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Periodic unique beta-expansions: the Sharkovskiĭ ordering

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To O. M. Sharkovskiĭ on the occasion of his 70th birthday

Abstract. Let $\beta \in (1, 2)$. Each $x \in [0, 1/(\beta - 1)]$ can be represented in the form

$$x = \sum_{k=1}^{\infty} \varepsilon_k \beta^{-k},$$

where $\varepsilon_k \in \{0, 1\}$ for all k (a β-expansion of x). If $\beta > (1 + \sqrt{5})/2$, then, as is well known, there always exist $x \in (0, 1/(\beta - 1))$ which have a unique β-expansion. We study (purely) periodic unique β-expansions and show that for each $n \geq 2$ there exists $\beta_n \in [(1 + \sqrt{5})/2, 2)$ such that there are no unique periodic β-expansions of smallest period n for $\beta \leq \beta_n$ and at least one such expansion for $\beta > \beta_n$. Furthermore, we prove that $\beta_k < \beta_m$ if and only if k is less than m in the sense of the Sharkovskiĭ ordering. We give two proofs of this result, one of which is independent, and the other one links it to the dynamics of a family of trapezoidal maps.

1. *History of the problem and formulation of results*

This paper continues the line of research related to the combinatorics of representations of real numbers in non-integer bases [**12, 13, 15, 21**].

Let $\beta \in (1, 2)$ be our parameter and let $x \in I_\beta := [0, 1/(\beta - 1)]$. Then x has at least one representation of the form

$$x = \pi_\beta(\varepsilon_1, \varepsilon_2, \dots) := \sum_{k=1}^{\infty} \varepsilon_k \beta^{-k}, \quad \varepsilon_k \in \{0, 1\}, \tag{1.1}$$

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(use, e.g., the greedy algorithm) which we call *a β-expansion of x* and write $x \sim (\varepsilon_1, \varepsilon_2, \dots)_\beta$.

Let us recall some key results regarding β-expansions. First, if $1 < \beta < G := (1+\sqrt{5})/2$, then each $x \in (0, 1/(\beta-1))$ has a continuum of β-expansions [**12**]. On the other hand, for any $\beta > G$, there exist infinitely many x which have a unique β-expansion (see [**9, 13**]), although almost all $x \in I_\beta$ still have a continuum of β-expansions [**21**].

More specifically, put $x \sim (010101\dots)_\beta = 1/(\beta^2-1)$. Then both x and βx have a unique β-expansion [**13**].

Let X_β denote the set of x which have a unique β-expansion (these numbers are sometimes called univoque numbers with respect to β, see [**10**] for example). Denote by

$$(\mathfrak{m}_k)_{k=0}^{\infty} = 0110\ 1001\ 0110\ 1001\dots$$

the Thue–Morse sequence, i.e. the fixed point of the morphism $0 \to 01$, $1 \to 10$ (see, e.g., the survey paper [**6**] for the many wonderful properties of this famous sequence). Now let $\beta_{KL} \approx 1.787\,23$ denote the *Komornik–Loreti constant*, i.e. the unique positive solution of the equation

$$\sum_{k=1}^{\infty} \mathfrak{m}_k x^{-k} = 1.$$

This constant proves to be the smallest β such that $1 \in X_\beta$; see [**14**]. Note that in [**4**] it was shown that β_{KL} is transcendental.

The main result of [**13**] asserts that the set X_β is:

(1) infinite countable if $G < \beta < \beta_{KL}$;

(2) a continuum of zero Hausdorff dimension if $\beta = \beta_{KL}$; and

(3) a continuum of Hausdorff dimension strictly between zero and one if $\beta_{KL} < \beta < 2$.

Note that for $\beta > \beta_{KL}$ the set X_β is not necessarily a Cantor set and may have a somewhat complicated topology; see [**15**] for more details and also for the case of an arbitrary $\beta > 1$.

Let Σ_β denote the set of all 0–1 sequences which are unique β-expansions, and σ_β denote the shift on Σ_β, i.e. $\sigma_\beta(\varepsilon_1, \varepsilon_2, \varepsilon_3, \dots) = (\varepsilon_2, \varepsilon_3, \dots)$. Let $\pi_\beta : \Sigma_\beta \to X_\beta$ be the projection given by (1.1). It is obvious that π_β is a bijection.

Since

$$\pi_\beta(\{\varepsilon : \varepsilon_1 = 0\}) \subset \left[0, \frac{1}{\beta}\right]$$

and

$$\pi_\beta(\{\varepsilon : \varepsilon_1 = 1\}) \subset \left[\frac{1}{\beta(\beta-1)}, \frac{1}{\beta-1}\right],$$

the set X_β has an empty intersection with the middle interval $[1/\beta, 1/(\beta(\beta-1))]$. Thus, we have the following commutative diagram:

$$\begin{array}{ccc} \Sigma_\beta & \xrightarrow{\sigma_\beta} & \Sigma_\beta \\ \downarrow{\scriptstyle \pi_\beta} & & \downarrow{\scriptstyle \pi_\beta} \\ X_\beta & \xrightarrow{F_\beta} & X_\beta \end{array}$$

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where

$$F_\beta(x) = \begin{cases} \beta x, & 0 \le x < \dfrac{1}{\beta}, \\ \beta x - 1, & \dfrac{1}{\beta(\beta-1)} < x \le \dfrac{1}{\beta-1}. \end{cases}$$

To state the main theorem of the present paper, we recall that the *Sharkovskiĭ ordering* on the natural numbers is as follows:

$$\begin{array}{ccccccccccccc}
 & 3 & \rhd & 5 & \rhd & 7 & \rhd & \cdots & \rhd & 2m+1 & \rhd & \cdots \\
\rhd & 2\cdot 3 & \rhd & 2\cdot 5 & \rhd & 2\cdot 7 & \rhd & \cdots & \rhd & 2\cdot(2m+1) & \rhd & \cdots \\
\rhd & 4\cdot 3 & \rhd & 4\cdot 5 & \rhd & 4\cdot 7 & \rhd & \cdots & \rhd & 4\cdot(2m+1) & \rhd & \cdots \\
 & \vdots & & \vdots & & \vdots & & & & \vdots & & \\
\rhd & 2^n\cdot 3 & \rhd & 2^n\cdot 5 & \rhd & 2^n\cdot 7 & \rhd & \cdots & \rhd & 2^n\cdot(2m+1) & \rhd & \cdots \\
 & \vdots & & \vdots & & \vdots & & & & \vdots & & \\
 & & & \cdots & \rhd & 8 & \rhd & 4 & \rhd & 2 & \rhd & 1,
\end{array}$$

where the relation $a \rhd b$ indicates that a comes before b in the ordering.

Remark 1.1. This is a complete ordering on $\mathbb{N} := \{1, 2, 3, \ldots\}$ since $(n, m) \mapsto 2^n(2m+1)$ is a bijection $(\mathbb{N} \cup \{0\})^2 \to \mathbb{N}$.

THEOREM 1.2. (Sharkovskiĭ's theorem) *Let f be a continuous map of the real line. If $k \rhd l$ in Sharkovkiĭ's ordering and if f has a point of smallest period k, then f also has a point of smallest period l.*

This was originally proved in [**20**], see also [**11**].

Now we are ready to state the main theorem of the present paper. Put

$$\begin{aligned} U_n &= \{\beta \in (1, 2) \mid \Sigma_\beta \text{ contains a periodic sequence of smallest period } n\} \\ &= \{\beta \in (1, 2) \mid F_\beta : X_\beta \to X_\beta \text{ has an } n\text{-cycle}\}. \end{aligned}$$

(By the result quoted above, $U_2 = (G, 2)$, for instance.)

THEOREM 1.3. *There exist real numbers β_n in $(1, 2)$ such that $U_n = (\beta_n, 2)$ for any $n \ge 2$. Furthermore, $\beta_n < \beta_m$ if and only if $n \lhd m$ in the sense of the Sharkovskiĭ ordering.*

Remark 1.4. We are going to give explicit formulae for the β_n via the fragments of the Thue–Morse sequence as well as the first n-cycle to appear: see Proposition 2.16 below. Note that for $n = 2^k$ this result is essentially contained in [**13**], where it is also shown that $\beta_{2^k} \nearrow \beta_{KL}$.

We make use *inter alia* of results in combinatorics on words. In particular, a set of binary sequences studied in [**2**] and denoted by Γ (see (2.2)) will play a role in some proofs. In this respect one could compare the combinatorial part of the main result of [**13**] with the 'Théorème' and 'Corollaire' on [**1**, p. 37].

The structure of this paper is as follows: the next section is devoted to the proof of Theorem 1.3. In §3 we discuss possible links of this claim with the classical theory of one-dimensional continuous maps. (Note that in [**5**] a link was shown between kneading sequences of unimodal continuous maps and unique β-expansions of 1.)

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Our central result of that section is in the negative direction: we show that if $\beta > \beta_4$, then any continuous extension S_β of $F_\beta : X_\beta \to X_\beta$ has a k-cycle for any $k \geq 2$ provided S_β is monotonic on $[0, 1/\beta]$.

Section 4 is devoted to a different proof of our Theorem 1.3 via the classical Sharkovskiĭ theorem applied to a family of trapezoidal maps and, again, some combinatorics on words and properties of the set Γ.

2. *Proof of the main theorem*

Definition 2.1. Put $\tau_\beta x = \beta x \bmod 1$, and $\varepsilon_n = [\beta \tau_\beta^{n-1}(x)]$ for $x \in [0, 1]$ and $n \geq 1$. Then $(\varepsilon_1 \varepsilon_2 \dots)$ is called the *greedy expansion* of x in base β. Put $\Sigma = \{0, 1\}^{\mathbb{N}}$ and let $d(\beta) \in \Sigma$ denote the greedy expansion of 1 in base β. If $d(\beta)$ is finite, i.e. of the form $d(\beta) = \varepsilon_1 \dots \varepsilon_{n-1}\, 10^\infty$, then $d'(\beta) := (\varepsilon_1 \dots \varepsilon_{n-1}\, 0)^\infty$: the *quasi-greedy expansion of* 1.

