Proc. Camb. Phil. Soc. (1972), **71**, 545
With 2 text-figures
PCPS 71–60
Printed in Great Britain

High frequency diffraction by a soft circular disc

II. The plane wave at oblique incidence

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(*Received* 6 *May* 1971)

Abstract. Initially a general incident field is considered and the equations are split into Fourier components. Each Fourier component gives rise to an integral equation similar to that obtained when investigating diffraction of a normally incident plane wave. After the oblique wave has been specified an analysis similar to that used for the normal wave leads to a solution of the problem containing an infinite number of constants. It is shown, however, that these constants do not affect the leading terms of the high frequency scattering coefficient.

Introduction. The high frequency diffraction by a soft circular disc of a normally incident plane scalar wave has been discussed by the author in an earlier paper (6). In the present work a general incident field is considered and the equations split into Fourier components. Each resulting integral equation is of a similar form to that obtained for the normally incident wave but contains n arbitrary constants, where n is the Fourier suffix. After specifying the obliquely incident plane wave a similar contour deformation to that used for the normally incident plane wave produces extensive cancellation within the work. This deformation also yields a closed line integral round a finite branch line and it is such an integral which is fundamental to the solution of each of the integral equations.

The n constants in each Fourier component are shown to be solutions of n linear equations but solving these equations does not appear to be practicable. However, it is demonstrated that these constants do not affect the leading terms of the far-scattered field and in particular the high-frequency scattering coefficient. These terms have been predicted by Levine and Wu (5), though their method is based upon two approximation.

1. *The integral equations.* It will be assumed that the problem is normalised in a cylindrical polar co-ordinate system so that the soft disc occupies the region $z = 0$, $0 \leqslant r \leqslant 1$. A general small amplitude harmonic scalar wave may be represented by $u_0(r, \phi, z)$ with time dependence $e^{i\omega t}$ being understood and omitted throughout. The total field $u(\mathbf{R})$ at some point $\mathbf{R}$ is then given by

$$u(\mathbf{R}) = u_0(\mathbf{R}) + u_s(\mathbf{R}) \tag{1}$$

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https://doi.org/10.1017/S0305004100050830 Published online by Cambridge University Press

where $u_s(\mathbf{R})$ is the field produced by scattering from the disc. A direct application of Green's theorem shows that

$$u_s(\mathbf{R}) = -\frac{1}{4\pi}\int_S f(r_1, \phi_1)\frac{e^{-i\alpha|\mathbf{R}-\mathbf{R}_1|}}{|\mathbf{R}-\mathbf{R}_1|}dS, \tag{2}$$

where the surface S is the unit circle in the plane $z = 0$ with centre at the origin.

$$f(r_1, \phi_1) = \left[\frac{\partial u(r_1, \phi_1, z_1)}{\partial z_1}\right]_{z_1=0+} - \left[\frac{\partial u(r_1, \phi_1, z_1)}{\partial z_1}\right]_{z_1=0-} \tag{3}$$

is the discontinuity in the normal derivative of the field across the disc. α is the product of wave number and disc radius, and is large at high frequencies.

If $\mathbf{R}$ is also a point of S,

$$u_0(r, \phi, 0) = \frac{1}{4\pi}\int_S f(r_1, \phi_1)\frac{e^{-i\alpha|\mathbf{R}-\mathbf{R}_1|}}{|\mathbf{R}-\mathbf{R}_1|}dS, \tag{4}$$

giving an integral equation for the unknown $f(r_1, \phi_1)$.

In order to reduce the problem to one involving a single variable in the unknown expand both $u_0(r, \phi, 0)$ and $f(r_1, \phi_1)$ in Fourier series,

$$u_0(r, \phi, 0) = \sum_{n=-\infty}^{\infty} a_n(r)e^{in\phi}, \tag{5}$$

$$f(r_1, \phi_1) = \sum_{n=-\infty}^{\infty} c_n(r_1)e^{in\phi_1}. \tag{6}$$

Then
$$\sum_{n=-\infty}^{\infty} a_n(r)e^{in\phi} = \frac{1}{4\pi}\sum_{n=-\infty}^{\infty}\int_0^1 r_1 c_n(r_1)\int_0^{2\pi}\frac{e^{-i\alpha|\mathbf{R}-\mathbf{R}_1|}}{|\mathbf{R}-\mathbf{R}_1|}e^{in\phi_1}d\phi_1 dr_1, \tag{7}$$

and since
$$\int_0^{2\pi}\frac{e^{-i\alpha|\mathbf{R}-\mathbf{R}_1|}}{|\mathbf{R}-\mathbf{R}_1|}e^{in\phi_1}d\phi_1 = e^{in\phi}\int_0^{2\pi}\frac{e^{-i\alpha(r^2+r_1^2-2rr_1\cos\theta)^{\frac{1}{2}}}}{(r^2+r_1^2-2rr_1\cos\theta)^{\frac{1}{2}}}\cos n\theta\, d\theta$$

(7) gives an infinite number of equations of the form

$$a_n(r) = \frac{1}{4\pi}\int_0^1 r_1 c_n(r_1)\int_0^{2\pi}\frac{e^{-i\alpha(r^2+r_1^2-2rr_1\cos\theta)^{\frac{1}{2}}}}{(r^2+r_1^2-2rr_1\cos\theta)^{\frac{1}{2}}}\cos n\theta\, d\theta\, dr_1. \tag{8}$$

This equation is of the same form as an equation considered by Jones(2) if $n \geqslant 0$. From this it can be deduced that the problem depends upon an equation

$$\int_0^1 w f_n(w)\left\{\frac{e^{-i\alpha(w+v)}}{w+v} + \frac{e^{-i\alpha(w-v)}}{w-v}\right\}dw = G_n(v) \tag{9}$$

where
$$\left(\frac{d}{w\,dw}\right)^n f_n(w) = (-1)^n\frac{c_n(w)}{w^n}, \tag{10}$$

$$G_n(v) = (2\pi\alpha)^{\frac{1}{2}}\frac{d}{dv}\int_0^v \frac{\cos\{\alpha(v^2-x^2)^{\frac{1}{2}}\}}{(v^2-x^2)^{\frac{1}{2}}}x\lambda_n(x)dx \tag{11}$$

and

$$\lambda_n(x) = 2\frac{d}{dx}\int_0^x \frac{I_{-\frac{1}{2}}\{\alpha(x^2-r^2)^{\frac{1}{2}}\}}{(x^2-r^2)^{\frac{1}{2}}}\left\{\sum_{m=0}^{n-1} b_{n,m}r^{2m} - \int_0^r \frac{(u^2-r^2)^{n-1}}{(n-1)!\,2^{n-1}}\frac{a_n(u)}{u^{n-1}}du\right\}r\,dr \quad (n \geqslant 1),$$

$$= 2\frac{d}{dx}\int_0^x \frac{I_{-\frac{1}{2}}\{\alpha(x^2-r^2)^{\frac{1}{2}}\}}{(x^2-r^2)^{\frac{1}{2}}}a_n(r)r\,dr \quad (n = 0). \tag{12}$$

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By letting
$$\int_0^1 w f_n(w) \frac{e^{-i\alpha(w-v)}}{w-v}\,dw = F_n(v) \tag{13}$$
Jones(1) has converted (9) to
$$F_n(v) = G_n(v) - \frac{1}{\pi}\left(\frac{v}{1+v}\right)^{\frac{1}{2}} e^{-i\alpha v}\int_0^1 \left(\frac{1-t}{t}\right)^{\frac{1}{2}} \frac{F_n(t)}{t+v} e^{-i\alpha t}\,dt, \tag{14}$$
an integral equation of the second kind for $F_n(v)$. Solution of (13) and application of the condition that $f_n(w)$ should be bounded at $w = 0$ shows that
$$w f_n(w) = \frac{1}{\pi^2}\left(\frac{w}{1-w}\right)^{\frac{1}{2}} e^{i\alpha w}\int_0^1 \left(\frac{1-v}{v}\right)^{\frac{1}{2}} \frac{F(v)}{w-v} e^{-i\alpha v}\,dv. \tag{15}$$

2. *The known function.* If the angle of incidence is β, the obliquely incident plane wave may be represented by
$$u_0(r, \phi, z) = e^{-i\alpha(r\sin\beta\cos\phi + z\cos\beta)} \tag{16}$$
and so
$$a_n(r) = (-i)^n J_n(\alpha r \sin\beta). \tag{17}$$
In this case $a_{-n}(r) = a_n(r)$ so that the restriction $n \geqslant 0$ can be accommodated.

Substitution of (17) in (12), the expression for $\lambda_n(x)$, gives an integral
$$\begin{aligned}
&\int_0^r u(u^2-r^2)^{n-1}\frac{J_n(\alpha u\sin\beta)}{u^n}\,du \\
&\quad= \frac{(-1)^n}{(\alpha\sin\beta)^n}\int_0^r u(u^2-r^2)^{n-1}\left(\frac{d}{u\,du}\right)^n J_0(\alpha u\sin\beta)\,du \\
&\quad= \frac{(-1)^n}{(\alpha\sin\beta)^n}\left\{\sum_{m=1}^{n-1}(-1)^{n-1}2^{n-m-1}\frac{(n-1)!}{m!}(u^2-r^2)^m(\alpha\sin\beta)^m\frac{J_m(\alpha u\sin\beta)}{u^m}\right|_0^r \\
&\qquad + (-1)^{n-1}2^{n-1}(n-1)!\,(J_0(\alpha r\sin\beta)-1)\Bigg\}
\end{aligned}$$
after integrating by parts n times to reduce the differential on the Bessel function. The summation term vanishes at the upper limit and at the lower limit only the first term of the series expansion of the Bessel function remains. Hence
$$\begin{aligned}
&\int_0^r u(u^2-r^2)^{n-1}\frac{J_n(\alpha u\sin\beta)}{u^n}\,du \\
&\qquad= \frac{1}{(\alpha\sin\beta)^n}\left\{\sum_{m=0}^{n-1}(-1)^m 2^{n-2m-1}\frac{(n-1)!}{(m!)^2}(\alpha r\sin\beta)^{2m}\right. \\
&\qquad\quad \left. - 2^{n-1}(n-1)!\,J_0(\alpha r\sin\beta)\right\}.
\end{aligned} \tag{18}$$
With this in (12), the expression for $\lambda_n(x)$ becomes
$$\begin{aligned}
\lambda_n(x) &= 2\frac{d}{dx}\int_0^x \frac{I_{-\frac{1}{2}}\{\alpha(x^2-r^2)^{\frac{1}{2}}\}}{(x^2-r^2)^{\frac{1}{2}}}\left\{\frac{J_0(\alpha r\sin\beta)}{(i\alpha\sin\beta)^n}\right. \\
&\quad \left. + \sum_{m=0}^{n-1}\left(b_{n,m} - \frac{(-1)^m}{(i\alpha\sin\beta)^m}\frac{(\alpha\sin\beta)^{2m}}{2^{2m}(m!)^2}\right) r^{2m}\right\} r\,dr \quad (n \geqslant 1), \\
&= 2\frac{d}{dx}\int_0^x \frac{I_{-\frac{1}{2}}\{\alpha(x^2-r^2)^{\frac{1}{2}}\}}{(x^2-r^2)^{\frac{1}{2}}} J_0(\alpha r\sin\beta)\,r\,dr \quad (n = 0).
\end{aligned}$$

