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Convergence groups and semiconjugacy

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Abstract. We study a problem that arises from the study of Lorentz surfaces and Anosov flows. For a non-decreasing map of degree one $h : \mathbb{S}^1 \to \mathbb{S}^1$, we are interested in groups of circle diffeomorphisms that act on the complement of the graph of h in $\mathbb{S}^1 \times \mathbb{S}^1$ by preserving a volume form. We show that such groups are semiconjugate to subgroups of $\mathrm{PSL}(2, \mathbb{R})$ and that, when $h \in \mathrm{Homeo}(\mathbb{S}^1)$, we have a topological conjugacy. We also construct examples where h is not continuous, for which there is no such conjugacy.

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1. *Introduction*

1.1. *Semiconjugacy.* We say that a map $h : \mathbb{S}^1 \to \mathbb{S}^1$ is *non-decreasing of degree one* if it is non-constant and it admits a non-decreasing lift $\widetilde{h} : \mathbb{R} \to \mathbb{R}$ such that $\widetilde{h}(x + 1) = \widetilde{h}(x) + 1$ for all $x \in \mathbb{R}$. Two representations $\rho_1, \rho_2 : \Gamma \to \mathrm{Homeo}(\mathbb{S}^1)$ are *semiconjugate* if there is a non-decreasing map of degree one $h : \mathbb{S}^1 \to \mathbb{S}^1$ such that $\rho_2(\gamma) \circ h = h \circ \rho_1(\gamma)$ for all $\gamma \in \Gamma$. We will say that (ρ_1, ρ_2, h) is a *semiconjugate triple*.

It was shown by Ghys in [**Ghy87b**] that semiconjugacy is an equivalence relation: if (ρ_1, ρ_2, h) is a semiconjugate triple, then there is $h^* : \mathbb{S}^1 \to \mathbb{S}^1$ non-decreasing of degree one such that (ρ_2, ρ_1, h^*) is a semiconjugate triple.

If $h : \mathbb{S}^1 \to \mathbb{S}^1$ is non-decreasing of degree one, then we denote by $G(h) \subset \mathbb{S}^1 \times \mathbb{S}^1$ the union of the graph of h and the vertical segments joining discontinuities. If (ρ_1, ρ_2, h) is

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a semiconjugate triple, then the map $(x, y) \mapsto (\rho_1(\gamma)(x), \rho_2(\gamma)(y))$ preserves $G(h)$ for all $\gamma \in \Gamma$.

Values at discontinuities of a semiconjugacy are not particularly interesting. If we change h by the function that is left continuous (or right continuous) and equal to h except eventually at discontinuity points, then we still have a semiconjugacy. The important object in a semiconjugate triple (ρ_1, ρ_2, h) is not the map h, but its 'graph' $G(h)$.

1.1.1. *Collapsing non-wandering intervals.* When considering a representation $\rho : \Gamma \to \mathrm{Homeo}(\mathbb{S}^1)$ that preserves a Cantor set $K \subset \mathbb{S}^1$, such that orbits of points in K are dense in K (called an exceptional minimal set), it is standard to consider the collapsed action $\mathrm{Col}(\rho)$ defined by collapsing the connected components of $\mathbb{S}^1 \setminus K$ to points. More precisely, we can consider a continuous map $h : \mathbb{S}^1 \to \mathbb{S}^1$ that is non-decreasing of degree one such that the intervals where h is constant are exactly the connected components of $\mathbb{S}^1 \setminus K$. It induces a unique representation $\mathrm{Col}(\rho) : \Gamma \to \mathrm{Homeo}(\mathbb{S}^1)$ such that $(\rho, \mathrm{Col}(\rho), h)$ is a semiconjugate triple. The interest in considering this collapsed action is that the orbits of $\mathrm{Col}(\rho)$ are dense in $\mathbb{S}^1$. It is well defined up to a conjugacy in $\mathrm{Homeo}(\mathbb{S}^1)$.

If ρ has values in $\mathrm{Diff}(\mathbb{S}^1)$, it is not clear whether $\mathrm{Col}(\rho)$ can also be asked to be differentiable. However, if ρ has values in $\mathrm{PSL}(2, \mathbb{R})$ and acts projectively on $\mathbb{S}^1 = \mathbb{RP}^1$, then the collapsed action $\mathrm{Col}(\varphi)$ still has the convergence property, i.e. all sequences have north/south dynamics. Using a theorem of Gabai and Casson and Jungreis [**Gab92**, **CJ94**], this implies that $\mathrm{Col}(\rho)$ is topologically conjugate to the projective action of a subgroup of $\mathrm{PSL}(2, \mathbb{R})$ (we will give more details on convergence groups in §2). More precisely, the class of *topologically Fuchsian* representations, i.e. representations $\rho : \Gamma \to \mathrm{Homeo}(\mathbb{S}^1)$ that are topologically conjugate to the projective action of a subgroup of $\mathrm{PSL}(2, \mathbb{R})$, is stable under collapsing.

1.1.2. *Opening orbits.* It is also possible to go backwards. Given a representation $\rho : \Gamma \to \mathrm{Homeo}(\mathbb{S}^1)$ and a point $x_0 \in \mathbb{S}^1$, we can open the orbit of x to intervals on which the action of $\mathrm{Stab}(x_0)$ can be chosen. Indeed, choose an increasing map of degree one $h : \mathbb{S}^1 \to \mathbb{S}^1$ whose points of discontinuity are exactly the points of the orbit of x. For every $x \in \Gamma \cdot x_0$, let I_x be the interval between the left and right limits of h at x. Fix an action $\alpha : \mathrm{Stab}(x_0) \to \mathrm{Homeo}(I_{x_0})$. For every $x \in \Gamma \cdot x_0$, we choose δ_x such that $x = \rho(\delta_x)(x_0)$ and a homeomorphism $h_x : I_{x_0} \to I_x$. We define $\hat{\rho} : \Gamma \to \mathrm{Homeo}(\mathbb{S}^1)$ such that:

(1) $\hat{\rho}(\gamma)(h(x)) = h(\rho(\gamma)(x))$ if $x \notin \Gamma \cdot x_0$;
(2) $\hat{\rho}(\gamma)(y) = \alpha(\gamma)(y)$ if $y \in I_{x_0}$ and $\gamma \in \mathrm{Stab}(x_0)$;
(3) $\hat{\rho}(\gamma)(y) = h_{\rho(\gamma)(x)}(\alpha(\delta^{-1}_{\rho(\gamma)(x)}\gamma\delta_x)(h_x^{-1}(y)))$ for $y \in I_x$ and $x \in \Gamma \cdot x_0$.

This defines an action $\hat{\rho} : \Gamma \to \mathrm{Homeo}(\mathbb{S}^1)$ such that $(\rho, \hat{\rho}, h)$ is a semiconjugate triple.

Note that if ρ has dense orbits, then $\mathrm{Col}(\hat{\rho}) = \rho$ (because the collapsed action is unique, but there are different possibilities for opening orbits).

Once again, if ρ has values in $\mathrm{Diff}(\mathbb{S}^1)$, then it is not easy to construct $\hat{\rho}$ in a differentiable way. A well-known construction of Denjoy (see [**Den32**] for the original construction, see also [**Her79**, Ch. X] and [**Nav11**, Ch. 4]) shows that if we start with an irrational rotation, then we can open an orbit in a C^1 way. However, Denjoy also showed that it is not possible to obtain a C^2 diffeomorphism. A theorem of Matsumoto

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(see [**Mat87**]) states that if Γ_g is the fundamental group of the closed oriented surface Σ_g of genus $g \geq 2$, and $\rho : \Gamma_g \to \mathrm{PSL}(2, \mathbb{R})$ is the representation given by a hyperbolic metric on Σ_g, then it is not possible to open an orbit with a C^2 action (because the Euler number stays the same).

1.2. *Semiconjugacy and area preserving actions.* Given a semiconjugate triple (ρ_1, ρ_2, h), the underlying group Γ acts on $M_h = \mathbb{S}^1 \times \mathbb{S}^1 \backslash G(h)$ through the maps $(x, y) \mapsto (\rho_1(\gamma)(x), \rho_2(\gamma)(y))$ for $\gamma \in \Gamma$. We will say that a semiconjugate triple (ρ_1, ρ_2, h) is *area preserving* if there is a continuous volume form ω on M_h preserved by the action of Γ.

Notice that this implies that ρ_1 and ρ_2 are differentiable (they take values in $\mathrm{Diff}^{k+1}(\mathbb{S}^1)$ if ω is C^k, $k \geq 0$). If we write $\omega = \omega(x, y)\, dx \wedge dy$, then the fact that the triple (ρ_1, ρ_2, h) preserves ω is equivalent to the following formula:

$$\text{for all } \gamma \in \Gamma, \text{ for all } (x, y) \in M_h, \quad \rho_1(\gamma)'(x)\rho_2(\gamma)'(y) = \frac{\omega(x, y)}{\omega(\rho_1(\gamma)(x), \rho_2(\gamma)(y))}.$$

In [**Mon14a**], we studied the case where $h = \mathrm{Id}$ (hence $\rho_1 = \rho_2$). The main example is given by the projective action of $\mathrm{PSL}(2, \mathbb{R})$ on $\mathbb{S}^1 \approx \mathbb{RP}^1$ and the volume form $(4/(x - y)^2)\, dx \wedge dy$. We proved that if $(\rho, \rho, \mathrm{Id})$ is area preserving, then ρ is topologically Fuchsian.

This result is very close to one of Navas [**Nav06**, Proposition 1.1] on the group of circle homeomorphisms that preserve a stable geodesic current (a special type of measure on $\mathbb{S}^1 \times \mathbb{S}^1 \backslash \Delta$). However, all volume forms are not stable geodesic currents, and the converse is also false. The strategies are similar, yet the techniques differ.

We wish to extend our result to the general case of semiconjugacy. The original motivation for this problem is the study of Lorentz surfaces.

1.2.1. *Lorentzian interpretation.* If the semiconjugate triple (ρ_1, ρ_2, h) preserves a volume form $\omega(x, y)\, dx \wedge dy$ on M_h, then it also preserves the pseudo-Riemannian metric $\omega(x, y)\, dx\, dy$, which is Lorentzian, i.e. of signature $(1, 1)$. Notice that for any semiconjugate triple (ρ_1, ρ_2, h) (i.e. not necessarily area preserving), the action on M_h preserves the conformal class of the flat metric $dxdy$. However, preserving a Lorentzian conformal class in dimension two is not a rigid notion (the conformal group may be infinite dimensional). The isometry group of a Lorentz surface is always a Lie group of dimension at most three (not necessarily connected), and we will show that only subgroups of $\mathrm{PSL}(2, \mathbb{R})$ can appear for metrics in the conformal class $[dxdy]$ on M_h.

The conformal classes $(M_h, [dxdy])$ share an important property: they are globally hyperbolic (this roughly means that the space of isotropic lines is a smooth manifold). Moreover, they are spatially compact (isotropic lines form a compact manifold). The study of isometry groups of spatially compact Lorentz surfaces was started in [**Mon14b**].

The problem can be formulated in terms of h and ω: we denote by G_ω the group of orientation and time orientation preserving (i.e. preserving the set of tangent vectors (u, v) such that $u < 0$ and $v > 0$) isometries of the Lorentz metric associated to ω. If $\varphi \in G_\omega$, then there are $f, g \in \mathrm{Diff}(\mathbb{S}^1)$ such that $\varphi(x, y) = (f(x), g(y))$ for all $(x, y) \in M_h$ (because φ

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preserves the horizontal and vertical lines, which are the isotropic directions). We will set $\rho_1(\varphi) = f$ and $\rho_2(\varphi) = g$. This defines two representations $\rho_1, \rho_2 : G_\omega \to \mathrm{Diff}(\mathbb{S}^1)$ such that the semiconjugate triple (ρ_1, ρ_2, h) preserves the volume form ω.

However, no Lorentzian background will be required in this paper, as we will deal with metrics that are only continuous, and all the classical tools (curvature, geodesics, exponential map ...) are only defined when the metric is at least C^2.

1.2.2. *Results.* As mentioned earlier, if $(\rho, \rho, \mathrm{Id})$ is area preserving, then ρ is topologically conjugate to a representation in $\mathrm{PSL}(2, \mathbb{R})$. This result can be extended to the more general setting of topological conjugacy.

THEOREM 1.1. *Let (ρ_1, ρ_2, h) be a semiconjugate triple, where $h \in \mathrm{Homeo}(\mathbb{S}^1)$. If (ρ_1, ρ_2, h) is area preserving, then ρ_1 is topologically Fuchsian.*

This can be slightly reformulated in terms of actions on the circle: if $\rho_1 : \Gamma \to \mathrm{Diff}(\mathbb{S}^1)$ is such that there are ρ_2 and $h \in \mathrm{Homeo}(\mathbb{S}^1)$ forming an area preserving triple (ρ_1, ρ_2, h), then ρ_1 is topologically Fuchsian.