We say that the sequence (respectively, the finite word) $\varepsilon = \varepsilon_1 \varepsilon_2 \dots$ is *lexicographically less* than the sequence (respectively, the finite word with the same length) ε' if $\varepsilon_k < \varepsilon'_k$ for the least k such that $\varepsilon_k \neq \varepsilon'_k$. Notation: we write $\varepsilon \preceq \varepsilon'$ if either $\varepsilon \prec \varepsilon'$ or $\varepsilon = \varepsilon'$.

We use the following simple remark.

Remark 2.2. The relation $\preceq$ is a total order on the set of infinite sequences (respectively, on the set of words of given length). Furthermore, let ε and ε' be two infinite sequences. Let u and u' be their respective prefixes of length say ℓ, then $u \prec u'$ implies $\varepsilon \prec \varepsilon'$, and $\varepsilon \preceq \varepsilon'$ implies $u \preceq u'$.

We need the following auxiliary results on greedy and quasi-greedy expansions.

LEMMA 2.3. *We have the following results.*

(1) *If $d'(\beta)$ is defined, then it is also an expansion of* 1 *in base β.*

(2) *The equation $1 = \sum_{i=1}^{\infty} \varepsilon_j x^{-j}$, for some fixed $\varepsilon = \varepsilon_1 \varepsilon_2 \varepsilon_3 \dots \in \Sigma$, with at least one zero and at least two ones, always has a unique solution $\beta \in (1, 2)$.*

(3) (Monotonicity) *Let $\beta, \tilde{\beta} \in (1, 2)$. Then $\beta > \tilde{\beta}$ if, and only if $d(\beta) \succ d(\tilde{\beta})$.*

(4) *Let $\beta, \tilde{\beta} \in (1, 2)$ and assume that $d'(\beta)$ and $d'(\tilde{\beta})$ are both defined and of the same smallest period. Then $\beta > \tilde{\beta}$ if, and only if $d'(\beta) \succ d'(\tilde{\beta})$.*

(5) *Assume $\varepsilon \in \Sigma$ satisfies*

$$\sigma^j \varepsilon \begin{cases} \prec \varepsilon & \text{if } j \not\equiv 0 \bmod n; \\ = \varepsilon & \text{if } j \equiv 0 \bmod n. \end{cases}$$

Then there exists $\beta \in (1, 2)$ such that $d'(\beta)$ is defined and equal to ε.

Proof. The proof of (1) is a trivial exercise, while (2) follows from letting

$$f(x) := \sum_{i=1}^{\infty} \frac{\varepsilon_i}{x^i} - 1$$

and observing that $f(1) > 0$, $f(2) < 0$, and $f'(x) < 0$, for all $x \in (1, 2)$; result (3) is proved in [**19**] and (4) is an easy consequence of it.

The proof of (5) follows from the fact that necessarily $\varepsilon_{pn} = 0$ for all $p \in \mathbb{N}$ (otherwise the condition in question is not satisfied), whence $\varepsilon_1 \dots \varepsilon_{n-1} 10^\infty$ is the greedy expansion of 1 in base β. □

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THEOREM 2.4. (Parry [**19**]) *Let $\varepsilon \in \Sigma$. Then $\varepsilon = d(\beta)$ for some $\beta \in (1, 2)$ if and only if $\sigma^j \varepsilon \prec \varepsilon$, for all $j \geq 1$.*

This is essentially proved in [**19**]. The following auxiliary lemmas are needed later on.

LEMMA 2.5. *There exist no $\beta, \tilde{\beta} \in (1, 2)$ such that $d'(\beta) \prec d(\tilde{\beta}) \prec d(\beta)$.*

Proof. Assume that $d'(\beta) := (\varepsilon_1 \dots \varepsilon_{n-1}0)^\infty$ is defined and let $d(\tilde{\beta}) := \delta_1\delta_2\delta_3 \dots$ and suppose that $d'(\beta) \prec d(\tilde{\beta}) \prec d(\beta)$. This immediately forces

$$\delta_1\delta_2\delta_3 \dots \delta_n = \varepsilon_1\varepsilon_2\varepsilon_3 \dots \varepsilon_{n-1}0.$$

By Theorem 2.4, $\delta_{n+1}\delta_{n+2} \cdots = \sigma^n d(\tilde{\beta}) \prec d(\tilde{\beta}) = \varepsilon_1 \dots \varepsilon_{n-1}0\delta_{n+1} \dots$. Hence,

$$\delta_{n+1} \dots \delta_{2n} \preceq \varepsilon_1 \dots \varepsilon_{n-1}0.$$

On the other hand, $(\varepsilon_1 \dots \varepsilon_{n-1}0)^\infty = d'(\beta) \prec d(\tilde{\beta})$ implies that

$$\delta_{n+1} \dots \delta_{2n} \succeq \varepsilon_1 \dots \varepsilon_{n-1}0,$$

and, hence,

$$\delta_{n+1} \dots \delta_{2n} = \varepsilon_1 \dots \varepsilon_{n-1}0.$$

By repeating this process we see that we are forced into the spurious conclusion that $d(\tilde{\beta}) = d'(\beta)$. □

Put

$$\mathcal{A}_\beta = X_\beta \cap \left(\frac{2-\beta}{\beta-1}, 1\right).$$

It is clear that $\mathcal{A}_\beta$ is invariant under F_β and, moreover, it is an attractor for F_β (see [**13**] for further details).

Let $\overline{\varepsilon}$ denote the mirror image of ε, i.e. $(\overline{\varepsilon})_n = 1 - \varepsilon_n$.

LEMMA 2.6. [**13**] *Let $\beta \in (1, 2)$. Then*

$$\Sigma_\beta^{\mathcal{A}_\beta} := \pi_\beta^{-1}(\mathcal{A}_\beta) = \{\varepsilon \in \Sigma \mid \overline{d(\beta)} \prec \sigma^j\varepsilon \prec d(\beta), \forall j \geq 0\} \tag{2.1}$$

if $d'(\beta)$ is not defined. If it is, replace $d(\beta)$ and $\overline{d(\beta)}$ with $d'(\beta)$ and $\overline{d'(\beta)}$, respectively, in the above.

Remark 2.7. In [**13**] it was shown that if $\varepsilon \in \Sigma_\beta$ is periodic, then $\pi_\beta(\varepsilon) \in \mathcal{A}_\beta$.

Put

$$\Gamma := \{\varepsilon \in \Sigma \mid \overline{\varepsilon} \preceq \sigma^k\varepsilon \preceq \varepsilon, \forall k \geq 0\}. \tag{2.2}$$

It is obvious that all sequences in Γ begin with 1 and, furthermore, if $\varepsilon \in \Gamma$ begins with 10, then $\varepsilon = (10)^\infty$. This set has been introduced and studied in detail in [**1, 2**], as well as the sets $\Gamma_\eta := \{\varepsilon \in \Sigma \mid \overline{\eta} \preceq \sigma^j\varepsilon \preceq \eta, \forall j \geq 0\}$ for $\eta \in \Gamma$. (Actually the sequences in the set Γ studied in [**1, 2**] satisfy the extra condition that they begin with 11, which only excludes from the present set Γ the sequence $(10)^\infty$.)

Define

$$V_n := \{\beta \in (1, 2) \mid d'(\beta) \text{ exists, has smallest period } n, \text{ and } d'(\beta) \in \Gamma\},$$

and let $\beta_n := \inf U_n$ and $\gamma_n := \min V_n$. (Provided that $U_n, V_n \neq \emptyset$.)

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PROPOSITION 2.8. *Let $n \geq 2$. Then U_n, $V_n \neq \emptyset$ and $\beta_n = \gamma_n$.*

Proof. For any $n \geq 2$, there always exists a $\beta \in (1, 2)$ such that $d'(\beta) = (1 \ldots 10)^\infty$ (of smallest period n) by Lemma 2.3(5) so $V_n \neq \emptyset$. Note also that V_n is finite, so min V_n is well defined.

Let $\beta \in (1, 2)$, $\beta > \gamma_n$. Then since $\gamma_n \in V_n$, $\sigma^j d'(\gamma_n) \preceq d'(\gamma_n) \prec d(\gamma_n) \prec d(\beta)$ (using monotonicity), and similarly $\sigma^j \overline{d'(\gamma_n)} \preceq d'(\gamma_n) \prec d(\gamma_n) \prec d(\beta)$, for all $j \geq 0$. So we have $\overline{d(\beta)} \prec \sigma^j d'(\gamma_n) \prec d(\beta)$, for all $j \geq 0$ whence $d'(\gamma_n) \in \Sigma_\beta^{\mathcal{A}_\beta} \subset \Sigma_\beta$ by Lemma 2.6 provided that $d'(\beta)$ does not exist. (In the event that $d'(\beta)$ does exist, this still holds via $\overline{d'(\beta)} \prec \sigma^j d'(\gamma_n) \prec d'(\beta)$, for all $j \geq 0$. Indeed if this were false, then $d'(\beta) \preceq \sigma^j d'(\gamma_n) \preceq d'(\gamma_n) \prec d(\gamma_n) \prec d(\beta)$, but this is rendered absurd by Lemma 2.5.) Hence, $d'(\gamma_n) \in \Sigma_\beta$, which implies that $\beta \in U_n$, because $d'(\gamma_n)$ is periodic with smallest period n and, therefore, $\gamma_n \geq \beta_n$, since β was arbitrary. Moreover, we now know that $U_n \neq \emptyset$.

We complete the proof by showing $\gamma_n \leq \beta_n$. Let $\beta \in U_n$, $\beta > \beta_n$. So there exists $\varepsilon \in \Sigma_\beta$ which is periodic with smallest period n. Moreover, because ε is periodic, it must represent some number (in base β) belonging to $\mathcal{A}_\beta$. That is, $\varepsilon \in \Sigma_\beta^{\mathcal{A}_\beta}$. Hence, by Lemma 2.6, $\overline{d(\beta)} \prec \sigma^j \varepsilon \prec d(\beta)$, for all $j \geq 0$. (If $d'(\beta)$ exists, use the fact that $d'(\beta) \prec d(\beta)$.)

Put $a = \max\{\sigma^j \varepsilon, \sigma^j \overline{\varepsilon} \mid 0 \leq j \leq n-1\}$. So $\overline{a} \preceq \sigma^j a \preceq a$, for all $j \geq 0$, that is, $a \in \Gamma$. Moreover, since a is periodic with smallest period n, we must have

$$\sigma^j a \begin{cases} \prec a & \text{if } j \not\equiv 0 \bmod n; \\ = a & \text{if } j \equiv 0 \bmod n. \end{cases}$$

Hence, $a = d'(\tilde{\beta})$, for some $\tilde{\beta}$ (by Lemma 2.3(5)), and so $\tilde{\beta} \in V_n$.

