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Exponential stability for small perturbations of steep integrable Hamiltonian systems

LAURENT NIEDERMAN

Université Paris XI, Topologie et Dynamique, UMR 8628 du CNRS, Bât 425, 91405 Orsay Cedex, France and IMCCE, Astronomie et Systèmes Dynamiques, UMR 8028 du CNRS, 77 avenue Denfert-Rochereau, 75014 Paris, France
(e-mail: Laurent.Niederman@math.u-psud.fr)

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Abstract. In the 1970s, Nekhorochev proved that for an analytic nearly integrable Hamiltonian system with a perturbation of size ε, the actions linked to the unperturbed Hamiltonian vary only by the order of ε^b over a time of the order of $\exp(C\varepsilon^{-a})$ for some positive constants a, b and C, provided that the unperturbed Hamiltonian meets some *generic* transversality condition known as *steepness*. Among steep systems, convex or quasiconvex systems are easier to describe since the use of energy conservation allows the proof of exponential estimates of stability to be shortened. In this case, Lochak–Neishtadt and Poschel have independently obtained the stability exponents $a = b = 1/2n$ for systems of n degrees of freedom—especially the time exponent (a) is expected to be optimal (see P. Lochak, J.-P. Marco and D. Sauzin. *Preprint*. Institut de Máthematique de Jussieu, 1999; J.-P. Marco and D. Sauzin. *Preprint*. Publ. Math. Inst. Hautes Etudes Science, 2001). Moreover, Lochak's study relies on simultaneous Diophantine approximation which gives a very transparent proof.

However, the proof in the steep case has rarely been studied since Nekhorochev's original work despite various physical examples where the model Hamiltonian is only steep. Here, we combine the original scheme with a simultaneous Diophantine approximation as in Lochak's proof. This yields significant simplifications with respect to Nekhorochev's reasoning: it also allows the exponents $a = b = (2np_1 \cdots p_{n-1})^{-1}$ where $(p_1 \cdots p_{n-1})$ are the steepness indices of the considered Hamiltonian to be obtained. In the quasiconvex case, the steepness indices are all equal to one and we find the same exponents $1/2n$ as Lochak–Neishtadt and Poschel, whose results are thus generalized to the steep case.

1. *Introduction*

1.1. *Set-up.* We look at a perturbed integrable Hamiltonian system whose Hamiltonian is of the form $\mathcal{H}(I, \varphi) = h(I) + \varepsilon f(I, \varphi)$ and is *real analytic* in the action-angle variables

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of the unperturbed system $(I, \varphi) \in \mathcal{P} \times \mathbb{T}^n$, $\mathcal{P} \subseteq \mathbb{R}^n$ denotes an open set and $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. Moreover, we assume that the integrable Hamiltonian is *steep*.

Namely, a numerical function h real analytic over a domain of $\mathbb{R}^n$ is said to be steep at a point $I \in \mathbb{R}^n$ on an affine subspace Λ containing I if there are constants $C > 0$, $\delta > 0$ and $p > 0$ such that along any regular curve γ on Λ connecting I and a point at a distance $d < \delta$ the function $h(\gamma(t))$ has its slope greater than Cd^p at some point. The constants (C, δ) and p are respectively called the coefficients and the index of steepness.

A function h of n variables is said to be steep at the point $I \in \mathbb{R}^n$ if I is not a critical point for h and if, for every $r \in \{1, \ldots, n-1\}$, there exist positive constants C_r, δ_r and p_r such that h is steep at I on any affine subspace Λ_r of dimension r containing I with coefficients (C_r, δ_r) and the index p_r.

Finally, a function h is steep over a domain $\mathcal{P} \subseteq \mathbb{R}^n$ with the steepness coefficients $(C_1, \ldots, C_{n-1}, \delta_1, \ldots, \delta_{n-1})$ and the steepness indices $(p_1, \ldots, p_{n-1})$ if there are no critical points for h in $\mathcal{P}$ and h is steep uniformly over $\mathcal{P}$ with these coefficients and indices.

The notion of steepness will be specified in §2 but the simplest example of a steep function is given by convex functions where all the steepness indices are equal to *one*.

Going back to Hamiltonian dynamics, Nekhorochev's proof is based on a partition of the action space in open sets over which the resonance properties of the action variables allow the local resonant normal forms to be built up to an exponentially small remainder. Then, the steepness of the integrable Hamiltonian allows only a drift of the normalized actions into sets associated to resonances of lower multiplicity than the initial one. Eventually, after a short distance, the orbits reach a resonance-free area (i.e. the Fourier expansion of the perturbation admits only non-resonant harmonics up to an exponentially small remainder) where the local normal form is integrable and yields the confinement of the action variables over the desired amount of time.

This yields the following stability exponents :

$$a = \frac{2}{12\xi + 3n + 14}; \quad b = \frac{3a}{2p_{n-1}}$$

where $\xi = [p_1(p_2 \cdots (p_{n-3}(p_{n-2}n + n - 2) + n - 3) + \cdots + 2) + 1] - 1$ for $n > 2$ and $\xi = 1$ for $n = 2$. Hence, we have $\xi \geq n(n-1)/2$ with equality in the convex case, where $a/2 = b/3 = (6n^2 - 3n + 14)^{-1}$. This proof has been resumed in the convex case by Benettin *et al* [**3**], where they have obtained the exponents $a = b = (n^2 + n)^{-1}$.

Benettin and Gallavotti [**2**] have proved that the quasiconvexity of the unperturbed Hamiltonian (i.e. the system $\nabla h(I) \cdot \eta = \nabla^2 h(I)\eta \cdot \eta = 0$ admits no non-trivial solutions at any point I) allows the proof of these theorems to be simplified significantly. Indeed, after normalization as in the steep case, the quasiconvexity provides an approximate Liapounov function over exponentially long time intervals, which allows the actions to be confined to the initial set where the considered orbit was located. Hence, we only need a local construction instead of a global construction as in Nekhorochev's proof. In the late 1980s and the beginning of the 1990s, the convex case was studied most. In particular, Lochak–Neishtadt [**11**] and Poschel [**17**] have independently obtained the global exponents $a = b = 1/2n$ for systems of n degrees of freedom and $a = b = 1/2(n-m)$ for trajectories

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starting in an area linked to a resonance of multiplicity m. The local time exponent (a) is actually expected to be *optimal* in certain cases [**10, 12**]. Poschel's result comes from a study of the geometry of the resonances which gives an accurate partition of the action space in open sets where the action variables are confined. However, Lochak's proof [**9**] is based on normalization around the periodic orbits of the integrable Hamiltonian. In this way, Lochak gets open sets over which exponential stability holds. Then, Dirichlet's theorem about simultaneous Diophantine approximation ensures that these open sets cover the whole action space and yields the global result.

However, previous studies do not cover the cases of a time-dependent perturbation or a perturbed steep integrable Hamiltonian, despite their importance in physics. For instance, a time-periodic perturbation of a convex Hamiltonian can be reduced to the time-independent perturbation of a quasiconvex Hamiltonian in the extended phase space but this is not the case for a general time-dependent perturbation, where the conservation of energy cannot be used. This problem has been studied by Giorgilli and Zehnder [**5**] in connection with the dynamics of a particle in a time-dependent potential with a high kinetic energy. An arbitrary time-dependent perturbation of a convex Hamiltonian also occurs in the study of the long-time dynamics of an asteroid in the solar system and exponential stability results have been proved in this context by Guzzo and Morbidelli [**6**]. The two latter studies are based on [**3**] and give the same exponents of stability $(n^2+n)^{-1}$. Finally, Nekhorochev's theory also allows the stability around an elliptic equilibrium point in a Hamiltonian system to be studied (see [**4, 13, 16, 18**]). But the most famous example of such an equilibrium in astronomy, namely the stability of an asteroid located at the top of an equilateral triangle with the Sun and Jupiter (the Lagrangian points L4, L5), cannot be tackled with the previous theorems in the convex case. Benettin *et al* [**1**] have shown that, with the actual masses of the Sun and Jupiter, the problem of the stability of the Lagrangian points could be reduced to a study of a small perturbation of an integrable Hamiltonian of three degrees of freedom whose 3-jet satisfies a condition which implies steepness. They proved a theorem with the latter assumptions and when this was applied to the astronomical problem it gave a confinement over very long time intervals for asteroids located close enough to the Lagrangian points L4 or L5.

Nevertheless, no general study of the stability of steep Hamiltonians has been carried out since Nekhorochev's original paper and this is precisely the goal of this paper.

