbb{R}^+$. Laplace's method yields the stated formula. The estimate of the contribution of orbits of length smaller than $(1 - 2\varepsilon)T$ is again given by Lemma 3.4, the above proof being also valid with the new function h_ξ.

Acknowledgements. We are indebted to F. Dal'bo for valuable discussions about Lalley's method for counting closed geodesics and to the referee for helping us in clarifying a few points.

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Appendix. The proof of Proposition 2.1

This proof is essentially contained in the book by Parry and Pollicott [**PP**], to which we refer for notation and details.

In order to code a hyperbolic flow (ϕ_t) and to keep track of its periodic orbits, Bowen [**Bo73**], following the analogous work by Manning for hyperbolic transformations, introduced a family $(\Sigma_{A_i}^{r_i}; \sigma_t^i)$, $i = 0, 1, \ldots, q$, of suspensions of subshifts of finite type with the following properties.

(i) There exist Hölder continuous maps $\pi_i : \Sigma_{A_i}^{r_i} \to M$, bounded-to-one intertwining σ_t^i with ϕ_t. Among these, only π_0 is surjective.

(ii) There exist numbers $\varepsilon_0 = 1, \varepsilon_i \in \{-1, 1\}, i = 1, \ldots, q$, such that for any $\wp$ in $\mathcal{P}'$ and any real number τ,

$$\sum_{i=0}^{q} \varepsilon_i \#\{\wp_i \in \mathcal{P}_i'; \pi_i(\wp_i) = \wp, l(\wp_i) = \tau\} = \begin{cases} 1 & \text{if } \tau = l(\wp) \\ 0 & \text{if not.} \end{cases}$$

Now let F be an $\mathbb{R}^d$-valued Hölder continuous function on M, $F_i = F \circ \pi_i$ be its lift to $\Sigma_{A_i}^{r_i}$ and $f_i : \Sigma_{A_i} \to \mathbb{R}^d$ be defined by

$$f_i(x) = \int_0^{r_i(x)} F_i(\sigma_t^i(x, 0))\, dt.$$

From the above properties, it follows that, for $u \in \mathbb{R}^d$ and σ large enough,

$$\sum_{\wp \in \mathcal{P}'} e^{-\sigma l(\wp) + \langle u | \int_\wp F \rangle} = \sum_{i=0}^{q} \varepsilon_i \sum_{\wp \in \mathcal{P}_i'} e^{-\sigma l(\wp) + \langle u | \int_\wp F_i \rangle}.$$

Therefore,

$$Z(\sigma, u) = \sum \varepsilon_i Z_i(\sigma, u),$$

where the corresponding functions Z_i associated with the flow (σ_t^i) and the function F_i can also be written, using the symbolic dynamics, as

$$Z_i(\sigma, u) = \sum_{n \geq 1} \frac{1}{n} \sum_{x : \sigma_i^n(x) = x} e^{-\sigma S_n r_i(x) + \langle u | S_n f_i(x) \rangle}.$$

Fix θ small enough so that all functions considered above are at least Hölder continuous with exponent $\sqrt{\theta}$ and for g in the space F_θ^i of real θ-Hölder continuous functions on the one sided subshift $\Sigma_{A_i}^+$, define the pressure $p_i(g)$ as the logarithm of the highest eigenvalue of the Ruelle–Perron–Frobenius operator:

$$\mathcal{L}_g^i : \psi \in F_\theta^i \to \mathcal{L}_g^i \psi(x) = \sum_{\sigma_i y = x} e^{g(y)} \psi(y).$$

When g is a $\sqrt{\theta}$-Hölder function on the two-sided subshift, $p_i(g)$ is defined as the pressure of any function g' in F_θ^i homologous to g. Then, for fixed u, the function $\sigma \to p_i(-\sigma r_i + \langle u | f_i \rangle)$ is strictly decreasing from $+\infty$ to $-\infty$ and vanishes for a unique real number σ denoted by $P_i(u)$. Let us now recall that:

(iii) $P_0(u)$ only depends on the original hyperbolic flow (ϕ_t) and equals $P(u)$ as defined in §1;

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(iv) P_i is continuous and $P_i(u) < P(u)$ for $i = 1, \dots, q$;
(v) $Z_i(\sigma, u)$ converges when $\sigma > P_i(u)$ and thus extends analytically on the domain $\Omega_i = \{(s, z) \in H(\tilde{A}); \operatorname{Re} s > P_i(\operatorname{Re} z)\}$.

