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Complex potential equations
II. An application to general relativity

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Abstract

The results of a previous paper are applied to a study of the class of Kerr–Schild metrics in general relativity. These metrics have the form

$$\mathbf{g} = \boldsymbol{\eta} + 2m\mathbf{l} \otimes \mathbf{l},$$

where $\boldsymbol{\eta}$ is the flat (Minkowski) space-time metric, m is an arbitrary real number, and $\mathbf{l}$ is a null covector. It is already known that for a certain restricted subclass of these metrics, the vacuum Einstein field equations, viz.

$$\mathbf{Ric}(\mathbf{g}) = \mathbf{0}$$

can be written in the form

$$(\nabla\gamma)^2 = \gamma^4$$

$$\nabla^2\gamma = 0$$

where γ is a complex potential. Using the methods developed in a previous paper, such space-times are characterized by means of a special family of complex surfaces in three-dimensional Euclidean space, and the exact solutions for the metric $\mathbf{g}$ are consequently recovered. It is also shown that the field equations for a much wider class of Kerr–Schild metrics can be expressed in terms of a potential formalism, not only in the vacuum case, but also for many electrovacuum solutions.

1. *Introduction*

The Kerr–Schild metrics of general relativity are those of the form

$$\mathbf{g} = \boldsymbol{\eta} + 2m\mathbf{l} \otimes \mathbf{l} \tag{1·1}$$

where $\boldsymbol{\eta}$ is a flat (Minkowski) space-time metric (signature $- + + +$), $\mathbf{l}$ is a covector which is null with respect to $\mathbf{g}$ (or, equivalently, with respect to $\boldsymbol{\eta}$) and m is a real constant. When the vacuum field equations of general relativity hold, viz.

$$R_{\alpha\beta} = 0 \quad (\alpha, \beta = 0, 1, 2, 3),$$

these metrics are known to admit a certain Killing vector $\mathbf{k} = k^\mu(\partial/\partial x^\mu)$, where x^μ are the usual 'Cartesian' coordinates in the auxiliary flat space-time, M, and the k^μ are constant (2). In M, the vector $\mathbf{k}$ acts as a translational Killing vector, which may be

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timelike, spacelike or null. In the case where it is timelike, one can orient axes and normalize so that $\mathbf{k} = \partial/\partial t$, where $t \equiv x^0$; the subset of Kerr–Schild metrics which admit $\partial/\partial t$ as a Killing vector has recently been examined in a remarkable paper by Schiffer, Adler, Mark and Sheffield (10). It was discovered that for such metrics the vacuum Einstein field equations ($R_{\alpha\beta} = 0$) could be expressed in terms of a complex potential, γ, in the form

$$(\nabla\gamma)^2 = \gamma^4 \tag{1·2a}$$

and

$$\nabla^2\gamma = 0. \tag{1·2b}$$

It is important to note that although $\partial/\partial t$ is a timelike Killing vector in the auxiliary space, M, it is not necessarily timelike in the curved Kerr–Schild space-time, and consequently the metrics under discussion are not necessarily stationary, contrary to the claims in (10).

Moreover, since division is made by a quantity $L+A$ in equation (4·3) of (10), the analysis presented therein should be restricted to the case $L+A \not\equiv 0$. However, a minor modification shows that the case $L+A \equiv 0$ can be treated, and that it gives rise to Minkowski space-time. For equation (4·2) of (10) with $L+A \equiv 0$ implies $MM^T l_0^2 = 0$, and this requires either $l_0^2 = 0$ or the matrix N in (4·7) of (10) is zero, in which case M is zero. Thus either the vector $\mathbf{l}$ is zero, or, using (3·15) of (10) successively with $\mu = \nu = 0$ and $\mu = 0 \neq \nu$ shows that $\mathbf{l}$ is constant. Whichever possibility holds, the metric is that of Minkowski space-time. Alternatively, a simple calculation shows that the case $L+A \equiv 0$ corresponds to the vanishing of the expansion rates of the null geodesic congruence defined by $\mathbf{l}$, and, as can be seen by referring to the results of (2), this necessarily requires a Minkowski metric. In fact, the analysis of (2) makes the express assumption that

(*) the expansion and rotation rates of the null geodesic congruence do not both vanish.