1.2. *Main result.* Our proof of stability relies on Nekhorochev's mechanism since it analyses the dynamics around resonances of any multiplicity and uses local resonant normal forms (see [**14, 17**]). But we substitute the original global construction with a semi-local construction along each trajectory: this construction is based on the approximation of the frequencies $\nabla h(I(t))$ at certain times by periodic vectors—as in the proof of exponential stability, by Lochak, in the convex case [**9**].

In this way, we obtain the following theorem.

THEOREM 1.1. *Under the previous assumptions, for a small enough perturbation* ($\varepsilon \leq \varepsilon_0$) *and for any orbit with initial conditions* $(I(t_0), \varphi(t_0)) \in \mathcal{P} \times \mathbb{T}^n$ *far enough from the*

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boundary of $\mathcal{P}$, we have

$$\|I(t) - I(t_0)\| \leq R(\varepsilon) \quad \textit{for } |t| \leq \mathcal{T}(\varepsilon), \tag{1.1}$$

where $R(\varepsilon)$ and $\mathcal{T}(\varepsilon)$ are of order ε^b and $\exp(C/\varepsilon^a)$ respectively for a certain constant $C > 1$ and

$$a = b = \frac{1}{(2n-1)p_1 \cdots p_{n-1} + 1} \geq \frac{1}{2np_1 \cdots p_{n-1}}.$$

Remark. In the quasiconvex case, $p_1 = \cdots = p_{n-1} = 1$ and we find the same exponents $1/2n$ as Lochak–Neishtadt and Poschel, whose results are thus generalized to the steep case.

2. *Preliminaries and a description of our proof*

2.1. *Preliminaries.* In this section, we describe our assumptions and the concept of steepness given in the previous section.

As before, we consider a nearly integrable Hamiltonian $\mathcal{H}(I, \theta) = h(I) + \varepsilon f(I, \theta)$ where $(I, \theta) \in \mathbb{R}^n \times \mathbb{T}^n$ with $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ are action-angle variables of the integrable Hamiltonian h. We assume that $\mathcal{H}$ is real *analytic* around a fixed complex neighbourhood $V_{R,S}\mathcal{P} \subset \mathbb{C}^{2n}$ of a real domain $\mathcal{P} \times \mathbb{T}^n \subset \mathbb{R}^n \times \mathbb{T}^n$ such that

$$V_{R,S}\mathcal{P} = V_R(\mathcal{P}) \times W_S(\mathbb{T}^n) = \left\{ \begin{array}{l} (I, \theta) \in \mathbb{C}^{2n} \text{ such that } \operatorname{dist}(I, \mathcal{P}) \leq R \text{ and} \\ \operatorname{Re}(\theta) \in \mathbb{T}^n;\ \operatorname{Max}_{j \in \{1,\ldots,n\}} |\operatorname{Im}(\theta_j)| \leq S \end{array} \right\}$$

with $1 > R > 0$, $S > 0$ and the distance to $\mathcal{P}$ given by the Euclidean norm in $\mathbb{C}^n$.

Let $\|\cdot\|_{R,S}$ be the sup norm (L^∞) for numerical or vector-valued functions defined over $V_{R,S}\mathcal{P}$. We assume that $\|f\|_{R,S} \leq 1$ and that ε is a small parameter.

The Hessian matrix is assumed to be uniformly bounded with respect to the norm on the operators, also denoted by $\|\cdot\|$, induced by the Euclidean norm: $\|\partial_I^2 h(I)\|_{R,S} \leq M$ for all $I \in V_R(\mathcal{P})$ for some constant $M > 1$.

We also assume the following *non-degeneracy* condition: $\det(\partial_I^2 h(I)) \neq 0$ for all $I \in U_R\mathcal{P} = V_R\mathcal{P} \cap \mathbb{R}^n$ and, more specifically, we assume that $\|(\partial_I^2 h(I))^{-1}\|_{R,S} \leq M$ for all $I \in U_R(\mathcal{P})$ with the same constant $M > 1$.

Finally, steepness ensures that there is no critical point for the Hamiltonian h over the domain $U_R\mathcal{P} = V_R\mathcal{P} \cap \mathbb{R}^n$ and we suppose that $(1/M) \leq \|\partial_I h(I)\| \leq M$ for all $I \in U_R(\mathcal{P})$.

Of course, we also assume that the integrable Hamiltonian h is *steep* with the coefficients $(C_1, \ldots, C_{n-1}, \delta_1, \ldots, \delta_{n-1})$ and the indices $(p_1, \ldots, p_{n-1})$ over the domain $U_R\mathcal{P} = V_R\mathcal{P} \cap \mathbb{R}^n$ with the following definitions.

Let f be a real function, analytic over a domain of $\mathbb{R}^n$. We denote by $\mathcal{T}_{x_0}\mathcal{F}$ the tangent plane to the level manifold $\mathcal{F} = f^{-1}(f_0)$ for a regular value f_0 at a point $x_0 \in \mathcal{F}$ and $\mathcal{T}_{x_0}\mathcal{F}$ will be identified with an affine subspace of $\mathbb{R}^n$.

Definition 2.1. The function f is steep at the point $x_0 \in \mathbb{R}^n$ if for each $r \in \{1, \ldots, n-1\}$ there exist two positive coefficients (C_r, δ_r) and an index p_r such that for any affine subspace $\Lambda_r \subseteq \mathcal{T}_{x_0}\mathcal{F}$ with $\operatorname{Dim}(\Lambda_r) = r$, we have, for $\eta < \delta_r$,

$$\operatorname{Max}_{0 \leq \xi \leq \eta} (\operatorname{Min}_{x \in S_\xi(x_0) \cap \Lambda_r} \|\nabla f_{|\Lambda_r}(x)\|) > C_r \eta^{p_r},$$

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where $S_\xi(x_0)$ is the sphere of radius ξ centred at x_0, $f_{|\Lambda_r}$ is the restriction of f to Λ_r and $\nabla f_{|\Lambda_r}(x) = \nabla(f_{|\Lambda_r})(x)$ is the projection of $\nabla f(x)$ onto Λ_r.

Definition 2.2. The function f is steep over the domain $\mathcal{P} \subseteq \mathbb{R}^n$ with the positive coefficients $(C_1, \ldots, C_{n-1}, \delta_1, \ldots, \delta_{n-1})$ and indices $(p_1, \ldots, p_{n-1})$ if there are no critical points for f in $\mathcal{P}$ and f is steep at any point $x \in \mathcal{P}$ with uniform coefficients and indices.

Remark.

(i) Note that we do not need uniform growth of the projection of the gradient.

(ii) With no loss of generality, we will assume that the coefficients $(C_1, \ldots, C_{n-1})$ are less than or equal to one.

Steep functions were first defined by Nekhorochev [**13, 14, 15**], who proved the *genericity* of this class of functions among $\mathcal{C}^\infty$ ones. In the analytic case, Ilyashenko [**8**] proved that a complex function, holomorphic in a domain $\mathcal{P}$ of $\mathbb{C}^n$, is steep if its restriction to any affine subspace of $\mathbb{C}^n$ which crosses $\mathcal{P}$ admits only a *finite* number of critical points (counting multiplicity). In this case, for $r \in \{1, \ldots, n-1\}$, the steepness index p_r is bounded by the maximal number of critical points (counting multiplicity) for the restriction of f to any affine subspace of dimension r which crosses $\mathcal{P}$. For instance, the restriction of a quasiconvex function to any affine subspace can only admit at most one critical point; hence, the steepness indices are all equal to *one* for a quasiconvex function.

Steepness can also be detected by a condition on the jets of order three at any point $x \in \mathcal{P}$ which generalizes quasiconvexity (see [**1, 13**]). Namely, we assume that the system $\nabla h(x).\eta = \nabla^2 h(x)\eta.\eta = \nabla^3 h(x)\eta\eta\eta = 0$ admits no non-trivial solutions at any point x, this condition is generic for the functions of three variables and is also used by Benettin *et al* to study the stability of the Lagrangian equilibria [**1**].

2.2. *Description of our proof.* In order to avoid cumbersome expressions, we use the notations $u \lessdot\!* v$ (respectively $u *\!\gtrdot v$, $u =\!* v$, $u *\!= v$) if there is $0 < \mathbf{C} \leq 1$ such that $u < \mathbf{C}v$ (respectively $u\mathbf{C} < v$, $u = \mathbf{C}v$ or $u\mathbf{C} = v$) and the constant $\mathbf{C}$ depends only on the dimension n, the steepness coefficients $(C_1, \ldots, C_{n-1})$ and M but not on the small parameter ε.