In the general case, we obtain a semiconjugacy.

THEOREM 1.2. *Let (ρ_1, ρ_2, h) be an area preserving triple. Then ρ_1 is semiconjugate to a representation in* $\mathrm{PSL}(2, \mathbb{R})$.

This result is almost a particular case of [**Mon14b**, Theorem 1.2], with the exception that we no longer assume that the preserved volume form is smooth.

1.3. *A new class of 'semiconvergence' groups.* A natural question is to ask if the result of Theorem 1.1 is true under the assumptions of Theorem 1.2. In §4, we will describe elements of G_ω and observe that there are examples where an element $\rho_1(\varphi)$ has three fixed points yet is not trivial; therefore, it cannot be topologically conjugate to an element of $\mathrm{PSL}(2, \mathbb{R})$. More importantly, we will see that these examples can occur in a situation where G_ω contains a free group.

THEOREM 1.3. *There are faithful representations $\rho_1, \rho_2 : \mathbb{F}_3 \to \mathrm{Diff}(\mathbb{S}^1)$ and $h : \mathbb{S}^1 \to \mathbb{S}^1$ non-decreasing of degree one such that (ρ_1, ρ_2, h) is area preserving and such that $\rho_2(\mathbb{F}_3)$ contains non-trivial elements with three fixed points.*

Such examples will be obtained by considering a representation $\rho_1 : \mathbb{F}_3 \to \mathrm{PSL}(2, \mathbb{R})$ associated to a certain hyperbolic metric on the twice-punctured torus (whose fundamental group is $\mathbb{F}_3$), and perturbing the representation in $\mathrm{Diff}(\mathbb{S}^1)$ by adding a third fixed point to one of the generators. The action ρ_2 can be seen as a different opening of an orbit of the collapsed representation $\mathrm{Col}(\rho_1)$.

1.4. *Transverse structure of Anosov flows.* Let (M, φ^t) be an Anosov flow on a compact 3-manifold. It is known that the quotient space Q^φ of lifts of orbits to the universal cover is diffeomorphic to $\mathbb{R}^2$ (see [**Bar95**]). The stable and unstable foliations determine two transversal one-dimensional foliations of Q^φ and hence a conformal Lorentz structure (whose isotropic lines are the leaves of these foliations).

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The fundamental group $\pi_1(M)$ acts naturally on Q^φ by preserving this structure. In fact, Anosov flows up to topological equivalence tend to be classified by this conformal action of the fundamental group [**Bar95**].

If φ^t preserves a volume form, then we get a Lorentz metric on Q^φ which is in general $C^{1+\text{Zygmund}}$ (and thus $C^{1+\alpha}$ for any $0 < \alpha < 1$). Higher regularity (C^2 or even $C^{1+\text{Zygmund}}$) implies not only rigidity for the Lorentz space Q^φ (the curvature is constant), but also rigidity for the flow itself (smooth conjugacy with a reparametrization of an algebraic flow, see [**HK90**] and [**Ghy87a**]). The fundamental group acts isometrically for this Lorentz metric.

Examples of such flows are abundant and exist for instance on many hyperbolic manifolds (see [**FH13**]). These examples belong to a special subcategory of Anosov flows: they are $\mathbb{R}$-covered [**Bar01**]. It means that the quotient space of Q^φ by one of the foliations defined by the stable or unstable foliation (and automatically by both) is Hausdorff and hence homeomorphic to $\mathbb{R}$. This is equivalent to saying that the Lorentz space Q^φ is globally hyperbolic.

In these examples, the Lorentz metric is defined on a subset of $\mathbb{R} \times \mathbb{R}$ and the action of $\pi_1(M)$ gives diffeomorphisms of the real line. However, in some examples, circle diffeomorphisms arise, mostly for geodesic flows on negatively curved surfaces, and for the action of the fundamental group of a Seifert piece in a graph manifold (see [**Bar96**]).

1.5. *Overview.* We will start by recalling the necessary notions of circle dynamics, namely convergence groups, minimal invariant sets and semiconjugacy. In §3, we will give the proofs of Theorems 1.2 and 1.1. Section 4 gives a description of the possible elements of G_ω and §5 is devoted to the proof of Theorem 1.3.

2. *Circle dynamics*

2.1. *Convergence groups.* Finding a conjugacy between a subgroup of $\text{Homeo}(\mathbb{S}^1)$ and a subgroup of $\text{PSL}(2, \mathbb{R})$, when it exists, is a rather complicated exercise. But there is a characterization of the existence of such a conjugacy that does not require us to find it explicitly.

Definition 2.1. A sequence $(f_n)_{n\in\mathbb{N}} \in \text{Homeo}(\mathbb{S}^1)^{\mathbb{N}}$ has the convergence property if there are $a, b \in \mathbb{S}^1$ such that, up to a subsequence, $f_n(x) \to b$ for all $x \neq a$.

A group $G \subset \text{Homeo}(\mathbb{S}^1)$ is a convergence group if every sequence in G either satisfies the convergence property or has an equicontinuous subsequence.

The classical definition of a convergence group also implies the sequence of the inverses f_n^{-1} and locally uniform convergence, but it is not necessary in the case of $\mathbb{S}^1$.

We say that a group $G \subset \text{Homeo}(\mathbb{S}^1)$ is *topologically Fuchsian* if there is $h \in \text{Homeo}(\mathbb{S}^1)$ such that $h^{-1}Gh \subset \text{PSL}(2, \mathbb{R})$. Note that we do not ask for G to be discrete. Subgroups of $\text{PSL}(2, \mathbb{R})$ all satisfy the convergence property and, since this property is invariant under topological conjugacy, so do topologically Fuchsian groups. This happens to be an equivalence.

THEOREM 2.2. (Gabai [**Gab92**], Casson and Jungreis [**CJ94**]) *A convergence group $G \subset \text{Homeo}(\mathbb{S}^1)$ is topologically Fuchsian.*

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2.2. *Minimal invariant sets.* An important object in the study of groups of circle homeomorphisms is a minimal closed invariant set. Given a group $G \subset \mathrm{Homeo}(\mathbb{S}^1)$, exactly one of the following conditions is satisfied (see [**Ghy01**] for a proof and more detail):

(1) G has a finite orbit;
(2) all orbits of G are dense;
(3) there is a compact G-invariant subset $K \subset \mathbb{S}^1$, which is infinite and different from $\mathbb{S}^1$, such that the orbits of points of K are dense in K.

In the first case, all finite orbits have the same cardinality. In the third case, the set K is unique and it is homeomorphic to a Cantor set. We will call a group $G \subset \mathrm{Homeo}(\mathbb{S}^1)$ *non-elementary* if it does not have any finite orbit, and use L_G to denote $\mathbb{S}^1$ in the second case and the G-invariant compact set K in the third case.

2.3. *Semiconjugacy.* Recall that $h : \mathbb{S}^1 \to \mathbb{S}^1$ is non-decreasing of degree one if it is non-constant and it has a lift $\widetilde{h} : \mathbb{R} \to \mathbb{R}$ such that $\widetilde{h}(x+1) = \widetilde{h}(x) + 1$ for all $x \in \mathbb{R}$. We say that $\rho_1, \rho_2 : \Gamma \to \mathrm{Homeo}(\mathbb{S}^1)$ are semiconjugate, or that (ρ_1, ρ_2, h) is a semiconjugate triple, if $\rho_2(\gamma) \circ h = h \circ \rho_1(\gamma)$ for all $\gamma \in \Gamma$.

If $a, b \in \mathbb{S}^1$, then we denote by $]a, b[$ the set $\{x \in \mathbb{S}^1 \mid a < x < b \leq a\}$ (and define $[a, b]$, $[a, b[$, $]a, b]$ in a similar way). If $h : \mathbb{S}^1 \to \mathbb{S}^1$ is non-decreasing of degree one, then we denote by h_l (respectively h_r) the map that is left (respectively right) continuous and equal to h except at points where h is not left (respectively right) continuous. We define $G(h) \subset \mathbb{S}^1 \times \mathbb{S}^1$ as the union of the segments $\{x\} \times [h_l(x), h_r(x)]$ for all $x \in \mathbb{S}^1$. Recall that the set of discontinuity points of h is at most countable, so $[h_l(x), h_r(x)] = \{h(x)\}$ except on a countable set.

Let us recall a few results of [**Ghy87b**] on semiconjugacy. First, as mentioned in the introduction, the advantage of the definition of semiconjugacy that we use (contrary to the standard definition, where h is asked to be continuous) is that semiconjugacy is an equivalence relation.

If $\Gamma = \mathbb{Z}$, then $f, g \in \mathrm{Homeo}(\mathbb{S}^1)$ are semiconjugate if and only if they have the same rotation number (the rotation number of $f \in \mathrm{Homeo}(\mathbb{S}^1)$ can actually be defined as the unique $\alpha \in \mathbb{S}^1$ such that f is semiconjugate to the rotation R_α).

For elementary groups, we have a simple characterization of semiconjugacy.

PROPOSITION 2.3. *Let $\rho : \Gamma \to \mathrm{Homeo}(\mathbb{S}^1)$ have a finite orbit $E \subset \mathbb{S}^1$ with at least two elements. A representation $\tau : \Gamma \to \mathrm{Homeo}(\mathbb{S}^1)$ is semiconjugate to ρ if and only if it has a finite orbit $F \subset \mathbb{S}^1$ such that there is a cyclic order preserving bijection from E to F which is equivariant under the actions of Γ.*

Consequently, if ρ and τ are semiconjugate, then ρ is elementary if and only if τ is elementary. If ω is a continuous volume form on M_h, and if $G \subset G_\omega$ is a subgroup, then $\rho_1(G)$ is elementary if and only if $\rho_2(G)$ is elementary.

Definition 2.4. Let ω be a continuous volume form on M_h. We say that a subgroup $G \subset G_\omega$ is elementary if $\rho_1(G)$ is elementary.

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3. *Convergence property for ρ_1 and ρ_2*

3.1. *Faithfulness of the actions.* It is a classical result in pseudo-Riemannian geometry that an isometry with a fixed point where the derivative is the identity must be the identity everywhere (provided that the manifold is connected). In other words, isometries are defined by their 1-jet at any point. This is a crucial fact, since it is the very definition of a rigid geometric structure (in the sense of Gromov). However, the standard proof uses the local existence and uniqueness of geodesics, which is not true for continuous metrics. We will see that this result remains true in our case, by using the fact that we can still define isotropic geodesics and that they are sufficient in order to get a linearization.

LEMMA 3.1. *Let ω be a continuous volume form on $M_h = \mathbb{S}^1 \times \mathbb{S}^1 \backslash G(h)$. If $\varphi \in G_\omega \backslash \{\mathrm{Id}\}$ fixes a point $(x_0, y_0) \in M_h$, then $\rho_1(\varphi)'(x_0)\rho_2(\varphi)'(y_0) = 1$ and $\rho_1(\varphi)'(x_0) \neq 1$.*

Proof. Write $f = \rho_1(\varphi)$ and $g = \rho_2(\varphi)$.

The identity $\omega(f(x), g(y))f'(x)g'(y) = \omega(x, y)$ considered at (x_0, y_0) shows that $f'(x_0)g'(y_0) = 1$. Assume that $f'(x_0) = 1$ (hence $g'(y_0) = 1$). Since $\varphi \neq \mathrm{Id}$, let us assume that $f \neq \mathrm{Id}$ (the case where $g \neq \mathrm{Id}$ is similar).

Let $x(t)$ be a maximal solution of the Cauchy problem

$$\begin{cases} x'(t) = \dfrac{1}{\omega(x(t), y_0)}, \\ x(0) = x_0. \end{cases}$$

Not only does x exist (Cauchy–Peano theorem), but it is also unique (so are solutions to all equations $y' = F(y)$ in $\mathbb{R}$, where $F > 0$). It should be seen as a parametrization of the geodesic $(\mathbb{S}^1 \times \{y_0\}) \cap M_h$ for the pseudo-Riemannian metric associated to ω. Since $x' > 0$, it is a diffeomorphism from an open interval $I \subset \mathbb{R}$ onto its image $\mathbb{S}^1 \backslash K$, where K is the set of points $x \in \mathbb{S}^1$ such that $(x, y_0) \in G(h)$. Let $\alpha = x^{-1} \circ f \circ x$. A simple calculation shows that $\alpha'(t) = 1$ for all $t \in I$. Since $\alpha(0) = 0$, we see that $\alpha = \mathrm{Id}$ and $f(x) = x$ for all x such that $(x, y_0) \in M_h$.