Finally, since $d'(\tilde{\beta}) = a \prec d(\beta)$, we must have $\tilde{\beta} \leq \beta$ by Lemma 2.5. That is, for all $\beta \in U_n$, $\beta > \beta_n$, there exists $\tilde{\beta} \in V_n$ such that $\tilde{\beta} \leq \beta$. Hence, $\gamma_n \leq \beta_n$, as claimed. □

COROLLARY 2.9. *We have $U_n = (\beta_n, 2)$.*

Proof. In the above it was shown that, if $\beta \in (1, 2)$ with $\beta > \gamma_n$, then $\beta \in U_n$. Moreover, it was shown that $\gamma_n = \beta_n = \inf U_n$, hence $U_n = (\beta_n, 2)$ or $[\beta_n, 2)$. We now discount the second case. Let $\beta \in U_n$. So there exists a periodic sequence, with smallest period n, $\varepsilon \in \Sigma$ such that $\pi_\beta(\varepsilon) \in X_\beta$. Moreover, the fact that ε is periodic ensures that $\pi_\beta(\varepsilon) \in \mathcal{A}_\beta$ (see [**13**]). Hence,

$$\overline{d(\beta)} \prec \sigma^j \varepsilon \prec d(\beta) \quad \text{for all } j \in [0, n-1]$$

by Lemma 2.6.

Putting $d(\beta) := d_1 d_2 d_3 \ldots$, let k be the smallest number such that

$$\overline{d_1 d_2 d_3 \ldots d_k} 1^\infty \prec \sigma^j \varepsilon \prec d_1 d_2 d_3 \ldots d_k 0^\infty \quad \text{for all } j \in [0, n-1].$$

Now by Theorem 2.4, there exists $\beta' \in (1, 2)$ such that $d(\beta') = d_1 d_2 d_3 \ldots d_k 0^\infty$, whence

$$\overline{d(\beta')} \prec \sigma^j \varepsilon \prec d(\beta') \quad \text{for all } j \in [0, n-1],$$

that is, $\beta' \in U_n$ by Lemma 2.6. Now note that $d(\beta') \prec d(\beta)$ implies $\beta' < \beta$ by monotonicity. □

We now introduce a result that allows us to make the connection to the Sharkovskiĭ ordering.

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PROPOSITION 2.10. *If a_k denote the lexicographically smallest sequence in Γ of smallest period k, then*

$$\begin{array}{ccccccccccccc}
 & a_3 & \succ & a_5 & \succ & a_7 & \succ & \cdots & \succ & a_{2m+1} & \succ & \cdots \\
\succ & a_{2\cdot 3} & \succ & a_{2\cdot 5} & \succ & a_{2\cdot 7} & \succ & \cdots & \succ & a_{2\cdot(2m+1)} & \succ & \cdots \\
\succ & a_{4\cdot 3} & \succ & a_{4\cdot 5} & \succ & a_{4\cdot 7} & \succ & \cdots & \succ & a_{4\cdot(2m+1)} & \succ & \cdots \\
 & \vdots & & \vdots & & \vdots & & & & \vdots & & \\
\succ & a_{2^n\cdot 3} & \succ & a_{2^n\cdot 5} & \succ & a_{2^n\cdot 7} & \succ & \cdots & \succ & a_{2^n\cdot(2m+1)} & \succ & \cdots \\
 & \vdots & & \vdots & & \vdots & & & & \vdots & & \\
 & & & & & \cdots & \succ & a_8 & \succ & a_4 & \succ & a_2.
\end{array}$$

Proof. This proposition is a consequence of Proposition 2.15 and Lemma 2.12(d) and (f): see below. □

Now by Lemma 2.3(5), $a_k = d'(\beta)$ for some $\beta \in (1, 2)$. Furthermore, by Lemma 2.3(4), $\beta = \beta_k$. Hence, invoking monotonicity allows us to apply the required ordering on the β_k. This reduces the proof of Theorem 1.3 to proving Proposition 2.10.

To calculate β_k explicitly, given a_k, we need to solve $a_k = d'(\beta)$ for β. That is, if $a_k = \alpha_1^{(k)}\alpha_2^{(k)}\ldots$, then β_k is the unique root in $(1, 2)$ of the polynomial

$$x^k - \alpha_1^{(k)}x^{k-1} - \alpha_2^{(k)}x^{k-2} - \cdots - \alpha_{k-1}^{(k)}x - 1.$$

In order to prove Proposition 2.10, we give a construction of the sequences (a_k). This construction was suggested in [**1**] (see also [**2**]). For the sake of completeness, we give here a self-contained proof extracted from these two references. We begin with a definition and two lemmas.

Definition 2.11. We denote by μ the map defined on Σ by: if $\varepsilon = (\varepsilon_n)_{n\geq 1}$ belongs to Σ, then

$$(\mu(\varepsilon))_1 := 1 \quad \text{and for all } n \geq 1, \quad \begin{cases} (\mu(\varepsilon))_{2n} & := \varepsilon_n, \\ (\mu(\varepsilon))_{2n+1} & := 1 - \varepsilon_n. \end{cases}$$

We denote by $L := (\mathfrak{m}_n)_{n\geq 1}$ the sequence obtained by shifting the (complete) Thue–Morse sequence $(\mathfrak{m}_n)_{n\geq 0}$.

LEMMA 2.12. *The following properties of the map μ and of the sequence L hold.*

(a) *For all $k \geq 0$, $\sigma^{2k+1}\mu = \sigma\mu\sigma^k$.*

(b) *For any sequence ε in Σ we have $\sigma(\overline{\mu(\varepsilon)}) = \sigma(\mu(\overline{\varepsilon})) = \overline{\sigma(\mu(\varepsilon))}$.*

(c) *Let ε be a sequence in Σ. If ε is periodic with smallest period $T > 0$ and $\varepsilon_T = 0$, then $\mu(\varepsilon)$ is periodic with smallest period $2T$. In particular, if ε is a sequence in Σ such that $\mu(\varepsilon)$ is periodic with smallest period U, then U cannot be odd, hence U is even, say $U = 2T$, and ε is periodic with smallest period T and satisfies $\varepsilon_T = 0$.*

(d) *If the sequences $\varepsilon = (\varepsilon_n)_{n\geq 1}$ and $\varepsilon' = (\varepsilon'_n)_{n\geq 1}$ in Σ satisfy $\varepsilon \prec \varepsilon'$, then $\mu(\varepsilon) \prec \mu(\varepsilon')$ and $\sigma(\mu(\varepsilon)) \prec \sigma(\mu(\varepsilon'))$.*

(e) *The sequence L is a fixed point of the map μ. Also, for any sequence $\varepsilon := (\varepsilon_n)_{n\geq 1}$, we have that $\mu^\infty(\varepsilon) := \lim_{n\to\infty}\mu^n(\varepsilon)$ exists and $\mu^\infty(\varepsilon) = L$.*

(f) *Let $\varepsilon := (\varepsilon_n)_{n\geq 1}$ be a sequence in Σ. If $\varepsilon \prec L$, then $\varepsilon \prec \mu(\varepsilon)$. If $\varepsilon \succ L$, then $\varepsilon \succ \mu(\varepsilon)$.*

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Proof. The proofs of (a)–(d) are easy and left to the reader.

Let us prove (e). It is straightforward to see that the sequence $0(\mu(\varepsilon))_{n\geq 1}$ is the image of the sequence $0(\varepsilon_n)_{n\geq 1}$ under the morphism φ defined by $\varphi(0) = 01$, $\varphi(1) = 10$, introduced above. In particular, the sequence $L = (\mathfrak{m}_n)_{n\geq 1}$ is a fixed point of μ, since the complete Thue–Morse sequence $(\mathfrak{m}_n)_{n\geq 0}$ is a fixed point of the morphism φ. Also we have, for any sequence ε in Σ and any $k \geq 0$, the equality $0\mu^k(\varepsilon) = \varphi^k(0\varepsilon)$. Hence, $0\mu^\infty(\varepsilon)$ exists and is equal to $\varphi^\infty(0\varepsilon)$ which is precisely the (complete) Thue–Morse sequence $(\mathfrak{m}_n)_{n\geq 0}$.

In order to prove (f), note that, if $\varepsilon \prec \mu(\varepsilon)$, then, using (d), $\mu(\varepsilon) \prec \mu(\mu(\varepsilon))$. Hence, $\varepsilon \prec \mu(\varepsilon) \prec \mu^2(\varepsilon)$, and by induction $\varepsilon \prec \mu(\varepsilon) \prec \mu^k(\varepsilon)$ for all $k \geq 2$. Letting k tend to infinity and using (e) gives that $\varepsilon \prec \mu(\varepsilon) \preceq L$, hence $\varepsilon \prec L$. Reversing the inequalities show that $\varepsilon \succ \mu(\varepsilon)$ implies that $\varepsilon \succ L$, which proves (f). □

LEMMA 2.13. *The set* Γ *has the following properties.*

(a) *Let* ε *be a sequence in* Γ. *Suppose that there exists* $d \geq 1$ *such that*

$$\varepsilon_{d+1}\, \varepsilon_{d+2} \cdots \varepsilon_{2d} = \overline{\varepsilon_1\, \varepsilon_2 \cdots \varepsilon_d}$$

then the sequence ε *is periodic of period* $2d$, *i.e. we have*

$$\varepsilon = (\varepsilon_1\, \varepsilon_2 \cdots \varepsilon_d \overline{\varepsilon_1\, \varepsilon_2 \cdots \varepsilon_d})^\infty.$$

(b) *Let* ε *be a sequence in* Σ. *Then* ε *belongs to* Γ *if and only if* $\varepsilon_1 = 1$ *and* $\mu(\varepsilon)$ *belongs to* Γ.

(c) *If* ε *is a sequence in* Γ *such that* $\varepsilon \neq (10)^\infty$ *and* $\varepsilon \preceq 1(10)^\infty$, *then there exists a sequence* ε' *also in* Γ *such that* $\varepsilon = \mu(\varepsilon')$.