First, we begin with an algebraic property: a vector $\omega_0 \in \mathbb{R}^n$ is said to be rational and T-periodic if it is a multiple of an integer vector with $T = \mathrm{Inf}\{t > 0 \text{ such that } t\omega_0 \in \mathbb{Z}^n\}$. In that case, the scalar products $|k \cdot \omega_0|$ for $k \in \mathbb{Z}^n$ such that $k \cdot \omega_0 \neq 0$ admits a lower bound $\ell_0 > 0$. Hence, let $\omega_0 \in \mathbb{R}^n$ be a rational vector and K a positive constant, then there exists a small neighbourhood V_0 of ω_0 and a lower bound $\ell_0 > \ell_1 > 0$ depending on K such that $|k \cdot \omega| \geq \ell_1$ for any $\omega \in V_0$ and all $k \in \mathbb{Z}^n \backslash \langle\omega_0\rangle^\perp$ with $\|k\| \leq K$. If we find a second rational vector ω_1 in the neighbourhood V_0, then the scalar products $|k \cdot \omega_1|$ admit a uniform lower bound for $k \in \mathbb{Z}^n \backslash \{\langle\omega_0\rangle^\perp \cap \langle\omega_1\rangle^\perp\}$ and $\|k\| \leq K$. Moreover, if ω_0 and ω_1 are linearly independent, we have $\mathrm{Dim}(\langle\omega_0\rangle^\perp \cap \langle\omega_1\rangle^\perp) = n - 2$. By a simple generalization, we can ensure that if we find a sequence $(\omega_0, \ldots, \omega_{n-1})$ of close enough rational vectors which are linearly independent (i.e. $(\omega_0, \ldots, \omega_{n-1})$ form a basis of $\mathbb{R}^n$), then all the scalar products $|k \cdot \omega_{n-1}|$ admit a uniform lower bound for $k \in \mathbb{Z}^n$ and $\|k\| < K$ with $K > 0$.

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Now, consider a trajectory of the perturbed system starting at a time t_0 and admitting an increasing sequence of times $\{t_0, \dots, t_{n-1}\} \subset [t_0, t_0 + \exp(cK)]$ such that, for some constants $c > 0$ and $K > 0$, each frequency vector $\nabla h(I(t_k))$ is close to a rational vector ω_k for each $k \in \{0, \dots, n-1\}$ where $(\omega_0, \dots, \omega_{n-1})$ is a basis of $\mathbb{R}^n$ composed of rational vectors which are close enough one to one another to satisfy the previous algebraic property with the constant K. Then, $I(t_{n-1})$ is located in a resonance-free area up to some finite order and a local integrable normal form can be built up to an exponentially small remainder, this allows the actions to be confined.

In §4, we assume that there is a high enough density of rational vectors and prove that Nekhorochev's mechanism together with the steepness of the integrable Hamiltonian precisely ensure that any trajectory of the perturbed system satisfies one of the two following properties. Either, it admits a projection onto the action space over exponentially long time intervals included in a small area or the previous sequence of times $(t_0, \dots, t_{n-1})$ and the basis $(\omega_0, \dots, \omega_{n-1})$ can be built iteratively. The closeness of $\nabla h(I(t_k))$ to ω_k for $k \in \{0, \dots, n-1\}$ is given by an application of a theorem by Dirichlet about a simultaneous Diophantine approximation, which yields a minimal rate of approximation of an arbitrary vector by periodic vectors. This last argument gives an upper bound on the order K of the normalization which can be carried out and imposes our value of the stability exponents, $a = b = ((2n-1)p_1 \cdots p_{n-1} + 1)^{-1}$.

It should be noted that the improved stability theorems found by Lochak and Poschel in the resonant areas cannot be proved with our method in the steep case. Indeed, in the convex case, a resonant (non-integrable) normal form up to an exponentially small remainder is sufficient to prove a stability result and this allows the higher density of periodic vectors in some given directions around resonant areas to be used. Here, we have to build a basis of rational vectors which prevents us from using the latter property.

Finally, we also recall that Nekhorochev [**14**] and Herman [**7**] gave examples of an integrable Hamiltonian which are close to steep but are still non-steep and where a drift of the action variables up to infinity appears under some appropriate perturbation (but no drift speed was computed). This should, perhaps, give a hint for checking the optimality of our estimates in the steep case.

3. *Normal forms and non-resonant areas*

3.1. *Normal forms.* Let Λ be a sublattice of $\mathbb{Z}^n$. A subset $\mathcal{D} \subset \mathcal{P} \subset \mathbb{R}^n$ is said to be (α, K)-non-resonant modulo Λ if, at every point $I \in \mathcal{D}$, we have

$$|k \cdot \partial_I h(I)| = |k \cdot \omega(I)| \geq \alpha \quad \text{for all } I \in \mathcal{D} \text{ and } k \in \mathbb{Z}^n_K \backslash \Lambda$$

where $\mathbb{Z}^n_K = \{k \in \mathbb{Z}^n \text{ such that } \|k\| \leq K\}$ with a fixed $K > 0$.

In the neighbourhood of such a set $\mathcal{D}$, the perturbed Hamiltonian $\mathcal{H}$ can be put in a Λ-resonant normal form $h + g + f_*$ where the Fourier expansion of g contains only harmonics in $\mathbb{Z}^n_K \cap \Lambda$ while the remainder f_* is a small general term.

LEMMA 3.1. (Normal form [**17**]) *Suppose that $\mathcal{D} \subset \mathcal{P}$ is (α, K)-non-resonant modulo Λ and that the following inequalities hold:*

$$\varepsilon \lessdot \frac{\alpha r}{K}; \quad r \lessdot \operatorname{Min}\left(\frac{\alpha}{K}, R\right); \quad 1 \ll K.$$

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Then we can define a real analytic, symplectic diffeomorphism $\Phi : V_{r_*,s_*}\mathcal{D} \longmapsto V_{r,s}\mathcal{D}$ *where* $r_* = r/2$, $s_* = s/6$ *such that the pull-back of* $\mathcal{H}$ *by* Φ *is a* Λ*-resonant normal form* $\mathcal{H} \circ \Phi = h + g + f_*$ *up to a remainder* f_* *with*

$$\|g\|_{r_*,s_*} \ll \varepsilon \quad \text{and} \quad \|f_*\|_{r_*,s_*} \ll \varepsilon \exp(-cK),$$

where c *is some positive constant.*

Moreover, $\|\Pi_I \circ \Phi - Id_I\|_{r_*,s_*} \lll r/6$ *uniformly over* $V_{r_*,s_*}\mathcal{D}$ *where* Π_I *denotes the projection onto the action space and* Id_I *is the identity in the action space. Hence,*

$$V_{r/3}\mathcal{D} \subset \Pi_I(\Phi(V_{r_*,s_*}\mathcal{D})) \subset V_{2r/3}\mathcal{D}.$$

COROLLARY 3.2. *Consider a solution of the normalized system governed by* $\mathcal{H} \circ \Phi$ *and* $(I_0, t_0, \delta_0) \in \mathcal{D} \times \mathbb{R} \times \mathbb{R}^+$ *such that* $\|I(t_0) - I_0\| \leq \delta_0$. *Let* λ_0 *be the affine subspace* $I_0 + \Lambda \otimes \mathbb{R}$ *where* $\Lambda \otimes \mathbb{R}$ *is the vector space spanned by* Λ. *We have*

$$\operatorname{dist}(I(t), \lambda_0) = \|I(t) - \lambda_0\| \ll \delta_0 + \varepsilon \quad \text{for } |t - t_0| \lll \exp(cK) \text{ and } |t| < \mathcal{T}_*$$

where $\mathcal{T}_*$ *is the time of escape of* $V_{r_*,s_*}\mathcal{D}$.

Proof. We denote by $\mathcal{Q}$ the orthogonal projection on $\langle\Lambda\rangle^\perp$. Since $H \circ \Phi$ is in Λ-resonant normal form, we have $(d/dt)\mathcal{Q}(I(t)) = -\mathcal{Q}(\partial_\theta f_*)$ and

$$\begin{aligned}\|I(t) - \lambda_0\| &\leq \|I(t_0) - I_0\| + \|\mathcal{Q}(I(t) - I(t_0))\| \\ &\leq \delta_0 + \|\mathcal{Q}(\partial_\theta f_*(I, \theta))\|_{r_*,s_*}|t - t_0| \ll \varepsilon + \delta_0\end{aligned}$$

provided that $|t - t_0| \lll \exp(cK)$. □

3.2. *Periodic tori.* A vector $\omega \in \mathbb{R}^n$ is said to be *rational* if there is $t > 0$ such that $t\omega \in \mathbb{Z}^n$, in which case $T = \operatorname{Inf}\{t > 0 / t\omega \in \mathbb{Z}^n\}$ is called the *period* of ω.

LEMMA 3.3. *Let* Ω *be the hyperplane* $\langle\omega\rangle^\perp$ *for a rational vector* ω *of period* T. *Then, for all* $k \in \mathbb{Z}^n \backslash \Omega$, *we have* $|k \cdot \omega| \geq 1/T$.