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The special case $n = 0$ may be included in the general case if $b_{0,m}$ is defined so that the summation vanishes when $n = 0$. This will be assumed. A further simplification can be made if another set of unknown constants, $B_{n,m}$, are defined in terms of the first set by

$$B_{n,m} = (i\alpha \sin\beta)^n b_{n,m} - (-1)^m \frac{(\alpha \sin\beta)^{2m}}{2^{2m}(m!)^2}. \tag{19}$$

The expression for $\lambda_n(x)$ can then be written

$$\lambda_n(x) = \frac{2}{(i\alpha \sin\beta)^n} \frac{d}{dx} \int_0^x \frac{I_{-\frac{1}{2}}\{\alpha(x^2-r^2)^{\frac{1}{2}}\}}{(x^2-r^2)^{\frac{1}{2}}} \left\{ rJ_0(\alpha r \sin\beta) + \sum_{m=0}^{n-1} B_{n,m} r^{2m+1} \right\} dr. \tag{20}$$

The integrals can be evaluated by applying Sonine's second finite integral theorem to the term involving the Bessel function and Sonine's first finite integral theorem to the summation. Then

$$\lambda_n(x) = \frac{2}{(i\alpha \sin\beta)^n} \left\{ (x\cos\beta)^{\frac{1}{2}} I_{-\frac{1}{2}}(\alpha x \cos\beta) + \sum_{m=0}^{n-1} B_{n,m} \frac{2^m m!}{\alpha^m} x^{m+\frac{1}{2}} I_{m-\frac{1}{2}}(\alpha x) \right\}. \tag{21}$$

Finally, this expression in (11) shows that

$$\begin{aligned} G_n(v) = \frac{(2\pi\alpha)^{\frac{1}{2}}}{(i\alpha \sin\beta)^n} \Bigg\{ & (\cos\beta)^{\frac{1}{2}} \frac{d}{dv} 2 \int_0^v \frac{\cos\{\alpha(v^2-x^2)^{\frac{1}{2}}\}}{(v^2-x^2)^{\frac{1}{2}}} x^{\frac{3}{2}} I_{-\frac{1}{2}}(\alpha x \cos\beta)\, dx \\ & + \sum_{m=0}^{n-1} B_{n,m} \frac{2^m m!}{\alpha^m} \frac{d}{dv} 2 \int_0^v \frac{\cos\{\alpha(v^2-x^2)^{\frac{1}{2}}\}}{(v^2-x^2)^{\frac{1}{2}}} x^{m+\frac{3}{2}} I_{m-\frac{1}{2}}(\alpha x)\, dx \Bigg\}. \end{aligned} \tag{22}$$

This relationship for $G_n(v)$ has features which were present in the equivalent function, $G(v)$, in the author's earlier paper (6) on the normally incident plane wave. It might be expected, therefore, that a similar analysis to that used previously would also be applicable to the present case.

3. *Contour representations.* Following the analysis of the earlier paper Bessel function identities (e.g. Watson (7)) are used to write

$$x^{\frac{3}{2}} I_{-\frac{1}{2}}(\alpha x \cos\beta) = \{e^{-\alpha x \cos\beta} \eta_0(\alpha x \cos\beta) - e^{\alpha x \cos\beta} \eta_0(-\alpha x \cos\beta)\} \tag{23}$$

and

$$x^{m+\frac{3}{2}} I_{m-\frac{1}{2}}(\alpha x) = \{e^{-\alpha x} \eta_m(\alpha x) - e^{\alpha x} \eta_m(-\alpha x)\} \tag{24}$$

where

$$\eta_m(ax) = \frac{(-1)^m}{\pi} x^{m+\frac{3}{2}} e^{ax} K_{m-\frac{1}{2}}(ax). \tag{25}$$

These expressions are substituted in (22). The first integral can then be written

$$\begin{aligned} & 2 \int_0^v \frac{\cos\{\alpha(v^2-x^2)^{\frac{1}{2}}\}}{(v^2-x^2)^{\frac{1}{2}}} x^{\frac{3}{2}} I_{-\frac{1}{2}}(\alpha x \cos\beta)\, dx \\ & = \int_0^v \frac{e^{i\alpha\{(v^2-x^2)^{\frac{1}{2}} + ix\cos\beta\}}}{(v^2-x^2)^{\frac{1}{2}}} \eta_0(\alpha x \cos\beta)\, dx \\ & \quad + \int_0^v \frac{e^{-i\alpha\{(v^2-x^2)^{\frac{1}{2}} - ix\cos\beta\}}}{(v^2-x^2)^{\frac{1}{2}}} \eta_0(\alpha x \cos\beta)\, dx \\ & \quad - \int_0^v \frac{e^{i\alpha\{(v^2-x^2)^{\frac{1}{2}} - ix\cos\beta\}}}{(v^2-x^2)^{\frac{1}{2}}} \eta_0(-\alpha x \cos\beta)\, dx \\ & \quad - \int_0^v \frac{e^{-i\alpha\{(v^2-x^2)^{\frac{1}{2}} + ix\cos\beta\}}}{(v^2-x^2)^{\frac{1}{2}}} \eta_0(-\alpha x \cos\beta)\, dx. \end{aligned}$$

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A change of variable is made in each integral to reduce the exponential to the form $e^{-i\alpha y}$. In the first integral, therefore, put

$$(v^2-x^2)^{\frac{1}{2}}+ix\cos\beta=-y.$$

Then, when $x=0$, $y=-v$ and when $x=v$, $y=-iv\cos\beta$. In addition

$$\frac{dx}{(v^2-x^2)^{\frac{1}{2}}}=\frac{-idy}{(y^2-v^2\sin^2\beta)^{\frac{1}{2}}}$$

and

$$x=\frac{i}{\sin^2\beta}\{(y^2-v^2\sin^2\beta)^{\frac{1}{2}}-y\cos\beta\}$$

where there is a branch line between $-v\sin\beta$ and $+v\sin\beta$ and the fractional index is given the conventional interpretation. Hence

$$\int_0^v \frac{e^{i\alpha\{(v^2-x^2)^{\frac{1}{2}}+ix\cos\beta\}}}{(v^2-x^2)^{\frac{1}{2}}}\eta_0(\alpha x\cos\beta)\,dx$$
$$=-i\int_{-v}^{-iv\cos\beta}\frac{e^{-i\alpha y}}{(y^2-v^2\sin^2\beta)^{\frac{1}{2}}}\eta_0\left\{\frac{i\alpha\cos\beta}{\sin^2\beta}((y^2-v^2\sin^2\beta)^{\frac{1}{2}}-y\cos\beta)\right\}dy.$$

Similar changes of variable in each of the other three integrals show that

$$2\int_0^v \frac{\cos\{\alpha(v^2-x^2)^{\frac{1}{2}}\}}{(v^2-x^2)^{\frac{1}{2}}}x^{\frac{1}{2}}J_{-\frac{1}{2}}(\alpha x\cos\beta)\,dx$$
$$=\left\{-i\int_{-v}^{-iv\cos\beta}+i\int_{v}^{-iv\cos\beta}-i\int_{-v}^{iv\cos\beta}+i\int_{v}^{iv\cos\beta}\right\}\frac{e^{-i\alpha y}}{(y^2-v^2\sin^2\beta)^{\frac{1}{2}}}$$
$$\times\eta_0\left\{\frac{i\alpha\cos\beta}{\sin^2\beta}((y^2-v^2\sin^2\beta)^{\frac{1}{2}}-y\cos\beta)\right\}dy. \tag{26}$$

The complete contour is shown in Fig. 1.

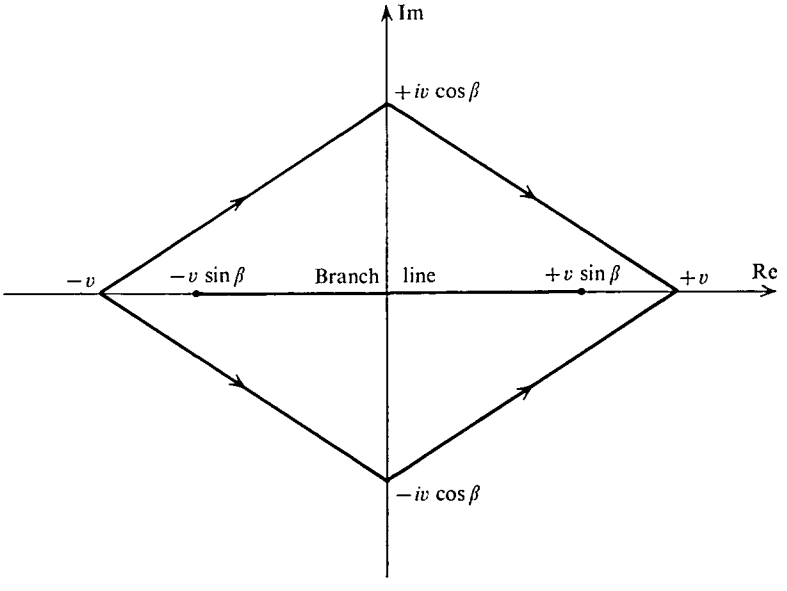


Fig. 1. Contours of equation (26) in the complex y-plane round the branch line.

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There are two paths of integration, one above and one below the branch line. This is analogous to the normally incident plane wave case. Indeed, if $\beta \to 0$, the two contours are identical.

Following the approach used at normal incidence, complete the contour to $-i\infty$ by adding

$$-2i\int_v^{-i\infty} \frac{e^{-i\alpha y}}{(y^2-v^2\sin^2\beta)^{\frac{1}{2}}}\eta_0\left\{\frac{i\alpha\cos\beta}{\sin^2\beta}((y-v^2\sin^2\beta)^{\frac{1}{2}}-y\cos\beta)\right\}dy$$

$$= -2\int_0^{\infty} \frac{e^{-i\alpha(v^2-x^2)^{\frac{1}{2}}}}{(v^2-x^2)^{\frac{1}{2}}}\frac{x^{\frac{3}{2}}}{\pi}K_{-\frac{1}{2}}(\alpha x\cos\beta)\,dx \qquad (27)$$

in the original co-ordinate system. The combined contours of (26) and (27) may be deformed in a similar manner to those in the earlier paper. The result is illustrated in Fig. 2.

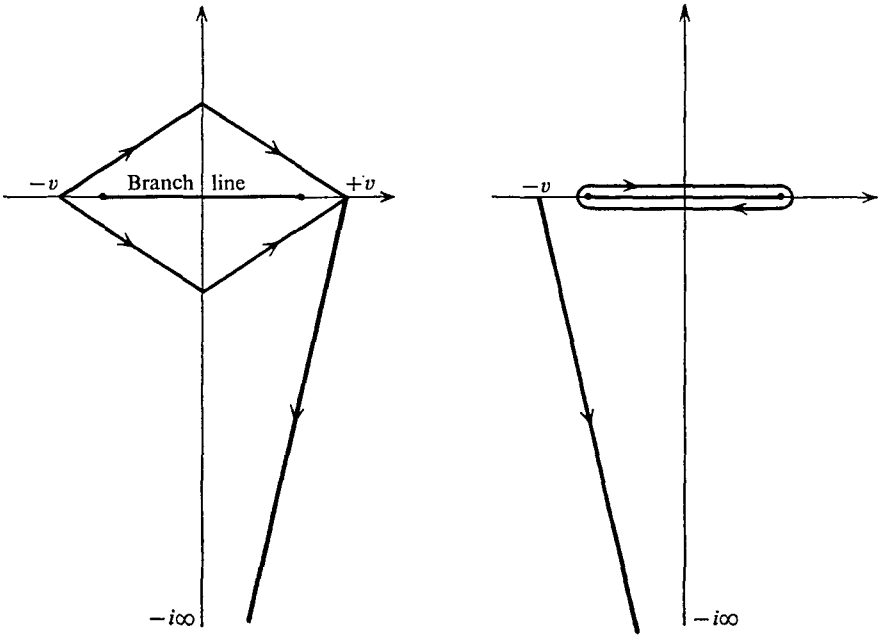


Fig. 2. The combined contours of equations (26) and (27) and their deformation.