No attempt was made in (10) to determine the general solution to the potential equations (1·2) but it was pointed out that the Schwarzschild metric is generated by the solution $\gamma = 1/r = \{x^2+y^2+z^2\}^{-\frac{1}{2}}$, and that the Kerr solution is generated from this by an imaginary translation of the origin. Once solutions for the potential γ are determined, the metric $\mathbf{g}$ in (1·1) can be calculated using the formulae

$$\boldsymbol{\lambda} = \frac{\nabla\omega + \nabla\bar{\omega} - i(\nabla\omega \wedge \nabla\bar{\omega})}{1 + \nabla\omega \cdot \nabla\bar{\omega}}, \tag{1·3}$$

$$l_0^2 = \kappa \mathscr{R}\gamma \tag{1·4}$$

and

$$\mathbf{l} = l_0(1, \boldsymbol{\lambda}) \tag{1·5}$$

where $\omega \equiv \gamma^{-1}$, a bar denotes complex conjugation, $\boldsymbol{\lambda}$ is a unit 3-dimensional vector, and κ is an arbitrary real constant which makes l_0^2 positive in (1·4). Equation (1·3) is valid provided γ is not identically zero. The case $\gamma \equiv 0$ corresponds to Minkowski space-time ($l_0 \equiv 0$) as discussed above. Thus *the simplest solution* $\gamma \equiv 0$ *to the potential problem* (1·2) *generates Minkowski space.*

The next most immediate solution is $\gamma = 1/r$, where $r^2 = x^2+y^2+z^2$ and (x, y, z) are

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ordinary Cartesian coordinates. *This potential* ($\gamma = 1/r$) *generates the static spherically symmetric Schwarzschild metric*, expressed in the coordinates of Eddington (3):

$$ds^2 = -d\tau^2 + dx^2 + dy^2 + dz^2 + \frac{2m}{r}\left(d\tau + \frac{x\,dx + y\,dy + z\,dz}{r}\right)^2$$

$$= -d\tau^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta\,d\phi^2) + \frac{2m}{r}(d\tau + dr)^2,$$

where (r, θ, ϕ) are ordinary spherical polar coordinates. Eddington's coordinates are related to the usual Schwarzschild coordinates, in which the metric takes the form

$$ds^2 = -\left(1 - \frac{2m}{r}\right)dt^2 + \frac{dr^2}{\left(1 - \frac{2m}{r}\right)} + r^2(d\theta^2 + \sin^2\theta\,d\phi^2),$$

by the transformation $\tau = t + 2m\ln((r/2m) - 1)$, which is valid if $r > 2m$, or, in a formal sense, if $r < 2m$.

Any solution generated from $\gamma = 1/r$ by a real translation of the origin in Euclidean space will simply generate the Schwarzschild metric, with a corresponding shift of origin. On the other hand, if an imaginary translation is made, e.g.

$$\gamma = [x^2 + y^2 + (z - ia)^2]^{-\frac{1}{2}} = [r^2 - 2iar\cos\theta - a^2]^{-\frac{1}{2}},$$

where a is real, and a suitable orientation of axes has been made so that the shift occurs in the z-direction, it is discovered that *this new potential* ($\gamma = [r^2 - 2iar\cos\theta - a^2]^{-\frac{1}{2}}$) *generates the stationary axisymmetric solution of Kerr* (4), and moreover the magnitude, a, of the translation is *precisely* the value of the parameter a that measures the angular momentum per unit mass, as observed at infinity by a stationary observer. This exhibits in a very transparent manner a relationship between Kerr and Schwarzschild geometries originally considered by Newman and Janis (8), cf. Newman (5). It also has a bearing on the wider issue of the role of complex analysis in general relativity, such as that currently being explored by Newman (6, 7) and Penrose (9).