Let $K = [a_1, a_2]$ and let $[b_1, b_2] = [h_l(x_0), h_r(x_0)]$. If $a_1 = a_2$, then f is the identity on a dense subset of $\mathbb{S}^1$, so $f = \mathrm{Id}$. If $a_1 \neq a_2$, then $(a_1, b_1) \in M_h$ is a fixed point of φ such that $f'(a_1) = 1$, so f is the identity on all points x such that $(x, b_1) \in M_h$, which includes $[a_1, a_2]$, so $f = \mathrm{Id}$. □

The first consequence of this rigidity is the faithfulness of the actions ρ_1 and ρ_2.

COROLLARY 3.2. *Let ω be a continuous volume form on M_h. The representations $\rho_1, \rho_2 : G_\omega \to \mathrm{Diff}(\mathbb{S}^1)$ are faithful.*

Proof. Assume that $\rho_1(\varphi) = \mathrm{Id}$. Then $g = \rho_2(\varphi)$ satisfies $g \circ h = h$, which implies that g has fixed points. If $g(y) = y$, then choose $x \in \mathbb{S}^1$ such that $(x, y) \notin G(h)$. Then (x, y) is a fixed point of φ such that $\rho_1(\varphi)'(x) = 1$. Lemma 3.1 implies that $\varphi = \mathrm{Id}$. □

3.2. *Elementary groups.* We will now give a proof of Theorem 1.2 for elementary groups. We start with stabilizers of points.

Recall that the affine group $\mathrm{Aff}(\mathbb{R})$ can be realized as a subgroup of $\mathrm{PSL}(2, \mathbb{R})$ as the stabilizer of a point in $\mathbb{S}^1$.

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LEMMA 3.3. *Let h be a non-decreasing map of degree one of $\mathbb{S}^1$ and let ω be a continuous volume form on $M_h = \mathbb{S}^1 \times \mathbb{S}^1 \backslash G(h)$. Let $x_0 \in \mathbb{S}^1$ and set $G = \{\varphi \in G_\omega \mid \rho_1(\varphi)(x_0) = x_0\}$. There is a representation $\rho : G \to \mathrm{Aff}(\mathbb{R}) \subset \mathrm{PSL}(2, \mathbb{R})$ that is semiconjugate to the restriction of ρ_1 to G.*

Proof. Let us assume that G is non-trivial.

Let $b = h(x_0)$. If $\varphi \in G$, then $\rho_2(\varphi)(b) = b$, i.e. φ preserves the horizontal line $\mathbb{S}^1 \times \{b\}$. Fix $a \in \mathbb{S}^1$ such that $(a, b) \in M_h$ and let x be a maximal solution of the Cauchy problem

$$\begin{cases} x'(t) = \dfrac{1}{\omega(x(t), b)}, \\ x(0) = a. \end{cases}$$

Just as in Lemma 3.1, it is a parametrization of the horizontal geodesic passing through (a, b), and it is a diffeomorphism from an open interval $I \subset \mathbb{R}$ onto $\mathbb{S}^1 \backslash \overline{h^{-1}(\{b\})}$. If $\varphi \in G$, then a simple calculation shows that $(x^{-1} \circ \rho_1(\varphi) \circ x)'(t) = 1/(\rho_2(\varphi)'(b))$ for all $t \in \mathbb{R}$ such that $x(t)$ is defined. This shows that x conjugates the action of G on $\mathbb{S}^1 \backslash \overline{h^{-1}(\{b\})}$ with a subgroup of $\mathrm{Aff}(\mathbb{R})$.

Since G is non-trivial, the interval I has non-trivial affine diffeomorphisms, so I is $\mathbb{R}$ or affinely equivalent to $]-\infty, 0[$ or $]0, +\infty[$, in which case the action of the affine group is differentially conjugate to the action of $\mathbb{R}$ on itself by translations. Therefore, up to changing I and x (while preserving the affine structure), we can assume that $I = \mathbb{R}$.

Let $\psi : \mathbb{S}^1 \to \mathbb{S}^1 = \mathbb{R} \cup \{\infty\}$ be defined by $\psi = x^{-1}$ on $\mathbb{S}^1 \backslash \overline{h^{-1}(\{b\})}$ and $\psi \equiv \infty$ on $\overline{h^{-1}(\{b\})}$. It provides a semiconjugacy between $\rho_1(G)$ and a representation $\rho : G \to \mathrm{Aff}(\mathbb{R})$. □

PROPOSITION 3.4. *Let h be a non-decreasing map of degree one of $\mathbb{S}^1$ and let ω be a continuous volume form on $M_h = \mathbb{S}^1 \times \mathbb{S}^1 \backslash G(h)$. Assume that $G \subset G_\omega$ is elementary. Then there is a representation $\rho : G \to \mathrm{PSL}(2, \mathbb{R})$ that is semiconjugate to ρ_1.*

Proof. Let $L_1 \subset \mathbb{S}^1$ be a finite orbit for $\rho_1(G)$. If $\sharp L_1 = 1$, then Lemma 3.3 applies. If $\sharp L_1 = k \geq 2$, then let $L_1 = \{x_{\overline{1}}, \ldots, x_{\overline{k}}\}$ (where the indices are in $\mathbb{Z}/k\mathbb{Z}$ and $x_{\overline{1}} < \cdots < x_{\overline{k}} < x_{\overline{1}}$). Since elements of $\rho_1(G)$ preserve the cyclic order, there is a morphism $\sigma : G \to \mathbb{Z}/k\mathbb{Z}$ such that $\rho_1(\varphi)(x_i) = x_{i+\sigma(\varphi)}$ for all $i \in \mathbb{Z}/k\mathbb{Z}$ and $\varphi \in G$. Since G acts transitively on L_1, we necessarily have $\sigma(G) = \mathbb{Z}/k\mathbb{Z}$. Then $\varphi \mapsto R_{(\sigma(\varphi))/k}$ is a representation of G in $\mathrm{SO}(2, \mathbb{R}) \subset \mathrm{PSL}(2, \mathbb{R})$ that is semiconjugate to ρ_1 by Proposition 2.3. □

Note that the reason why we had to start with stabilizers of points is that the semiconjugacy defined in the proof of Proposition 3.4 is a constant map in the case where $\sharp L_1 = 1$ and hence does not satisfy our definition of a non-decreasing map of degree one.

3.3. *The general case.* Let us reformulate Theorem 1.2 in terms of the group G_ω.

THEOREM 3.5. *Let h be a non-decreasing map of degree one of $\mathbb{S}^1$ and let ω be a continuous volume form on $\mathbb{S}^1 \times \mathbb{S}^1 \backslash G(h)$. The group $\rho_1(G_\omega)$ is semiconjugate to a subgroup of* $\mathrm{PSL}(2, \mathbb{R})$.

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Proof. If h has a finite number of values, then G_ω is elementary, and we can apply Proposition 3.4. We can now assume that h is not finite valued.

Let U_1 be the union of the open intervals where h is constant and let U_2 be the union of open intervals between discontinuities of h. The complement of U_1 (respectively of U_2) is a closed ρ_1-invariant (respectively ρ_2-invariant) set.

Let $p_1, p_2 : \mathbb{S}^1 \to \mathbb{S}^1$ be continuous non-decreasing maps of degree one such that the intervals where p_i is constant are exactly the connected components of U_i. They induce representations $\hat{\rho}_1, \hat{\rho}_2 : G_\omega \to \mathrm{Homeo}(\mathbb{S}^1)$ such that $\hat{\rho}_i \circ p_i = p_i \circ \rho_i$, and we now have a homeomorphism $\hat{h}$ such that $\hat{h} \circ \hat{\rho}_1 = \hat{\rho}_2 \circ \hat{h}$. We are going to show that $\hat{\rho}_1$ and $\hat{\rho}_2$ are topologically Fuchsian, i.e. that they satisfy the convergence property.

Let $(\varphi_n)_{n\in\mathbb{N}} \in G_\omega^{\mathbb{N}}$ be a sequence such that $\hat{\rho}_1(\varphi_n)$ has no equicontinuous subsequence.

To simplify the notation, we will set $f_n = \rho_1(\varphi_n)$, $g_n = \rho_2(\varphi_n)$, $\hat{f}_n = \hat{\rho}_1(\varphi_n)$ and $\hat{g}_n = \hat{\rho}_2(\varphi_n)$.

We are first going to show that the sequences $\hat{f}_n(x)$ have at most two distinct limit points. Indeed, assume that there are three distinct points $\hat{\alpha} < \hat{\beta} < \hat{\gamma} < \hat{\alpha}$ in $\mathbb{S}^1$ and $\hat{a}, \hat{b}, \hat{c} \in \mathbb{S}^1$ such that $\hat{f}_n(\hat{a}) \to \hat{\alpha}$, $\hat{f}_n(\hat{b}) \to \hat{\beta}$ and $\hat{f}_n(\hat{c}) \to \hat{\gamma}$. We consider a subsequence such that $f_n(a) \to \alpha$, $f_n(b) \to \beta$ and $f_n(c) \to \gamma$ for some lifts $a, b, c, \alpha, \beta, \gamma$ with respect to p_1.

We also set $\hat{\alpha}' = \hat{h}(\hat{\alpha})$, $\hat{a}' = \hat{h}(\hat{a}), \ldots$ and choose lifts $a', b', c', \alpha', \beta', \gamma'$ with respect to p_2.

First, let us assume that $g_n'(a') \to 0$. Let K be a compact interval of $\mathbb{S}^1 \backslash \{\alpha\}$ that contains β and γ in its interior. There is $n_0 \in \mathbb{N}$ such that $f_n(x) \in K$ for all $x \in [b, c]$ and all $n \geq n_0$. This implies that the sequence $(f_n(x), g_n(a'))$ stays in a compact set of M_h and

$$f_n'(x) = \frac{1}{g_n'(a')} \frac{\omega(x, a')}{\omega(f_n(x), g_n(a'))} \to +\infty.$$

Fatou's lemma implies that $\int_b^c f_n'(x)\, dx \to \infty$, which is impossible because $\int_{\mathbb{S}^1} f_n'(x)\, dx = 1$. This shows that the sequence $g_n'(a')$ cannot converge to 0, nor can any subsequence, and there is a constant $C > 0$ such that $g_n'(a') \geq C$ for all $n \in \mathbb{N}$. Since a, b and c have similar roles, we can also assume that $g_n'(b') \geq C$ and $g_n'(c') \geq C$.

We now see that $f_n'(x) = (1/g_n'(a'))(\omega(x, a')/\omega(f_n(x), g_n(a')))$ is uniformly bounded on $[b, c]$; therefore, f_n is equicontinuous on this interval and, up to a subsequence, we can assume that f_n converges uniformly on $[b, c]$ (Ascoli's theorem). Since b and c have similar roles to a, there is a subsequence that converges uniformly on $[c, a]$ and on $[a, b]$, therefore on all of $\mathbb{S}^1$, which is impossible because we assumed that $\hat{f}_n$ has no equicontinuous subsequence.

We now know that there are at most two possible limits for f_n, say $\hat{\alpha}$ and $\hat{\beta}$ (we keep similar notation for $\hat{a}, \hat{a}', a, a', b \ldots$). Let A (respectively B) be the set of points $x \in \mathbb{S}^1$ such that $\hat{f}_n(x) \to \hat{\alpha}$ (respectively $\hat{f}_n(x) \to \hat{\beta}$).

If $x, y \in A$, then one of the two intervals $[\hat{f}_n(x), \hat{f}_n(y)]$ and $[\hat{f}_n(y), \hat{f}_n(x)]$ shrinks to $\{\hat{\alpha}\}$. This implies that one of the intervals $[x, y]$ and $[y, x]$ is included in A, hence A is connected, and it is an interval of $\mathbb{S}^1$. The same goes for B.

Assume that neither A nor B is reduced to a point or void. Let $\hat{y} \in \hat{h}(\mathring{B})$ and $y \in p_2^{-1}(\hat{y})$. First, assume that $g_n'(y) \to 0$. Then $f_n'(x) \to \infty$ for all $x \in p_1^{-1}(A)$,

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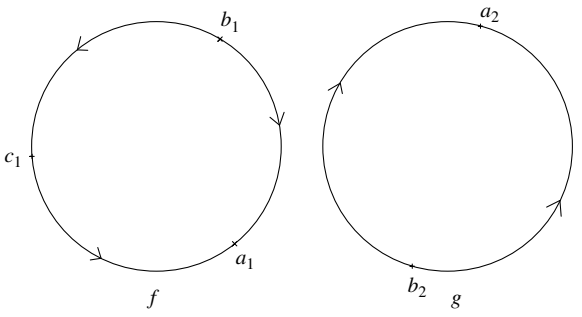


FIGURE 1. Dynamics of f and g.

and $\int_{p_1^{-1}(A)} f_n'(x)\,dx \to \infty$, which is absurd. This shows that there is $C > 0$ such that $g_n'(y) \geq C$ for all $n \in \mathbb{N}$. Then f_n' is uniformly bounded on $p_1^{-1}(A)$, and the sequence $\hat{f}_n$ is equicontinuous on A. Similarly, the sequence $\hat{g}_n$ is equicontinuous on $\hat{h}(B)$, and $\hat{f}_n$ is equicontinuous on B. Since we can choose a subsequence such that $A \cup B = \mathbb{S}^1$ (by making f_n convergent on a dense countable subset of $\mathbb{S}^1$), this implies that $\hat{f}_n$ is equicontinuous, which is absurd. Therefore, A or B contains at most one point and the sequence $\hat{f}_n$ satisfies the convergence property. □

As a corollary of the proof, we get the convergence property for ρ_1 and ρ_2 when $h \in \mathrm{Homeo}(\mathbb{S}^1)$, i.e. Theorem 1.1.