Hence

$$2\int_0^{v} \frac{\cos\{\alpha(v^2-x^2)^{\frac{1}{2}}\}}{(v^2-x^2)^{\frac{1}{2}}}x^{\frac{3}{2}}I_{-\frac{1}{2}}(\alpha x\cos\beta)\,dx - 2\int_0^{\infty} \frac{e^{-i\alpha(v^2-x^2)^{\frac{1}{2}}}}{(v^2-x^2)^{\frac{1}{2}}}\frac{x^{\frac{3}{2}}}{\pi}K_{-\frac{1}{2}}(\alpha x\cos\beta)\,dx$$

$$= -2i\int_{-v}^{-i\infty} \frac{e^{-i\alpha y}}{(y^2-v^2\sin^2\beta)^{\frac{1}{2}}}\eta_0\left\{\frac{i\alpha\cos\beta}{\sin^2\beta}((y^2-v^2\sin^2\beta)^{\frac{1}{2}}-y\cos\beta)\right\}dy$$

$$-i\oint \frac{e^{-i\alpha y}}{(y^2-v^2\sin^2\beta)^{\frac{1}{2}}}\eta_0\left\{\frac{i\alpha\cos\beta}{\sin^2\beta}((y^2-v^2\sin^2\beta)^{\frac{1}{2}}-y\cos\beta)\right\}dy \qquad (28)$$

where $\oint$ is taken to mean a closed line integral in a clockwise direction round the branch line between $-v\sin\beta$ and $+v\sin\beta$.

The second integral in (22) is treated in a similar way, the analysis being very close

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to that of the earlier paper. The contour is completed to $-i\infty$ again by the addition of another term and the deformation performed. The original integral and the additional term are

$$2\int_0^v \frac{\cos\{\alpha(v^2-x^2)^{\frac{1}{2}}\}}{(v^2-x^2)^{\frac{1}{2}}} x^{m+\frac{3}{2}} I_{m-\frac{1}{2}}(\alpha x)\,dx - 2\int_0^\infty \frac{e^{-i\alpha(v^2-x^2)^{\frac{1}{2}}}}{(v^2-x^2)^{\frac{1}{2}}} \frac{(-1)^m}{\pi} x^{m+\frac{3}{2}} K_{m-\frac{1}{2}}(\alpha x)\,dx$$

$$= -2i\int_{-v}^{-i\infty} \frac{e^{-iay}}{y} \eta_m\left\{\frac{i\alpha}{2y}(y^2-v^2)\right\} dy - i\oint \frac{e^{-iay}}{y} \eta_m\left\{\frac{i\alpha}{2y}(y^2-v^2)\right\} dy \quad (29)$$

after the deformation, where $\oint$ indicates a line integral round the pole at $y = 0$ in a clockwise direction.

In view of the foregoing results, define a new function $\mathscr{G}_n(v)$ by

$$\mathscr{G}_n(v) = -\frac{(2\pi\alpha)^{\frac{1}{2}}}{(i\alpha\sin\beta)^n}\left\{(\cos\beta)^{\frac{1}{2}} \frac{d}{dv} 2\int_0^\infty \frac{e^{-i\alpha(v^2-x^2)^{\frac{1}{2}}}}{(v^2-x^2)^{\frac{1}{2}}} \frac{x^{\frac{3}{2}}}{\pi} K_{-\frac{1}{2}}(\alpha x\cos\beta)\,dx \right.$$

$$\left. + \sum_{m=0}^{n-1} B_{n,m} \frac{2^m m!}{\alpha^m} \frac{d}{dv} 2\int_0^\infty \frac{e^{-i\alpha(v^2-x^2)^{\frac{1}{2}}}}{(v^2-x^2)^{\frac{1}{2}}} \frac{(-1)^m}{\pi} x^{m+\frac{3}{2}} K_{m-\frac{1}{2}}(\alpha x)\,dx\right\}. \quad (30)$$

The addition of $\mathscr{G}_n(v)$ to $G_n(v)$ completes the contour to $-i\infty$ and permits the deformations outlined above. Then

$$G_n(v) + \mathscr{G}_n(v) = \gamma_n(v) - \mathscr{G}_n(-v) \quad (31)$$

where $\gamma_n(v)$ is defined by

$$\gamma_n(v) = \frac{(2\pi\alpha)^{\frac{1}{2}}}{(i\alpha\sin\beta)^n}\left\{(\cos\beta)^{\frac{1}{2}} \frac{d}{dv}(-i)\oint \frac{e^{-iay}}{(y^2-v^2\sin^2\beta)^{\frac{1}{2}}} \right.$$

$$\times \eta_0\left\{\frac{i\alpha\cos\beta}{\sin^2\beta}((y^2-v^2\sin^2\beta)^{\frac{1}{2}} - y\cos\beta)\right\} dy$$

$$\left. + \sum_{m=0}^{n-1} B_{n,m} \frac{2^m m!}{\alpha^m} \frac{d}{dv}(-i)\oint \frac{e^{-iay}}{y} \eta_m\left\{\frac{i\alpha}{2y}(y^2-v^2)\right\} dy\right\}. \quad (32)$$

The integral equation to be solved is (14) and following the method used at normal incidence define a new unknown $\mathscr{F}_n(v)$ by

$$\mathscr{F}_n(v) = F_n(v) - G_n(v) - \mathscr{G}_n(v). \quad (33)$$

The integral equation is then transformed into

$$\mathscr{F}_n(v) = g_n(v) - \frac{1}{\pi}\left(\frac{v}{1+v}\right)^{\frac{1}{2}} e^{-iav}\int_0^1 \left(\frac{1-t}{t}\right)^{\frac{1}{2}} \frac{\mathscr{F}_n(t)}{t+v} e^{-iat}\,dt \quad (34)$$

where

$$g_n(v) = -\mathscr{G}_n(v) - \frac{1}{\pi}\left(\frac{v}{1+v}\right)^{\frac{1}{2}} e^{-iav}\int_0^1 \left(\frac{1-t}{t}\right)^{\frac{1}{2}} \frac{e^{-iat}}{t+v}\{G_n(t) + \mathscr{G}_n(t)\}\,dt. \quad (35)$$

In view of the similarities between the normal and oblique cases it is reasonable to anticipate that $g_n(v)$ may be more amenable than $G_n(v)$ as a known function in the integral equation.

4. *The new known function.* Using (31) the new known function can be written

$$g_n(v) = -\mathscr{G}_n(v) - \frac{1}{\pi}\left(\frac{v}{1+v}\right)^{\frac{1}{2}} e^{-iav}\int_0^1 \left(\frac{1-t}{t}\right)^{\frac{1}{2}} \frac{e^{-iat}}{t+v}\{\gamma_n(t) - \mathscr{G}_n(-t)\}\,dt. \quad (36)$$

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Consider therefore

$$\mathscr{G}_n(v) = \frac{(2\pi\alpha)^{\frac{1}{2}}}{(i\alpha\sin\beta)^n}\left\{(\cos\beta)^{\frac{1}{2}}\frac{d}{dv}(-2i)\int_v^{-i\infty}\frac{e^{-iay}}{(y^2-v^2\sin^2\beta)^{\frac{1}{2}}}\right.$$
$$\times\eta_0\left\{\frac{i\alpha\cos\beta}{\sin^2\beta}((y^2-v^2\sin^2\beta)^{\frac{1}{2}}-y\cos\beta)\right\}dy$$
$$\left.+\sum_{m=0}^{n-1}B_{n,m}\frac{2^m m!}{\alpha^m}\frac{d}{dv}(-2i)\int_v^{-i\infty}\frac{e^{-iay}}{y}\eta_m\left\{\frac{i\alpha}{2y}(y^2-v^2)\right\}dy\right\}. \tag{37}$$

From (25)
$$\eta_0(ax) = \frac{x}{(2\pi a)^{\frac{1}{2}}}$$

and
$$\eta_m(ax) = \frac{(-1)^m}{(2\pi a)^{\frac{1}{2}}}\sum_{r=0}^{m-1}\frac{(m-1+r)!}{r!(m-1-r)!}\frac{x^{m+1-r}}{(2a)^r} \quad (m\geqslant 1),$$

using the series expansion of the modified Bessel function. Then, after some manipulation, we may write

$$\mathscr{G}_n(v) = -2v\int_v^{-i\infty}\mathscr{H}_n(y,v)\,e^{-iay}\,dy \tag{38}$$

where

$$\mathscr{H}_n(y,v) = \frac{1}{(i\alpha\sin\beta)^n}\left\{\frac{y\cos\beta}{(y^2-v^2\sin^2\beta)^{\frac{3}{2}}}\right.$$
$$\left.+\sum_{m=0}^{n-1}B_{n,m}\frac{2^m m!}{\alpha^m}(-1)^m\sum_{r=0}^{m-1}{}'\frac{(m-1+r)!}{r!(m-1-r)!}\frac{(m+1-r)}{(2\alpha)^r}\left\{\frac{i}{2y}(y^2-v^2)\right\}^{m-r}\frac{1}{y^2}\right\} \tag{39}$$

and
$$\sum_{r=0}^{m-1}{}' = \sum_{r=0}^{0} \quad (m=0)$$
$$= \sum_{r=0}^{m-1} \quad (m\geqslant 1).$$

If y is replaced by $x+v$,

$$\mathscr{G}_n(v) = -2ve^{-iav}\int_0^{-i\infty}\mathscr{H}_n(x+v,v)\,e^{-iax}\,dx. \tag{40}$$

$\mathscr{G}_n$ also occurs inside an integral in (36), which may be written

$$\int_0^1\left(\frac{1-t}{t}\right)^{\frac{1}{2}}\frac{e^{-iat}}{t+v}\mathscr{G}_n(-t)\,dt = \int_0^{-i\infty}e^{-iax}\,2\int_0^1\frac{(1-t)^{\frac{1}{2}}t^{\frac{1}{2}}}{t+v}\mathscr{H}_n(x-t,-t)\,dt\,dx, \tag{41}$$

after using (40) and changing the order of integration. Now, $\mathscr{H}_n(x-t,-t)$ has properties similar to the corresponding function in the normally incident case in that it is singular at $t=x$ and has a finite branch line. The singularities and the branch line are in the lower half of the complex t plane. In addition, $\mathscr{H}_n(x-t,-t)$ is $O(1/t^2)$ as $t\to\pm\infty$. The result obtained in the appendix of the earlier paper then shows that

$$2\int_0^1\frac{(1-t)^{\frac{1}{2}}t^{\frac{1}{2}}}{t+v}\mathscr{H}_n(x-t,-t)\,dt = -2\pi(1+v)^{\frac{1}{2}}v^{\frac{1}{2}}\mathscr{H}_n(x+v,v)\{H(-1-v)+H(v)\}$$
$$+\oint\frac{(1-t)^{\frac{1}{2}}t^{\frac{1}{2}}}{t+v}\mathscr{H}_n(x-t,-t)\,dt. \tag{42}$$

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Substitution of this result in (41) and some manipulation of the contour integrals indicates

$$\int_0^1 \left(\frac{1-t}{t}\right)^{\frac{1}{2}} \frac{e^{-i\alpha t}}{t+v} \mathscr{G}_n(-t)\,dt = \pi \left(\frac{1+v}{v}\right)^{\frac{1}{2}} e^{i\alpha v} \mathscr{G}_n(v) + \int_0^{-i\infty} \left(\frac{1-t}{t}\right)^{\frac{1}{2}} \frac{e^{-i\alpha t}}{t+v} \gamma_n(t)\,dt \quad (43)$$

and so, from (26), after extensive cancellation,

$$g_n(v) = \frac{1}{\pi}\left(\frac{v}{1+v}\right)^{\frac{1}{2}} e^{-i\alpha v} \int_1^{1-i\infty} \left(\frac{1-t}{t}\right)^{\frac{1}{2}} \frac{e^{-i\alpha t}}{t+v} \gamma_n(t)\,dt. \quad (44)$$

As expected, the new known function, $g_n(v)$, is more suitable than $G_n(v)$ and contains $\gamma_n(t)$, which may be termed the fundamental function for oblique incidence. $\gamma_n(t)$ is not quite as simple as the equivalent function, $\gamma(t)$, for normal incidence since it contains an integral round a branch line in addition to poles.