In a previous paper ((1), part I) the present author has investigated the real and complex solutions of partial differential equations of the form

$$(\nabla\gamma)^2 = f(\gamma) \quad \text{and} \quad \nabla^2\gamma = g(\gamma).$$

It is a simple matter to specialize the results of Theorems 3·1 and 4·1 of part I to the case $f(\gamma) \equiv \gamma^4$, $g(\gamma) \equiv 0$. In this case, we recover the general solution (2) of the system (1·2). Thus we obtain

THEOREM 1·1. *The solutions to the system*

$$(\nabla\gamma)^2 = \gamma^4 \quad \textit{and} \quad \nabla^2\gamma = 0$$

are

(i) $\gamma \equiv 0$,

(ii) $\gamma = \pm 1/R$, *where* $R = [(x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2]^{\frac{1}{2}}$, *and* x_1, y_1 *and* z_1 *are arbitrary constants, and*

(iii) $\gamma = \pm (F_\beta)^{-1}$ *where* $F(\beta, x, y, z) \equiv \phi(\beta) + \frac{1}{2}[(1 - \beta^2)x + i(1 + \beta^2)y - 2\beta z] = 0$, $\phi(\beta)$ *is an arbitrary analytic function with the property* $\phi'''(\beta) \not\equiv 0$.

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Proof. It follows immediately from Part I that the only constant solution is $\gamma \equiv 0$. If γ is not identically constant, then, choosing $\nu = -1/\gamma$ as in Theorems 3·1 and 4·1 we have

$$(\nabla\nu)^2 = 1 \quad \text{and} \quad \nabla^2\nu = 2/\nu.$$

Thus, in the terminology used in part I, we obtain either $\nu = \mp R$, where

$$R = [(x-x_1)^2 + (y-y_1)^2 + (z-z_1)^2]^{\frac{1}{2}},$$

or $\nu = \mp F_\beta$ where

$$F(\beta, x, y, z) \equiv \phi(\beta) + \tfrac{1}{2}[(1-\beta^2)x + i(1+\beta^2)y - 2\beta z] = 0$$

and $\phi(\beta)$ is an arbitrary analytic function satisfying $\phi''' \not\equiv 0$. Substituting $\gamma = -1/\nu$ we obtain the required result. ∎

Remark. Solution (i) generates Minkowski space-time, solution (ii) generates the Schwarzschild and Kerr metrics, and solution (iii) generates the remaining class of vacuum Kerr–Schild metrics (with $\partial/\partial t$ as a timelike Killing vector in the auxiliary Minkowski space-time and under the assumption of condition (*)). From this and previous results (1) we have

Corollary. Let S be the set whose elements are the families of (real or complex) non-null equipotential surfaces in 3-dimensional Euclidean space which possess the properties that their normal congruences are straight lines and they are surfaces of constant principal normal curvature.

The class of vacuum Kerr–Schild metrics (with $\partial/\partial t$ as a timelike Killing vector in the auxiliary space-time, and under the assumption (*)) consists of Minkowski space-time together with a set of metrics which is in 1:1 correspondence with the set S. The metrics are those of Schwarzschild and Kerr if and only if the expansion tensor, $\theta_{\alpha\beta}$, of the normal congruence to the equipotential surfaces possesses three linearly independent non-null eigenvectors. If they are neither Schwarzschild nor Kerr metrics, then $\theta_{\alpha\beta}$ possesses only one non-null eigenvector together with one null eigenvector.

This result has the peculiar feature that, with regard to the eigenvectors of $\theta_{\alpha\beta}$, the Schwarzschild and Kerr metrics appear to be more *general* than the other Kerr–Schild metrics, whereas, according to the Petrov classification (which can likewise be accomplished by consideration of the eigenvectors of a complex symmetric matrix), general Kerr–Schild metrics are of type II, *except* for the Schwarzschild and Kerr metrics, which belong to the more *special* type D.

2. *Generalization of the potential formalism*

The potential formalism described in the previous section is valid for a much wider class of metrics then considered in (10). In this section we discussed the extension to other vacuum and electrovacuum Kerr–Schild metrics.