From now on, we fix an arbitrary compact set K_0 in $\mathbb{R}^d$ together with a compact neighborhood K of $(0, 0)$ in $\mathbb{C} \times A^*$. We are interested in the behavior of $Z(s, z) = \sum_{i=0}^{q} \varepsilon_i Z_i(s, z)$ when (s, z) varies in a neighborhood in $H(\tilde{A})$ of the set

$$X = \{(P(u) + it, u + iv); u \in K_0, (t, v) \in K\}.$$

By property (iv), we can find $\delta > 0$ such that $P_i(u) < P(u) - \delta$ if $u \in K_0$ and $1 \le i \le q$. Hence, for $i \ge 1$, Z_i is analytic on a domain containing X so that the singular behavior of Z is only due to the component Z_0.

The limiting behavior of $Z_0(s, z)$ as $\operatorname{Re} s$ approaches $P(\operatorname{Re} z)$ is directed by the eigenvalues of the associated complex Ruelle operator. In the following, we drop the index 0 for the main suspension. First, observe that by [**PP**, Proposition 5.2], we may assume that the function f takes values in A. Then, for (s, z) in $H(\tilde{A})$, consider the complex Ruelle operator $\mathcal{L}_{s,z}$ associated with the function $-sr + \langle z | f \rangle$ acting on the space of complex θ-Hölder functions on Σ^+. For $(s, z) \in X$, the spectral radius of $\mathcal{L}_{s,z}$ is less or equal to 1 and we have the following properties.

(a) Denote by $X_0 = \{(P(u), u); u \in K_0\}$ the critical subset of X. Then, there exists $\theta_0 < 1$ such that for $(s, z) \in X_0$, the spectrum of $\mathcal{L}_{s,z}$ is the union of $\{1\}$ and a closed subset contained in the open disk $D(0, \theta_0)$. By perturbation theory, there exists a tubular neighborhood U_0 of X_0 in $H(\tilde{A})$ such that for $(s, z) \in U_0$, the operator $\mathcal{L}_{s,z}$ has an eigenvalue of highest modulus, $k_0(s, z)$ say, and the rest of the spectrum of $\mathcal{L}_{s,z}$ is also contained in $D(0, \theta_0)$. The function $(s, z) \to k_0(s, z)$ is analytic on U_0.

(b) By Assumption (A), when $(t, v) \neq (0, 0)$, i.e. when $(s, z) \in X \backslash X_0$, 1 cannot be an eigenvalue of $\mathcal{L}_{s,z}$ [**PP**, Theorem 4.5]. However, for $u \in K_0$, there might be a closed subset K_u of $\mathbb{R} \times A^* - \{(0, 0)\}$ with the property that for (t, v) in K_u, the spectral radius of $\mathcal{L}_{P(u)+it,u+iv}$ equals 1. Let X_1 be the closed subset of X defined by $X_1 = \{(P(u) + it, u + iv); u \in K_0, (t, v) \in K_u \cap K\}$. For $(s, z) \in X_1$, $\mathcal{L}_{s,z}$ has again a unique eigenvalue of modulus 1, and the rest of the spectrum lies inside a disc of radius $\theta_1 < 1$ [**PP**, Proposition 4.5]. Therefore, there exists a neighborhood U_1 of X_1 disjoint from U_0 such that for $(s, z) \in U_1$, the operator $\mathcal{L}_{s,z}$ has an eigenvalue of highest modulus, $k_1(s, z)$, and the rest of the spectrum of $\mathcal{L}_{s,z}$ is contained in $D(0, \theta_1)$. The function $(s, z) \to k_1(s, z)$ is analytic on U_1.

(c) On $X_2 = X \backslash (U_0 \cup U_1)$, the spectral radius of $\mathcal{L}_{s,z}$ is less than 1. Thus, there exist a neighborhood U_2 of X_2 and $\theta_2 < 1$ such that for $(s, z) \in U_2$, the spectral radius of $\mathcal{L}_{s,z}$ is less that θ_2.

Let

$$Z_n(s, z) = \sum_{x : \sigma^n(x) = x} e^{-sS_n r(x) + \langle z | S_n f(x) \rangle}.$$