5. *The fundamental function.* This function, $\gamma_n(v)$, is defined by (32). The first integral may be evaluated to give

$$\oint \frac{e^{-i\alpha y}}{(y^2 - v^2\sin^2\beta)^{\frac{1}{2}}} \eta_0\left\{\frac{i\alpha\cos\beta}{\sin^2\beta}\left((y^2 - v^2\sin^2\beta)^{\frac{1}{2}} - y\cos\beta\right)\right\} dy = \frac{2\pi i v\cos\beta}{(2\pi\alpha\cos\beta)^{\frac{1}{2}}\sin\beta} J_1(\alpha v\sin\beta).$$

The integrals in the summation are evaluated by residues at $y = 0$. Therefore

$$\oint \frac{e^{-i\alpha y}}{y} \eta_m\left\{\frac{i\alpha}{2y}(y^2 - v^2)\right\} dy = \frac{(-1)^m}{(2\pi\alpha)^{\frac{1}{2}}} {\sum_{r=0}^{m-1}}' \frac{(m-1+r)!}{r!(m-1-r)!} \frac{1}{(2\alpha)^r} \oint \frac{e^{-i\alpha y}}{y}\left\{\frac{i}{2y}(y^2 - v^2)\right\}^{m+1-r} dy$$

and

$$\oint \frac{e^{-i\alpha y}}{y}\left\{\frac{i}{2y}(y^2 - v^2)\right\}^{m+1-r} dy = -\left(\frac{i}{2}\right)^{m+1-r} \frac{2\pi i}{(m+1-r)!}\left(\frac{d}{dy}\right)^{m+1-r} \left\{(y^2 - v^2)^{m+1-r} e^{-i\alpha y}\right\}\Big|_{y=0}.$$

Substitution of these results in (32) shows that in the summation v only occurs in odd powers from 1 up to $2n-1$. The coefficients of v contain the unknown constants $B_{n,m}$. In view of these remarks the fundamental function may be written

$$\gamma_n(v) = \frac{2\pi\alpha\cos\beta}{(i\alpha\sin\beta)^n}\left\{v J_0(\alpha v\sin\beta) + \sum_{m=0}^{n-1} A_{n,m} v^{2m+1}\right\} \quad (45)$$

where $A_{n,m}$ are unknown constants. These constants may be determined from the edge conditions.

The integral equation (34) is of the same form as the equation in the earlier paper and so the iteration scheme may be written down.

$$\mathscr{F}_n(v) \sim \sum_{k=1}^{\infty} \int_0^{\infty} \Psi_{n,k}(w) \frac{e^{-i\alpha(w-v)}}{w-v}\,dw, \quad (46)$$

$$\Psi_{n,0}(w) = \frac{i}{\pi}\gamma_n(w), \quad (47)$$

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$$\psi_{n,k+1}(v) = \int_1^{1-i\infty} \left(\frac{w-1}{w}\right)^{\frac{1}{2}} \frac{e^{-i\alpha w}}{w+v} \Psi_{n,k}(w)\, dw, \tag{48}$$

$$\Psi_{n,k+1}(w) = \frac{i\alpha}{4\pi} \int_0^{\infty} \left(\frac{v}{1+v}\right)^{\frac{1}{2}} \psi_{n,k+1}(v)\, e^{-i\alpha v} M(v,w)\, dv, \tag{49}$$

$$f_n(w) \sim \sum_{k=0}^{\infty} f_{n,k}(w) \tag{50}$$

$$f_{n,k}(w) = \frac{1}{w}\Psi_{n,k}(w) + \frac{\psi_{n,k+1}(-w)\, e^{i\alpha w}}{\pi w^{\frac{1}{2}}(1-w)^{\frac{1}{2}}}. \tag{51}$$

The solution therefore depends upon the function $\gamma_n(w)$ which contains n arbitrary constants.

6. *The edge conditions.* The edge condition (Jones (3)) which determines the constants is that the field on the disc must behave like $(1-w)^{-\frac{1}{2}}$ as $w \to 1$ at the edge of the disc. The field on the disc has been expanded as a Fourier series in (6) with unknown coefficients $c_n(w)$. A new function $f_n(w)$ was defined in (10) and in view of this relationship the edge condition on $f_n(w)$ is that as $w \to 1$ it must behave like $(1-w)^{n-\frac{1}{2}}$. To find the precise form of this condition $f_n(w)$ must be expressed as a power series in $(1-w)$.

From (51) and (48)

$$f_{n,k}(w) = \frac{1}{w}\Psi_{n,k}(w) + \frac{e^{i\alpha w}}{\pi w^{\frac{1}{2}}(1-w)^{\frac{1}{2}}} \int_1^{1-i\infty} \left(\frac{t-1}{t}\right)^{\frac{1}{2}} \frac{e^{-i\alpha t}}{t-w} \Psi_{n,k}(t)\, dt. \tag{52}$$

For the immediate present let

$$\frac{1}{w^{\frac{1}{2}}} \Psi_{n,k}(w)\, e^{-i\alpha w} = N_0(w), \tag{53}$$

then

$$f_{n,k}(w) = \frac{e^{i\alpha w}}{\pi w^{\frac{1}{2}}(1-w)^{\frac{1}{2}}} \left\{ \pi(1-w)^{\frac{1}{2}} N_0(w) + \int_1^{1-i\infty} N_0(t) \frac{(t-1)^{\frac{1}{2}}}{t-w}\, dt \right\}. \tag{54}$$

Now

$$\frac{1}{t-w} = \frac{1}{t-1}\left(1 + \frac{w-1}{t-w}\right)$$

and so

$$\begin{aligned}
&\int_1^{1-i\infty} N_0(t) \frac{(t-1)^{\frac{1}{2}}}{t-w}\, dt \\
&= \int_1^{1-i\infty} \frac{N_0(t)}{(t-1)^{\frac{1}{2}}}\, dt + (w-1) \int_1^{1-i\infty} \frac{N_0(t)}{(t-1)^{\frac{1}{2}}(t-w)}\, dt \\
&= \int_1^{1-i\infty} \frac{N_0(t)}{(t-1)^{\frac{1}{2}}}\, dt + (w-1) N_0(1) \int_1^{1-i\infty} \frac{dt}{(t-1)^{\frac{1}{2}}(t-w)} + (w-1) \int_1^{1-i\infty} \frac{N_0(t) - N_0(1)}{(t-1)^{\frac{1}{2}}(t-w)}\, dt \\
&= \int_1^{1-i\infty} \frac{N_0(t)}{(t-1)^{\frac{1}{2}}}\, dt + (w-1) N_0(1) \frac{\pi}{(1-w)^{\frac{1}{2}}} + (w-1) \int_1^{1-i\infty} N_1(t) \frac{(t-1)^{\frac{1}{2}}}{t-w}\, dt.
\end{aligned}$$

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The final integral is of the same form as the original and so by repeating the process

$$\int_1^{1-i\infty} N_0(t)\frac{(t-1)^{\frac{1}{2}}}{t-w}dt = \sum_{r=0}^{n}(w-1)^r\int_1^{1-i\infty}\frac{N_r(t)}{(t-1)^{\frac{1}{2}}}dt$$
$$-\pi(1-w)^{\frac{1}{2}}\sum_{r=0}^{n}(w-1)^r N_r(1)+(w-1)^{n+1}\int_1^{1-i\infty}N_{n+1}(t)\frac{(t-1)^{\frac{1}{2}}}{t-w}dt, \tag{55}$$

where
$$N_{r+1}(t) = \frac{N_r(t)-N_r(1)}{t-1}.$$

From this last relationship
$$N_r(t) = N_r(1)+(t-1)N_{r+1}(t)$$

and so
$$N_0(t) = N_0(1)+(t-1)N_1(t)$$
$$= \sum_{r=0}^{n}(t-1)^r N_r(1)+(t-1)^{n+1}N_{n+1}(t). \tag{56}$$

Use of this result, together with (55) and (54) shows that

$$f_{n,k}(w) = \frac{e^{i\alpha w}}{\pi w^{\frac{1}{2}}(1-w)^{\frac{1}{2}}}\left\{\sum_{r=0}^{n}(w-1)^r\int_1^{1-i\infty}\frac{N_r(t)}{(t-1)^{\frac{1}{2}}}dt+O(w-1)^{n+1}\right\}. \tag{57}$$

Now we may write

$$\int_1^{1-i\infty}\frac{N_r(t)}{(t-1)^{\frac{1}{2}}}dt = \int_1^{1-i\infty}\frac{(t-1)^r N_r(t)}{(t-1)^{r+\frac{1}{2}}}dt$$
$$= \frac{\pi^{\frac{1}{2}}}{(r-\frac{1}{2})!}\int_1^{1-i\infty}\frac{\{(t-1)^r N_r(t)\}^{(r)}}{(t-1)^{\frac{1}{2}}}dt$$

after repeated integration by parts. Differentiating (56) r times with $n = r-1$ gives

$$\{(t-1)^r N_r(t)\}^{(r)} = N_0^{(r)}(t)$$

and so
$$\int_1^{1-i\infty}\frac{N_r(t)}{(t-1)^{\frac{1}{2}}}dt = \frac{\pi^{\frac{1}{2}}}{(r-\frac{1}{2})!}\int_1^{1-i\infty}\frac{N_0^{(r)}(t)}{(t-1)^{\frac{1}{2}}}dt.$$

Hence

$$f_{n,k}(w) = \frac{e^{i\alpha w}}{\pi w^{\frac{1}{2}}(1-w)^{\frac{1}{2}}}\left\{\sum_{r=0}^{n}(w-1)^r\frac{\pi^{\frac{1}{2}}}{(r-\frac{1}{2})!}\int_1^{1-i\infty}\frac{N_0^{(r)}(t)}{(t-1)^{\frac{1}{2}}}dt+O(w-1)^{n+1}\right\}. \tag{58}$$

Finally, expanding the remaining terms,

$$f_{n,k}(w) \sim \frac{e^{i\alpha}}{\pi}\sum_{s=0}^{\infty}(1-w)^{s-\frac{1}{2}}\sum_{r=0}^{s}\int_1^{1-i\infty}\left\{\frac{1}{t^{\frac{1}{2}}}\Psi_{n,k}(t)e^{-i\alpha t}\right\}^{(r)}\frac{dt}{(t-1)^{\frac{1}{2}}}$$
$$\times\frac{(-1)^r\pi^{\frac{1}{2}}}{(r-\frac{1}{2})!}\sum_{p=0}^{s-r}\frac{(-i\alpha)^p}{p!}\frac{(s-p-r-\frac{1}{2})!}{\pi^{\frac{1}{2}}(s-p-r)!}, \tag{59}$$

and
$$f_n(w) \sim \sum_{k=0}^{\infty} f_{n,k}(w). \tag{60}$$

The edge condition is that $f_n(w)$ must be $O(1-w)^{n-\frac{1}{2}}$, and so all terms for which $s < n$ must vanish.