It is shown in (2) that the general† solution for the Kerr–Schild metrics of the vacuum Einstein equations (and also for a subclass of the general solution to the Einstein–Maxwell equations) is expressible in the form

$$ds^2 = 2\,d\zeta\,d\bar{\zeta} + 2du\,dv + P^{-3}[m(Z+\bar{Z}) - \psi\bar{\psi}P^{-1}Z\bar{Z}](du + \bar{Y}\,d\zeta + Y\,d\bar{\zeta} - Y\bar{Y}\,dv)^2, \quad (2{\cdot}1)$$

† Assuming condition (*).

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where m is a real constant and the coordinates $(\zeta, \bar{\zeta}, u, v)$ are null coordinates in Minkowski space-time, related to ordinary Cartesian coordinates (t, x, y, z) by $\zeta = 1/(\sqrt{2})(x+iy)$, $\bar{\zeta} = 1/(\sqrt{2})(x-iy)$, $u = 1/(\sqrt{2})(z+t)$ and $v = 1/(\sqrt{2})(z-t)$. The metric (2·1) is then related to the expression (1·1) by identifying the covector $\mathbf{l}$ with the term

$$\left|\tfrac{1}{2}P^{-3}\left[Z+\bar{Z}-\frac{1}{m}\psi\bar{\psi}P^{-1}Z\bar{Z}\right]\right|^{\frac{1}{2}}(du+\overline{Y}\,d\zeta+Y\,d\bar{\zeta}-Y\overline{Y}\,dv), \tag{2·2}$$

for $m \neq 0$ if $\psi \not\equiv 0$. The function Y is a complex variable, given implicitly in terms of $(\zeta, \bar{\zeta}, u, v)$ by

$$F(Y,\zeta,\bar{\zeta},u,v) \equiv \phi(Y)+(qY+c)(\zeta-Yv)-(pY+\bar{q})(u+Y\bar{\zeta}) = 0, \tag{2·3}$$

and $Z = -PF_{\overline{Y}}^{-1}$ where $P = pY\overline{Y}+qY+\bar{q}\overline{Y}+c$. Further, $\phi(Y)$ and $\psi(Y)$ are arbitrary analytic functions of Y; p and c are real constants, and q is a complex constant (not all of p, c and q are zero). The electromagnetic field vanishes if and only if $\psi(Y) \equiv 0$. The metric (2·2) admits (at least) a one-parameter continuous isometry group, with Killing vector $\mathbf{k} = c(\partial/\partial u)+\bar{q}(\partial/\partial\zeta)+q(\partial/\partial\bar{\zeta})-p(\partial/\partial v)$; Y and Z satisfy

$$\mathbf{K}(Y) = \mathbf{K}(Z) = \mathbf{0}.$$

In the situation so far studied in the present paper we have $p = c = 2^{-\frac{1}{2}}$, $q = 0$, $Y \equiv \beta$ and $\psi \equiv 0$. The two prescriptions for the metric provided in (2) and (10) then show that we should identify the function γ with the term $\pm Z/P = \mp F_{\overline{Y}}^{-1}$. Since we have been concerned with differential equations for γ considered as a potential, we now generalize by considering differential equations in the general case for the quantity $\Gamma = -Z/P = F_{\overline{Y}}^{-1} \not\equiv 0$. We shall use the relationships

$$(\nabla u)^2 = (\nabla v)^2 = (\nabla\zeta)^2 = (\nabla\bar{\zeta})^2 = \nabla u\cdot\nabla\zeta = \nabla u\cdot\nabla\bar{\zeta} = \nabla v\cdot\nabla\zeta = \nabla v\cdot\nabla\bar{\zeta} = 0,$$

$$\nabla u\cdot\nabla v = \nabla\zeta\cdot\nabla\bar{\zeta} = 1, \quad \nabla Y\cdot\nabla u = Y_v, \quad \nabla Y\cdot\nabla v = Y_u, \quad \nabla Y\cdot\nabla\zeta = Y_{\bar{\zeta}}$$

and

$$\nabla Y\cdot\nabla\zeta = Y_\zeta.$$

Evaluating partial derivatives using (2·3), we thus obtain $(\nabla Y)^2 = 2(Y_\zeta Y_{\bar{\zeta}}+Y_u Y_v) = 0$ and, putting $\Omega = \Gamma^{-1} = F_Y$, we have

$$\nabla\Omega = F_{YY}\nabla Y-(p\nabla u+c\nabla v+\bar{q}\nabla\bar{\zeta}-q\nabla\zeta)-2Y(q\nabla v+p\nabla\bar{\zeta});$$

it follows that $(\nabla\Omega)^2 = 2(pc-q\bar{q})$. Moreover, $\nabla^2 Y = 2(Y_{uv}+Y_{\zeta\bar{\zeta}}) = 0$. This means that $\nabla^2\Omega = 4\epsilon\Gamma$ where for brevity we write $\epsilon = pc-q\bar{q}$.