(d) *If* ε *is a periodic sequence in* Γ *such that* $\varepsilon \preceq 1(10)^\infty$, *then its smallest period is even.*

Proof. We first prove (a). Define, for $j \geq 0$, the *block* (or *word*) z_j by

$$z_j := (\varepsilon_{jd+1}\, \varepsilon_{jd+2} \cdots \varepsilon_{(j+1)d})$$

so that we can write

$$\varepsilon = (\varepsilon_1\, \varepsilon_2 \cdots \varepsilon_d)(\varepsilon_{d+1}\, \varepsilon_{d+2} \cdots \varepsilon_{2d}) \cdots (\varepsilon_{jd+1}\, \varepsilon_{jd+2} \cdots \varepsilon_{(j+1)d}) \cdots = z_0\, z_1\, z_2 \cdots .$$

We prove by induction on $j \geq 0$ that $z_{2j} = z_0$ and $z_{2j+1} = \overline{z_0}$. The case $j = 0$ is exactly the hypothesis in (a). Suppose that the result is true for some $j \geq 0$, i.e. that

$$\varepsilon = (z_0\, \overline{z_0})^{j+1}\, z_{2j+2}\, z_{2j+3} \cdots .$$

Now

$$z_{2j+2}\, z_{2j+3} \cdots = \sigma^{2d(j+1)}(\varepsilon) \preceq \varepsilon = z_0\, \overline{z_0} \cdots \tag{2.3}$$

hence $z_{2j+2} \preceq z_0$, and

$$\overline{z_0}\, z_{2j+2}\, z_{2j+3} \cdots = z_{2j+1}\, z_{2j+2}\, z_{2j+3} \cdots = \sigma^{d(2j+1)}(\varepsilon) \succeq \overline{\varepsilon} = (\overline{z_0}\, z_0)^{j+1} \cdots$$

hence $\overline{z_0}\, z_{2j+2} \succeq \overline{z_0}\, z_0$, which gives $z_{2j+2} \succeq z_0$, hence $z_{2j+2} = z_0$. However, the inequality (2.3) now implies $z_{2j+3} \preceq \overline{z_0}$. On the other hand

$$z_{2j+3}\, z_{2j+4} \cdots = \sigma^{d(2j+3)}(\varepsilon) \succeq \overline{\varepsilon} = \overline{z_0} \cdots$$

hence $z_{2j+3} \succeq \overline{z_0}$ which finally implies that $z_{2j+3} = \overline{z_0}$.

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We now prove (b). Suppose that ε belongs to Γ. Since $\overline{\varepsilon} \preceq \varepsilon$, we have $\varepsilon_1 = 1$. Applying Lemma 2.12(b) and (d) to the inequalities $\overline{\varepsilon} \preceq \sigma^k(\varepsilon) \preceq \varepsilon$, for all $k \geq 0$, we obtain

$$\sigma(\overline{\mu(\varepsilon)}) = \sigma(\mu(\overline{\varepsilon})) \preceq \sigma(\mu(\sigma^k(\varepsilon))) \preceq \sigma(\mu(\varepsilon)).$$

Hence, from Lemma 2.12(a),

$$\sigma(\overline{\mu(\varepsilon)}) \preceq \sigma^{2k+1}(\mu(\varepsilon)) \preceq \sigma(\mu(\varepsilon)). \tag{2.4}$$

Since $(\mu(\varepsilon))_1 = 1$, we have $\sigma(\mu(\varepsilon)) \preceq \mu(\varepsilon)$ and $\overline{\mu(\varepsilon)} \preceq \sigma(\overline{\mu(\varepsilon)})$. Hence, the above inequalities yield

$$\overline{\mu(\varepsilon)} \preceq \sigma^{2k+1}(\mu(\varepsilon)) \preceq \mu(\varepsilon).$$

It remains to prove that, for every $k \geq 0$,

$$\overline{\mu(\varepsilon)} \preceq \sigma^{2k}(\mu(\varepsilon)) \preceq \mu(\varepsilon).$$

If $(\sigma^{2k}(\mu(\varepsilon)))_1 = 0$, then $\sigma^{2k}(\mu(\varepsilon)) \prec \mu(\varepsilon)$ as $\mu(\varepsilon)_1 = 1$. On the other hand, (2.4) implies that $\sigma(\overline{\mu(\varepsilon)}) \preceq \sigma(\sigma^{2k}(\mu(\varepsilon)))$. Hence, $\overline{\mu(\varepsilon)} \preceq \sigma^{2k}(\mu(\varepsilon))$, since $(\overline{\mu(\varepsilon)})_1 = (\sigma^{2k}(\mu(\varepsilon)))_1$ $(= 0)$.

If $(\sigma^{2k}(\mu(\varepsilon)))_1 = 1$, then $\overline{\mu(\varepsilon)} \prec \sigma^{2k}(\mu(\varepsilon))$ as $\overline{\mu(\varepsilon)}_1 = 0$. On the other hand, (2.4) implies that $\sigma(\sigma^{2k}(\mu(\varepsilon))) \preceq \sigma(\mu(\varepsilon))$. Hence $\sigma^{2k}(\mu(\varepsilon)) \preceq \mu(\varepsilon)$, since $(\sigma^{2k}(\mu(\varepsilon)))_1$ $= (\mu(\varepsilon))_1$ $(= 1)$.

Now suppose that $\mu(\varepsilon)$ belongs to Γ, and that $\varepsilon_1 = 1$. We clearly have $\overline{\varepsilon} \prec \varepsilon$. It thus suffices to prove that, for any $k \geq 1$, both inequalities $\sigma^k(\varepsilon) \preceq \varepsilon$ and $\overline{\varepsilon} \preceq \sigma^k(\varepsilon)$ hold.

- Let us prove the first inequality. Let $k \geq 1$. If $\varepsilon_{k+1} = 0$, we have $\sigma^k(\varepsilon) = \varepsilon_{k+1}$ $\varepsilon_{k+2} \cdots \prec \varepsilon$. If $\varepsilon_{k+1} = 1$, then either $\varepsilon_j = 1$ for all $j \in [1, k+1]$ and $\sigma^k(\varepsilon) \preceq \varepsilon$ since $\sigma^k(\varepsilon)$ begins with less ones than ε, or there exists $\ell \in [2, k+1]$ such that $\varepsilon_j = 1$ for all $j \in [\ell, k+1]$ and $\varepsilon_{\ell-1} = 0$. However, then

$$\begin{aligned} 1\, \varepsilon_\ell\, \overline{\varepsilon_\ell}\, \varepsilon_{\ell+1}\, \overline{\varepsilon_{\ell+1}} \cdots &= \overline{\varepsilon_{\ell-1}}\, \varepsilon_\ell\, \overline{\varepsilon_\ell}\, \varepsilon_{\ell+1}\, \overline{\varepsilon_{\ell+1}} \cdots \\ &= \sigma^{2\ell-2}(\mu(\varepsilon)) \preceq \mu(\varepsilon) = 1\, \varepsilon_1\, \overline{\varepsilon_1}\, \varepsilon_2\, \overline{\varepsilon_2} \cdots, \end{aligned}$$

  hence

$$\varepsilon_\ell\, \overline{\varepsilon_\ell}\, \varepsilon_{\ell+1}\, \overline{\varepsilon_{\ell+1}} \cdots \preceq \varepsilon_1\, \overline{\varepsilon_1}\, \varepsilon_2\, \overline{\varepsilon_2} \cdots,$$

  which easily implies

$$\sigma^{\ell-1}(\varepsilon) = \varepsilon_\ell\, \varepsilon_{\ell+1} \cdots \preceq \varepsilon_1\, \varepsilon_2 \cdots = \varepsilon.$$

  This in turn implies $\sigma^k(\varepsilon) \preceq \varepsilon$, since the sequence $\sigma^k(\varepsilon)$, beginning with less ones than $\sigma^{\ell-1}(\varepsilon)$, is smaller than $\sigma^{\ell-1}(\varepsilon)$.
- Let us prove the second inequality. Let $k \geq 1$. If $\varepsilon_{k+1} = 1$, we have $\sigma^k(\varepsilon) = \varepsilon_{k+1}$ $\varepsilon_{k+2} \cdots \succ \overline{\varepsilon}$. If $\varepsilon_{k+1} = 0$, then there exists $\ell \in [2, k+1]$ such that $\varepsilon_j = 0$ for all $j \in [\ell, k+1]$ and $\varepsilon_{\ell-1} = 1$ (remember that $\varepsilon_1 = 1$). However, then

$$\begin{aligned} 0\, \varepsilon_\ell\, \overline{\varepsilon_\ell}\, \varepsilon_{\ell+1}\, \overline{\varepsilon_{\ell+1}} \cdots &= \overline{\varepsilon_{\ell-1}}\, \varepsilon_\ell\, \overline{\varepsilon_\ell}\, \varepsilon_{\ell+1}\, \overline{\varepsilon_{\ell+1}} \cdots \\ &= \sigma^{2\ell-2}(\mu(\varepsilon)) \succeq \overline{\mu(\varepsilon)} = 0\, \overline{\varepsilon_1}\, \varepsilon_1\, \overline{\varepsilon_2}\, \varepsilon_2 \cdots, \end{aligned}$$

  hence

$$\varepsilon_\ell\, \overline{\varepsilon_\ell}\, \varepsilon_{\ell+1}\, \overline{\varepsilon_{\ell+1}} \cdots \succeq \overline{\varepsilon_1}\, \varepsilon_1\, \overline{\varepsilon_2}\, \varepsilon_2 \cdots,$$

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which easily implies

$$\sigma^{\ell-1}(\varepsilon) = \varepsilon_\ell\, \varepsilon_{\ell+1} \cdots \succeq \overline{\varepsilon_1}\, \overline{\varepsilon_2} \cdots = \overline{\varepsilon}.$$

This in turn implies $\sigma^k(\varepsilon) \succeq \overline{\varepsilon}$, since the sequence $\sigma^k(\varepsilon)$, beginning with less zeros than $\sigma^{\ell-1}(\varepsilon)$, is larger than $\sigma^{\ell-1}(\varepsilon)$.