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For $n = 0$ none of the terms need vanish and so there is no equation. This is expected since for $n = 0$ there are no unknown constants.

For $n = 1$ the term $s = 0$ must vanish and so

$$\sum_{k=0}^{\infty} \int_1^{1-i\infty} \left\{ \frac{1}{t^{\frac{1}{2}}} \Psi_{1,k}(t)\, e^{-i\alpha t} \right\} \frac{dt}{(t-1)^{\frac{1}{2}}} = 0.$$

For $n = 2$ the terms $s = 0, 1$ must vanish and in general the edge condition for the nth Fourier component give rise to n equations

$$\sum_{k=0}^{\infty} \int_1^{1-i\infty} \left\{ \frac{1}{t^{\frac{1}{2}}} \Psi_{n,k}(t) \right\}^{(r)} \frac{e^{-i\alpha t}}{(t-1)^{\frac{1}{2}}} dt = 0 \quad [r = 0, 1, \ldots, (n-1)]. \tag{61}$$

7. *The arbitrary contents.* From (45) and (47)

$$\Psi_{n,0}(w) = \frac{2i\alpha \cos\beta}{(i\alpha \sin\beta)^n} \left\{ w J_0(\alpha w \sin\beta) + \sum_{m=0}^{n-1} A_{n,m} w^{2m+1} \right\}. \tag{62}$$

It is convenient to write this as

$$\Psi_{n,0}(w) = \Psi_{0,0}(w) + \sum_{m=0}^{n-1} A_{n,m} \Phi_{m,0}(w) \tag{63}$$

where

$$\Phi_{m,0}(w) = \frac{2i\alpha \cos\beta}{(i\alpha \sin\beta)^n} w^{2m+1}. \tag{64}$$

In this way the constants can be separated out and the iteration scheme becomes

$$(\psi_{0,k+1}(v), \phi_{m,k+1}(v)) = \int_1^{1-i\infty} \left(\frac{w-1}{w} \right)^{\frac{1}{2}} \frac{e^{-i\alpha w}}{w+v} (\Psi_{0,k}(w),\ \Phi_{m,k}(w))\, dw, \tag{65}$$

$$(\Psi_{0,k+1}(w), \Phi_{m,k+1}(w)) = \frac{i\alpha}{4\pi} \int_0^{\infty} \left(\frac{v}{1+v} \right)^{\frac{1}{2}} (\psi_{0,k+1}(v), \phi_{m,k+1}(v))\, M(v,w)\, e^{-i\alpha v} dv. \tag{66}$$

The iteration scheme and the edge conditions together show that the constants may be determined from n linear equations

$$\sum_{k=0}^{\infty} \int_1^{1-i\infty} \left\{ \frac{1}{t^{\frac{1}{2}}} \Psi_{0,k}(t) \right\}^{(r)} \frac{e^{-i\alpha t}}{(t-1)^{\frac{1}{2}}} dt + \sum_{m=0}^{n-1} A_{n,m} \sum_{k=0}^{\infty} \int_1^{1-i\infty} \left\{ \frac{1}{t^{\frac{1}{2}}} \Phi_{m,k}(t) \right\}^{(r)} \frac{e^{-i\alpha t}}{(t-1)^{\frac{1}{2}}} dt = 0. \tag{67}$$

As n increases the task of finding $A_{n,m}$ for each m becomes more difficult. For very large n the task is considerable unless, for instance, the coefficients decrease with n or m, so that $A_{n,m}$ decreases. To determine whether this is so an estimate of the magnitude of $A_{n,m}$ is found.

For the nth Fourier component the n linear equations may be written

$$A\mathbf{x} = \mathbf{c}$$

where

$$A = \{a_{ij}\}, \quad a_{ij} = \sum_{k=0}^{\infty} \int_1^{1-i\infty} \left\{ \frac{1}{t^{\frac{1}{2}}} \Phi_{j-1,k}(t) \right\}^{(i-1)} \frac{e^{-i\alpha t}}{(t-1)^{\frac{1}{2}}} dt,$$

$$\mathbf{x} = \begin{pmatrix} A_{n,0} \\ A_{n,1} \\ \vdots \\ A_{n,n-1} \end{pmatrix}, \quad \mathbf{c} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$$

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and
$$c_i = -\sum_{k=0}^{\infty} \int_1^{1-i\infty} \left\{ \frac{1}{t^{\frac{1}{2}}} \Psi_{0,k}(t) \right\}^{(i-1)} \frac{e^{-i\alpha t}}{(t-1)^{\frac{1}{2}}} dt.$$

The leading term in each case is given by the zero iterate $k = 0$. An asymptotic evaluation of the integrals for this iterate shows that all a_{ij} are $O(\alpha^{\frac{1}{2}-n})$ and the c_i are $O(\alpha^{i-n-1})$. To a first approximation $c_i = 0$, $i \neq n$, and $c_n = O(\alpha^{-1})$. Then, if it is assumed that A has an inverse,
$$\mathbf{x} = A^{-1}\mathbf{c} = \frac{\operatorname{adj} A}{|A|}\mathbf{c} = O(\alpha^{n-\frac{3}{2}}).$$

This does not depend upon i or j and indicates that the constants increase in magnitude as n increases, unless some internal cancellation takes place.

As a check a few of the constants have been calculated.

$$A_{1,0} \sim \frac{i}{(2\pi \sin\beta)^{\frac{1}{2}}} \left(\frac{i}{\alpha}\right)^{\frac{1}{2}} \left\{ \frac{e^{i\alpha \sin\beta}}{(1-\sin\beta)^{\frac{1}{2}}} + i \frac{e^{-i\alpha \sin\beta}}{(1+\sin\beta)^{\frac{1}{2}}} \right\} + O(\alpha^{-\frac{3}{2}}),$$

$$A_{2,0} \sim \frac{1}{2} \left(\frac{\alpha \sin\beta}{2\pi}\right)^{\frac{1}{2}} e^{i\frac{1}{4}\pi} \left\{ \frac{e^{i\alpha \sin\beta}}{(1-\sin\beta)^{\frac{1}{2}}} - i \frac{e^{-i\alpha \sin\beta}}{(1+\sin\beta)^{\frac{1}{2}}} \right\} + O(\alpha^{-\frac{1}{2}}),$$

$$A_{2,1} \sim -\frac{1}{2} \left(\frac{\alpha \sin\beta}{2\pi}\right)^{\frac{1}{2}} e^{i\frac{1}{4}\pi} \left\{ \frac{e^{i\alpha \sin\beta}}{(1-\sin\beta)^{\frac{1}{2}}} - i \frac{e^{-i\alpha \sin\beta}}{(1+\sin\beta)^{\frac{1}{2}}} \right\} + O(\alpha^{-\frac{1}{2}}).$$

These are in agreement with the estimate.

The increasing magnitude of the $A_{n,m}$ as n increases makes solving the n equations impractical.

Having now obtained an estimate of the magnitude of the constants consider the fundamental function, $\gamma_n(w)$, given by
$$\gamma_n(w) = \frac{2\pi\alpha \cos\beta}{(i\alpha \sin\beta)^n} \left\{ wJ_0(\alpha w \sin\beta) + \sum_{m=0}^{n-1} A_{n,m} w^{2m+1} \right\}.$$

The increasing nature of $A_{n,m}$ means that $\gamma_n(w)$ appears to be dominated by the terms involving the arbitrary constants, rather than by the Bessel function. Intuitively this is unlikely since the oblique case may be reduced to the normal case by considering $n = 0$ and letting $\beta \to 0$. If this is done there are no constants and it is the Bessel function which gives $\gamma(w)$, the fundamental function for the normally incident wave.

If $\gamma_n(w)$ is not to be dominated by the constants, then the constants must be self cancelling or become eliminated in some other way. There are features in the problem which make this hypothesis worth further consideration. There are no constants in the integral equation (8) for the Fourier coefficients, c_n, of the field on the disc. A new unknown, f_n, is introduced, which is essentially the nth integral of c_n (cf. (10)). The resulting equation for f_n is integrated n times (Jones (2)) and the n constants $b_{n,m}$ appear. Further manipulation yields the Jones integral equation (9) for f_n. This is the equation which is solved. However, it is c_n, and not f_n, which gives the field on the disc. It may appear therefore that the constants are somewhat artificially introduced to facilitate the solution of the integral equation and perhaps differentiate out when c_n

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is found by differentiating f_n. To see that this may be worth investigating write

$$c_n(w) \sim \sum_{k=0}^{\infty} c_{n,k}(w) \tag{68}$$

then
$$c_{n,k}(w) = (-1)^n w^n \left(\frac{d}{w\,dw}\right)^n f_{n,k}(w) \tag{69}$$

so that
$$c_{n,k}(w) = (-1)^n w^n \left(\frac{d}{w\,dw}\right)^n \left\{\frac{1}{w}\Psi_{n,k}(w)\right\} + (-1)^n w^n \left(\frac{d}{w\,dw}\right)^n \left\{\frac{\psi_{n,k+1}(-w)\,e^{i\alpha w}}{\pi w^{\frac{1}{2}}(1-w)^{\frac{1}{2}}}\right\}. \tag{70}$$

Now,
$$\frac{1}{w}\Psi_{n,0}(w) = \frac{2i\alpha\cos\beta}{(i\alpha\sin\beta)^n}\left\{J_0(\alpha w\sin\beta) + \sum_{m=0}^{n-1} A_{n,m} w^{2m}\right\}$$

and so
$$(-1)^n w^n \left(\frac{d}{w\,dw}\right)^n \left\{\frac{1}{w}\Psi_{n,0}(w)\right\} = 2i\alpha\cos\beta\, i^{-n} J_n(\alpha w\sin\beta).$$

In this particular term the constants $A_{n,m}$ have been differentiated out. While by no means conclusive this indicates that further consideration may be worthwhile.