THEOREM 3.6. *Let $h \in \mathrm{Homeo}(\mathbb{S}^1)$ and let ω be a continuous volume form on $\mathbb{S}^1 \times \mathbb{S}^1 \backslash G(h)$. The group $\rho_1(G_\omega)$ is topologically Fuchsian.*

Proof. If $h \in \mathrm{Homeo}(\mathbb{S}^1)$, then $U_1 = \emptyset$ and $U_2 = \emptyset$ in the proof of Theorem 3.5. This implies that $\hat{\rho}_1 = \rho_1$, so $\rho_1(G_\omega) = \hat{\rho}_1(G_\omega)$ is topologically Fuchsian. □

4. *Elements of G_ω*

All elements of $\mathrm{PSL}(2, \mathbb{R})$ can appear (because of the form $\omega_0 = (4\,dx \wedge dy)/(x - y)^2$ on $M_{\mathrm{Id}} = \mathbb{S}^1 \times \mathbb{S}^1 \backslash \Delta$ for which $\rho_1(G_{\omega_0}) = \mathrm{PSL}(2, \mathbb{R})$), so we are now going to focus on elements of $\rho_1(G_\omega)$ that are not conjugate in $\mathrm{Homeo}(\mathbb{S}^1)$ to elements in $\mathrm{PSL}(2, \mathbb{R})$. We will construct three types of examples: first introducing a parabolic fixed point in a hyperbolic element of $\mathrm{PSL}(2, \mathbb{R})$, then opening the fixed point of a parabolic element of $\mathrm{PSL}(2, \mathbb{R})$ and finally considering the lift of a parabolic element of $\mathrm{PSL}(2, \mathbb{R})$ to the two-sheeted cover $\mathrm{PSL}_2(2, \mathbb{R})$.

4.1. *Hyperbolic elements.*

Let $f, g \in \mathrm{Diff}(\mathbb{S}^1)$ be such that:

- f has three fixed points $a_1 < b_1 < c_1 < a_1$;
- g has two fixed points a_2, b_2;
- $f'(a_1)g'(b_2) = 1$ and $f'(b_1)g'(a_2) = 1$;
- $f'(a_1) < 1$ and $f'(b_1) > 1$.

This defines f and g up to topological conjugacy (see Figure 1).

Let $h : \mathbb{S}^1 \to \mathbb{S}^1$ be defined by:

- $h(x) = b_2$ for $x \in [b_1, c_1[$;

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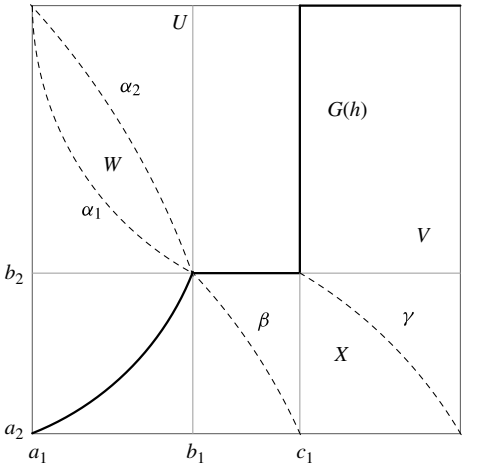


FIGURE 2. Constructing ω on M_h.

- $h(x) = a_2$ for $x \in [c_1, a_1[$;
- $h : [a_1, b_1] \to [a_2, b_2]$ is an orientation preserving homeomorphism (or a non-decreasing map such that $h(a_1) = a_2$ and $h(b_1) = b_2$) such that $g \circ h = h \circ f$.

PROPOSITION 4.1. *The map* $(x, y) \mapsto \varphi(x, y) = (f(x), g(y))$ *preserves a continuous volume form on* M_h.

Proof. We start by dividing M_h into several subsets, as shown in Figure 2. Let $\alpha_1, \alpha_2 : [a_1, b_1] \to [b_2, a_2]$, $\beta : [b_1, c_1] \to [a_2, b_2]$ and $\gamma : [c_1, a_1] \to [a_2, b_2]$ be decreasing homeomorphisms whose graphs are invariant under φ (i.e. that conjugate f and g). We choose α_1 and α_2 so that $b_2 < \alpha_1(x) < \alpha_2(x) < a_2 < b_2$ for all $x \in]a_1, b_1[$.

Let U, V, W and X be the open subsets of M_h delimited by $G(h)$ and the graphs of α_1, α_2, β and γ:

$$\begin{aligned}
U &= \{(x, y) \in]a_1, b_1] \times \mathbb{S}^1 \mid y \in]\alpha_2(x), h(x)[\} \\
&\quad \cup \{(x, y) \in [b_1, c_1[\times \mathbb{S}^1 \mid y \in]b_2, \beta(x)[, \} \\
V &= \{(x, y) \in]c_1, a_1] \times \mathbb{S}^1 \mid y \in]\gamma(x), a_2[\} \\
&\quad \cup \{(x, y) \in [a_1, b_1[\times \mathbb{S}^1 \mid y \in]h(x), \alpha_1(x)[\}, \\
W &= \{(x, y) \in]a_1, b_1[\times \mathbb{S}^1 \mid y \in]\alpha_1(x), \alpha_2(x)[\}, \\
X &= \{(x, y) \in]b_1, c_1] \times \mathbb{S}^1 \mid y \in]\beta(x), b_2[\} \\
&\quad \cup \{(x, y) \in [c_1, a_1[\times \mathbb{S}^1 \mid y \in]a_2, \gamma(x)[\}.
\end{aligned}$$

Consider the linearization maps $\tau^f_{a_1} :]c_1, b_1[\to \mathbb{R}$, $\tau^f_{b_1} :]a_1, c_1[\to \mathbb{R}$, $\tau^g_{a_2} : \mathbb{S}^1 \backslash \{b_2\} \to \mathbb{R}$ and $\tau^g_{b_2} : \mathbb{S}^1 \backslash \{a_2\} \to \mathbb{R}$. They are smooth maps such that

$$\begin{aligned}
\tau^f_{a_1} \circ f \circ (\tau^f_{a_1})^{-1}(x) &= f'(a_1)x, \\
\tau^f_{b_1} \circ f \circ (\tau^f_{b_1})^{-1}(x) &= f'(b_1)x, \\
\tau^g_{a_2} \circ g \circ (\tau^g_{a_2})^{-1}(x) &= g'(a_2)x, \\
\tau^g_{b_2} \circ g \circ (\tau^g_{b_2})^{-1}(x) &= g'(b_2)x.
\end{aligned}$$

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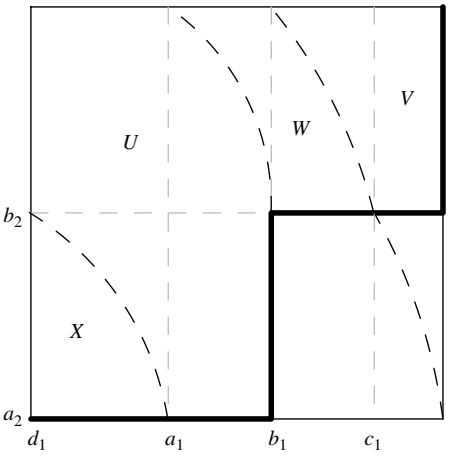


FIGURE 3. Example with four fixed points.

The map $(x, y) \mapsto (\tau^f_{b_1}(x), \tau^g_{a_2}(y))$ sends U to an open set of $\mathbb{R}^2$ and conjugates φ with $(x, y) \mapsto (f'(b_1)x, g'(a_2)y)$. Since $f'(b_1)g'(a_2) = 1$, this map preserves $dx \wedge dy$ on $\mathbb{R}^2$, and φ preserves the pull-back ω_U on U.

The map $(x, y) \mapsto (\tau^f_{a_1}(x), \tau^g_{b_2}(y))$ sends V to an open set of $\mathbb{R}^2$ and conjugates φ with $(x, y) \mapsto (f'(a_1)x, g'(b_2)y)$. Since $f'(a_1)g'(b_2) = 1$, this map preserves $dx \wedge dy$ on $\mathbb{R}^2$, and φ preserves the pull-back ω_V on V.

To extend ω to W, we notice that the action of φ on $]a_1, b_1[\times]b_2, a_2[$ is conjugate to a translation in $\mathbb{R}^2$, so the quotient is diffeomorphic to a cylinder. The images of U and V on the cylinder are open sets $\hat{U}$, $\hat{V}$ each bounded by a curve on which volume forms $\hat{\omega}_U$, $\hat{\omega}_V$ are defined. Simply consider a continuous volume form $\hat{\omega}$ on the cylinder that is equal to $\hat{\omega}_U$ on $\hat{U}$ and equal to $\hat{\omega}_V$ on $\hat{V}$ (this is possible because $\hat{\omega}_U$ and $\hat{\omega}_V$ can be defined on open sets larger than U and V). This lifts to a volume form ω on $]a_1, b_1[\times]b_2, a_2[$ that is invariant under φ and that is equal to ω_U in a neighborhood of the axes $\{a_1\} \times]b_2, a_2[\cup]a_1, b_1[\times \{b_2\}$ and to ω_V on a neighborhood of $\{b_1\} \times]b_2, a_2[\cup]a_1, b_1[\times \{a_2\}$.

We now only have to extend ω to X. If $x \in]a_2, b_2[$, then $]b_1, a_1[\times [x, g(x)[$ is a fundamental domain for the action of φ on $]b_1, a_1[\times]a_2, b_2[$, which shows that the quotient is diffeomorphic to the cylinder, and we can extend ω to X in the same way that we did for W. □

The same could also be done for f with four fixed points $a_1 < b_1 < c_1 < d_1 < a_1$ such that b_1 and d_1 are parabolic, a_1 is attracting and c_1 repelling. We then choose g with two hyperbolic fixed points a_2, b_2 such that $f'(a_1)g'(b_2) = 1$ and $f'(c_1)g'(a_1) = 1$. We set $h \equiv a_2$ on $[d_1, b_1[$ and $h \equiv b_2$ on $[b_1, d_1[$. The same kind of division of M_h into four invariant open sets U, V, W, X gives a (f, g)-invariant volume form on M_h (see Figure 3).

4.2. *Parabolic elements.* Let $\gamma \in \mathrm{PSL}(2, \mathbb{R})$ be parabolic and let $x_0 \in \mathbb{S}^1$ be its fixed point. We will denote by ω_0 the volume form on $\mathbb{S}^1 \times \mathbb{S}^1 \backslash \Delta$ preserved by $\mathrm{PSL}(2, \mathbb{R})$.

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Let $I \subset \mathbb{S}^1$ be a compact interval and define a continuous function $h : \mathbb{S}^1 \to \mathbb{S}^1$ such that $h(I) = \{x_0\}$ and $h : \mathbb{S}^1 \backslash I \to \mathbb{S}^1 \backslash \{x_0\}$ is an affine diffeomorphism. It is non-decreasing of degree one.

We can define $f \in \mathrm{Diff}^{1,1}(\mathbb{S}^1)$ (i.e. differentiable with a Lipschitz continuous derivative) such that the restriction to I is an orientation preserving diffeomorphism with $f' = 1$ at the end points and $h^{-1} \circ \gamma \circ h$ on the complement of I. We have $h \circ f = \gamma \circ h$, so the map (f, γ) acts on M_h.

PROPOSITION 4.2. *The map* $(x, y) \mapsto (f(x), \gamma(y))$ *preserves a continuous volume form on* M_h.

Proof. First, define ω on $M_h \backslash (\mathring{I} \times \mathbb{S}^1)$ to be $\omega(x, y) = \omega_0(h(x), y)$. It is a continuous volume form (even Lipschitz), and let us show that $(x, y) \mapsto (f(x), \gamma(y))$ preserves ω. If $x \notin \mathring{I}$, then we get

$$\begin{aligned}
\omega(f(x), \gamma(y)) f'(x)\gamma'(y) &= \omega_0(h \circ f(x), \gamma(y)) f'(x)\gamma'(y) \\
&= \omega_0(\gamma(h(x)), \gamma(y))\gamma'(h(x))\gamma'(y)\frac{f'(x)}{\gamma'(h(x))} \\
&= \omega_0(h(x), y)\frac{f'(x)}{\gamma'(h(x))} \\
&= \omega(x, y)\frac{h'(x)}{h'(f(x))} \\
&= \omega(x, y).
\end{aligned}$$

To extend ω to M_h, we simply notice that the quotient of $I \times \mathbb{S}^1 \backslash G(h)$ by (f, γ) is diffeomorphic to a cylinder, on which we have defined a volume form on the boundary. We can extend it to a volume form on the cylinder and then lift it to a continuous volume form on $I \times \mathbb{S}^1 \backslash G(h)$ that is (f, γ)-invariant and equal to ω on the boundary. In this way, we have defined a continuous volume form on M_h that is (f, γ)-invariant. □

Remark. In all of the other examples, the invariant volume forms can actually be constructed in a smooth way. However, in the case of the opening of the fixed point of a parabolic element of $\mathrm{PSL}(2, \mathbb{R})$, it is not clear whether it is possible to find a smooth volume form.