To finish with the proof of Lemma 2.13 it suffices to prove (c): namely (d) is a consequence of (c) and of Lemma 2.12(c). So, let ε be a sequence in Γ with $\varepsilon \preceq 1(10)^\infty$. Since $1(10)^\infty = \mu(1^\infty)$, we may suppose that $\varepsilon \prec 1(10)^\infty$. We thus have

$$0(01)^\infty \prec \overline{\varepsilon} \preceq \sigma^k(\varepsilon) \preceq \varepsilon \prec 1(10)^\infty$$

for all $k \geq 0$. This implies in particular that ε cannot contain three consecutive ones nor three consecutive zeros. The sequence ε must begin with one. If $\varepsilon = 10\cdots$, then $\varepsilon = (10)^\infty$ from Lemma 2.13 (a), which is excluded. Thus, $\varepsilon = 11\cdots$, hence $\varepsilon = 110\cdots$. From the inequality $\varepsilon \prec 1(10)^\infty$, there is a maximal integer $i_1 \geq 1$ such that $\varepsilon = 1(10)^{i_1}\cdots$. Hence, $\varepsilon = 1(10)^{i_1}01\cdots$ (recall that ε does not contain three consecutive zeros). Then there exists a maximal integer $j_1 \geq 1$ such that $\varepsilon = 1(10)^{i_1}(01)^{j_1}\cdots$. However, $0(01)^{j_1}\cdots = \sigma^{2i_1}\varepsilon \succeq \overline{\varepsilon} = 0(01)^{i_1}(10)^{j_1}\cdots$. This implies that $j_1 \leq i_1$ and the next term in $\sigma^{2i_1}\varepsilon$ must be 1, i.e. $\sigma^{2i_1}\varepsilon = 0(01)^{j_1}1\cdots$. Finally $\varepsilon = 1(10)^{i_1}(01)^{j_1}1\cdots$, with $0 \leq j_1 \leq i_1$. A similar reasoning can be applied to $\sigma^{2i_1+2j_1}\varepsilon = 11\cdots$, yielding $\sigma^{2i_1+2j_1}\varepsilon = 1(10)^{i_2}(01)^{j_2}1\cdots$ with $1 \leq i_2 \leq i_1$ and $0 \leq j_2 \leq i_1$. Thus, iterating, we obtain

$$\varepsilon = 1(10)^{i_1}(01)^{j_1}(10)^{i_2}(01)^{j_2}(10)^{i_3}(01)^{j_3}\cdots$$

with $1 \leq i_k \leq i_1$ and $0 \leq j_k \leq i_1$. This implies that

$$\varepsilon = \mu(1^{i_1}0^{j_1}1^{i_2}0^{j_2}1^{i_3}0^{j_3}\cdots).$$

The sequence $1^{i_1}0^{j_1}1^{i_2}0^{j_2}1^{i_3}0^{j_3}\cdots$ belongs to Γ from Lemma 2.13(b). □

Remark 2.14. In Lemma 2.13(a) is [**1**, Lemme 2 b, p. 27]. The 'only if' part of (b) is on in [**2**, pp. 26–06]. The proof of Part (c) is inspired by the proof of the partly more general [**1**, 'Lemme 1' on p. 47].

We are now ready to present and prove the following construction of the sequences (a_k).

PROPOSITION 2.15. *Denote by a_k the smallest periodic sequence belonging to Γ whose smallest period is equal to $k \geq 1$. Let $k = 2^n(2m+1)$, then*

$$a_k = \begin{cases} 1^\infty & \text{if } m = n = 0 \text{ (i.e., } k = 1), \\ \mu^n(0^\infty) & \text{if } m = 0 \text{ and } n \geq 1, \\ \mu^n((1(10)^m)^\infty) & \text{if } m \geq 1. \end{cases}$$

Proof. We split the proof into multiple parts.

- The case $m = n = 0$ is trivial.
- Let us address the case $m = 0$. For $n = 1$, we have $k = 2$, and it is easy to see that $a_2 = (10)^\infty = \mu(0^\infty)$. Suppose that $a_{2^n} = \mu^n(0^\infty)$ for some $n \geq 1$. Then $\mu(a_{2^n})$ has smallest period 2^{n+1}, hence $a_{2^{n+1}} \preceq \mu(a_{2^n}) = \mu^{n+1}(0^\infty)$. Now $0^\infty \prec L \prec 1(10)^\infty$, hence $a_{2^{n+1}} \preceq \mu^{n+1}(0^\infty) \prec \mu^{n+1}(L) = L \prec 1(10)^\infty$. This implies, from Lemma 2.13(c), the existence of a sequence z in Σ such that $a_{2^{n+1}} = \mu(z)$. We

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have that z is periodic, and its smallest period is 2^n. Furthermore, $\mu(z) \preceq \mu^{n+1}(0^\infty)$, hence $z \preceq \mu^n(0^\infty)$. This forces $z = \mu^n(0^\infty)$, thus $a_{2^{n+1}} = \mu(z) = \mu^{n+1}(0^\infty)$.

- Let us prove the result for $m \geq 1$ by induction on $n \geq 0$. Take first $n = 0$. Let ε be a sequence in Γ with smallest period $(2m+1)$ for some $m \geq 1$. From Lemma 2.13(d), we must have $\varepsilon \succ 1(10)^\infty$. Hence, the prefix of ε of length $2m+1$ must be larger than or equal to the prefix of $1(10)^\infty$ of length $2m+1$, i.e. $1(10)^m$. The sequence ε being periodic with smallest period $(2m+1)$, this implies that $\varepsilon \succeq (1(10)^m)^\infty$. However, this last sequence clearly belongs to Γ, which implies that it is the smallest sequence in Γ that has smallest period $2m+1$.
- Now, suppose that the result is true for some $n \geq 0$. Let ε be the smallest sequence in Γ whose smallest period is $2^{n+1}(2m+1)$. Since the sequence $\mu^{n+1}((1(10)^m)^\infty)$ is in Γ (use Lemma 2.13 (b)) and has smallest period $2^{n+1}(2m+1)$ (use Lemma 2.12 (c); note that we *need* $m \neq 0$), we have $\varepsilon \preceq \mu^{n+1}((1(10)^m)^\infty)$. Since this clearly implies $\varepsilon \preceq 1(10)^\infty$, Lemma 2.13(c) gives the existence of a sequence ε' in Γ such that $\varepsilon = \mu(\varepsilon')$. The inequality $\mu(\varepsilon') = \varepsilon \preceq \mu^{n+1}((1(10)^m)^\infty)$ implies, using Lemma 2.12(d), that $\varepsilon' \preceq \mu^n((1(10)^m)^\infty)$. However, ε' belongs to Γ and has smallest period $2^n(2m+1)$ (use Lemma 2.12(c)). Hence, from the induction hypothesis, $\varepsilon' = \mu^n((1(10)^m)^\infty)$. Thus, $\varepsilon = \mu(\varepsilon') = \mu^{n+1}((1(10)^m)^\infty)$. □

We conclude the section with an explicit formula for the a_k via fragments of the Thue–Morse sequence.

PROPOSITION 2.16. *The sequences (a_k) are related to the Thue–Morse sequence as follows. Let $k = 2^n(2m+1)$. Then*

$$a_k = \begin{cases} \mathfrak{m}_1^\infty & \text{if } m = n = 0 \text{ (i.e., } k = 1), \\ (\mathfrak{m}_1\mathfrak{m}_2\mathfrak{m}_3 \cdots \mathfrak{m}_{2^n-1}\overline{\mathfrak{m}_{2^n}})^\infty & \text{if } m = 0 \text{ and } n \geq 1, \\ (\mathfrak{m}_1\mathfrak{m}_2\mathfrak{m}_3 \cdots \mathfrak{m}_{3.2^n}(\mathfrak{m}_1 \cdots \overline{\mathfrak{m}_{2^{n+1}}})^{m-1})^\infty & \text{if } m \geq 1. \end{cases}$$

Proof. We may assume that $(m, n) \neq (0, 0)$. We then note that (see the proof of Lemma 2.12 (e)), for any sequence ε in Σ and for any $n \geq 0$, we have $0\mu^n(\varepsilon) = \varphi^n(0\varepsilon)$. Hence, to prove the proposition, it suffices to show that, for any $n \geq 0$,

$$\varphi^n(0^\infty) = 0(\mathfrak{m}_1\mathfrak{m}_2\mathfrak{m}_3 \cdots \mathfrak{m}_{2^n-1}\overline{\mathfrak{m}_{2^n}})^\infty,$$
$$\varphi^n(0(1(10)^m)^\infty) = 0(\mathfrak{m}_1\mathfrak{m}_2\mathfrak{m}_3 \cdots \mathfrak{m}_{3.2^n}(\mathfrak{m}_1 \cdots \overline{\mathfrak{m}_{2^{n+1}}})^{m-1})^\infty \quad \text{for all } m \geq 1.$$

Remembering that $\mathfrak{m}_n$ is nothing but the parity of the sum of the binary digits of n, we clearly have $\mathfrak{m}_{2^n} = 1$, $0(\mathfrak{m}_1\mathfrak{m}_2\mathfrak{m}_3 \cdots \mathfrak{m}_{2^n-1}\overline{\mathfrak{m}_{2^n}})^\infty = (\mathfrak{m}_0\mathfrak{m}_2\mathfrak{m}_3 \cdots \mathfrak{m}_{2^n-1})^\infty$, and

$$\begin{aligned} \mathfrak{m}_1\mathfrak{m}_2\mathfrak{m}_3 \cdots \mathfrak{m}_{3.2^n} &= \mathfrak{m}_1\mathfrak{m}_2\mathfrak{m}_3 \cdots \mathfrak{m}_{2^{n+1}}\mathfrak{m}_{2^{n+1}+1} \cdots \mathfrak{m}_{2^{n+1}+2^n} \\ &= \mathfrak{m}_1\mathfrak{m}_2\mathfrak{m}_3 \cdots \mathfrak{m}_{2^{n+1}-1}\overline{\mathfrak{m}_0\mathfrak{m}_1 \cdots \mathfrak{m}_{2^n-1}}\,0. \end{aligned}$$

Denoting by W_n the word (of length 2^n) $W_n := \mathfrak{m}_0\mathfrak{m}_1 \cdots \mathfrak{m}_{2^n-1}$, what we have to prove boils down to

$$\varphi^n(0^\infty) = W_n^\infty$$
$$\varphi^n(0(1(10)^m)^\infty) = (W_{n+1}\overline{W_n}\,W_{n+1}^{m-1})^\infty \quad \text{for all } m \geq 1.$$

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TABLE 1. The table of β_n for small values of n.