By the work of Haydn [**Ha90**], see also [**PP**, Chapter 10], we may write for $(s, z) \in U_i$, $i = 0, 1, 2$,

$$Z_n(s, z) = k_i(s, z)^n + r_n^i(s, z),$$

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where $k_i(s,z)$ is defined as above for $i=0,\ i=1$, k_2 is 0 on U_2 and $|r_n^i(s,z)| \le C_i\theta_i^n$ for some constant C_i. Now, $R^i(s,z) = \sum_n r_n^i(s,z)/n$ converges absolutely on U_i and defines an analytic function on U_i. On the other hand, when $\operatorname{Re} s > P(\operatorname{Re} z)$, $|k_i(s,z)| < 1$ and $\sum_n k_i^n(s,z)/n = -\log(1-k_i(s,z))$. Since $k \to \log(1-k)$ extends analytically on a domain containing $\{k; \operatorname{Re} k < 1\}$, the function

$$(t,v) \to -\log(1-k_i(P(u)+it, u+iv))$$

is well defined and is the limit of $-\log(1-k_i(\sigma+it,u+iv))$ as σ decreases to $P(u)$ for all values of $(t,v) \neq (0,0)$. If we choose a partition of unity $\rho_0,\ \rho_1,\ \rho_2$ of class C^N associated to $U_0,\ U_1,\ U_2$, such that $\rho_0(\sigma+it,u+iv) = \rho'(\sigma - P(u))\rho(t,v)$, we have obtained that for $u \in K_0$, if $(t,v) \in K\backslash\{(0,0)\}$,

$$\lim_{\sigma \searrow P(u)} Z(\sigma+it,u+iv) = -\rho(t,v)\log(1-k_0(P(u)+it,u+iv)) + B_u(t,v), \quad (*)$$

where

$$B_u(t,v) = \left(\sum_{i=0,1,2} \rho_i R^i - \rho_1 \log(1-k_1) + \sum_{i=1}^{q} \varepsilon_i Z_i\right)(P(u)+it, u+iv)$$

is some function of class C_N depending continuously on u.

The last step to obtain Proposition 2.1 is to apply the Weierstrass preparation theorem [**Hor**, Theorem 7.5.1] to the analytic function $(s,z) \to 1-k_0(s,z)$ in the neighborhood of $(P(u),u)$. We get

$$1-k_0(s,z) = a_u(s,z)(s-b_u(z)),$$

where both a_u and b_u are analytic functions. Let m_u be the Gibbs state associated with $-P(u)r + \langle u|f\rangle$. From the fact that

$$a_u(P(u),u) = -\frac{\partial k_0}{\partial s}(P(u),u) = \int_\Sigma r\, dm_u$$

is positive and bounded away from 0 uniformly on $u \in K_0$, it follows that one can find a neighborhood U of $(0,0)$ and $\delta > 0$ such that $\operatorname{Re}(a_u(\sigma+it,u+iv)) > \delta$ if $u \in K_0$, $\sigma \ge P(u)$ and $(t,v) \in U$. Moreover, $b_u(u+iv)$ coincides with $P(u)$ when $v=0$ by the very definition of $P(u)$. Thus, $P(u+iv) := b_u(u+iv)$ provides an analytic extension of P on a complex neighborhood of K_0. This gives for $u \in K_0$, $(t,v) \in U$,

$$\log(1-k_0(P(u)+it,u+iv)) = \log(P(u)+it-P(u+iv)) + C_u(t,v). \quad (**)$$

Assertions $(*)$ and $(**)$ together imply the first statement of Proposition 2.1. We now have that for some constants $A > 0$ and $B > 0$, and for (t,v) close to $(0,0)$,

$$\lim_{\sigma \searrow P(u)} |Z(\sigma+it,u+iv)| \le A + B\log|P(u)+it-P(u+iv)|.$$

When $d=0$, the right-hand side is locally integrable on $\mathbb{R}$. When $d \ge 1$, we follow Lemme (2.13) in [**Ba88**] and write

$$\begin{aligned}
&\log|P(u)+it-P(u+iv)| \\
&\le \frac{1}{|P(u)+it-P(u+iv)|} \\
&\le \frac{1}{|\operatorname{Re}(P(u)+it-P(u+iv))|^{\varepsilon}\,|\operatorname{Im}(P(u)+it-P(u+iv))|^{1-\varepsilon}}
\end{aligned}$$

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for any $0 \leq \varepsilon \leq 1$. Since $P(u+iv) = P(u)+i\langle \nabla P(u)|u\rangle - P''(u)(v,v)+o(v^2)$, we have that $|\operatorname{Re}(P(u)+it-P(u+iv))| \geq c|v|^2$ for some $c > 0$, and $\operatorname{Im}(P(u)+it-P(u+iv)) = t+\phi(v)$ for some function ϕ. The local integrability of $\lim_{\sigma \searrow P(u)} |Z(\sigma+it, u+iv)|$ follows by noticing that $1/|t+\phi(v)|^{1-\varepsilon}|v|^{2\epsilon}$ is integrable at $(0,0)$ as soon as $0 < \varepsilon < d/2$. The upper bounds can be easily derived from the above arguments, thereby finishing the proof of Proposition 2.1.

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