We have therefore obtained functions Y and Γ which satisfy the equations $(\nabla Y)^2 = \nabla^2 Y = 0$,

$$(\nabla\Gamma)^2 = 2\epsilon\Gamma^4 \tag{2·4a}$$

and

$$\nabla^2\Gamma = 0. \tag{2·4b}$$

In the case considered in section 1, equations (2·4) reduce to (1·2) on identifying Γ with γ.

In the general case, there seems to be no straightforward way to determine the covector $\mathbf{l}$ from a knowledge of Γ alone. We can determine the quantity $l_u P$ from the identification formula

$$2ml_u^2 \equiv \left|\frac{m}{P^3}(Z+\bar{Z})-\frac{\psi\bar{\psi}Z\bar{Z}}{P^4}\right| \neq 0;$$

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this is obtained using equation (2·2). Thus

$$(l_u P)^2 = \left| \mathscr{R}e\,\Gamma + \frac{\psi\bar{\psi}\Gamma\bar{\Gamma}}{2m} \right| \quad (m \neq 0 \quad \text{if} \quad \psi \neq 0).$$

In the case where $\mathbf{K} = \partial/\partial t$, we can recover the formulae given in (10). We consider solutions $\Gamma \not\equiv 0$ (ignoring Minkowski space-time) and we examine first the case when $\nabla\Omega + \nabla\bar{\Omega}$ and $i(\nabla\Omega - \nabla\bar{\Omega})$ are non-zero linearly independent real vectors. A direct calculation shows that since $\mathbf{l}\cdot\nabla Y = 0$, it follows that $\mathbf{l}\cdot\nabla\Omega = -l_u P$. Thus

$$\mathbf{l}\cdot[i(\nabla\Omega - \nabla\bar{\Omega})] = 0,$$

and since $\mathbf{l}$ is null this requires $i(\nabla\Omega - \nabla\bar{\Omega})$ to be spacelike or null. If $i(\nabla\Omega - \nabla\bar{\Omega})$ is null, $\mathbf{l} = Ai(\nabla\Omega - \nabla\bar{\Omega})$ for some scalar A. But then $l_u P = 0$ (since $\mathbf{K}\cdot\mathbf{l} = l_u P$, and $\mathbf{K}\cdot\nabla Y = \mathbf{K}\cdot\nabla\Omega = \mathbf{k}\cdot\nabla\bar{\Omega} = 0$), which yields Minkowski space-time, a contradiction. Hence $i(\nabla\Omega - \nabla\bar{\Omega})$ is spacelike and so $\nabla\Omega - \nabla\bar{\Omega} > 1$. But

$$(\nabla\Omega + \nabla\bar{\Omega})^2 = 8\epsilon + [i(\nabla\Omega - \nabla\bar{\Omega})]^2 > 0,$$

implying that $\nabla\Omega + \nabla\bar{\Omega}$ is spacelike. Thus in the real 3-dimensional Euclidean space orthogonal to $\mathbf{K}$, we can expand vectors in terms of the three real linearly independent vectors $\nabla\Omega + \nabla\bar{\Omega}$, $i(\nabla\Omega - \nabla\bar{\Omega})$ and $i(\nabla\Omega \wedge \nabla\bar{\Omega})$, where $\wedge$ denotes the usual vector product. Thus we write

$$\mathbf{l} = A\mathbf{K} + B(\nabla\Omega + \nabla\bar{\Omega}) + Ci(\nabla\Omega - \nabla\bar{\Omega}) + Di(\nabla\Omega \wedge \nabla\bar{\Omega})$$