Proof. We have $|k \cdot \omega| = (1/T)|k \cdot T\omega| = (1/T)|k \cdot \alpha|$ for some $\alpha \in \mathbb{Z}^n$ and $|k \cdot \alpha| \neq 0$ since $k \notin \langle\alpha\rangle^\perp = \langle\omega\rangle^\perp$. Hence, $|k \cdot \alpha| \geq 1$ and $|k \cdot \omega| \geq 1/T$. □

COROLLARY 3.4. *Let* $I \in \mathcal{P}$ *such that* $\omega = \partial_I h(I)$ *is rational of period* T. *Consider* $K > 0$ *and* $\varrho > 0$ *such that* $2K\varrho T \lll 1$. *Then the ball* $B(I, \varrho)$ *is* $(1/2T, K)$*-non-resonant modulo* Λ *where* Λ *is the* $\mathbb{Z}$*-module spanned by* $\mathbb{Z}^n_K \cap \langle\omega\rangle^\perp$.

Proof. Since ω is rational, we can ensure that $\dim(\Lambda) = n - 1$ and for all $J \in B(I, \varrho)$, and $k \in \mathbb{Z}^n_K \backslash \Lambda$, we have

$$|k \cdot \partial_I h(J)| \geq |k \cdot \partial_I h(I)| - |k| M\varrho \geq \frac{1}{T} - MK\varrho \gg \frac{1}{2T}.$$ □

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3.3. *Set-up.* For an integer $m \in \{0, \dots, n-1\}$, consider a decreasing sequence of positive real numbers $(\varrho_0, \dots, \varrho_m)$ and $m+1$ periodic actions $(I_0, \dots, I_m) \in \mathcal{P}^{m+1}$ of respective periods $(T_0, \dots, T_m)$ such that $\|I_{j+1} - I_j\| \leq \varrho_j$ for $j \in \{0, \dots, m-1\}$.

For $j \in \{0, \dots, m\}$, we denote by Ω_j the hyperplanes $\langle \nabla h(I_j) \rangle^{\perp}$, by $\mathcal{I}_j$ the sets $\Omega_0 \cap \cdots \cap \Omega_j$ and by τ_j the numbers $\mathrm{Max}_{0 \leq i \leq j}(T_i)$.

Consider a positive constant $K > 0$ then the $\mathbb{Z}$-module (respectively the $\mathbb{R}$-vector space) spanned by $\mathbb{Z}^n_K \cap \mathcal{I}_j$ is denoted by Λ_j (respectively $\Lambda_j \otimes \mathbb{R}$).

LEMMA 3.5. *With the previous notation, assume that* $B(I_0, (n+1)\varrho_0) \subset \mathcal{P}$ *and that the frequencies* $\omega_j = \nabla h(I_j)$ *for* $j \in \{0, \dots, m\}$ *are linearly independent.*

Then, we can ensure that $\dim(\Lambda_j \otimes \mathbb{R}) \leq n - j - 1$ *for all* $j \in \{0, \dots, m\}$ *and if* $2(n-j+1)K\varrho_j\tau_j <\!\!* 1$ $(\forall j \in \{0, \dots, m\})$, *then the balls* $B(I_j, (n-j+1)\varrho_j)$ *are* $(1/2\tau_j, K)$*-non-resonant modulo* Λ_j.

Proof. Let $j \in \{0, \dots, m\}$. Since the sequence $(\varrho_l)_{0 \leq l < j}$ is decreasing, we have $\|I_l - I_j\| \leq \varrho_l + \cdots + \varrho_{j-1} \leq (j-1-l)\varrho_l$ $(\forall l \in \{0, \dots, j-1\})$. Hence, $I \in B(I_j, (n-j+1)\varrho_j)$ yields $\|I - I_l\| \leq (n-l)\varrho_l$ for all $l \in \{0, \dots, j-1\}$.

Then, with the argument of the previous lemma, we have $\forall I \in B(I_j, (n-j+1)\varrho_j)$, $\forall l \in \{0, \dots, j\}$ and $\forall k \in \mathbb{Z}^n_K \backslash \Omega_l$

$$|k \cdot \nabla h(I)| \geq \frac{1}{\tau_l} - M(n-l+1)\varrho_l \not\geq \frac{1}{2\tau_l}$$

with the thresholds of Lemma 3.5.

Hence, for all $k \in \bigcup_{l=0}^{j} \mathbb{Z}^n_K \backslash \Omega_l = \mathbb{Z}^n_K \backslash \bigcap_{l=0}^{j} \Omega_l = \mathbb{Z}^n_K \backslash \Lambda_j$, we have

$$|k \cdot \nabla h(I)| \not\geq \frac{1}{2\tau_j} \quad \text{for all } I \in B(I_j, (n-j+1)\varrho_j). \qquad \square$$

4. *Proof of the main theorem*

The goal now is to build the sequence described in §2 along any trajectory and to get a periodic action located in an area which is free of resonances up to an exponentially small order. In this case, the main theorem of the paper is proved. We will first build it formally with Nekhorochev's mechanism and, in the next section, the use of a simultaneous Diophantine approximation theorem of Dirichlet will prove that our formal scheme is convergent for an arbitrary trajectory.

Consider an arbitrary trajectory $(I(t), \varphi(t))$ of the perturbed system. Let $(I_0, \dots, I_m)$ be a sequence of periodic action of respective periods $(T_0, \dots, T_m)$ and frequencies $(\omega_0, \dots, \omega_m)$. As before, for $j \in \{0, \dots, m\}$ define $\tau_j = \mathrm{Max}_{0 \leq i \leq j}(T_i)$, $\Omega_j = \langle \omega_j \rangle^{\perp}$ and $\mathcal{I}_j = \Omega_0 \cap \cdots \cap \Omega_j$; let also Λ_j be the $\mathbb{Z}$-module spanned by $\mathbb{Z}^n_K \cap \mathcal{I}_j$, and let $d_j = \dim_{\mathbb{R}}(\Lambda_j \otimes \mathbb{R})$.

Definition 4.1. (Fitted sequence) The sequence of periodic action $(I_0, \dots, I_m)$ is a fitted sequence if there are non-decreasing real numbers $t_0, \dots, t_m$, decreasing positive real numbers $r_0, \dots, r_m$ and constants $\kappa > 1$, $K > 0$, $c > 0$, $M > 0$ such that the following six properties hold.

(i) $t_j < t_0 + \exp(cK)$ and $\|I(t_j) - I_j\| \leq Mr_j$ for $j \in \{1, \dots, m\}$.

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(ii) $B(I(t_j), (m+1-j)\kappa r_j + r_j) \subset \mathcal{P}$ and $\|I_{j+1} - I_j\| \leq \kappa r_j$ for $j \in \{0, \ldots, m\}$.
(iii) $r_j < r_{j-1}^{p_{d_{j-1}}}/12M^2$ for $j \in \{0, \ldots, m\}$, with $r_{-1} = 6R/M$, $p_{d_{-1}} = 1$.
(iv) The dimensions $(d_0, \ldots, d_m)$ satisfy $d_0 > d_1 > \cdots > d_j > \cdots > d_m = 0$.
(v) The constant $K > 0$ satisfies the thresholds of Lemma 3.1 with the parameters

$$\left(\frac{1}{4\tau_j}, r_j\right) \text{ for all } j \in \{0, \ldots, m\} \quad \text{and} \quad \left(\frac{1}{4\tau_j}, \frac{M}{3} r_j^{p_{d_j}}\right) \text{ for all } j \in \{0, \ldots, m-1\}.$$

(vi) The thresholds of Lemma 3.5 are satisfied with $(K, \kappa r_j, 2T_j)$ for all $j \in \{0, \ldots, m\}$.

Remark. The last assumptions make it possible to build a normalizing transformation $\mathcal{C}^{(j)}$ on the domain $V_{r_j}(B_j)$ where $B_j = B(I_j, (m+1-j)\kappa r_j)$ for all $j \in \{0, \ldots, m\}$.

4.1. *Formal construction of a fitted sequence.*

THEOREM 4.2. *Consider an arbitrary trajectory $(I(t), \varphi(t))$ of the perturbed system. If the perturbation is small enough, one of the following properties holds:*
(i) *$I(t)$ is bounded over exponentially long time intervals or*
(ii) *assuming that there are periodic actions satisfying properties (i) and (iii) of Definition 4.1, then the trajectory has a fitted sequence of actions with $\kappa = 3M^2$.*

Remark.
(i) In the two cases, the actions are bounded over exponentially long times.
(ii) We make our formal construction going forward in time but we have the same results backward in time.