β_n	$d(\beta_n)$	Minimal polynomial	Numerical value	Below β_{KL}?
$n=2$	11	x^2-x-1	1.618 03	Yes
$n=3$	111	x^3-x^2-x-1	1.839 29	No
$n=4$	1101	x^3-2x^2+x-1	1.754 88	Yes
$n=5$	11011	$x^5-x^4-x^3-x-1$	1.812 40	No
$n=6$	110101	$x^6-x^5-x^4-x^2-1$	1.788 54	No
$n=7$	1101011	$x^6-2x^5+x^4-x^3-1$	1.805 09	No
$n=8$	11010011	$x^5-2x^4+x^2-1$	1.784 60	Yes

Using the easily proven relation $W_{n+1}=\varphi(W_n)$ we have $W_n=\varphi^n(0)$, $\overline{W_n}=\varphi^n(1)$, and $W_{n+1}=W_n\overline{W_n}$, thus

$$\varphi^n(0^\infty)=(\varphi^n(0))^\infty=W_n^\infty$$

and

$$\begin{aligned}\varphi^n(0(1(10)^m)^\infty)&=\varphi^n(0)\varphi^n[(1(10)^m)^\infty]=\varphi^n(0)[\varphi^n(1)(\varphi^n(10))^m]^\infty\\&=\varphi^n(0)[\varphi^n(1)(\varphi^n(1)\varphi^n(0))^m]^\infty\\&=W_n[\overline{W_n}\,(\overline{W_n}W_n)^m]^\infty=W_n[\overline{W_n}\,(\overline{W_n}W_n)^{m-1}\overline{W_n}W_n]^\infty\\&=(W_n\overline{W_n}\,(\overline{W_n}W_n)^{m-1}\overline{W_n})^\infty=(W_n\overline{W_n}\,\overline{W_n}(W_n\overline{W_n})^{m-1})^\infty\\&=(W_{n+1}\overline{W_n}W_{n+1}^{m-1})^\infty.\end{aligned}$$

□

For the first eight values of β_n see Table 1.

3. *Impossibility of continuous extension of F_β*

Recall that the map F_β acts on a nowhere dense subset of I_β. It would be tempting to try to explain the Sharkovskiĭ order in our model via some extension of F_β to the whole interval and then applying the classical Sharkovskiĭ theorem to that extended map. In this section we show that this is, in fact, impossible.

THEOREM 3.1. *Let, as above, $\beta_4\approx 1.754\,88$ denote the unique root of $x^3=2x^2-x+1$ lying in $(1,2)$. Assume that we have $\beta\in(\beta_4,2)$; then any continuous map $S_\beta: I_\beta\to I_\beta$ such that $S_\beta|_{X_\beta}=F_\beta$ has a k-cycle for all $k\in\mathbb{N}$ provided that S_β is monotonic on $[0,1/\beta]$.*

Remark 3.2. In particular, this means that any map of the form

$$S_\beta(x)=\begin{cases}\beta x, & 0\le x<\dfrac{1}{\beta},\\ G(x), & \dfrac{1}{\beta}\le x\le\dfrac{1}{\beta(\beta-1)}\\ \beta x-1, & \dfrac{1}{\beta(\beta-1)}<x\le\dfrac{1}{\beta-1},\end{cases}$$

where G is continuous, and

$$G\left(\frac{1}{\beta}\right)=1,\quad G\left(\frac{1}{\beta(\beta-1)}\right)=\frac{2-\beta}{\beta-1},$$

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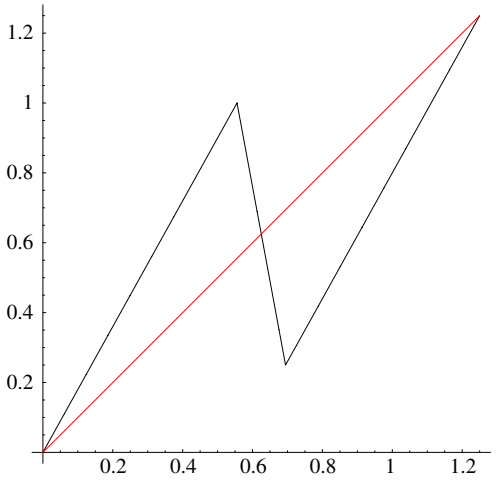


FIGURE 1. A continuous extension of F_β for $\beta = 1.8$.

has cycles of any length provided $\beta > \beta_4$: see Figures 1 and 2. On the other hand, as we know, if $\beta < \beta_8$, then F_β itself has only cycles of length 2 and 4. This means that there is no immediate connection between the classical Sharkovskiĭ theorem and our Theorem 1.3. (See §4 for a less immediate connection.)

Proof. Since $\beta > \beta_4$, we have the following 4-cycle in Σ_β:

$$x_1 \sim (0011)^\infty, \quad x_2 \sim (0110)^\infty,$$
$$x_3 \sim (1100)^\infty, \quad x_4 \sim (1001)^\infty.$$

Consequently, we have a 4-cycle for F_β which we denote by $\{x_1, x_2, x_3, x_4\}$ as well. Note that x_1 and x_2 lie in the left-hand side interval, while x_3 and x_4 belong to the right-hand side interval.

Suppose that such a map S_β exists; then we have $S_\beta^3(x_1) = x_4 > x_1$, $S_\beta^3(x_2) = x_1 < x_2$. By the mean value theorem, the map S_β^3 has a fixed point x_* between x_1 and x_2. Since $F_\beta(x_1) > x_1$ and $F_\beta(x_2) > x_2$ and our assumption on the monotonicity of S_β between x_1 and x_2, we conclude that $S_\beta(x_*) > x_*$, whence x_* is a period-three point for S_β. By the classical Sharkovskiĭ theorem, this implies that S_β has cycles of all possible lengths for $\beta > \beta_4$. □

4. *Another proof of our main theorem using Sharkovskiĭ's classical theorem for trapezoidal maps*

In the previous section we established the fact that there is no natural extension of F_β to the whole interval which preserves the delicate structure of the Sharkovskiĭ ordering on the periodic orbits of F_β. In this section we modify F_β by flipping the right branch, which leads to the family of *trapezoidal maps* (see, e.g., [**16**]) and links our result to the classical Sharkovskiĭ theorem.

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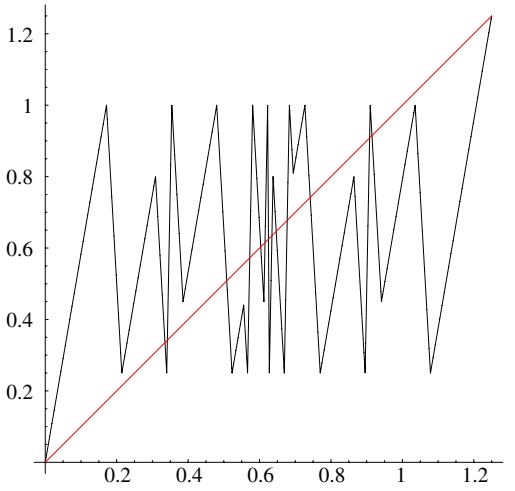


FIGURE 2. The third iterate of the map from Figure 1: observe all the 'parasite' fixed points.

More precisely, define the map $T_\beta : I_\beta \to I_\beta$ as follows:

$$T_\beta(x) = \begin{cases} \beta x, & 0 \le x < \dfrac{1}{\beta}, \\ 1, & \dfrac{1}{\beta} \le x \le \dfrac{1}{\beta(\beta-1)}, \\ \dfrac{\beta}{\beta-1} - \beta x, & \dfrac{1}{\beta(\beta-1)} < x \le \dfrac{1}{\beta-1}. \end{cases}$$

Following the standard notation, we denote the corresponding intervals by L, C and R, respectively; see Figure 3. (Here $C = [1/\beta, 1/(\beta(\beta-1))]$.) We write the itineraries of points under T_β using this notation.

Our goal is to present another proof of the Sharkovskiĭ theorem for the family $(\Sigma_\beta, \sigma_\beta)$ using the classical Sharkovskiĭ theorem for T_β.

Define the map $h : \Sigma \to \{L, R\}^{\mathbb{N}}$ as follows (from here on $*$ denotes an arbitrary, but fixed, tail):

- $h(0*) = Lh(*)$;
- $h(1^a0^b1*) = RL^{a-1}RL^{b-1}h(1*)$ for $a, b \ge 1$;
- $h(1^a0^\infty) = RL^{a-1}RL^\infty$;
- $h(1^\infty) = RL^\infty$.

It is clear that h is well defined and is one-to-one, with $h^{-1}(RL^aRL^b) = 1^{a+1}0^{b+1}$ for $a, b \ge 0$ (on blocks). We claim that h, in fact, maps the orbits of σ_β into the orbits of T_β which do not fall into C.

More precisely, put $\rho_\beta : \{L, R\} \to I_\beta$, where $\rho_\beta(\xi) = x$ such that ξ is the itinerary of x under T_β.

LEMMA 4.1. *For $\varepsilon = 1^\ell 0*$ with $\ell \ge 0$ we have*

$$F_\beta^{\ell+1}(x) = T_\beta^{\ell+1}(x),$$

where $x = \rho_\beta h(\varepsilon)$.

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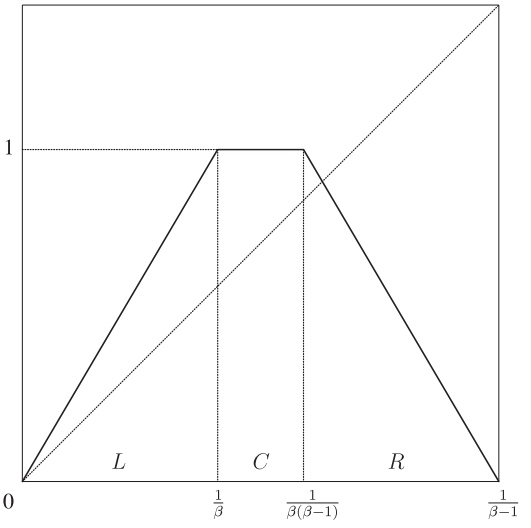


FIGURE 3. The trapezoidal map T_β for $\beta = 1.7$.