8. *The Fourier coefficients.* The coefficients c_n are given by (68) and (70). It has been shown that for the iterate $k = 0$ the first term of $c_{n,k}$ is independent of the constants. The second term requires a more detailed analysis. Now

$$\left(\frac{d}{w\,dw}\right)\left\{\frac{e^{i\alpha w}}{\pi w^{\frac{1}{2}}(1-w)^{\frac{1}{2}}}\int_1^{1-i\infty}\left(\frac{t-1}{t}\right)^{\frac{1}{2}}\frac{e^{-i\alpha t}}{t-w}\Psi_{n,k}(t)\,dt\right\}$$
$$= \frac{e^{i\alpha w}}{\pi w^{\frac{3}{2}}(1-w)^{\frac{1}{2}}}\left\{\int_1^{1-i\infty}\frac{t^{\frac{3}{2}}(t-1)^{\frac{1}{2}}}{t-w}\left(\frac{d}{t\,dt}\right)\left\{\frac{1}{t}\Psi_{n,k}(t)\right\}e^{-i\alpha t}\,dt\right.$$
$$-\frac{1}{2w}\int_1^{1-i\infty}\left(\frac{t-1}{t}\right)^{\frac{1}{2}}\left\{\frac{1}{t}\Psi_{n,k}(t)\right\}e^{-i\alpha t}\,dt$$
$$\left.+\frac{1}{2(1-w)}\int_1^{1-i\infty}\left(\frac{t}{t-1}\right)^{\frac{1}{2}}\left\{\frac{1}{t}\Psi_{n,k}(t)\right\}e^{-i\alpha t}\,dt\right\}$$

after some manipulation. In view of the form of this relationship consider an integral

$$\mathfrak{I}_r(w) = \frac{e^{i\alpha w}}{\pi w^{r+\frac{1}{2}}(1-w)^{\frac{1}{2}}}\int_1^{1-i\infty}\frac{t^{r+\frac{1}{2}}(t-1)^{\frac{1}{2}}}{t-w}\left(\frac{d}{t\,dt}\right)^r\left\{\frac{1}{t}\Psi_{n,k}(t)\right\}e^{-i\alpha t}\,dt. \tag{71}$$

Differentiation then shows that

$$\left(\frac{d}{w\,dw}\right)\mathfrak{I}_r(w) = \mathfrak{I}_{r+1}(w) + \mathfrak{R}_r(w) \tag{72}$$

where

$$\mathfrak{R}_r(w) = \frac{e^{i\alpha w}}{\pi w^{r+\frac{3}{2}}(1-w)^{\frac{1}{2}}}\left\{-\frac{(r+\frac{1}{2})}{w}\int_1^{1-i\infty}\left(\frac{t-1}{t}\right)^{\frac{1}{2}}t^r e^{-i\alpha t}\left(\frac{d}{t\,dt}\right)^r\left\{\frac{1}{t}\Psi_{n,k}(t)\right\}dt\right.$$
$$\left.+\frac{1}{2(1-w)}\int_1^{1-i\infty}\left(\frac{t}{t-1}\right)^{\frac{1}{2}}t^r e^{-i\alpha t}\left(\frac{d}{t\,dt}\right)^r\left\{\frac{1}{t}\Psi_{n,k}(t)\right\}dt\right\}. \tag{73}$$

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It is noted that the second term of $c_{n,k}$ can be written

$$(-1)^n w^n \left(\frac{d}{w\,dw}\right)^n \mathfrak{F}_0(w)$$

and use of (72) shows that

$$\left(\frac{d}{w\,dw}\right)^n \mathfrak{F}_0(w) = \mathfrak{F}_n(w) + \sum_{r=0}^{n-1} \left(\frac{d}{w\,dw}\right)^r \mathfrak{R}_{n-r-1}(w). \tag{74}$$

When $k = 0$ the arbitrary constants in $\mathfrak{F}_n(w)$ are differentiated out and so if the series on the right-hand side is of lower order than the other terms a first approximation to $c_n(w)$ may be obtained which will not contain any of the arbitrary constants. A more detailed consideration of the series is therefore necessary. $\mathfrak{R}_r(w)$ is given by (73) and so each term may be evaluated. However, this would appear to lead to considerable labour and in order to avoid this rewrite (74) as

$$\sum_{r=0}^{n-1} \left(\frac{d}{w\,dw}\right)^r \mathfrak{R}_{n-r-1}(w) = \left(\frac{d}{w\,dw}\right)^n \mathfrak{F}_0(w) - \mathfrak{F}_n(w) \tag{75}$$

and consider each term on the right-hand side. Using (55) and (56)

$$\mathfrak{F}_0(w) \sim \frac{e^{i\alpha w}}{\pi w^{\frac{1}{2}}(1-w)^{\frac{1}{2}}} \left\{ \sum_{r=0}^{\infty} (w-1)^r \frac{\pi^{\frac{1}{2}}}{(r-\frac{1}{2})!} \int_1^{1-i\infty} \left\{ \frac{1}{t^{\frac{1}{2}}} \Psi_{n,k}(t)\, e^{-i\alpha t} \right\}^{(r)} \frac{dt}{(t-1)^{\frac{1}{2}}} \right.$$
$$\left. - \pi(1-w)^{\frac{1}{2}} \frac{1}{w^{\frac{1}{2}}} \Psi_{n,k}(w)\, e^{-i\alpha w} \right\}. \tag{76}$$

Similarly

$$\mathfrak{F}_n(w) \sim \frac{e^{i\alpha w}}{\pi w^{n+\frac{1}{2}}(1-w)^{\frac{1}{2}}} \left\{ \sum_{r=0}^{\infty} (w-1)^r \frac{\pi^{\frac{1}{2}}}{(r-\frac{1}{2})!} \int_1^{1-i\infty} \left\{ t^{n+\frac{1}{2}} e^{-i\alpha t} \left(\frac{d}{t\,dt}\right)^n \left(\frac{1}{t} \Psi_{n,k}(t)\right) \right\}^{(r)} \right.$$
$$\left. \times \frac{dt}{(t-1)^{\frac{1}{2}}} - \pi(1-w)^{\frac{1}{2}} w^{n+\frac{1}{2}} e^{-i\alpha w} \left(\frac{d}{w\,dw}\right)^n \left(\frac{1}{w} \Psi_{n,k}(w)\right) \right\}. \tag{77}$$

Hence

$$\sum_{r=0}^{n-1} \left(\frac{d}{w\,dw}\right)^n \mathfrak{R}_{n-r-1}(w) = \left(\frac{d}{w\,dw}\right)^n \mathfrak{F}_0(w) - \mathfrak{F}_n(w)$$
$$\sim \frac{i}{\pi} \sum_{r=0}^{\infty} \frac{\pi^{\frac{1}{2}}}{(r-\frac{1}{2})!} \left[\left(\frac{d}{w\,dw}\right)^n \left(\frac{e^{i\alpha w}}{w^{\frac{1}{2}}} (w-1)^{r-\frac{1}{2}} \right) \right.$$
$$\times \int_1^{1-i\infty} \left\{ \frac{1}{t^{\frac{1}{2}}} \Psi_{n,k}(t)\, e^{-i\alpha t} \right\}^{(r)} \frac{dt}{(t-1)^{\frac{1}{2}}} - \frac{e^{i\alpha w}}{w^{n+\frac{1}{2}}} (w-1)^{r-\frac{1}{2}}$$
$$\left. \times \int_1^{1-i\infty} \left\{ t^{n+\frac{1}{2}} e^{-i\alpha t} \left(\frac{d}{t\,dt}\right)^n \left(\frac{1}{t} \Psi_{n,k}(t)\right) \right\}^{(r)} \frac{dt}{(t-1)^{\frac{1}{2}}} \right]. \tag{78}$$

Consider the first series of terms given by

$$\frac{i}{\pi} \sum_{r=0}^{\infty} \frac{\pi^{\frac{1}{2}}}{(r-\frac{1}{2})!} \left(\frac{d}{w\,dw}\right)^n \left(w^{-\frac{1}{2}} (w-1)^{r-\frac{1}{2}} e^{i\alpha w} \right) \int_1^{1-i\infty} \left\{ \frac{1}{t^{\frac{1}{2}}} \Psi_{n,k}(t)\, e^{-i\alpha t} \right\}^{(r)} \frac{dt}{(t-1)^{\frac{1}{2}}}.$$

In the integral let $\qquad \dfrac{1}{t^{\frac{1}{2}}} \Psi_{n,k}(t) = \xi_{n,k}(t)$

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so that

$$\int_1^{1-i\infty}\left\{\frac{1}{t^{\frac{1}{2}}}\Psi_{n,k}(t)\,e^{-i\alpha t}\right\}^{(r)}\frac{dt}{(t-1)^{\frac{1}{2}}}=\sum_{s=n}^{r}\frac{r\,!}{s\,!(r-s)\,!}(-i\alpha)^{r-s}\int_1^{1-i\infty}\xi^{(s)}_{n,k}(t)\frac{e^{-i\alpha t}}{(t-1)^{\frac{1}{2}}}dt$$

after applying the edge conditions. Evaluation of the integral gives

$$\int_1^{1-i\infty}\xi^{(s)}_{n,k}(t)\frac{e^{-i\alpha t}}{(t-1)^{\frac{1}{2}}}dt\sim\alpha^{-\frac{1}{2}}e^{-i(\alpha+\frac{1}{4}\pi)}\sum_{j=0}^{\infty}\left(\frac{-i}{\alpha}\right)^{j}\xi^{(s+j)}_{n,k}(1)\frac{(j-\frac{1}{2})\,!}{j\,!}.$$

Now, we may write

$$\left(\frac{d}{w\,dw}\right)^{n}=\sum_{m=0}^{n-1}\epsilon_m\frac{1}{w^{n+m}}\left(\frac{d}{dw}\right)^{n-m}$$

where ϵ_m are numerical coefficients. It is then a simple matter to carry out the differentiation. Finally, collecting the various terms together,

$$\begin{aligned}
&\frac{i}{\pi}\sum_{r=0}^{\infty}\frac{\pi^{\frac{1}{2}}}{(r-\frac{1}{2})\,!}\left(\frac{d}{w\,dw}\right)^{n}(w^{-\frac{1}{2}}(w-1)^{r-\frac{1}{2}}e^{i\alpha w})\int_1^{1-i\infty}\left\{\frac{1}{t^{\frac{1}{2}}}\Psi_{n,k}(t)\,e^{-i\alpha t}\right\}^{(r)}\frac{dt}{(t-1)^{\frac{1}{2}}}\\
&\quad\sim\frac{i}{\pi}\frac{e^{i\alpha w}}{w^{n+\frac{1}{2}}(w-1)^{\frac{1}{2}}}\alpha^{-\frac{1}{2}}e^{-i(\alpha+\frac{1}{4}\pi)}\sum_{r=0}^{\infty}\frac{\pi^{\frac{1}{2}}}{(r-\frac{1}{2})\,!}\{\alpha(w-1)\}^{r}\sum_{m=0}^{n-1}\epsilon_m\\
&\qquad\times\sum_{p=0}^{n-m}\frac{(n-m)\,!}{p\,!(n-m-p)\,!}\frac{(-\frac{1}{2})\,!}{(-\frac{1}{2}-n+m+p)\,!}\sum_{l=0}^{\infty}\frac{(-n+p)\,!}{l\,!(-n+p-l)\,!}\alpha^{p-l}\\
&\qquad\times\sum_{j=0}^{\infty}\left(-\frac{i}{\alpha}\right)^{n+j}\xi^{(n+j)}_{n,k}(1)\{\alpha(w-1)\}^{l}(-1)^{n-p}\,i^{p-r}\\
&\qquad\times\sum_{s=0}^{r+p-n}\frac{r\,!(-1)^{s}}{(n+s-p)\,!(r+p-n-s)\,!}\frac{(j-s-\frac{1}{2})\,!}{(j-s)\,!}
\end{aligned}\tag{79}$$