Proof: first step. Consider an arbitrary solution of the perturbed system with some initial condition $(I(t_0), \varphi(t_0))$ such that $\|I(t_0) - I_0\| \leq Mr_0$ where $\omega_0 = \partial_I h(I_0)$ is a T_0-rational vector and

$$r_0 < \frac{r_{-1}^{p_{d_{-1}}}}{12M^2} = \frac{R}{2M^3}.$$

We choose $\kappa = 3M^2$ and assume that $B(I(t_0), (n+1)\kappa r_0) \subset \mathcal{P}$ and that $(K, r_0, 1/4T_0)$, $(K, 1/4T_0, (M/3)r_0^{p_{d_0}})$ (respectively $(K, \kappa r_0, 2T_0)$) satisfy the thresholds of Lemma 3.1 (respectively of Lemma 3.5), then we have the following lemma.

LEMMA 4.3. *Denoting the projections* Proj_{Ω_0} *(respectively* $\mathrm{Proj}_{\langle\omega_0\rangle}$*) by $\mathcal{Q}_0$ (respectively $\widetilde{\mathcal{Q}}_0$), if $\varepsilon \ll r_0^{\rho_1}$ with $\rho_1 = p_{n-1} = p_{d_0}$, we have*
(i) *either $\|\mathcal{Q}_0(\partial_I h(I(t_0)))\| \geq \frac{1}{2} r_0^{\rho_1}$; or*
(ii) *there is a symplectic analytic transformation defined around a periodic action Mr_0-close to $I(t_0)$ such that in the normalized coordinates, we have*
(∗) *either $\|I(t) - I(t_0)\| < \kappa r_0$ for all $t \in [t_0, t_0 + \exp(cK)]$; or*
(∗∗) *there is $t_1 \in]t_0, t_0 + \exp(cK)[$ such that $\|\mathcal{Q}_0(\partial_I h(I(t_1)))\| \geq \frac{1}{3} r_0^{\rho_1}$.*

Proof. (i) If we have $\|\mathcal{Q}_0(\partial_I h(I(t_0)))\| \geq \frac{1}{2} r_0^{\rho_1}$, the first claim of Lemma 4.3. is satisfied and we choose $t_1 = t_0$.

(ii) Otherwise, we have $\|\mathcal{Q}_0(\partial_I h(I(t_0)))\| \leq \frac{1}{2} r_0^{\rho_1}$ and

$$\|\partial_I h(I(t_0)) - \widetilde{\mathcal{Q}}_0(\partial_I h(I(t_0)))\| \leq \tfrac{1}{2} r_0^{\rho_1}.$$

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Then, our assumption on the non-degeneracy of the frequency map allows us to use the inverse function theorem and to obtain $\widetilde{I}_0 \in \mathcal{P}$ such that $\|I(t_0) - \widetilde{I}_0\| \leq \frac{1}{2} M r_0^{\rho_1} < M r_0$ (since $r_0 < 1$ and $\rho_1 \geq 1$) and $\widetilde{\omega}_0 = \partial_I h(\widetilde{I}_0) = \widetilde{\mathcal{Q}}_0(\partial_I h(I(t_0)))$. Moreover, we have

$$\|\omega_0 - \widetilde{\mathcal{Q}}_0(\partial_I h(I(t_0)))\| = \|\widetilde{\mathcal{Q}}_0(\partial_I h(I_0)) - \widetilde{\mathcal{Q}}_0(\partial_I h(I(t_0)))\| \leq M^2 r_0$$

hence, $\widetilde{\omega}_0 = (1+\xi)\omega_0$ where $|\xi| \leq M^2 r_0 / \|\omega_0\| \leq M^3 r_0 < R/2 < 1/2$.

Consequently, $\widetilde{\omega}_0$ is a rational vector of period $\widetilde{T}_0 = T_0/(1+\xi)$, hence, $2T_0/3 \leq \widetilde{T}_0 \leq 2T_0$ and we substitute $\widetilde{I}_0$ for I_0 as the new periodic action by using $2T_0$ instead of T_0 in the thresholds of Lemmas 3.1 and 3.5 (this is included in the definition of a fitted sequence).

Given this construction and our thresholds in the definition of a fitted sequence, there exists a normalization from $\mathcal{D}_0 = V_{r_0/2}(B(\widetilde{I}_0, (n+1)\kappa r_0))$ to $V_{r_0}(B(\widetilde{I}_0, (n+1)\kappa r_0))$.

Moreover, our refined thresholds in Lemma 3.1 with the parameters $(K, 1/4T_0, \frac{1}{3} M r_0^{\rho_1})$ show that the restriction $\widetilde{\mathcal{C}}^{(0)}$ of the normalizing transformation $\mathcal{C}^{(0)}$ on the smaller domain $\widetilde{\mathcal{D}}_0 = V_{\widetilde{r}_0/3}(B(\widetilde{I}_0, \widetilde{r}_0))$ with $\widetilde{r}_0 = \frac{1}{2} M r_0^{p_{d_0}}$ satisfies $\|\Pi_I \circ \widetilde{\mathcal{C}}^{(0)} - Id_I\|_{\widetilde{\mathcal{D}}_0} \leq^* \widetilde{r}_0/18$ uniformly over $\widetilde{\mathcal{D}}_0$. Indeed, the sets $\widetilde{\mathcal{D}}_0 \subset \mathcal{D}_0$ are $(1/4T_0, K)$-non-resonant modulo $\mathbb{Z}^n_K \cap \Omega_0$ and we use the same construction (given in [**17**]) to build the transformations $\mathcal{C}^{(0)}$ and $\widetilde{\mathcal{C}}^{(0)}$ but we obtain refined estimates for $\widetilde{\mathcal{C}}^{(0)}$ over $\widetilde{\mathcal{D}}_0$ due to our improved thresholds. Now, we have in the *normalized* coordinates (we will use the same notation for all the coordinate systems):

$$\|I(t_0) - \widetilde{I}_0\| \leq \|I(t_0) - {\mathcal{C}^{(0)}}^{-1}(I(t_0))\| + \|{\mathcal{C}^{(0)}}^{-1}(I(t_0)) - {\mathcal{C}^{(0)}}^{-1}(\widetilde{I}_0)\| + \|{\mathcal{C}^{(0)}}^{-1}(\widetilde{I}_0) - \widetilde{I}_0\|$$

hence,

$$\|I(t_0) - \widetilde{I}_0\| \leq \frac{M}{2} r_0^{\rho_1} + 2\frac{M}{18} r_0^{\rho_1} < M^2 r_0 = \frac{\kappa r_0}{3}. \tag{4.1}$$

Finally, Corollary 3.2, the initial distance from $I(t_0)$ to $\widetilde{I}_0$ and our threshold in Lemma 4.3 ($\varepsilon <^* r_0^{\rho_1}$) together with $r_0 < 1$, $\rho_1 \geq 1$ give the following estimate, for all $t \in [t_0, \mathrm{Inf}(t_0 + \exp(cK); t_0^*)]$ where t_0^* is the time of escape of $\mathcal{D}_0$:

$$\|I(t) - \widetilde{I}_0 - \mathcal{Q}_0(I(t) - \widetilde{I}_0)\| \leq \|I(t_0) - \widetilde{I}_0\| + \|I(t) - I(t_0) - \mathcal{Q}_0(I(t) - I(t_0))\|$$

hence,

$$\|I(t) - \widetilde{I}_0 - \mathcal{Q}_0(I(t) - \widetilde{I}_0)\| \leq \frac{2}{3} M r_0^{\rho_1} < M^2 r_0 = \frac{\kappa r_0}{3}. \tag{4.2}$$

($*$) If $\|\mathcal{Q}_0(I(t) - \widetilde{I}_0)\| < M^2 r_0 = \kappa r_0/3$ for all $t \in [t_0, t_0 + \exp(cK)]$, then using the previous estimates (4.1) and (4.2), we set

$$\|I(t) - \widetilde{I}_0\| < 2M^2 r_0 \quad \text{for all } t \in [t_0, t_0 + \exp(cK)].$$

Hence, we have $\|I(t) - I(t_0)\| \leq 3M^2 r_0 = \kappa r_0$. Moreover, $I(t) \in B(\widetilde{I}_0, \kappa r_0)$ for $|t - t_0| < \exp(cK)$ ensures that the trajectory remains in the normalizing domain for an exponentially long times and this proves the second claim of Lemma 4.3.