Proof. Assume first that $\ell \geq 1$; we have $h(\varepsilon) = RL^{\ell-1}Rh(*)$, whence by the definition of T_β,

$$\begin{aligned} T_\beta(x) &= \frac{\beta}{\beta-1} - \beta x, \\ T_\beta^\ell(x) &= \frac{\beta^\ell}{\beta-1} - \beta^\ell x, \\ T_\beta^{\ell+1}(x) &= \frac{\beta}{\beta-1} - \frac{\beta^{\ell+1}}{\beta-1} + \beta^{\ell+1}x \\ &= \beta^{\ell+1}x - \beta^\ell - \cdots - \beta^2 - \beta. \end{aligned}$$

It is easy to verify that the greedy β-expansion of x begins with $1^\ell 0$, whence by the definition of F_β (see §1), $F_\beta^{\ell+1}(x) = \beta^{\ell+1}x - \beta^\ell - \cdots - \beta^2 - \beta = T_\beta^{\ell+1}(x)$.

If $\ell = 0$, then $F_\beta(x) = T_\beta(x) = \beta x$. □

COROLLARY 4.2. *For $\varepsilon = 1^{a_1}0^{b_1}\ldots 1^{a_s}0*$ with $a_j \geq 1$, $b_j \geq 1$, we have*

$$F_\beta^{\ell+1}(x) = T_\beta^{\ell+1}(x),$$

where $x = \rho_\beta h(\varepsilon)$ and $\ell = \sum_{j=1}^{s-1}(a_j + b_j) + a_s$.

This result allows us to link the periodic orbits of F_β to those of T_β. Note that since h acts blockwise and does not alter the length of a block, a p-periodic orbit of F_β maps into a q-periodic orbit of F_β, where q divides p. In fact, we show that there are only two possibilities: either $q = p$ or $q = p/2$, see below.

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Example 4.3. If $\varepsilon = (1100)^\infty$, then $h(\varepsilon) = (RL)^\infty$. A more complicated example: $\varepsilon = (1101011\ 0010100)^\infty$ and $h(\varepsilon) = (RLRRRRL)^\infty$. Note that in both cases $\varepsilon = (v\overline{v})^\infty$ for some v. We will see later that this is always the case when h cuts a period in half.

Now we are ready to present an alternative proof of the main theorem of this paper. Since the case of powers of two is considered in [**13**], we assume $m \rhd k$ in the Sharkovskiĭ ordering and m is not a power of two. Suppose F_β has an m-cycle; then T_β has an m-cycle or an $m/2$-cycle. In either case, by the classical Sharkovskiĭ theorem, T_β has a k-cycle. We need to make sure, however, that such a cycle does not involve C. Let us call a cycle with this property an *L–R cycle* and prove a version of the Sharkovskiĭ theorem†.

PROPOSITION 4.4. *If T_β has an L–R m-cycle, then it has an L–R k-cycle for any $k \lhd m$.*

Proof. Let $\{x_1, \ldots, x_m\}$ be the cycle in question. Without loss of generality assume that x_1 is the point of this cycle closest to C. If $x_1 < 1/\beta$, we put

$$\widetilde{T}_\beta(x) = \begin{cases} \beta x, & 0 \le x < x_1, \\ \beta x_1, & x_1 \le x \le \dfrac{1}{\beta - 1} - x_1, \\ \dfrac{\beta}{\beta - 1} - \beta x, & \dfrac{1}{\beta - 1} - x_1 < x \le \dfrac{1}{\beta - 1}. \end{cases}$$

If $x_1 > 1/(\beta(\beta - 1))$, put

$$\widetilde{T}_\beta(x) = \begin{cases} \beta x, & 0 \le x < \dfrac{1}{\beta - 1} - x_1, \\ \dfrac{\beta}{\beta - 1} - \beta x_1, & \dfrac{1}{\beta - 1} - x_1 \le x \le x_1, \\ \dfrac{\beta}{\beta - 1} - \beta x, & x_1 < x \le \dfrac{1}{\beta - 1}. \end{cases}$$

In other words, $\widetilde{T}_\beta$ is a trapezoidal map whose graph is made out of the graph of T_β by 'sawing off' its top at the level $y = \beta x_1$ or $y = \beta/(\beta - 1) - \beta x_1$, respectively. Note that $\{x_1, \ldots, x_m\}$ is still an m-cycle for $\widetilde{T}_\beta$, whence, by the classical Sharkovskiĭ theorem, $\widetilde{T}_\beta$ has a k-cycle $\{y_1, \ldots, y_k\}$. It suffices to observe that $y_j \notin C$ for all j, because otherwise $\widetilde{T}_\beta(y_j) = \widetilde{T}_\beta(x_1)$ and, consequently, $\widetilde{T}_\beta^r(y_j) \notin C$ for any $r \ge 1$.

Hence, by our construction, $\{y_1, \ldots, y_k\}$ is a sought *L–R* cycle for T_β. □

Thus, T_β has an *L–R* k-cycle, whence by applying h^{-1}, we conclude that F_β (or σ_β) has a k-cycle or an ℓk-cycle for some $\ell \ge 2$. It suffices to discard the latter case.

Let $\varepsilon = u^\infty$, with

$$u = 1^{a_1}0^{b_1} \ldots 1^{a_{s-1}}0^{b_{s-1}}1^{a_s} \mid 0^{b_s}1^{a_{s+1}}0^{b_{s+1}} \ldots 1^{a_r}0^{b_r},$$

where | separates the two halves of u. Then

$$h(u) = RL^{a_1-1}RL^{b_1-1} \ldots RL^{a_s-1} \mid RL^{b_s-1}RL^{a_{s+1}-1}RL^{b_{s+1}-1} \ldots RL^{a_r-1}RL^{b_r-1}. \tag{4.1}$$

From (4.1) it is clear that if a word u is not a power of another word itself and $h(u)$ is a power, then it must be a square, that is, $\ell = 2$.

† The authors are grateful to Sebastian van Strien for the idea behind this proof.

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Suppose $h(u) = ww$ for some w. Then

$$RL^{a_1-1}RL^{b_1-1}\dots RL^{a_s-1} = RL^{b_s-1}RL^{a_{s+1}-1}RL^{b_{s+1}-1}\dots RL^{a_r-1}RL^{b_r-1},$$

that is, $a_1 = b_s$, $b_1 = a_{s+1}, \dots, a_s = b_r$. In other words, $u = v\overline{v}$ for $v = 1^{a_1}0^{b_1}\dots 1^{a_{s-1}}0^{b_{s-1}}1^{a_s}$.

Thus, if a sequence ε is of smallest period $2k$ and $h(\varepsilon)$ has smallest period k, then $\varepsilon = (v\overline{v})^\infty$ for some v. Our goal is to show that for such an ε one can find $\varepsilon' \in \Sigma_\beta$ of smallest period k, which will conclude the proof.

Analogously to the proof of Proposition 2.8, we consider all the shifts of ε and $\overline{\varepsilon}$ (which are all in Σ_β) and choose the maximal. Hence, without loss of generality, we may again assume $\varepsilon \in \Gamma$, where Γ is given by (2.2).

PROPOSITION 4.5. *Assume that $\varepsilon \in \Gamma$ is a sequence of smallest period $2k$, of the form $\varepsilon = (v\overline{v})^\infty$ for some v, where $|v| = k > 1$. Then there exists a sequence $\varepsilon' \in \Gamma$ such that ε' has smallest period k and $\varepsilon' \prec \varepsilon$.*

Proof. This result can be deduced from the combination of two results of [**1**] (namely Proposition 2 on p. 34 applied to the sequence ε and Proposition 1 on p. 32), but we give a direct proof. Since the sequence ε begins with 1, and since $\sigma^{2k-1}\varepsilon \prec \varepsilon$, the word $\overline{v}$ must end in 0, hence the word v must end in 1. Let $v := w1$, hence $v\overline{v} = w1\overline{w}0$.

The sequence $\varepsilon' := (w0)^\infty$ has period k. This is the smallest period of the sequence ε': if this were not the case, we would have $w0 = (z0)^\ell$ for some word z and some integer $\ell \geq 2$. Thus $w = (z0)^{\ell-1}z$. This would imply that $\varepsilon = (w1\overline{w}0)^\infty = ((z0)^{\ell-1}z1(\overline{z}1)^{\ell-1}\overline{z}0)^\infty$. However, then $\sigma^{2\ell-2}\varepsilon$ begins with $z1$ and the condition $\sigma^{2\ell-2}\varepsilon \preceq \varepsilon$ would not be satisfied.

We clearly have $\varepsilon' \prec \varepsilon$. To prove that ε' belongs to Γ, it clearly suffices to prove that if the word w is written as $w := xy$ (thus, $(w0)^\infty = (xy0)^\infty$), with the condition $0 < |x| < |w| = k - 1$, then

$$\overline{x}\,\overline{y}\,1 \prec y\,0\,x \prec x\,y\,0, \tag{4.2}$$

thus yielding $\overline{\varepsilon'} \prec \sigma^{|x|}\varepsilon' \prec \varepsilon'$ or $\overline{x}\,\overline{y}\,1 = y\,0\,x \prec x\,y\,0$, thus yielding $\overline{\varepsilon'} = \sigma^{|x|}\varepsilon' \prec \varepsilon'$.

Let us prove (4.2). Since $\varepsilon = (w\,1\,\overline{w}\,0)^\infty = (x\,y\,1\,\overline{x}\,\overline{y}\,0)^\infty$ belongs to Γ, we have $\sigma^{|x|}(\varepsilon) \preceq \varepsilon$, hence $y\,1\,\overline{x} \preceq x\,y\,1$. Note that $y\,1\,\overline{x}$ cannot be equal to $x\,y\,1$. Namely, if these two words were equal, this would imply $y\,1\,\overline{x}\,\overline{y}\,0\,x = x\,y\,1\,\overline{x}\,\overline{y}\,0$. This shows that the words x and $y\,1\,\overline{x}\,\overline{y}\,0$ would commute: from a theorem of Lyndon and Schützenberger [**17**] this would imply that there exist a word z and two positive integers a and b such that $x = z^a$ and $y\,1\,\overline{x}\,\overline{y}\,0 = z^b$. Hence, $w\,1\,\overline{w}\,0 = z^{a+b}$ and $2k$ would not be the smallest period of the sequence ε. We thus have $y\,1\,\overline{x} \prec x\,y\,1$, whence

$$y\,1\,\overline{x} \preceq x\,y\,0. \tag{4.3}$$

Obviously, $y\,0\,x \prec y\,1\,\overline{x}$, whence $y\,0\,x \prec x\,y\,0$, which proves the inequality on right-hand side of in (4.2).