4.3. *Lifts of parabolic elements to* $\mathrm{PSL}_2(2, \mathbb{R})$. The two-sheeted covering of the circle is still a circle, which induces a two-sheeted covering group $\mathrm{PSL}_2(2, \mathbb{R})$ of $\mathrm{PSL}(2, \mathbb{R})$. Lifts of parabolic elements have either two fixed points or two (not fixed) periodic points. Let $R \in \mathrm{PSL}_2(2, \mathbb{R})$ be a generator of the center (R is of order two).

If $\gamma \in \mathrm{PSL}_2(2, \mathbb{R})$ is a lift of a parabolic element without fixed points, then it has two periodic points x_0, $R(x_0)$. Let $U = \{(x, y) \mid y \in]x, R(x)[\}$. The quotient by the map $(x, y) \mapsto (R(x), R(y))$ of U endowed with the action of $\mathrm{PSL}(2, \mathbb{R})$ is equivariant to the diagonal action on $\mathbb{S}^1 \times \mathbb{S}^1 \backslash \Delta$ and therefore has a volume form preserved by $\mathrm{PSL}(2, \mathbb{R})$, which can be lifted to U as a volume form ω_0 invariant under $\mathrm{PSL}_2(2, \mathbb{R})$.

Let $h : \mathbb{S}^1 \to \mathbb{S}^1$ be defined by $h(x) = x_0$ if $x \in [x_0, R(x_0)[$ and $h(x) = R(x_0)$ if $x \in [R(x_0), x_0[$. It is non-decreasing of degree one and it commutes with γ.

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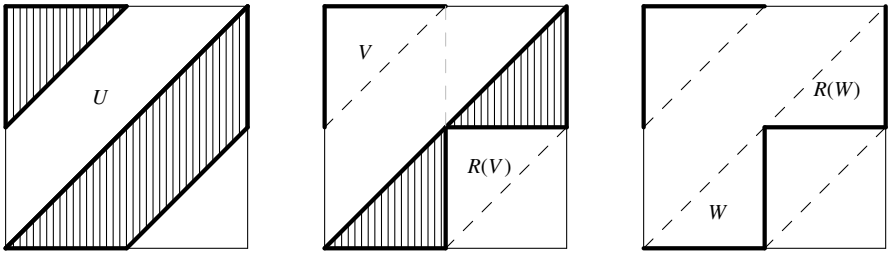


FIGURE 4. Invariant volume form for a parabolic element of $\mathrm{PSL}_2(2, \mathbb{R})$.

PROPOSITION 4.3. *The map* $(x, y) \mapsto (\gamma(x), \gamma(y))$ *preserves a continuous volume form on* M_h.

Proof. In order to extend ω_0 to M_h, we start by considering the open set $V = \{(x, y) \mid x \in]x_0, R(x_0)[, y \in]x, x_0[\}$ and its image $R(V)$ under the map $(x, y) \mapsto (R(x), R(y))$ (see Figure 4).

Since the action of γ on $V \cup R(V)$ is proper (the quotient is a cylinder), we can extend ω_0 from $U \cap (V \cup R(V))$ to a volume form ω on $U \cup V \cup R(V)$.

Similarly, we set $W = \{(x, y) \mid x \in]x_0, R(x_0)[, y \neq x_0\}$ and extend ω to $W \cup R(W)$, so that ω is now defined on M_h. □

4.4. *Classification of elements of* $\rho_1(G_\omega)$ *up to topological conjugacy.* We will now see that we have described all possible examples.

PROPOSITION 4.4. *Let* $h : \mathbb{S}^1 \to \mathbb{S}^1$ *be non-decreasing of degree one and let* ω *be a continuous volume form on* M_h. *For any* $\varphi \in G_\omega$, $\rho_1(\varphi)$ *is topologically conjugate to one of the examples described above, i.e. it satisfies one of the following propositions:*

- $\rho_1(\varphi)$ *is topologically conjugate to an element of* $\mathrm{PSL}(2, \mathbb{R})$*;*
- $\rho_1(\varphi)$ *is topologically conjugate to a parabolic element of* $\mathrm{PSL}_2(2, \mathbb{R})$*;*
- $\rho_1(\varphi)$ *has three fixed points* $a < b < c < a$ *such that* a, b *are hyperbolic and* c *is parabolic;*
- $\rho_1(\varphi)$ *has four fixed points* $a < b < c < d < a$ *such that* a, c *are hyperbolic and* b, d *are parabolic.*

Let $h : \mathbb{S}^1 \to \mathbb{S}^1$ be non-decreasing of degree one, let ω be a continuous volume form on M_h and let $\varphi = (f, g) \in G_\omega \backslash \{\mathrm{Id}\}$. We will classify them according to their number of fixed points.

As we saw in Lemma 3.1, fixed points of φ in M_h are hyperbolic. This implies that there cannot be too many of them.

LEMMA 4.5. *If* $a_1 \in \mathbb{S}^1$ *is fixed by* f*, then there is at most one fixed point* $b_2 \in \mathbb{S}^1$ *of* g *such that* $(a_1, b_2) \in M_h$.

Proof. Assume that there are two fixed points a_2, b_2 of g such that $(a_1, a_2) \in M_h$ and $(a_1, b_2) \in M_h$. According to Lemma 3.1, a_1 is hyperbolic, so, up to replacing φ by φ^{-1}, we can assume that a_1 is attracting for f. This implies that a_2 and b_2 are both repelling for

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g, so g has non-repelling fixed points $c_2 \in]a_2, b_2[$ and $d_2 \in]b_2, a_2[$. Since they are not repelling, $(a_1, c_2) \notin M_h$ and $(a_1, d_2) \notin M_h$. This is absurd because the set of $y \in \mathbb{S}^1$ such that $(a_1, y) \in M_h$ is connected. □

Proof of Proposition 4.4. We will distinguish different cases depending on the number of fixed points of f and g.

One fixed point: first, we assume that g has exactly one fixed point $y_0 \in \mathbb{S}^1$. The interval $h^{-1}(\{y_0\})$ is stabilized by f, so its end points are fixed by f. If f had a fixed point x outside of $h^{-1}(\{y_0\})$, then $h(x)$ is fixed by g, which is impossible; therefore, f has no fixed point outside of $h^{-1}(\{y_0\})$ and it is topologically conjugate to an example described in 4.2.

Two fixed points: now, assume that both f and g have at least two fixed points. Let $a_1 \neq b_1$ be fixed points of f and $a_2 \neq b_2$ be fixed points of g.

First, let us assume that φ has no fixed point in M_h. In this case, none of the four points (a_1, a_2), (a_1, b_2), (b_1, a_2), (b_1, b_2) are in M_h. This implies that h is constant on both intervals $]a_1, b_1[$ and $]b_1, a_1[$; therefore, f (as well as g) has exactly two fixed points, so either it has north/south dynamics (in which case it is topologically conjugate to a hyperbolic element of $\mathrm{PSL}(2, \mathbb{R})$) or it is topologically conjugate to a parabolic element of $\mathrm{PSL}_2(2, \mathbb{R})$.

We can now assume that φ has a fixed point in M_h. Up to renaming the fixed points, we can assume that $(a_1, b_2) \in M_h$. Lemma 3.1 implies that they are hyperbolic fixed points for f and g. If they each have only two fixed points, then they have north/south dynamics and are topologically conjugate to a hyperbolic element in $\mathrm{PSL}(2, \mathbb{R})$. Up to replacing φ by φ^{-1}, we can assume that a_1 is attracting for f and b_2 is repelling for g.

Assume that f has a third fixed point $c_1 \in]b_1, a_1[$. Because of Lemma 4.5, the points (b_1, b_2) and (c_1, b_2) are not in M_h, which implies that $h \equiv b_2$ on $]b_1, c_1[$. Up to replacing b_1 and c_1 by the edges of the interval $h^{-1}(\{b_2\})$ (which are fixed by f), we can assume that $h^{-1}(\{b_2\}) = [b_1, c_1]$. Lemma 4.5 implies that a_1 is the only fixed point of f in $]c_1, a_1[$ and that f has at most one fixed point in $]b_1, c_1[$ and, that if it exists, then it is hyperbolic. This implies that f has either three or four fixed points and is topologically conjugate to one of the examples described in 4.1.

Notice that in this case (where f has at least three fixed points), g has only two fixed points (an extra fixed point would generate a point in M_h, which cannot exist because of Lemma 4.5).

No fixed points: we have treated all cases where f and g have fixed points (note that since f and g are semiconjugate, if one has a fixed point, then so does the other). They share the same rotation number $\alpha \in \mathbb{R}/\mathbb{Z}$. If they have no fixed points, then $\alpha \neq 0$.

If $\alpha = p/q$ is rational, then the numbers of fixed points of f^q and g^q are at least q and are multiples of q, and $(f^q, g^q) \in G_\omega$. This shows that if $f^q \neq \mathrm{Id}$, then $\alpha = \frac{1}{2}$ (because either f^q or g^q has exactly two fixed points) and f^2 has north/south dynamics, has four fixed points or is topologically conjugate to a parabolic element of $\mathrm{PSL}_2(2, \mathbb{R})$. In the latter case, f is itself topologically conjugate to a parabolic element of $\mathrm{PSL}_2(2, \mathbb{R})$.

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If f^2 has north/south dynamics, let a_1, b_1 be its fixed points. Then f conjugates f^2 on a neighborhood of a_1 with f^2 on a neighborhood of a_2, which is absurd because one is attracting and the other is repelling.

If f^2 has some hyperbolic points (which is the case if it has four fixed points), then let a_1 be a hyperbolic periodic point of f and $b_1 = f(a_1)$. They satisfy $(f^2)'(a_1) = f'(b_1)f'(a_1) = (f^2)'(b_1)$. This is impossible, since one is attracting for f^2 and the other is repelling. Therefore, if f has no fixed points and $\alpha = p/q \in \mathbb{Q}$, then either $f^q = \mathrm{Id}$, and f is topologically conjugate to a rotation, or f is topologically conjugate to a parabolic element of $\mathrm{PSL}_2(2, \mathbb{R})$.

If $\alpha \notin \mathbb{Q}$, then there are two possibilities: either f is topologically conjugate to a rotation or f has an invariant Cantor set and is semiconjugate to a rotation (a Denjoy example, which cannot be the case if f is C^2, see [**Nav11**] for a thorough treatment of Denjoy diffeomorphisms). We are going to show that f and g are both topologically conjugate to a rotation.

Assume that f is a Denjoy example. Let $K \subset \mathbb{S}^1$ be the invariant Cantor set and let I be a connected component of $\mathbb{S}^1 \setminus K$. Using the fact that $f^n(I) \cap f^p(I) = \emptyset$ if $n \neq p$, a simple calculation (see [**Nav11**, Exercise 3.5.24]) shows that

$$\int_I \left(\sum_{n \in \mathbb{Z}} (f^n)'(x) \right) dx = \sum_{n \in \mathbb{Z}} |f^n(I)| \leq 1.$$

In particular, the set of points $x \in \mathbb{S}^1$ such that $(f^n)'(x) \to 0$ as $n \to \infty$ is a set of full Lebesgue measure in $\mathbb{S}^1 \setminus K$ and hence non-empty. The points $y \in \mathbb{S}^1$ such that the sequence $((g^n)'(y))_{n \in \mathbb{Z}}$ is bounded form a non-empty set [**Her79**, Ch. X] invariant under g.