($**$) Otherwise, there is a time of escape $t_e \in]t_0, t_0 + \exp(cK)[$ such that in the normalized coordinates we have $\|\mathcal{Q}_0(I(t_e) - \widetilde{I}_0)\| = M^2 r_0$. Then, the steepness of h yields

$$\exists t_1 \in [t_0, t_e] \text{ such that } \|\mathcal{Q}_0(\partial_I h(\widetilde{I}_0 + \mathcal{Q}_0(I(t_1) - \widetilde{I}_0)))\| \geq (M^2 r_0)^{\rho_1} \geq M^2 r_0^{\rho_1}$$

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since $M > 1$. Now, the lower bound on the projection of the gradient given by the steepness together with the estimate (4.2) yields

$$\|\partial_I h(I(t_1)) - \partial_I h(\widetilde{I}_0 + \mathcal{Q}_0(I(t_1) - \widetilde{I}_0))\| \leq \tfrac{2}{3}M^2 r_0^{\rho_1}$$
$$\Longrightarrow \|\mathcal{Q}_0(\partial_I h(I(t_1)))\| \geq \tfrac{1}{3}M^2 r_0^{\rho_1}.$$

Moreover, $\|I(t_1) - \widetilde{I}_0\| \leq \frac{2}{3}\kappa r_0$, which completes the proof of Lemma 4.3. □

Now, we assume that there is a T_1-periodic action I_1 such that $\|I(t_1) - I_1\| \leq M r_1$ with $r_1 \leq r_0^{\rho_1}/12M^2$.

Then, we can write, with $\omega_1 = \partial_I h(I_1)$,

$$\|\partial_I h(I(t_1)) - \omega_1\| \leq M\|I(t_1) - I_1\| \leq \frac{r_0^{\rho_1}}{12} \Longrightarrow \|\mathcal{Q}_0(\omega_1)\| \geq \frac{r_0^{\rho_1}}{4}$$

and $\|I_1 - I_0\| < \kappa r_0$.

Hence, (ω_0, ω_1) are linearly independent and we can ensure that $d_1 < d_0$.

In order to complete the proof of the first step of Theorem 4.2, we still take $\kappa = 3M^2$ and assume that $(K, r_1, 1/4\tau_1)$, $(K, (\frac{1}{3}M)r_1^{p_{d_1}}, 1/4\tau_1)$ (respectively $(K, \kappa r_1, 2\tau_1)$) with $\tau_1 = \mathrm{Max}(T_0, T_1)$ satisfy the thresholds of Lemma 3.1 (respectively of Lemma 3.5).

4.2. *Recurrence.* Assume that an increasing sequence of times $(t_0, \ldots, t_k)$ and a sequence of periodic actions $(I_0, \ldots, I_k)$ of respective periods $(T_0, \ldots, T_k)$ which satisfy the assumptions of a fitted sequence have been built up to order $k \in \{0, \ldots, m-1\}$. Under the previous assumptions, a repeat of the arguments in Lemma 4.3 give the following lemma.

LEMMA 4.4. *If $\varepsilon \lessdot\!\!\ast\, r_k^{p_{d_k}}$, we have*

(i) *either* $\|\mathrm{Proj}_{\Lambda_k \otimes \mathbb{R}}(\partial_I h(I(t_k)))\| \geq \frac{1}{2} r_k^{p_{d_k}}$*; or*

(ii) *there is a symplectic analytic transformation defined around a periodic action Mr_k-close to $I(t_k)$ such that in the normalized coordinates, we have*

(∗) *either $\|I(t) - I(t_k)\| < \kappa r_k$ for all $t \in [t_k, t_0 + \exp(cK)]$ or*

(∗∗) *there is $t_{k+1} \in]t_k, t_0 + \exp(cK)[$ such that $\|\mathrm{Proj}_{\Lambda_k \otimes \mathbb{R}}(\partial_I h(I(t_{k+1})))\| \geq \frac{1}{3} r_k^{p_{d_k}}$.*

Now, we assume that there exists a T_{k+1}-periodic action I_{k+1} such that

$$\|I(t_{k+1}) - I_{k+1}\| \leq M r_{k+1} \quad \text{with } r_{k+1} \leq \frac{r_k^{p_{d_k}}}{12M^2},$$

which implies that $\|\mathrm{Proj}_{\Lambda_k \otimes \mathbb{R}}(\omega_{k+1})\| \geq \frac{1}{4} r_k^{p_{d_k}}$ and $\|I_{k+1} - I_k\| < \kappa r_k$.

Hence, $\langle \omega_{k+1} \rangle^{\perp} \not\subset \Lambda_k \otimes \mathbb{R}$ and we can ensure that $d_{k+1} < d_k$.

To complete the iterative proof of Theorem 4.2, again we still set $\kappa = 3M^2$ and assume that $(K, r_{k+1}, 1/4\tau_{k+1})$, $(K, \frac{1}{3}M r_{k+1}^{p_{d_{k+1}}}, 1/4\tau_{k+1})$ (respectively $(K, \kappa r_{k+1}, 2\tau_{k+1})$) with $\tau_{k+1} = \mathrm{Max}_{0 \leq i \leq k+1}(T_i)$ satisfy the thresholds of Lemma 3.1 (respectively of Lemma 3.5).

Hence, the construction of a fitted sequence is completed for an arbitrary trajectory provided that the thresholds in Definition 4.1 and Lemmas 4.3 and 4.4 are satisfied, this completes the proof of Theorem 4.2. □

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4.3. *Last step.* We have located a resonant-free area $B_m = B(I_m, \kappa r_m)$ and, in order to confine the actions, we only need to build the integrable normal form over the domain $\mathcal{D}_m = V_{r_m}(B_m)$. This is possible provided that the parameters $(K, 1/4\tau_m, r_m)$ (respectively $(K, \kappa r_m, 2\tau_m)$) satisfy the thresholds of Lemma 3.1 (respectively of Lemma 3.5).

5. *Complete construction of a fitted sequence*

Now, we have to ensure the existence of a sequence of periodic action $(I_0, \dots, I_m)$ such that $\|I(t_k) - I_k\| < Mr_k$ for $k \in \{0, \dots, m\}$ with $r_k < r_{k-1}^{p_{d_{k-1}}}/12M^2$.

We denote $\rho_j = p_{n-j}$ and $\pi_k = \prod_{l=n-k}^{n-1} p_l$ for j and k in $\{1, \dots, n-1\}$.

We assume that $d_k = \mathrm{Dim}(\Lambda_k \otimes \mathbb{R}) = n - k - 1$; hence, $m = n - 1$, $p_{d_{-1}} = 1$ and $p_{d_k} = \rho_{k+1}$ for $k \in \{0, \dots, n-2\}$. With this assumption, we have the lowest rate of decrease of the dimension of the resonant module and, consequently, the worst estimates for the radius and the time of stability.

We introduce Dirichlet's theorem on simultaneous Diophantine approximation (see Schmidt [**19**]), which gives the lowest rate of approximation of an arbitrary vector by an integer one.

Let $x \in \mathbb{R}^n$ and $Q \in \mathbb{N}^*$, we can renumber the indices in such a way that $x = \xi(\pm 1, x')$ for some $x' \in \mathbb{R}^{n-1}$ and $\xi = \|x\|_\infty$ where $\|\cdot\|_\infty$ is the maximum of the components.

The question is now reduced to approximation in $\mathbb{R}^{n-1}$. Dirichlet's theorem yields $q \in \mathbb{N}^*$ and $l' \in \mathbb{Z}^{n-1}$ such that $1 \le q < Q$ and $\|qx' - l'\|_\infty \le Q^{-1/(n-1)}$.

If $x^* = \xi(\pm 1, l'/q)$ we have

$$\|x^* - x\|_\infty \le \frac{\xi}{q} Q^{-1/(n-1)} \Longrightarrow \|x^* - x\| \le \sqrt{n-1}\,\frac{\xi}{q} Q^{-1/(n-1)}$$

for the Euclidean norm.

One can easily check that x^* is a rational vector of period $T = \xi/q$ which satisfies

$$\frac{1}{M} \le \frac{1}{\xi} \le T \le \frac{q}{\xi} \le nMq < nMQ \quad \text{or} \quad 1 \ast\!\!\leq T \ast\!\!< Q$$

with our assumption $1/M \le \|\partial_I h(I)\| \le M$ for all $I \in U_R(\mathcal{P})$.

In this way, with $x_k = \partial_I h(I(t_k))$ for $k \in \{0, \dots, n-1\}$, we can find a T_k-periodic vector $x_k^* \in \mathbb{R}^n$ such that

$$\|x_k - x_k^*\| \le \frac{M}{T_k Q_k^{1/(n-1)}} \quad \text{with } 1 \ast\!\!< T_k \le Q_k$$

and the inversibility of the frequency map gives the desired sequence of T_k-periodic action.

With this ingredient, we now prove the following.