and use the properties

$$\mathbf{K}\cdot\nabla\Omega = \mathbf{K}\cdot\nabla\bar{\Omega} = \mathbf{l}\cdot\mathbf{l} = 0, \quad \mathbf{K}\cdot\mathbf{K} = -1 \quad \text{and} \quad \mathbf{l}\cdot\nabla\Omega = \mathbf{l}\cdot\nabla\bar{\Omega} = -\mathbf{K}\cdot\mathbf{l} = -l_u P$$

to obtain $A = -\mathbf{K}\cdot\mathbf{l}$, $B = -(\mathbf{K}\cdot\mathbf{l})/(1 + \nabla\Omega\cdot\nabla\bar{\Omega})$, $C = 0$ and $D = \pm(\mathbf{K}\cdot\mathbf{l})/(1 + \nabla\Omega\cdot\nabla\bar{\Omega})$. Writing $\mathbf{l} = l_0(1, \boldsymbol{\lambda})$, where $\boldsymbol{\lambda}$ is a unit vector, and $\mathbf{K} = (1, \mathbf{0})$, we obtain

$$\boldsymbol{\lambda} = \frac{\nabla\Omega + \nabla\bar{\Omega} \mp i(\nabla\Omega \wedge \nabla\bar{\Omega})}{1 + \nabla\Omega\cdot\nabla\bar{\Omega}} \tag{2·5}$$

with

$$l_0^2 = \left| \mathscr{R}e\Gamma + \frac{\psi\bar{\psi}\Gamma\bar{\Gamma}}{2m} \right| \quad (m \neq 0 \quad \text{if} \quad \psi \neq 0)$$

which, when no electromagnetic field is present ($\psi \equiv 0$), is seen to agree with (1·3)–(1·5) on identifying Γ with γ and $\Omega = \Gamma^{-1}$ with $\omega = \gamma^{-1}$. In the case where $\nabla\Omega + \nabla\bar{\Omega} = \mathbf{0}$, we have $\nabla\Omega\cdot\nabla\bar{\Omega} < 0$, which is impossible. If $i(\nabla\Omega - \nabla\bar{\Omega}) = \mathbf{0}$, it follows that $\mathbf{l}\cdot(\mathbf{K} + \nabla\Omega) = 0$ and $(\mathbf{K} + \nabla\Omega)^2 = 0$. Since $\mathbf{K} + \nabla\Omega$ and $\mathbf{l}$ are real, it follows that either $\mathbf{l} = A(\mathbf{K} + \nabla\Omega)$, for some non-zero scalar, A, or that $\mathbf{K} + \nabla\Omega = \mathbf{0}$. This last alternative is impossible, for then $\mathbf{K}\cdot\mathbf{K}$ would be zero. As before it follows that $A = -\mathbf{K}\cdot\mathbf{l}$, and the more general formula (2·3) derived above is still valid.

In all other cases for which our potential formalism leading to (2·4) is valid, namely in the set of Kerr–Schild metrics that satisfy the vacuum Einstein field equations for which condition (*) is valid and for which $\mathbf{K}$ is null or spacelike (together with a subclass of the corresponding electrovacuum solutions), no simple prescription along the

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same lines as (2·5) seems to arise. The situation is made particularly complicated by the facts that:

(i) When $\mathbf{K}$ is null, the particular solution $\mathbf{k} = -\nabla\Omega$ does not seem to yield any formula for the generation of the covector $\mathbf{l}$, without resorting to solving the equations $(\nabla Y)^2 = \nabla^2 Y = 0$ for Y first,

(ii) When $\mathbf{K}$ is spacelike, the vector $\nabla\Omega + \nabla\bar{\Omega}$ seems to be capable of being spacelike, timelike or null. This yields a complicated trichotomy in the formulation of further relationships.

(iii) If $\mathbf{K}$ is either spacelike or null, the 3-spaces factored out by $\mathbf{K}$ are not positive definite, and so basic Euclidean vectors and vector operations are inappropriate.

Note added in proof: R. J. Finkelstein (*J. Math. Phys.* **16** (1975), 1271–7) has used a different approach to consider the problem of generalising the potential formalism to a special electrovacuum case.

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