Note that $G(g \circ h) \subset M_h$. Indeed, if h is continuous at x, then $g(h(x)) \neq h(x)$ and, if h is discontinuous at x, then the interval $]h_l(x), h_r(x)[$ is wandering for g and therefore cannot intersect its image $g(]h_l(x), h_r(x)[)$. Since $\varphi^2 \in G_\omega$, we also have $G(g^2 \circ h) \subset M_h$. Up to replacing φ by φ^{-1}, we can assume that $K = \{(x, y) \in \mathbb{S}^1 \times \mathbb{S}^1 \mid y \in [g(h_l(x)), g^2(h_r(x))]\}$ is a compact subset of M_h, invariant under φ, with non-empty interior. Let $x \in \mathbb{S}^1$ be such that $(f^n)'(x) \to 0$ as $n \to \infty$ and let $y \in \mathbb{S}^1$ be such that the sequence $((g^n)'(y))_{n \in \mathbb{Z}}$ is bounded and $(x, y) \in K$. Let $M = \max_K \omega$ and $m = \min_K \omega$. We then have

$$\underbrace{(f^n)'(x)(g^n)'(y)}_{\to 0} = \frac{\omega(x, y)}{\omega(f^n(x), g^n(y))} \in \left[\frac{m}{M}, \frac{M}{m} \right].$$

This is a contradiction; therefore, f cannot be a Denjoy example, so it is topologically conjugate to a rotation, which is an element of $\mathrm{PSL}(2, \mathbb{R})$. □

5. *Non-elementary examples*

This section is dedicated to the proof of Theorem 1.3. The main idea in constructing a non-elementary example consists in starting with an appropriately chosen representation $\rho_1 : \mathbb{F}_3 \to \mathrm{PSL}(2, \mathbb{R})$ and then considering $\rho_2 : \mathbb{F}_3 \to \mathrm{Diff}(\mathbb{S}^1)$, where we modified only one of the generators by introducing a parabolic fixed point (just as in the elementary case). Proposition 4.1 shows that the images of the generators by (ρ_1, ρ_2) each preserve a volume form, and the difficulty consists in showing that we can find one that is preserved by all three. The proof is almost identical to the proof of [**Mon14a**, Theorem 1.4].

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One of the main ingredients in this proof is the construction of a flow on a 3-manifold (a deformation of the geodesic flow on $\mathrm{T}^1\mathbb{H}^2/\rho_1(\Gamma)$) that has a transverse structure given by (ρ_1, ρ_2). This construction follows an idea of Ghys used in two different settings. The first one, found in [**Ghy93**], was to show a rigidity theorem for actions of surface groups on the circle, and the second was the construction of (the only) exotic Anosov flows with smooth weak stable and weak unstable foliations on 3-manifolds in [**Ghy92**], called quasi-Fuchsian flows. However, Ghys used a local construction (given a certain atlas on $\mathrm{T}^1\mathbb{H}^2/\rho_1(\Gamma)$), whereas we will take a global approach.

5.1. *Hyperbolic flows.* Let us recall a few basic notions of hyperbolic flows. Let φ^t be a complete flow generated by a vector field X on a manifold M. We say that a compact invariant set $K \subset M$ is *hyperbolic* if there are positive constants C, λ and a decomposition of tangent spaces $T_x M = E^s_x \oplus E^u_x \oplus \mathbb{R}.X$ for each $x \in K$ such that

$$\text{for all } x \in K, \text{ for all } v \in E^s_x, \text{ for all } t \geq 0, \quad \|D\varphi^t_x(v)\| \leq Ce^{-\lambda t}\|v\|,$$
$$\text{for all } x \in K, \text{ for all } v \in E^u_x, \text{ for all } t \leq 0, \quad \|D\varphi^t_x(v)\| \leq Ce^{\lambda t}\|v\|.$$

The norm $\|\cdot\|$ denotes the norm given by any Riemannian metric on M (since K is compact, the definition does not depend on the choice of a Riemannian metric). If the whole manifold M is a hyperbolic set, then we say that φ^t is an Anosov flow.

Let φ^t be a smooth flow on a manifold M. If $K \subset M$ is a compact hyperbolic set and $x \in K$, then we define the *stable and unstable manifolds* through x:

$$W^s(x) = \left\{ z \in M \;\middle|\; d(\varphi^t(x), \varphi^t(z)) \underset{t\to+\infty}{\longrightarrow} 0 \right\},$$
$$W^u(x) = \left\{ z \in M \;\middle|\; d(\varphi^t(x), \varphi^t(z)) \underset{t\to-\infty}{\longrightarrow} 0 \right\}.$$

The stable manifold theorem states that they are submanifolds of M tangent to E^s and E^u at x (see [**HP69**]).

The most important fact for us is that the convergence $d(\varphi^t(x), \varphi^t(z)) \to 0$ occurs at a an exponential rate: for all compact set A and all $\varepsilon > 0$, there is a constant $C' > 0$ such that

$$\text{for all } x \in K, \text{ for all } z \in W^s(x) \cap A, \text{ for all } t \geq 0, \quad d(\varphi^t(x), \varphi^t(z)) \leq C'e^{-(\lambda-\varepsilon)t},$$
$$\text{for all } x \in K, \text{ for all } z \in W^u(x) \cap A, \text{ for all } t \leq 0, \quad d(\varphi^t(x), \varphi^t(z)) \leq C'e^{(\lambda-\varepsilon)t}.$$

We will denote by $W^s(K)$ (respectively $W^u(K)$) the union $W^s(K) = \bigcup_{x\in K} W^s(x)$ (respectively $W^u(K) = \bigcup_{x\in K} W^u(x)$).

5.2. *A cohomological reformulation.* Searching for an invariant volume form is equivalent to solving a cohomological equation. Let ω_0 be a volume form on a smooth manifold M and let $\rho : \Gamma \to \mathrm{Diff}(M)$ be a group representation. Any other volume form on M is a multiple of ω_0; hence, if $\gamma \in \Gamma$, then we can write $\rho(\gamma)^*\omega_0 = e^{-\alpha_\gamma}\omega_0$. The chain rule shows that α_γ satisfies the cocycle relation $\alpha_{\gamma'\gamma} = \alpha_{\gamma'} \circ \rho(\gamma) + \alpha_\gamma$.

Let $\omega = e^\sigma \omega_0$ be a volume form on M. We can compute the pull-back $\rho(\gamma)^*\omega = e^{\sigma\circ\rho(\gamma)}\rho(\gamma)^*\omega_0 = e^{\sigma\circ\rho(\gamma)-\sigma-\alpha_\gamma}\omega$; hence, ω is preserved by Γ if and only if $\sigma \circ \rho(\gamma) - \sigma = \alpha_\gamma$ for all $\gamma \in \Gamma$. In other words, we wish to show that the cocycle α_γ is a coboundary.

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The issue with this formulation of the problem is that cohomological equations for the action (ρ_1, ρ_2) on M_h are difficult to solve. We will now see how we can translate the problem to a cohomological equation for a hyperbolic flow, which is a much more simple situation. In this setting, a cocycle is a smooth function $\alpha : M \to \mathbb{R}$ (where M is the manifold on which we study a flow φ^t), and we look for a smooth function $\sigma : M \to \mathbb{R}$ such that $\sigma(\varphi^t(x)) - \sigma(x) = \int_0^t \alpha(\varphi^s(x))\, ds$ for all $(x, t) \in M \times \mathbb{R}$.

There is a first necessary condition for the existence of a solution: if $x \in \mathrm{Per}(\varphi)$, i.e. if there is $T > 0$ such that $\varphi^T(x) = x$, then $\int_0^T \alpha(\varphi^s(x))\, ds = 0$. Livšic's theorem states that this condition is sufficient in order to find a solution on a compact hyperbolic set.

THEOREM 5.1. *Let φ^t be a smooth flow on a manifold M and let K be a compact hyperbolic set such that the action on K has a dense orbit. If $\alpha : K \to \mathbb{R}$ is a Hölder continuous function such that $\int_0^T \alpha(\varphi^s(x))\, ds = 0$ for all $x \in K$ such that $\varphi^T(x) = x$, then there is a unique Hölder continuous function $\sigma : K \to \mathbb{R}$ such that $\sigma(\varphi^t(x)) - \sigma(x) = \int_0^t \alpha(\varphi^s(x))\, ds$ for all $(x, s) \in K \times \mathbb{R}$.*

As stated, the proof can be found in [**KH95**] (Livšic's work in [**Liv71**] deals with Anosov flows on compact manifolds).

However, Livšic's theorem will not be of any use in the proof of Theorem 1.3, because we will already have a solution on the hyperbolic set. Instead, we will show that given a solution on a compact hyperbolic set K, we can extend it to $W^s(K) \cup W^u(K)$. When translating the problem back to the action on M_h, this will give a volume form invariant at points of $L_{\rho_1(\Gamma)} \times \mathbb{S}^1 \cup \mathbb{S}^1 \times L_{\rho_2(\Gamma)}$, and there will still be some work involved in order to extend the solution to M_h (which is the content of §5.9).

5.3. *Convex cocompact groups and geodesic flows.* Let $\Gamma \subset \mathrm{PSL}(2, \mathbb{R})$ be a discrete non-elementary subgroup such that the limit set L_Γ is a Cantor set. The *convex hull* of Γ is the subset C_Γ of $\mathbb{H}^2$ bounded by geodesics joining fixed points of hyperbolic elements of Γ. We say that Γ is *convex cocompact* if C_Γ / Γ is compact. A particular case of Ahlfors' finiteness theorem (see [**Ahl64**] or [**Ber65**]) states that any finitely generated discrete subgroup of $\mathrm{PSL}(2, \mathbb{R})$ with only hyperbolic elements is convex cocompact.

If $\Gamma \subset \mathrm{PSL}(2, \mathbb{R})$ is convex cocompact, then denote by φ^t the geodesic flow on $\mathrm{T}^1\mathbb{H}^2/\Gamma$ (we remark that even if $\mathbb{H}^2/\Gamma$ is not a manifold, the unit bundle $\mathrm{T}^1\mathbb{H}^2/\Gamma$ always is when Γ is discrete).

The *non-wandering set* Ω_φ of a flow is the set of points x such that there are sequences $x_n \to x$ and $t_n \to \infty$ satisfying $\varphi^{t_n}(x_n) \to x$. For the geodesic flow, Ω_φ can be described as follows: its lift to $\mathrm{T}^1\mathbb{H}^2$ is the set of vectors tangent to a geodesic that lies entirely in C_Γ. The important property of φ^t is that it is an Axiom A flow: Ω_φ is a compact hyperbolic set for φ^t, and it is equal to the closure of periodic orbits $\mathrm{Per}(\varphi)$ (Axiom A flows are a generalization of Anosov flows that can be defined even on non-compact manifolds). We will now use a presentation of the geodesic flow that is particularly convenient when we define perturbations.

Let $\Sigma_3 = \{(x_-, x_0, x_+) \in (\mathbb{S}^1)^3 \mid x_- < x_0 < x_+ < x_-\}$ be the set of ordered triples of $\mathbb{S}^1$. We can identify $\mathrm{T}^1\mathbb{H}^2$ and Σ_3 in the following way: given a unit vector $v \in \mathrm{T}^1\mathbb{H}^2$, we consider x_- and x_+, the limits at $-\infty$ and $+\infty$ of the geodesic given by v, and x_0 is the

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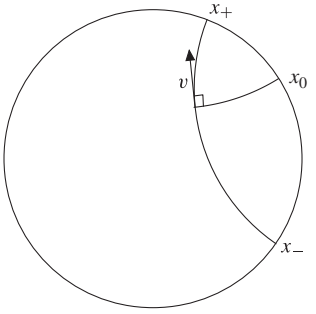


FIGURE 5. Identification between $\mathrm{T}^1\mathbb{H}^2$ and Σ_3.

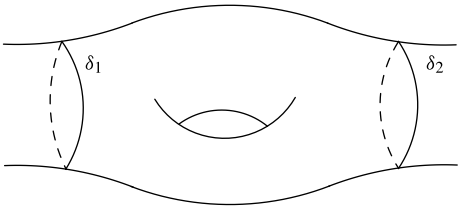


FIGURE 6. The twice-punctured torus.

limit at $+\infty$ of the geodesic passing through the base point of v in an orthogonal direction, oriented to the right of v (see Figure 5).

On Σ_3, the geodesic vector field is a rescaling of the constant vector field $(0, 1, 0)$, and the action α of $\mathrm{PSL}(2, \mathbb{R})$ is the diagonal action. The geodesic flow φ^t is defined on the quotient manifold $M = \Sigma_3/\alpha(\Gamma) \approx \mathrm{T}^1\mathbb{H}^2/\Gamma$. The image of a point (x_-, x_0, x_+) in M is in Ω_φ if and only if $(x_-, x_+) \in L_\Gamma \times L_\Gamma$, and it is in $\mathrm{Per}(\varphi)$ if and only if (x_-, x_+) is the pair of fixed points of an element $\gamma \in \Gamma$.

5.4. *Choice of ρ_1 and construction of ρ_2.* Let T be the twice-punctured torus. Its fundamental group is the free group on three generators $\mathbb{F}_3 = \langle a, b, c\rangle$. Given a complete hyperbolic structure on T such that neighborhoods of the omitted points have infinite volume, we obtain a convex cocompact representation $\rho_1 : \mathbb{F}_3 \to \mathrm{PSL}(2, \mathbb{R})$.