THEOREM 5.1. *There exists a constant $c_0 > 1$ sufficiently large such that with*

$$K_0 \ll c_0^{-\pi_{n-1}},\ a = (1 + (2n-1)\pi_{n-1})^{-1} \ge (2n\pi_{n-1})^{-1} \quad (\pi_{n-1} = p_1 \cdots p_{n-1})$$

and ε sufficiently small, there is a fitted sequence for an arbitrary trajectory with the parameters $K = K_0\varepsilon^{-a}$, $\tau_l \ast\!\!\leq (c_0\varepsilon^a)^{-(n-1)\pi_l}$, $\kappa = 3M^2$ and $r_l \le \widetilde{c}_0 \varepsilon^{a\pi_l}$ for all $l \in \{0, \dots, n-1\}$ with $\pi_0 = 1$ and a positive constant $\widetilde{c}_0$.

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5.1. *Summary of the thresholds.* With $\tau_k = \mathrm{Max}_{0\le l\le k}(T_l)$, $r_{-1} = R$ and $\rho_0 = 1$, for $k \in \{0, \dots, n-1\}$, we need

(i) $4\varepsilon\tau_k K \lessapprox r_k$,

(ii) $K\tau_k r_k \lessapprox 1$,

(iii) $1 \lessapprox K$, and

(iv) $r_k \le r_{k-1}^{\rho_k}/12M^2$.

Moreover, for $k \in \{0, \dots, n-2\}$, we need

(v) $\varepsilon \lessapprox r_k^{\rho_{k+1}}$,

(vi) $4\varepsilon\tau_k K \lessapprox \frac{1}{3}Mr_k^{\rho_{k+1}}$, and

(vii) $K\tau_k\frac{1}{3}Mr_k^{\rho_{k+1}} \lessapprox 1$.

Remark. An important ingredient for the completion of this proof follows from Lemma 4.3 where it was shown that the use of Dirichlet's theorem to find a periodic action in an r_k-neighbourhood of $I(t_k)$ also ensures the existence of a refined normal form in an $r_k^{\rho_{k+1}}$-neighbourhood of $I(t_k)$. Indeed, the thresholds needed for the construction of a normal form in an $r_k^{\rho_{k+1}}$-neighbourhood cannot be satisfied since the period of an action given by Dirichlet's theorem in this small neighbourhood grows too rapidly. This can already be checked at the first step of our iterative lemma.

5.2. *The iterative scheme.* As in Lochak's paper [**9**] we start with $r_0 = c_0(\varepsilon^a/T_0)$ with $a > 0$ and $c_0 > 1$ a positive constant which will be chosen *implicitly sufficiently large* to ensure that all the following thresholds are satisfied. Hence, we need

$$\|x_0 - x_0^*\| \le \frac{M}{T_0 Q_0^{(1/n-1)}} < Mc_0\frac{\varepsilon^a}{T_0} \quad \text{where } 1 \lessapprox T_0 \le Q_0$$

and we choose $Q_0 = (c_0\varepsilon^a)^{-n+1}$.

Here, thresholds (i)–(vii) are ensured for ε sufficiently small if we choose a and K such that

$$0 < a \le \frac{1}{(1+\rho_1)n} \quad \text{and} \quad K_0 \eqslantless c_0^{-\rho_1} \quad \text{since } K = K_0\varepsilon^{-a}.$$

Then, for $l \in \{1, \dots, n-1\}$, we assume the existence of a sequence of T_k-periodic actions $(I_0, \dots, I_{l-1})$ such that $\|I(t_k) - I_k\| \le Mr_k$ with the parameters (K, τ_k, r_k) which satisfy the thresholds (i)–(vii) for $k \in \{0, \dots, l-1\}$.

Let $\mathcal{T}_0 = T_0$ and $\mathcal{T}_l = \mathrm{Max}_{0\le k\le l-1}(T_k^{\pi_l/\pi_k}) = \mathrm{Max}_{0\le k\le l-1}(T_k^{\rho_l\cdots\rho_{k+1}})$ for $l \ge 1$ (with $\pi_0 = 1$). The considered sequence is built in the following way:

- if there is a T_l-periodic action such that $\|I(t_k) - I_k\| \le r_{l-1}^{\rho_l}/12M$ and $T_l \le \mathcal{T}_l$, then we set $r_l = r_{l-1}^{\rho_l}/12M^2$;
- otherwise, we use Dirichlet's theorem to find a T_l-periodic action I_l such that $\|I(t_k) - I_k\| \le r_{l-1}^{\rho_l}/12M$ with $\mathcal{T}_l < T_l \le Q_l$.

Let $j_0 = 0$ and let $(j_1, \dots, j_m)$ be the subsequence of $\{1, \dots, n-1\}$ where $T_{j_k} > \mathcal{T}_{j_k}$. Then, for $l \in \{j_k, \dots, j_k - 1\}$, we have $T_l \le \mathcal{T}_l$. Hence, $r_l = r_{l-1}^{\rho_l}/12M^2$ and

$$r_{j_{k+1}-1} = \frac{1}{12M^2}\left(\frac{1}{12M^2}\right)^{\pi_{j_{k+1}-1}/\pi_{j_{k+1}-2}} \cdots \left(\frac{1}{12M^2}\right)^{\pi_{j_{k+1}-1}/\pi_{j_k+1}} r_{j_k}^{\pi_{j_{k+1}-1}/\pi_{j_k}}.$$

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We want

$$r_{j_{k+1}} \leq \frac{1}{12M^2}\left(\frac{1}{12M^2}\right)^{\pi_{j_{k+1}}/\pi_{j_{k+1}-1}} \cdots \left(\frac{1}{12M^2}\right)^{\pi_{j_{k+1}}/\pi_{j_k+1}} r_{j_k}^{\pi_{j_{k+1}}/\pi_{j_k}}.$$

Since we use Dirichlet's theorem at the j_kth and j_{k+1}th steps of the iterative scheme, we need

$$\frac{1}{T_{j_{k+1}} Q_{j_{k+1}}^{1/(n-1)}} \lessapprox \left(\frac{1}{T_{j_k} Q_{j_k}^{1/(n-1)}}\right)^{\pi_{j_{k+1}}/\pi_{j_k}}$$

$$\iff \frac{1}{Q_{j_{k+1}}^{1/(n-1)}} \lessapprox \frac{T_{j_{k+1}}}{T_{j_k}^{\rho_{j_{k+1}}\cdots\rho_{j_k+1}}}\left(\frac{1}{Q_{j_k}^{1/(n-1)}}\right)^{\pi_{j_{k+1}}/\pi_{j_k}}.$$

Since $T_{j_k}^{\rho_{j_{k+1}}\cdots\rho_{j_k+1}} \leq \mathcal{T}_{j_{k+1}} < T_{j_{k+1}}$, we set $Q_{j_{k+1}} \eqslantless Q_{j_k}^{\pi_{j_{k+1}}/\pi_{j_k}}$ and $Q_0 = (c_0\varepsilon^a)^{-n+1}$ gives

$$Q_{j_k} \eqslantless (*Q_{j_{k-1}})^{\pi_{j_k}/\pi_{j_{k-1}}} = (*Q_{j_{k-2}})^{\pi_{j_k}/\pi_{j_{k-2}}} = \cdots = Q_0^{\pi_{j_k}}$$

and the bounds $Q_{j_k} \eqslantless (c_0\varepsilon^a)^{-(n-1)\pi_{j_k}}$ for $k \in \{1, \ldots, m\}$ are sufficient to ensure that $r_{j_k} \leq r_{j_k-1}^{\rho_{j_k}}/12M^2$.

The previous inequality is also satisfied by construction for $l \in \{0, \ldots, n-1\}\backslash\{j_0, \ldots, j_m\}$.

Then, we should check the other thresholds.

Case 1. Let $l \in \{0, \ldots, n-2\}$ such that $\mathcal{T}_l < T_l$; hence, $l \in \{j_0, \ldots, j_m\}$ and for $j \in \{0, \ldots, l-1\}$, we have $T_l \geq T_j^{\rho_l \cdots \rho_{j+1}} \geq T_j T_j^{\rho_l \cdots \rho_{j+1}-1}$ then $T_j \gtrapprox 1$ gives $T_l \gtrapprox T_j$ and $T_l \gtrapprox \tau_l$.

Hence, $\varepsilon T_l K \lessapprox r_l$ implies threshold (i), since $Q_l \eqslantless (c_0\varepsilon^a)^{-(n-1)\pi_l}$: this is ensured if $\varepsilon T_l^2 K \lessapprox (c_0\varepsilon^a)^{\pi_l}$ which imposes $a \leq (1 + (2n-1)\pi_l)^{-1}$.

Threshold (ii) is implied by $K T_l r_l \lessapprox 1$, then the value of Q_l gives the previous inequality if we impose $K_0 \lessapprox c_0^{-\pi_l}$.