Let us prove that $\overline{x}\,\overline{y}\,1 \prec y\,0\,x$ or, equivalently, that

$$\overline{y}\,1\,\overline{x} \prec x\,y\,0. \tag{4.4}$$

We can write $xy = YX$, with $|X| = |x|$ and $|Y| = |y|$. Thus, $\varepsilon = (Y\,X\,1\,\overline{x}\,\overline{y}\,0)^\infty$. Since $\sigma^{2|x|+1+|y|}\varepsilon \preceq \varepsilon$, we have $\overline{y} \preceq Y$. Now, if $\overline{y} \prec Y$, then $\overline{y}\,1\,\overline{x} \prec Y\,X\,0 = x\,y\,0$, which is the sought inequality (4.4), and we are done.

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If $\overline{y} = Y$, then $\varepsilon = (Y\ X\ 1\ \overline{x}\ Y\ 0)^\infty$. We claim that $X\ 1\ \overline{x}$ cannot begin with 0, because if this were the case, say $X\ 1\ \overline{x} := 0\ t$, then $\varepsilon = (Y\ 0\ t\ Y\ 0)^\infty$. The inequality $\sigma^{|y|+|t|+1}\varepsilon \preceq \varepsilon$ would imply that $Y\ 0\ Y\ 0\ t \preceq Y\ 0\ t\ Y\ 0$, hence $Y\ 0\ t \preceq t\ Y\ 0$. On the other hand, $\sigma^{|y|+1}\varepsilon \preceq \varepsilon$ implies that $t\ Y\ 0 \preceq Y\ 0\ t$. We would thus have $t\ Y\ 0 = Y\ 0\ t$. In other words, $Y0$ and t commute, whence, as above, there exist a word z and two positive integers a and b, with $Y0 = z^a$ and $t = z^b$. Consequently, $\varepsilon = (z^{2a+b})^\infty$, which contradicts the minimality of the period of the sequence ε.

Therefore, $X\ 1\ \overline{x}$ must begin with 1, say, $X\ 1\ \overline{x} := 1\ t$. We have $\varepsilon = (Y\ 1\ t\ Y\ 0)^\infty$. Now $\overline{\varepsilon} \preceq \sigma^{|y|+1}\varepsilon$ implies that $\overline{Y}\ 0\ \overline{t} \preceq t\ Y\ 0 \prec t\ Y\ 1$. Hence, in view of $\overline{y} = Y$ and $\overline{X}\ 0\ x = 0\ \overline{t}$, we have

$$\begin{aligned} \overline{y}\ 1\ \overline{x}\ \overline{y}\ 0\ x &= Y\ 1\ \overline{Y}\ \overline{X}\ 0\ x = Y\ 1\ \overline{Y}\ 0\ \overline{t} \\ &\prec Y\ 1\ t\ Y\ 1 = Y\ X\ 1\ \overline{x}\ Y\ 1 \\ &= x\ y\ 1\ \overline{x}\ Y\ 1. \end{aligned} \tag{4.5}$$

This, in turn, implies $\overline{y}\ 1\ \overline{x} \preceq x\ y\ 1$. Note that if we had $\overline{y}\ 1\ \overline{x} = x\ y\ 1$, then (4.5) would imply $\overline{y}\ 0\ x \prec \overline{x}\ Y\ 1 = \overline{x}\ \overline{y}\ 1$, i.e., by barring everything, $x\ y\ 0 \prec y\ 1\ \overline{x}$—which clearly contradicts (4.3).

Hence, $\overline{y}\ 1\ \overline{x} \prec x\ y\ 1$, and thus, by (4.5), $\overline{y}\ 1\ \overline{x} \preceq x\ y\ 0$. Either $\overline{y}\ 1\ \overline{x} \prec x\ y\ 0$, which is precisely the required left-hand side inequality in (4.2), and we are done, or $\overline{y}\ 1\ \overline{x} = x\ y\ 0$, that is, $y\ 0\ x = \overline{x}\ \overline{y}\ 1$. This implies $\sigma^{|x|}\varepsilon' = (y\ 0\ x)^\infty = (\overline{x}\ \overline{y}\ 1)^\infty = \overline{\varepsilon'}$, which, together with the proven right-hand side of the inequality (4.2) yields $\overline{\varepsilon'} \in \Gamma$. □

Thus, we have constructed a periodic sequence $\varepsilon' \in \Gamma$ with smallest period k, which implies $\sigma^j\varepsilon' \preceq \varepsilon' \prec \varepsilon \prec d(\beta)$, and similarly, $\sigma^j\overline{\varepsilon'} \prec d(\beta)$. Hence, $\varepsilon' \in \Sigma_\beta$, and this concludes the second proof of Theorem 1.3 in the case when m is not a power of two.

5. *Final remarks and an open problem*

Remark 5.1. The case $m = 2^n$ is a bit more delicate: here h can map a periodic sequence with smallest period m into a periodic sequence with smallest period $m/2$, for instance, $h((1100)^\infty) = (RL)^\infty$. A direct inspection shows that a 2-cycle for T_β does appear at $\beta = \beta_2$, but it is $(RC)^\infty$, not $(RL)^\infty$. The latter cycle, in fact, appears at $\beta = \beta_4$, where the former case disappears.

We conjecture that the first 2^n-cycle to arise for T_β appears at $\beta = \beta_{2^n}$ but always involves C. The proof is left to the interested reader.

Remark 5.2. Bruin [**7**] proved various results on the Sharkovskiĭ theorem for unimodal maps.

Remark 5.3. Let, as above, $\tau_\beta : [0, 1) \to [0, 1)$ denote the (greedy) β-transformation, that is, $\tau_\beta x = \beta x \bmod 1$. Our remark consists of a simple observation that there is no Sharkovskiĭ theorem for the family $([0, 1), \tau_\beta)_{\beta\in(1,2)}$. Indeed, the set of admissible sequences here is

$$\widetilde{\Sigma}_\beta = \{\varepsilon \in \Sigma : \sigma^j\varepsilon \prec d(\beta), \forall j \geq 0\} \tag{5.1}$$

(see [**19**]), and it is obvious that the smallest periodic sequence ε with smallest period n such that $\varepsilon_1 = 1$ and $\sigma^j\varepsilon \preceq \varepsilon$ for all $j \geq 0$, is $(10^{n-1})^\infty$. Hence, the analogue of U_n is $\widetilde{U}_n = (q_n, 2)$, where q_n is the appropriate root of $x^n = x^{n-1} + 1$, that is, $d'(q_n) = (10^{n-1})^\infty$.

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Hence, $\widetilde{U}_n \subset \widetilde{U}_k$ if and only if $n < k$, which is not a particularly interesting result. Comparing (2.1) and (5.1), we see that the extra condition $\sigma^j \overline{\varepsilon} \prec d(\beta)$ makes all the difference.

Remark 5.4. Let $\prec_u$ denote the *unimodal order* on the itineraries of T_β, that is, $L \prec_u C \prec_u R$ and $\varepsilon \prec_u \varepsilon'$ if $\varepsilon_i \equiv \varepsilon'_i$, $1 \le i \le k$ and either $\varepsilon_{k+1} \prec_u \varepsilon'_{k+1}$ with $\#\{i \in [1, k] : \varepsilon_i = R\}$ even or $\varepsilon_{k+1} \succ_u \varepsilon'_{k+1}$ with $\#\{i \in [1, k] : \varepsilon_i = R\}$ odd (see, e.g., [**18**]).

PROPOSITION 5.5. *We have, for* $\varepsilon, \varepsilon' \in \Sigma$,

$$\varepsilon \prec \varepsilon' \iff h(\varepsilon) \prec_u h(\varepsilon').$$

Proof. Essentially this claim can be found in [**8**], but for the reader's convenience we give a sketch of the proof. Let

$$\varepsilon = 1^{a_1}0^{b_1} \dots 1^{a_s}0^{b_s}0 \dots,$$
$$\varepsilon' = 1^{a_1}0^{b_1} \dots 1^{a_s}0^{b_s}1 \dots.$$

Then

$$h(\varepsilon) = RL^{a_1-1}RL^{b_1-1} \dots RL^{a_s-1}RL^{b_s-1}L \dots$$
$$h(\varepsilon') = RL^{a_1-1}RL^{b_1-1} \dots RL^{a_s-1}RL^{b_s-1}R \dots,$$

whence $h(\varepsilon) \prec_u h(\varepsilon')$. The other cases are similar. □

Open problem. Let $\Omega = \{\boldsymbol{p}_1, \dots, \boldsymbol{p}_m\}$ be points in $\mathbb{R}^d$ and let $S_\Omega(\beta)$ denote the set of 'β-expansions', where the 'digits' are taken from the set Ω. More precisely, for $\beta > 1$, put

$$S_\Omega(\beta) = \left\{(\beta - 1)\sum_{n=1}^{\infty} \beta^{-n}\boldsymbol{a}_n \mid \boldsymbol{a}_n \in \Omega\right\}.$$

Clearly, $S_\Omega(\beta)$ is a subset of the convex hull of Ω, and each $\boldsymbol{x} \in S_\Omega(\beta)$ has at least one *address* $(\boldsymbol{a}_1, \boldsymbol{a}_2, \dots) \in \Omega^{\mathbb{N}}$. Similarly to our setting, one can define the set of points which have a unique address and enquire about possible periods for such points.

In [**22**, §4] the third author studied the case $d = 2$, $m = 3$ (with non-collinear points $\boldsymbol{p}_1$, $\boldsymbol{p}_2$, $\boldsymbol{p}_3$) and showed that the first period to appear is period three. It is also easy to show that the last period to appear is period two, so we have a reverse Sharkovskiĭ order here: at least at the endpoints. Other periods are much harder to deal with though, because of the holes in S_Ω, and it is not even clear whether U_n in this model is an interval for each n.

Obtaining a direct analogue of the Sharkovskiĭ theorem for the shift on the set of unique addresses would be intriguing.

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