The second group of terms in (78) is

$$\frac{i}{\pi}\sum_{r=0}^{\infty}\frac{\pi^{\frac{1}{2}}}{(r-\frac{1}{2})\,!}\frac{(w-1)^{r-\frac{1}{2}}}{w^{n+\frac{1}{2}}}e^{i\alpha w}\int_1^{1-i\infty}\left\{t^{n+\frac{1}{2}}\,e^{-i\alpha t}\left(\frac{d}{t\,dt}\right)^{n}\left(\frac{1}{t}\Psi_{n,k}(t)\right)\right\}^{(r)}\frac{dt}{(t-1)^{\frac{1}{2}}}.$$

Using the earlier notation we may write

$$\begin{aligned}
\left(\frac{d}{t\,dt}\right)^{n}\left(\frac{1}{t}\Psi_{n,k}(t)\right)&=\left(\frac{d}{t\,dt}\right)^{n}\left(\frac{1}{t^{\frac{1}{2}}}\xi_{n,k}(t)\right)\\
&=\sum_{m=0}^{n-1}\epsilon_m\sum_{p=0}^{n-m}\frac{(n-m)\,!}{p\,!(n-m-p)\,!}\frac{(-\frac{1}{2})\,!}{(-\frac{1}{2}-n+m+p)\,!}t^{-\frac{1}{2}-2n+p}\,\xi^{(p)}_{n,k}(t)
\end{aligned}$$

and the integral becomes

$$\begin{aligned}
&\int_1^{1-i\infty}\{t^{-n+p}\,\xi^{(p)}_{n,k}(t)\,e^{-i\alpha t}\}^{(r)}\frac{dt}{(t-1)^{\frac{1}{2}}}\\
&\quad\sim\alpha^{-\frac{1}{2}}e^{-i(\alpha+\frac{1}{4}\pi)}\sum_{l=0}^{\infty}\frac{(-n+p)\,!}{l\,!(-n+p-l)\,!}\sum_{s=0}^{r}\frac{r\,!}{s\,!(r-s)\,!}(-i)^{r}\\
&\qquad\times\sum_{j=0}^{\infty}\frac{1}{j\,!}\xi^{(p+j)}_{n,k}(1)\,(-i)^{j+l}\frac{(j+l)\,!}{(j+l-s)\,!}\alpha^{r-j-l}(-\tfrac{1}{2}+j+l-s)\,!.
\end{aligned}$$

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The edge conditions are now put into a form applicable to the present case. The edge conditions are given by (61) and asymptotic evaluation of the integral shows that

$$\int_1^{1-i\infty}\left\{\frac{1}{t^{\frac{1}{2}}}\Psi_{n,k}(t)\right\}^{(r)}\frac{e^{-i\alpha t}}{(t-1)^{\frac{1}{2}}}dt = \int_1^{1-i\infty}\xi^{(r)}_{n,k}(t)\frac{e^{-i\alpha t}}{(t-1)^{\frac{1}{2}}}dt$$

$$\sim e^{-i(\alpha+\frac{1}{4}\pi)}\sum_{m=0}^{\infty}\frac{1}{m!}\xi^{(r+m)}_{n,k}(1)\,(-i)^m\frac{(m-\frac{1}{2})!}{\alpha^{m+\frac{1}{2}}}$$

$$= 0 \quad [r = 0, 1, \ldots, (n-1)].$$

These are n equations from which an expression for $\xi^{(p)}_{n,k}(1)$, $p < n$, may be obtained in terms of $\xi^{(n+m)}_{n,k}(1)$, $m \geqslant 0$. Hence

$$\xi^{(p)}_{n,k}(1) \sim -\frac{(i\alpha)^p}{\pi^{\frac{1}{2}}}\frac{(n-p-\frac{3}{2})!}{(n-p-1)!\,\pi^{\frac{1}{2}}}\sum_{m=0}^{\infty}\left(-\frac{i}{\alpha}\right)^{n+m}\xi^{(n+m)}_{n,k}(1)\frac{(m+\frac{1}{2})!}{m!}\frac{1}{(m+n-p)}$$

$$[p = 0, 1, \ldots, (n-1)].$$

Combining these results shows that the second group of terms can be written

$$\frac{i}{\pi}\sum_{r=0}^{\infty}\frac{\pi^{\frac{1}{2}}}{(r-\frac{1}{2})!}\frac{(w-1)^{r-\frac{1}{2}}}{w^{n+\frac{1}{2}}}e^{i\alpha w}\int_1^{1-i\infty}\left\{t^{n+\frac{1}{2}}e^{-i\alpha t}\left(\frac{d}{t\,dt}\right)^n\left(\frac{1}{t}\Psi_{n,k}(t)\right)\right\}^{(r)}\frac{dt}{(t-1)^{\frac{1}{2}}}$$

$$\sim \frac{i}{\pi}\frac{e^{i\alpha w}}{w^{n+\frac{1}{2}}(w-1)^{\frac{1}{2}}}\alpha^{-\frac{1}{2}}e^{-i(\alpha+\frac{1}{4}\pi)}\sum_{r=0}^{\infty}\frac{\pi^{\frac{1}{2}}}{(r-\frac{1}{2})!}\{\alpha(w-1)\}^r\sum_{m=0}^{n-1}\epsilon_m$$

$$\times\sum_{p=0}^{n-m}\frac{(n-m)!}{p!(n-m-p)!}\frac{(-\frac{1}{2})!}{(-\frac{1}{2}-n+m+p)!}\sum_{l=0}^{\infty}\frac{(-n-p)!}{l!(-n+p-l)!}$$

$$\times\sum_{j=0}^{\infty}\left(\frac{-i}{\alpha}\right)^{n+j}\xi^{(n+j)}_{n,k}(1)\,\alpha^{p-l}\sum_{s=0}^{\infty}\frac{r!(-1)^s i^{p-l-r}}{s!(r-s)!}\left\{\frac{(l+n-p+j)!}{(n-p+j)!}\frac{(l-s+n-p+j-\frac{1}{2})!}{(l-s+n-p+j)!}\right.$$

$$\left.-\frac{(j+\frac{1}{2})!}{j!}\sum_{q=0}^{n-p-1}\frac{(q+l)!}{q!}\frac{(q+l-s-\frac{1}{2})!}{(q+l-s)!\,\pi^{\frac{1}{2}}}\frac{(n-p-q-\frac{3}{2})!}{(n-p-q-1)!\,\pi^{\frac{1}{2}}}\frac{1}{(j+n-p-q)}\right\}. \tag{80}$$

Substitution of (79) and (80) in (75) produces cancellation of the first two orders of α. The remainder series

$$\sum_{r=0}^{n-1}\left(\frac{d}{w\,dw}\right)^n\mathfrak{R}_{n-r-1}(w)$$

is therefore two orders of magnitude in α lower than

$$\left(\frac{d}{w\,dw}\right)^n\mathfrak{F}_0(w) \quad \text{and} \quad \mathfrak{F}_n(w).$$

Now, from (70) and (74),

$$c_{n,k}(w) = (-1)^n w^n\left(\frac{d}{w\,dw}\right)^n\left\{\frac{1}{w}\Psi_{n,k}(w)\right\}$$

$$+(-1)^n w^n\frac{e^{i\alpha w}}{\pi w^{n+\frac{1}{2}}(1-w)^{\frac{1}{2}}}\int_1^{1-i\infty}\frac{t^{n+\frac{1}{2}}(t-1)^{\frac{1}{2}}}{t-w}\left(\frac{d}{t\,dt}\right)^n\left\{\frac{1}{t}\Psi_{n,k}(t)\right\}e^{-i\alpha t}dt$$

$$+\sum_{r=0}^{n-1}\left(\frac{d}{w\,dw}\right)^n\mathfrak{R}_{n-r-1}(w) \tag{81}$$

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where the summation is two orders of magnitude in α lower than the integral. The leading terms of $c_n(w)$ are given by the iterate $k = 0$ and so, neglecting the remainder series,

$$c_n(w) \sim c_{n,0}(w) \sim (-i)^n 2i\alpha \cos\beta \times \left\{ J_n(\alpha w \sin\beta) + \frac{e^{i\alpha w}}{\pi w^{\frac{1}{2}}(1-w)^{\frac{1}{2}}} \int_1^{1-i\infty} \frac{t^{\frac{1}{2}}(t-1)^{\frac{1}{2}}}{t-w} J_n(\alpha t \sin\beta)\, e^{-i\alpha t}\, dt \right\} \tag{82}$$

after substitution of the expression $\Psi_{n,0}(w) = (i/\pi)\gamma_n(w)$. In this form the arbitrary constants have been eliminated.

9. *The far scattered field.* The scattered field $u_s(\mathbf{R})$ is given by (2). If the field point $\mathbf{R}$ is assumed far from the disc and spherical polar co-ordinates are employed then

$$u_s(\mathbf{R}) = -\frac{1}{4\pi} \sum_{n=-\infty}^{\infty} \int_0^1 w c_n(w) \int_0^{2\pi} \frac{e^{-i\alpha|\mathbf{R}-\mathbf{R}_1|}}{|\mathbf{R}-\mathbf{R}_1|} e^{in\phi_1}\, d\phi_1\, dw$$
$$\sim \sum_{n=-\infty}^{\infty} \left\{ -\frac{1}{2}\frac{e^{-i\alpha R}}{R} i^n \int_0^1 w c_n(w) J_n(\alpha w \sin\theta)\, dw \right\} e^{in\phi} + O\left(\frac{1}{R^2}\right). \tag{83}$$

The integral is the significant part and following a notation similar to that used at normal incidence write

$$F_n(\theta) \sim \sum_{k=0}^{\infty} F_{n,k}(\theta) \tag{84}$$

where
$$F_{n,k}(\theta) = i^n \int_0^1 w c_{n,k}(w) J_n(\alpha w \sin\theta)\, dw$$
$$= (-i)^n \int_0^1 w^{n+1} \left(\frac{d}{w\,dw}\right)^n \left\{ \frac{1}{w} \Psi_{n,k}(w) \right\} J_n(\alpha w \sin\theta)\, dw$$
$$+ (-i)^n \int_0^1 w^{n+1} \left(\frac{d}{w\,dw}\right)^n \left\{ \frac{\psi_{n,k+1}(-w)\, e^{i\alpha w}}{\pi w^{\frac{1}{2}}(1-w)^{\frac{1}{2}}} \right\} J_n(\alpha w \sin\theta)\, dw. \tag{85}$$

The first integral may be split to give

$$\int_0^1 w^{n+1} J_n(\alpha w \sin\theta) \left(\frac{d}{w\,dw}\right)^n \left(\frac{1}{w}\Psi_{n,k}(w)\right) dw$$
$$= \int_0^{\infty} w^{n+1} J_n(\alpha w \sin\theta) \left(\frac{d}{w\,dw}\right)^n \left(\frac{1}{w}\Psi_{n,k}(w)\right) dw$$
$$- \int_1^{\infty} w^{n+1} J_n(\alpha w \sin\theta) \left(\frac{d}{w\,dw}\right)^n \left(\frac{1}{w}\Psi_{n,k}(w)\right) dw. \tag{86}$$