Threshold (iii) is ensured for ε small enough.

Threshold (v) and the value of Q_l imposes $\varepsilon T_l^{\rho_{l+1}} \lessapprox (c_0\varepsilon^a)^{\pi_l \rho_{l+1}}$ and we can ensure the previous inequality if we impose $a \leq (n\pi_{l+1})^{-1}$.

The threshold (vi) and the value of Q_l imposes $\varepsilon T_l^{1+\rho_{l+1}} K \lessapprox (c_0\varepsilon^a)^{\pi_l \rho_{l+1}}$ and we can ensure the previous inequality if we impose $a \leq (1 + n\pi_{l+1} + (n-1)\pi_l)^{-1}$.

Finally, the value of Q_l and $K_0 \lessapprox c_0^{-\rho_{l+1}}$ together with $T_l \gtrapprox 1$ ensure the threshold (vii).

Case 2. Let $l \in \{0, \ldots, n-2\}$ such that $T_l \leq \mathcal{T}_l$; hence, $l \in \{j_k + 1, \ldots, j_{k+1} - 1\}$ where $k \in \{0, \ldots, m\}$. With our construction, we have $r_l \eqslantgtr r_{j_k}^{\pi_l/\pi_{j_k}} \eqslantgtr (c_0\varepsilon^a)^{\pi_l} T_{j_k}^{-\pi_l/\pi_{j_k}}$ with $1 \lessapprox T_{j_k} \leq Q_{j_k}$ where $Q_{j_k}^{-\pi_l/\pi_{j_k}} \eqslantless (c_0\varepsilon^a)^{-(n-1)\pi_l}$.

LEMMA 5.2. *For $i \in \{j_k + 1, \ldots, l\}$, we have $T_i \leq \mathcal{T}_i = T_{j_k}^{\rho_i \cdots \rho_{j_k+1}} = T_{j_k}^{\pi_i/\pi_{j_k}}$.*

Proof. We have $\mathcal{T}_{j_k} < T_{j_k}$ which implies $T_{j_k}^{\rho_{j_k+1}} > T_l^{\rho_{j_k+1}\cdots\rho_{l+1}}$ for all $l \leq j_k - 1$; hence, $\mathcal{T}_{j_k+1} = T_{j_k}^{\rho_{j_k+1}}$ and $T_{j_k+1} \leq \mathcal{T}_{j_k+1}$ yields $T_{j_k+1} \leq \mathcal{T}_{j_k+1} T_{j_k}^{\rho_{j_k+1}}$.

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For $i \in \{j_k+1,\dots,l-1\}$, assume that we have $T_j \le \mathcal{T}_j = T_{j_k}^{\rho_{j}\cdots\rho_{j_k+1}}$ when $j \in \{j_k+1,\dots,i\}$ then $T_{j_k}^{\rho_{i+1}\rho_i\cdots\rho_{j_k+1}} \ge T_j^{\rho_{i+1}\rho_i\cdots\rho_{j+1}}$ for all $j \le i-1$ and $T_i^{\rho_{i+1}} \le \mathcal{T}_i^{\rho_{i+1}}$ implies $T_i^{\rho_{i+1}} \le T_{j_k}^{\rho_{i+1}\rho_i\cdots\rho_{j_k+1}}$; hence, $\mathcal{T}_{i+1} \le T_{j_k}^{\rho_{i+1}\rho_i\cdots\rho_{j_k+1}}$ and we have the equality with the definition of $\mathcal{T}_{i+1}$. Finally, $i+1 \le j_{k+1}-1$ implies $T_{i+1} \le \mathcal{T}_{i+1}$. □

As proved in the first case, since $T_{j_k} \ge \mathcal{T}_{j_k}$ we have $T_{j_k} >\!\!* \tau_{j_k}$ and, for $l \in \{j_k+1,\dots,j_{k+1}-2\}$, we can write $\tau_l *\!\!< \mathrm{Max}(T_{j_k}, T_{j_k+1},\dots,T_l)$. With Lemma 5.2, we have $T_{j_k}^{\pi_i/\pi_{j_k}} \ge T_i$ for all $i \in \{j_k+1,\dots,l\}$ and $\tau_l *\!\!< T_{j_k}^{\pi_l/\pi_{j_k}}$. Hence, with the values of r_l, Q_{j_k} and the upper bound on τ_l, the inequality $\varepsilon T_{j_k}^{2\pi_l/\pi_{j_k}} K <\!\!* (c_0\varepsilon^a)^{\pi_l}$ implies threshold (i): since $T_{j_k} *\!\!< (c_0\varepsilon^a)^{-(n-1)\pi_{j_k}}$ this is ensured if $a \le (1+(2n-1)\pi_l)^{-1}$.

Threshold (ii) is implied by $K_0\varepsilon^{-a}(c_0\varepsilon^a)^{\pi_l} <\!\!* 1$ and this is ensured if we impose $K_0 <\!\!* c_0^{-\pi_l}$.

Threshold (iii) is ensured for ε small enough.

Threshold (v), the value of r_l and Q_{j_k} together with $\pi_l\rho_{l+1} = \pi_{l+1}$ show that the previous inequality is satisfied for ε small enough if $a \le (n\pi_{l+1})^{-1}$.

The same ingredients give threshold (vi) if we impose $a \le (1+n\pi_{l+1}+(n-1)\pi_l)^{-1}$.

Finally, the inequality $K_0 <\!\!* c_0^{-\rho_{l+1}}$ together with $T_l >\!\!* 1$ ensure threshold (vii).

At the last step, $l = n-1$, we only need to build the integrable normal form linked to I_{n-1}; hence, we have to check the first three thresholds with the parameters τ_{n-1}, r_{n-1}, K.

If $\mathcal{T}_{n-1} < T_{n-1}$, then $\tau_{n-1} *\!\!< T_{n-1}$ and the inequality $\varepsilon T_{n-1}^2 K <\!\!* (c_0\varepsilon^a)^{\pi_{n-1}}$ implies threshold (i).

Since $T_{n-1} *\!\!< (c_0\varepsilon^a)^{-(n-1)\pi_{n-1}}$, this is ensured if $a \le (1+(2n-1)\pi_{n-1})^{-1}$.

In the same way, threshold (ii) is ensured if we impose $K_0 <\!\!* c_0^{-\pi_{n-1}}$.

Threshold (iii) is always ensured for ε small enough.

The sames bounds are also found in the case where $T_{n-1} \le \mathcal{T}_{n-1}$ with the previous reasoning.

All the bounds found in the proof of Theorem 5.1 are satisfied if we choose c_0 sufficiently large, $K_0 <\!\!* c_0^{-\pi_{n-1}}$, $a = (1+(2n-1)\pi_{n-1})^{-1}$ and ε sufficiently small. Finally, we have $r_l =\!\!* (c_0\varepsilon^a)^{\pi_l} T_{j_k}^{-\pi_l/\pi_{j_k}}$ for any $l \in \{j_k,\dots,j_{k+1}-1\}$ and $T_{j_k} \ge\!\!* 1$ implies that the radii r_l are bounded by $\widetilde{c}_0\varepsilon^{a\pi_l}$ with some positive constant $\widetilde{c}_0$ and this completes the proof of Theorem 5.1. □

Now, we can complete the proof of the main result (Theorem 1.1). Indeed an action can drift at most a distance κr_l at each step of our iterative process (Lemmas 4.3 and 4.4). If $\kappa r_{n-1} \ge \varepsilon$, an action can drift at most a distance κr_{n-1} at the last step of our process. Moreover, the size of each change of coordinates in our iterative scheme is bounded by $\frac{1}{6}r_l$ for $l \in \{0,\dots,n-1\}$. Since the threshold $\kappa r_{n-1} \ge \varepsilon$ is obviously satisfied, we have

$$\|I(t)-I(t_0)\| \le \kappa(r_0+r_1+\cdots+r_{n-1})+\tfrac{2}{6}(r_0+r_1+\cdots+r_{n-1}) \quad \text{if } |t-t_0| < \exp(cK).$$

with $K = K_0\varepsilon^{-a}$, $K_0 =\!\!* c_0^{-\pi_{n-1}}$, $\kappa = 3M^2$ and $r_l <\!\!* \widetilde{c}_0\varepsilon^{a\pi_l}$. We get

$$\|I(t)-I(t_0)\| \le (\kappa+\tfrac{1}{3})n\widetilde{c}_0\varepsilon^a \quad \text{if } |t-t_0| < \exp(c_1 c_0^{-\pi_{n-1}}\varepsilon^{-a}) = \exp(C\varepsilon^{-a})$$

for a some positive constant $c_1 \le c$. This complete the proof of Theorem 1.1. □

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