Let δ_1 and δ_2 be the simple loops going around the omitted points (see Figure 6). If I is a connected component of $\mathbb{S}^1 \backslash L_{\rho_1(\mathbb{F}_3)}$ and γ stabilizes I, then γ is conjugate to a power of δ_1 or δ_2. It is explained in [**But00**] how the generators can be chosen in such a way that $\delta_1 = abc$ and $\delta_2 = cba$. This shows that a, b, δ_1 freely generate $\pi_1(T)$.

Let $N, S \in \mathbb{S}^1$ be the fixed points of $\rho_1(\delta_1)$ such that $]N, S[$ is a connected component of $\mathbb{S}^1 \backslash L_{\rho_1(\mathbb{F}_3)}$. Let $f \in \mathrm{Diff}(\mathbb{S}^1)$ be equal to $\rho_1(a)$ on $[S, N]$, and have exactly one fixed point P_a in $]N, S[$. We define $\rho_2 : \mathbb{F}_3 \to \mathrm{Diff}(\mathbb{S}^1)$ by $\rho_2(a) = \rho_1(a)$, $\rho_2(b) = \rho_1(b)$ and $\rho_2(\delta_1) = f$.

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Since f coincides with $\rho_1(\delta_1)$ on the limit set, we see that $\rho_2(\gamma)$ and $\rho_1(\gamma)$ coincide on $L_{\rho_1(\mathbb{F}_3)}$ for all $\gamma \in \mathbb{F}_3$. More precisely, they are equal everywhere except on intervals that are bounded by images of N and S. Therefore, for any $\gamma \in \mathbb{F}_3$ that is not a power of a conjugate of δ_1, $\rho_2(\gamma)$ has two fixed points, which are hyperbolic.

LEMMA 5.2. $L_{\rho_2(\mathbb{F}_3)} = L_{\rho_1(\mathbb{F}_3)}$.

Proof. The compact set $L_{\rho_1(\mathbb{F}_3)}$ is invariant under ρ_2, which shows that $L_{\rho_2(\mathbb{F}_3)} \subset L_{\rho_1(\mathbb{F}_3)}$ because of the uniqueness of the minimal compact invariant set. Since the actions on $L_{\rho_1(\mathbb{F}_3)}$ are equal, the orbits are dense and $L_{\rho_2(\mathbb{F}_3)} = L_{\rho_1(\mathbb{F}_3)}$. □

We will denote this set by $L = L_{\rho_1(\mathbb{F}_3)} = L_{\rho_2(\mathbb{F}_3)}$.

LEMMA 5.3. *The representations ρ_1 and ρ_2 are semiconjugate by a map that is the identity on L.*

Proof. We start by setting h to be the identity on L. Let $I_1 =]N, S[$ be the connected component of $\mathbb{S}^1 \backslash L$ stabilized by δ_1.

Set $h : I_1 \to I_1$ to be a semiconjugacy between $\rho_1(\delta_1)$ and $\rho_2(\delta_1)$, and extend h on images $\rho_1(\gamma)(I_1)$ for $\gamma \in \mathbb{F}_3$ by $h_{/\rho_1(\gamma)(I_1)} = \rho_2(\gamma) \circ h_{/I_1} \circ \rho_1(\gamma^{-1})$.

Finally, set h to be the identity on the other connected components of $\mathbb{S}^1 \backslash L$. It provides a semiconjugacy between ρ_1 and ρ_2 that is the identity on L. □

Given $h : \mathbb{S}^1 \to \mathbb{S}^1$ non-increasing of degree one such that h is the identity on L and such that (ρ_1, ρ_2, h) is a semiconjugate triple, we wish to show that it is area preserving.

Since semiconjugacy is an equivalence relation, we can also consider $h^* : \mathbb{S}^1 \to \mathbb{S}^1$ non-decreasing of degree one such that $\rho_1 \circ h^* = h^* \circ \rho_2$. Since we can construct it in the same way as for h, we can assume that h^* is the identity on L.

Remark. Such a construction is not possible if we start with a Fuchsian representation of the fundamental group of a compact surface, because the limit set is the whole circle. This implies that we have to work with a surface of finite type, i.e. a closed surface of genus $g \geq 0$ with r omitted points, so that $2g + r > 2$ (to ensure the existence of a hyperbolic structure and that the representation in $\mathrm{PSL}(2, \mathbb{R})$ is non-elementary). In the construction of ρ_2, we used two technical properties of ρ_1. First, it is important to work with a free group, so that a choice of images of the generators in $\mathrm{Diff}(\mathbb{S}^1)$ always corresponds to a representation of the fundamental group considered. This is always the case with a non-compact surface of finite type (i.e. $r \geq 1$). The second important condition is that we can choose a system of generators so that one of them represents a simple loop around one of the omitted points. This is not possible with a simple punctured torus (where the fundamental group is $\mathbb{F}_2 = \langle a, b \rangle$ and the simple loop around the omitted point is the commutator $[a, b] = a^{-1}b^{-1}ab$). This is why we used the twice-punctured torus, but the sphere with three omitted points (i.e. a pair of pants) could also have been used. More generally, this construction works for a sphere with three or more removed points, or a closed surface of genus $g \geq 2$ with two or more removed points.

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5.5. *The flow associated to* (ρ_1, ρ_2). We consider the following 3-manifold:

$$\Sigma_h = \{(a, b, c) \in (\mathbb{S}^1)^3 \mid a < b < z < a \ \forall z \in h^{-1}(\{c\})\}.$$

The group $\mathbb{F}_3$ acts on Σ_h by $\gamma \cdot (a, b, c) = (\rho_1(\gamma)(a), \rho_1(\gamma)(b), \rho_2(\gamma)(c))$.

PROPOSITION 5.4. *The action of* $\mathbb{F}_3$ *on* Σ_h *is properly discontinuous.*

Proof. Assume that $\gamma_k \to \infty$ and that there is a sequence $(a_k, b_k, c_k) \to (a, b, c) \in \Sigma_h$ such that $\gamma_k \cdot (a_k, b_k, c_k) \to (u, v, w) \in \Sigma_h$. Up to a subsequence and up to replacing γ_k with γ_k^{-1}, we can assume that $\rho_1(\gamma_k)(x) \to v$ for all $x \neq u$.

Assume that a does not bound an interval where $\rho_1 \neq \rho_2$. In that case, one can find a compact interval $K \subset \mathbb{S}^1$ bounded by points of L such that $a \notin K$ and $c \in \mathring{K}$. The sequence of intervals $\rho_1(\gamma_k)(K)$ collapses to the point $\{v\}$. Since $\rho_2(\gamma_k)(K) = \rho_1(\gamma_k)(K)$ and $c_k \in K$, we see that $\rho_2(\gamma_k)(c_k) \to v$ and hence $w = v = h(v)$, which is absurd because $(u, v, w) \in \Sigma_h$.

We now know that $u = N$, where $I =]N, S[$ is an interval bounded by fixed points for a conjugate δ of δ_1 such that the third fixed point P is in I. If c_k were not in I for k large enough, then we could still find a compact interval K as above, which is impossible; hence, $c_k \in I$ for k large enough. This implies that γ_k stabilizes I, which gives us $v = S$ and $w = P$. This is also impossible, because $(N, S, P) \notin \Sigma_h$ by construction of h. □

We can now consider the quotient manifold $N_h = \Sigma_h / \mathbb{F}_3$.

5.6. *Invariant volume on the hyperbolic set.* The projection on N_h of the constant vector field $(0, 1, 0)$ on Σ_h can be reparametrized into a smooth flow ψ^t. Consider the map $\widetilde{H}^* : \Sigma_h \to \Sigma_3$ defined by $\widetilde{H}^*(x, y, z) = (x, y, h^*(z))$. It induces a map $H^* : N_h \to M = \mathrm{T}^1\mathbb{H}^2 / \rho_1(\mathbb{F}_3)$. Its restriction to Ω_ψ is a diffeomorphism onto Ω_φ that sends ψ^t to a reparametrization of φ^t. From this, we deduce that Ω_ψ is a compact hyperbolic set for ψ^t. If the image $x \in N_h$ of $(x_-, x_0, x_+) \in \Sigma_h$ is in Ω_ψ, then the stable (respectively unstable) manifold of x is the set of images of points (y_-, y_0, y_+) such that $y_+ = x_+$ (respectively $y_- = x_-$).

We will use this flow in order to extend the volume form to the stable and unstable manifolds of the non-wandering set.

LEMMA 5.5. *There is a continuous volume form* ω_1 *on* N_h *that is invariant under* ψ^t *at points of* $W^s(\Omega_\psi) \cup W^u(\Omega_\psi)$.

Proof. The differentiable conjugacy on the non-wandering set implies that there is a smooth volume form ω_0 on N_h that is preserved by the flow at points of the non-wandering set. Hence, if $\psi^{t*}\omega_0 = e^{-A(t,x)}\omega_0$ and $\alpha(x) = (\partial A / \partial t)(0, x)$, then $\alpha = 0$ on Ω_ψ. We will now construct a smooth function σ on N_h such that $\sigma(\psi^t(x)) - \sigma(x) = \int_0^t \alpha(\psi^s(x))\, ds$ for all $x \in W^s(\Omega_\psi) \cup W^u(\Omega_\psi)$, so that $\omega_1 = e^\sigma \omega_0$ meets our requirements.

If $x \in W^s(z)$ with $z \in \Omega_\psi$, and if we have found such a function σ, then $\sigma(\psi^t(x)) \approx \sigma(\psi^t(z)) = 0$ for t large enough; hence, $\sigma(x) = -\int_0^\infty \alpha(\psi^t(x))\, dt$. We will use this formula as a definition of σ. If it is well defined, then it satisfies the cohomological equation.

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Let $C > 0$ be such that $d(\psi^t(x), \psi^t(z)) \leq Ce^{-t}$ (locally C can be chosen independently from x and z). Let k be a Lipschitz constant for α in a neighborhood U of Ω_ψ. For t such that $\psi^t(x) \in U$ (which is locally uniform in x), we have

$$|\alpha(\psi^t(x))| \leq \underbrace{|\alpha(\psi^t(z))|}_{=0} + k \underbrace{d(\psi^t(x), \psi^t(z))}_{\leq Ce^{-t}}.$$

This gives us uniform convergence; hence, σ is well defined and continuous. By applying the same reasoning with negative times, we define σ on $W^u(\Omega_\psi)$. □

5.7. *Going back from the flow to M_h.* Now that we have found an invariant volume form on a larger set for the flow ψ^t, we need to translate it in terms of the action on M_h.

LEMMA 5.6. *If there is a continuous volume form v on N_h preserved by ψ^t at points of $W^s(\Omega_\psi) \cup W^u(\Omega_\psi)$, then there is a continuous volume form ω on M_h preserved by (ρ_1, ρ_2) at points of $L \times \mathbb{S}^1 \cup \mathbb{S}^1 \times L$.*

Proof. Let $\omega_1 = e^\sigma \omega_0$ be a continuous volume form on N_h that is invariant at points of $W^s(\Omega_\psi) \cup W^u(\Omega_\psi)$. Let $\widetilde{\omega}_1$ be its lift to Σ_h and write

$$\widetilde{\omega}_1 = \widetilde{\omega}_1(x_-, x_0, x_+)\, dx_- \wedge dx_0 \wedge dx_+.$$

If x_- or x_+ is in L, then the image in N_h is in $W^s(\Omega_\psi) \cup W^u(\Omega_\psi)$, and the invariance under the flow ψ^t gives us $\widetilde{\omega}_1(x_-, x_0, x_+) = \widetilde{\omega}_1(x_-, x_0', x_+)$ for all x_0' such that $(x_-, x_0', x_+) \in \Sigma_h$.

Choose a continuous map $i_0 : M_h \to \mathbb{S}^1$ such that $(x_-, i_0(x_-, x_+), x_+) \in \Sigma_h$ for all $(x_-, x_+) \in M_h$ and let $\omega_2(x_-, x_+) = \widetilde{\omega}_1(x_-, i_0(x_-, x_+), x_+)$ for $(x_-, x_+) \in M_h$. If x_- or x_+ is in L and $\gamma \in \mathbb{F}_3$, then the invariance under ψ^t gives us

$$\begin{aligned}
&\omega_2(\rho_1(\gamma)(x_-), \rho_2(\gamma)(x_+))\rho_1(\gamma)'(x_-)\rho_2(\gamma)'(x_+)\\
&\quad = \widetilde{\omega}_1(\rho_1(\gamma)(x_-), i_0(\rho_1(\gamma)(x_-), \rho_2(\gamma)(x_+)), \rho_2(\gamma)(x_+))\rho_1(\gamma)'(x_-)\rho_2(\gamma)'(x_+)\\
&\quad = \widetilde{\omega}_1(\rho_1(\gamma)(x_-), \rho_1(\gamma)(i_0(x_-, x_+)), \rho_2(\gamma)(x_+))\rho_1(\gamma)'(x_-)\rho_2(\gamma)'(x_+)\\
&\quad = \widetilde{\omega}_1(x_-, i_0(x_-, x_+), x_+)\\
&\quad = \omega_2(x_-, x_+).
\end{aligned}$$

We have defined a continuous volume form ω_2 on M_h that is (ρ_1, ρ_2)-invariant at points of $(L \times \mathbb{S}^1 \cup \mathbb{S}^1 \times L) \cap M_h$. □

5.8. *Horizontal strips.* The first step in extending ω to all of M_h is to extend it to horizontal strips delimited by elements of L, so that we only need to deal with invariance under one element of the group.