Similarly, write the second integral as

$$\int_0^1 w^{n+1} J_n(\alpha w \sin\theta) \left(\frac{d}{w\,dw}\right)^n \left(\frac{\psi_{n,k+1}(-w)\, e^{i\alpha w}}{\pi w^{\frac{1}{2}}(1-w)^{\frac{1}{2}}}\right) dw$$
$$= \frac{1}{\pi} \int_0^{i\infty} w^{n+1} J_n(\alpha w \sin\theta) \left(\frac{d}{w\,dw}\right)^n \left(\frac{\psi_{n,k+1}(-w)\, e^{i\alpha w}}{w^{\frac{1}{2}}(1-w)^{\frac{1}{2}}}\right) dw$$
$$- \frac{1}{\pi} \int_1^{1+i\infty} w^{n+1} J_n(\alpha w \sin\theta) \left(\frac{d}{w\,dw}\right)^n \left(\frac{\psi_{n,k+1}(-w)\, e^{i\alpha w}}{w^{\frac{1}{2}}(1-w)^{\frac{1}{2}}}\right) dw. \tag{87}$$

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An analysis similar to that employed at normal incidence shows

$$\begin{aligned}\int_0^\infty w^{n+1} J_n(\alpha w \sin\theta) \left(\frac{d}{w\,dw}\right)^n \left(\frac{1}{w}\Psi_{n,k}(w)\right) dw \\ = \frac{i}{\pi}\int_0^{-i\infty} v^{n+1} J_n(\alpha v \sin\theta) \left(\frac{d}{v\,dv}\right)^n \left(\frac{\psi_{n,k}(v)\, e^{-i\alpha v}}{v^{\frac{1}{2}}(1+v)^{\frac{1}{2}}}\right) dv \\ = -\frac{1}{\pi}\int_0^{i\infty} v^{n+1} J_n(\alpha v \sin\theta) \left(\frac{d}{v\,dv}\right)^n \left(\frac{\psi_{n,k}(-v)\, e^{i\alpha v}}{v^{\frac{1}{2}}(1-v)^{\frac{1}{2}}}\right) dv \end{aligned} \tag{88}$$

and so the first integrals on the right-hand sides of (86) and (87) will cancel down the iteration scheme. Hence

$$\begin{aligned} F_n(\theta) \sim (-i)^n \int_0^1 w^{n+1} J_n(\alpha w \sin\theta) \left(\frac{d}{w\,dw}\right)^n \left(\frac{1}{w}\Psi_{n,0}(w)\right) dw \\ -(-i)^n \frac{1}{\pi}\int_1^{1+i\infty} w^{n+1} J_n(\alpha w \sin\theta) \left(\frac{d}{w\,dw}\right)^n \left(\frac{\psi_{n,1}(-w)\, e^{i\alpha w}}{w^{\frac{1}{2}}(1-w)^{\frac{1}{2}}}\right) dw \\ +\sum_{k=1}^{\infty} \left\{ -(-i)^n \int_1^\infty w^{n+1} J_n(\alpha w \sin\theta) \left(\frac{d}{w\,dw}\right)^n \left(\frac{1}{w}\Psi_{n,k}(w)\right) dw \right. \\ \left. -(-i)^n \frac{1}{\pi}\int_1^{1+i\infty} w^{n+1} J_n(\alpha w \sin\theta) \left(\frac{d}{w\,dw}\right)^n \left(\frac{\psi_{n,k+1}(-w)\, e^{i\alpha w}}{w^{\frac{1}{2}}(1-w)^{\frac{1}{2}}}\right) dw \right\}, \end{aligned} \tag{89}$$

in which the contributions from the centre of the disc have vanished except for the first integral. However, this integral is

$$\begin{aligned} 2i\alpha\cos\beta \int_0^1 w J_n(\alpha w \sin\theta) J_n(\alpha w \sin\beta)\, dw \\ = \frac{1}{\alpha(\sin^2\beta - \sin^2\theta)} \{\sin\beta\, J_{n+1}(\alpha\sin\beta) J_n(\alpha\sin\theta) \\ -\sin\theta\, J_n(\alpha\sin\beta) J_{n+1}(\alpha\sin\theta)\}, \quad \sin\theta \neq \sin\beta, \\ = \tfrac{1}{2}\{J_n^2(\alpha\sin\beta) - J_{n-1}(\alpha\sin\beta) J_{n+1}(\alpha\sin\beta)\}, \quad \sin\theta = \sin\beta. \end{aligned}$$

In either case there is no contribution from the centre of the disc.

From (81) and (89) the leading terms of $F_n(\theta)$ are given by

$$\begin{aligned} F_n(\theta) \sim (-i)^n \int_0^1 w^{n+1} J_n(\alpha w \sin\theta) \left(\frac{d}{w\,dw}\right)^n \left(\frac{1}{w}\Psi_{n,0}(w)\right) dw \\ -(-i)^n \frac{1}{\pi}\int_1^{1+i\infty} w^{n+1} J_n(\alpha w \sin\theta) \frac{e^{i\alpha w}}{w^{n+\frac{1}{2}}(1-w)^{\frac{1}{2}}} \\ \times \int_1^{1-i\infty} \frac{t^{n+\frac{1}{2}}(t-1)^{\frac{1}{2}}}{t-w} \left(\frac{d}{t\,dt}\right)^n \left(\frac{1}{t}\Psi_{n,0}(t)\right) e^{-i\alpha t}\, dt \end{aligned} \tag{90}$$

and the far scattered field is

$$u_s(\mathbf{R}) \sim -\frac{1}{2}\frac{e^{-i\alpha R}}{R} F(\theta,\phi) \tag{91}$$

where

$$F(\theta,\phi) = \sum_{n=-\infty}^{\infty} F_n(\theta)\, e^{in\phi}. \tag{92}$$

https://doi.org/10.1017/S0305004100050830 Published online by Cambridge University Press

10. *The scattering coefficient.* The scattering cross-section is given by the expression (Jones (4))

$$-\frac{4\pi}{k}\operatorname{Im} A$$

where k is the wave number and A the amplitude of the scattered field in the direction of propagation of the incident field. The scattering coefficient is obtained from this expression by dividing it by the resolved area of the disc normal to the incident field. In the present normalized case $k = \alpha$ and the area of the disc is π. The resolved area is therefore $\pi \cos\beta$. From (91) the far scattered field amplitude function is

$$-\tfrac{1}{2}F(\theta, \phi)$$

so that the scattering coefficient, σ, is given by

$$\sigma \sim \frac{2}{\alpha \cos\beta}\operatorname{Im} F(\beta, 0). \tag{93}$$

From (90) and (92), after substituting for $\Psi_{n,0}(w)$ from (62),

$$\begin{aligned} F(\theta,\phi) \sim 2i\alpha\cos\beta\Bigg\{&\int_0^1 wJ_0\{\alpha w(\sin^2\theta+\sin^2\beta-2\sin\theta\sin\beta\cos\phi)^{\frac{1}{2}}\}\,dw \\ &-\frac{1}{\pi}\int_1^{1+i\infty}\left(\frac{w}{1-w}\right)^{\frac{1}{2}}e^{i\alpha w}\int_1^{1-i\infty}\frac{t^{\frac{1}{2}}(t-1)^{\frac{1}{2}}}{t-w}\,e^{-i\alpha t} \\ &\times J_0\{\alpha(w^2\sin^2\theta+t^2\sin^2\beta-2wt\sin\theta\sin\beta\cos\phi)^{\frac{1}{2}}\}\,dt\,dw\Bigg\} \end{aligned}$$

so that

$$\begin{aligned} F(\beta,0) \sim 2i\alpha\cos\beta\Bigg\{&\frac{1}{2}-\frac{1}{\pi}\int_1^{1-i\infty}t^{\frac{1}{2}}(t-1)^{\frac{1}{2}}\,e^{-i\alpha t} \\ &\times\int_1^{1+i\infty}\left(\frac{w}{1-w}\right)^{\frac{1}{2}}\frac{J_0\{\alpha\sin\beta(t-w)\}}{t-w}\,e^{i\alpha w}\,dw\,dt\Bigg\}. \end{aligned} \tag{94}$$

In view of the limits on the integrals the asymptotic form of the Bessel function cannot be employed. However, using an integral representation, we may write

$$\begin{aligned} &\int_1^{1+i\infty}\left(\frac{w}{1-w}\right)^{\frac{1}{2}}\frac{J_0\{\alpha\sin\beta(t-w)\}}{t-w}\,e^{i\alpha w}\,dw \\ &\quad=\int_1^{1+i\infty}\left(\frac{w}{1-w}\right)^{\frac{1}{2}}\frac{e^{i\alpha w}}{t-w}\frac{1}{\pi}\int_0^{\pi}e^{i\alpha\sin\beta(t-w)\cos\zeta}\,d\zeta\,dw \\ &\quad=\frac{1}{\pi}\int_0^{\pi}e^{i\alpha t\sin\beta\cos\zeta}\int_1^{1+i\infty}\left(\frac{w}{1-w}\right)^{\frac{1}{2}}\frac{e^{i\alpha w(1-\sin\beta\cos\zeta)}}{t-w}\,dw\,d\zeta \\ &\quad\sim-\frac{e^{i\frac{1}{4}\pi}}{\pi}\sum_{m=0}^{\infty}\frac{(\frac{1}{2})!}{m!(\frac{1}{2}-m)!}\,i^m(m-\tfrac{1}{2})!\,e^{i\alpha t}\int_0^{\pi}\int_{\alpha(1-\sin\beta\cos\zeta)}^{\infty}\frac{e^{-i(t-1)x}}{x^{m+\frac{1}{2}}}\,dx\,d\zeta \end{aligned} \tag{95}$$

after changing w to $1+iw$, expanding for small w and using the relationship

$$\int_0^{\infty}\frac{w^{\nu}e^{-\mu w}}{v-iw}\,dw = \nu!\,i\,e^{i\mu v}\int_{\mu}^{\infty}\frac{e^{-ivx}}{x^{\nu+1}}\,dx,$$

$$\nu > -1, \quad \mu > 0 \quad \text{and} \quad v > 0.$$

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Substitution of (95) in (94) and some further manipulation then leads to

$$F(\beta, 0) \sim i\alpha \cos\beta \left\{1 - \frac{i}{\pi\alpha}\int_0^\pi \frac{d\zeta}{(1-\sin\beta\cos\zeta)} - \frac{1}{4\pi\alpha^2}\int_0^\pi \frac{d\zeta}{(1-\sin\beta\cos\zeta)^2} + O(\alpha^{-\frac{5}{2}})\right\}$$

$$\sim i\alpha\cos\beta\left\{1 - \frac{i}{\alpha\cos\beta} - \frac{1}{4\alpha^2\cos^3\beta} + O(\alpha^{-\frac{5}{2}})\right\}. \tag{96}$$

Hence, from (93), the scattering coefficient for oblique incidence is given by

$$\sigma \sim 2\left\{1 - \frac{1}{4\alpha^2\cos^3\beta} + O(\alpha^{-\frac{5}{2}})\right\}. \tag{97}$$

The expression (90) for $F_n(\theta)$ contains just the iterate $k = 0$ and so only the first two terms of the integral in (94), giving $F(\beta, 0)$, can be included. Further terms would be of the same order of magnitude as those given by other iterates. The scattering coefficient is therefore correct as far as it goes but in order to obtain more terms other iterates would have to be considered. However, this would raise the problem of the arbitrary constants again.

The two terms found here in the scattering coefficient were predicted by Levine and Wu (5), though their method relied upon certain approximations. Such approximations have not been made in the present work.

The author would like to express his appreciation of the many helpful discussions he has had with Professor D. S. Jones while preparing the work for this paper.

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https://doi.org/10.1017/S0305004100050830 Published online by Cambridge University Press