LEMMA 5.7. *Let I be a connected component of $\mathbb{S}^1 \backslash L$ and let $\gamma \in \mathbb{F}_3$ be a generator of its stabilizer. There is a continuous volume form ω on $\mathbb{S}^1 \times \overline{I} \backslash G(h)$ that is invariant by $(\rho_1(\gamma), \rho_2(\gamma))$ and that is equal to ω_2 on $L \times \mathbb{S}^1 \cup \mathbb{S}^1 \times L$.*

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Proof. If γ is conjugate to δ_1, then Proposition 4.1 states that there is a continuous volume form ω_γ on M_h that is invariant under $(\rho_1(\gamma), \rho_2(\gamma))$. If γ is conjugate to δ_2, then [**Mon14a**, Proposition 1.7] gives the same result (the proof is almost identical to Proposition 4.1).

Let $a \in L\backslash\overline{I}$. The interval $[a, \rho_1(\gamma)(a)[$ is a fundamental domain for the action of $\rho_1(\gamma)$ on $\mathbb{S}^1\backslash\overline{I}$, i.e. for every $y \in \mathbb{S}^1\backslash\overline{I}$ there is a unique $n_y \in \mathbb{Z}$ such that $\rho_1(\gamma^{n_y})(y) \in [a, \rho_1(\gamma)(a)[$. We set $\omega = \omega_2$ on $[a, \rho_1(\gamma)(a)[\times \overline{I}$ and extend ω to $(\mathbb{S}^1\backslash\overline{I}) \times \overline{I}$ by using the equivariance formula:

$$\frac{\omega(x, y)}{\omega_2(\rho_1(\gamma^{n_y})(x), \rho_2(\gamma^{n_y})(y))} = \rho_1(\gamma^{n_y})'(x)\rho_2(\gamma^{n_y})'(y).$$

We have to show that ω is continuous. First, we remark that it is continuous at every point of $[a, \rho_1(\gamma)(a)[\times \overline{I}$: if $(x_n, y_n) \to (a, y)$ with $\rho_1(\gamma)(x_n) \in [a, \rho_1(\gamma)(a)[$, then, using the fact that $a \in L$ and because the volume ω_2 is preserved at (a, y), we get

$$\begin{aligned}\omega(x_n, y_n) &= \omega_2(\rho_1(\gamma)(x_n), \rho_1(\gamma)(y_n))\rho_1(\gamma)'(x_n)\rho_1(\gamma)'(y_n)\\ &\to \omega_2(\rho_1(\gamma)(a), \rho_1(\gamma)(y))\rho_1(\gamma)'(a)\rho_1(\gamma)'(y)\\ &= \omega_2(a, y) = \omega(a, y).\end{aligned}$$

If $(x_k, y_k) \in I \times \overline{I} \to (x, y) \in M_h$ with $x \in \partial I$, then set $n_k = n_{x_k}$, as well as $u_k = \rho_1(\gamma^{n_k})(x_k)$ and $v_k = \rho_1(\gamma^{n_k})(y_k)$. By definition, we have

$$\omega(x_k, y_k) = \omega_2(u_k, v_k)\rho_1(\gamma^{n_k})'(x_k)\rho_1(\gamma^{n_k})'(y_k).$$

Since ω_γ is invariant under $\rho_1(\gamma)$, we have

$$\rho_1(\gamma^{n_k})'(x_k)\rho_1(\gamma^{n_k})'(y_k) = \frac{\omega_\gamma(x_k, y_k)}{\omega_\gamma(u_k, v_k)}.$$

These two equalities give us

$$\omega(x_k, y_k) = \frac{\omega_2(u_k, v_k)}{\omega_\gamma(u_k, v_k)}\omega_\gamma(x_k, y_k).$$

The continuity of ω_γ gives us $\omega_\gamma(x_k, y_k) \to \omega_\gamma(x, y)$.

Since $x_k \to x \in \partial I$, we have $n_k \to \infty$ and $v_k \to v$, where v is the other extremal point of I. By using the uniform continuity of ω_2 and ω_γ on $[a, \rho_1(\gamma)(a)] \times \overline{I}$, we obtain

$$\omega(x_k, y_k) \sim \frac{\omega_2(u_k, v)}{\omega_\gamma(u_k, v)}\omega_\gamma(x, y).$$

We now only have to deal with the restrictions of ω_2 and ω_γ to the axes $\{x\} \times \mathbb{S}^1 \cup \mathbb{S}^1 \times \{v\}$ (see Figure 7), where continuous volume forms invariant under $(\rho_1(\gamma), \rho_2(\gamma))$ are unique up to multiplication by a constant: there is $\lambda > 0$ such that $\omega_2(s, t) = \lambda\omega_\gamma(s, t)$ whenever $s = x$ or $t = v$. We can finally conclude that

$$\omega(x_k, y_k) \to \lambda\omega_\gamma(x, y) = \omega_2(x, y) = \omega(x, y).$$

Finally, we can extend ω to $\mathbb{S}^1 \times \overline{I}\backslash G(h)$ by setting $\omega = \lambda\omega_\gamma$ on $\overline{I} \times \overline{I}\backslash G(h)$. □

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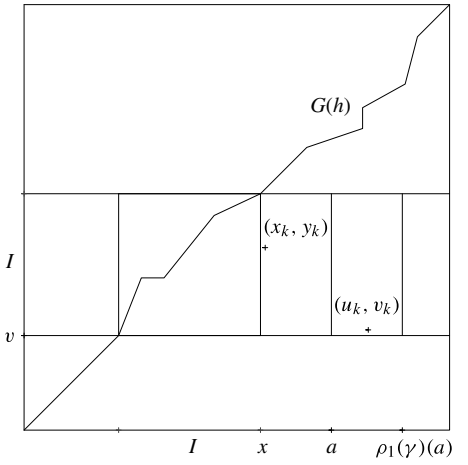


FIGURE 7. Defining ω on horizontal strips.

5.9. *Extending to M_h.* We can now extend ω to M_h. Getting an invariant volume form is not complicated; however, its regularity requires some work.

Our proof of the regularity of ω on horizontal strips relied on the existence of a continuous invariant form by any element of $\mathbb{F}_3$. To deal with the invariance under the whole group, we will need a different method.

PROPOSITION 5.8. *There is a continuous invariant form ω on M_h that is invariant under (ρ_1, ρ_2) and that is equal to ω_2 on $L \times \mathbb{S}^1 \cup \mathbb{S}^1 \times L$.*

Proof. The action of $\mathbb{F}_3$ on the set of connected components of $\mathbb{S}^1 \backslash L$ has two orbits. Let I_1, I_2 be the components preserved by δ_1 and δ_2. By Lemma 5.7, there is a continuous volume form ω on $\mathbb{S}^1 \times \overline{I}_i \backslash G(h)$ that is equal to ω_2 in restriction to $L \times \mathbb{S}^1 \cup \mathbb{S}^1 \times L$ and that is invariant under the stabilizer of I_i. If $\gamma \in \mathbb{F}_3$, then we define ω on $\mathbb{S}^1 \times \rho_2(\gamma)(\overline{I}_i) \backslash G(h)$ to be $(\rho_1(\gamma), \rho_2(\gamma))_* \omega$. This defines a volume form ω on M_h that is (ρ_1, ρ_2)-invariant, continuous on all horizontal strips $\mathbb{S}^1 \times \overline{I} \backslash G(h)$, where I is a connected component of $\mathbb{S}^1 \backslash L$, and equal to ω_2 on $L \times \mathbb{S}^1 \cup \mathbb{S}^1 \cup L$.

To show that ω is continuous, assume that $(x_k, y_k) \to (x, y)$ with $y \in L$ (if $y \notin L$, then there is a connected component I of $\mathbb{S}^1 \backslash L$ such that $y_k \in I$ for k large enough, which gives us $\omega(x_k, y_k) \to \omega(x, y)$). If $y_k \in L$ for all k, then $\omega(x_k, y_k) = \omega_2(x_k, y_k)$ and we already have the continuity; hence, we can assume that $y_k \notin L$ for all k. Up to considering two subsequences, we can assume that there is $\gamma_k \in \mathbb{F}_3$ such that $v_k = \rho_2(\gamma_k)(y_k) \in I_1$. By composing γ_k with an element of the stabilizer of I_1, we can take v_k in a compact interval $K \subset I_1$.

Let $u_k = \rho_1(\gamma_k)(x_k)$. The definition of ω is

$$\omega(x_k, y_k) = \omega(u_k, v_k) \rho_1(\gamma_k)'(x_k) \rho_2(\gamma_k)'(y_k).$$

We have already seen that ω is continuous on $\mathbb{S}^1 \times \overline{I}_1 \backslash G(h)$ and $v_k \in I_1$. The problem in finding the limit of $\omega(x_k, y_k)$ is the control of the Jacobian product $\rho_1(\gamma_k)'(x_k) \rho_2(\gamma_k)'(y_k)$. However, we know that ω is continuous on $L \times \mathbb{S}^1 \cup \mathbb{S}^1 \times L$. We will use this fact to

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get rid of the derivatives: if x'_k and y'_k are sequences in L such that $(x'_k, y'_k) \notin G(h)$, $(x'_k, y_k) \notin G(h)$ and $(x_k, y'_k) \notin G(h)$, then we set $u'_k = \rho_1(\gamma_k)(x'_k)$ and $v'_k = \rho_2(\gamma_k)(y'_k)$. The equivariance equation for ω gives us

$$\frac{\omega(x_k, y_k)}{\omega(x_k, y'_k)} \frac{\omega(x'_k, y'_k)}{\omega(x'_k, y_k)} = \frac{\omega(u_k, v_k)}{\omega(u_k, v'_k)} \frac{\omega(u'_k, v'_k)}{\omega(u'_k, v_k)}. \tag{1}$$

We are now looking for suitable points x'_k and y'_k. Let $I_1 =]a, b[$ and assume that u_k does not admit a as a limit point (up to considering two subsequences and replacing a by b in the following discussion, we can always assume that it is the case), i.e. that a_k lies in a compact interval $J \subset \mathbb{S}^1 \backslash \{a\}$. Let $v'_k = a$ and $y'_k = \rho_2(\gamma_k^{-1})(a) \to y$. If $x_k \in L$, then we choose $x'_k = x_k$. If $x_k \notin L$, then we set x'_k to be an extremal point of the connected component of $\mathbb{S}^1 \backslash L$ containing x_k in a way such that $u'_k = \rho_1(\gamma_k)(x'_k) \in J$.

We now have $y'_k \to y$ and $y_k \to y$, so we get

$$\frac{\omega(x_k, y_k)}{\omega(x_k, y'_k)} \frac{\omega(x'_k, y'_k)}{\omega(x'_k, y_k)} \sim \frac{\omega(x_k, y_k)}{\omega(x, y)} \frac{\omega(x'_k, y)}{\omega(x'_k, y)} = \frac{\omega(x_k, y_k)}{\omega(x, y)}.$$

We wish to show that this quantity converges to 1 as $k \to \infty$. The compact set $E = J \times \{b\} \cup \mathbb{S}^1 \backslash I_1 \times K$ of M_h contains the sequences (u_k, v_k), (u_k, v'_k), (u'_k, v_k) and (u'_k, v'_k). Consequently, the ratio (1) lies in a compact set of $]0, +\infty[$ and it is enough to see that its only possible limit is 1. If there is a subsequence such that the ratio (1) converges to $\lambda \in]0, +\infty[$, then, up to another subsequence, we can assume that the sequence γ_k has the convergence property: there are $N, S \in \mathbb{S}^1$ such that $\rho_1(\gamma_k)(z) \to N$ for all $z \neq S$. Since $\rho_1(\gamma_k^{-1})(z) \to x$ for all $z \in I_1$, we see that S is necessarily equal to x and hence the sequences v_k and v'_k converge to $N \in \mathbb{S}^1$. We get

$$\frac{\omega(u_k, v_k)}{\omega(u_k, v'_k)} \frac{\omega(u'_k, v'_k)}{\omega(u'_k, v_k)} \to \frac{\omega(u, N)}{\omega(u, N)} \frac{\omega(a, N)}{\omega(a, N)} = 1.$$

This shows that $\lambda = 1$; therefore, $\omega(x_k, y_k) \to \omega(x, y)$ and ω is continuous. □

We only showed that ω is continuous, but following [**Mon14a**] one can show that it is possible to obtain a smooth volume form.

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