Euro. Jnl of Applied Mathematics (2013), *vol.* 24, *pp.* 471–500. © Cambridge University Press 2013
doi:10.1017/S0956792513000028

Population dynamics with age-dependent diffusion and death rates

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(*Received 18 May 2011; revised 6 January 2013; accepted 14 January 2013; first published online 21 February 2013*)

In this paper we investigate the population dynamics of a species with age structure in the case where the diffusion and death rates of the matured population are both age-dependent. We develop a new application of the age-structure technique in terms of an integral equation. For unbounded spatial domains, we study the existence of travelling waves, whilst in bounded domains, we investigate the existence of positive steady-state solutions and their stability.

Key words: Steady-state solution; Travelling wave solution; Age-structured model; Birth function; 34K10, 35B35, 35K55, 92D25

1 Introduction

Spatial movement and temporal maturation are two important characters in most of biological systems; modelling the interaction between them has attracted considerable attention recently [1, 4–7, 10–12, 16–19, 21, 22, 25, 28]. One of the most important methods applied is the Smith-Thieme age-structure technique [21]. In this approach, species population is divided into two groups: mature and immature. At different ages, the standard model with age-structured and diffusion is incorporated (see [15]):

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial a} = D(a)\frac{\partial^2 u}{\partial x^2} - d(a)u. \tag{1.1}$$

Here $u(t, a, x)$ denotes the density of population of the species at time $t \geqslant 0$, age $a \geqslant 0$, and location $x \in \mathbb{R}$. $D(a)$ and $d(a)$ are the diffusion and death rates, respectively, at age a. Let $r \geqslant 0$ be the maturation time for the species and $A_l > 0$ be the life span of the species. Overall the matured population density at time t and location x is given by

$$w(t, x) = \int_r^{A_l} u(t, a, x)da. \tag{1.2}$$

† This work was supported by NSERC discovery grant.

https://doi.org/10.1017/S0956792513000028 Published online by Cambridge University Press

Since only the mature can reproduce, one can assume

$$u(t,0,x) = b(w(t,x)), \tag{1.3}$$

where $b(.)$ is the birth function.

In [22], So, Wu and Zou assume that the diffusion and death rates, $D(a)$ and $d(a)$, of the mature population are age independent. i.e.,

$$D(a) = D_m \quad \text{and} \quad d(a) = d_m.$$

Based on this assumption, they substitute (1.1) into (1.2) to derive

$$\frac{\partial w}{\partial t} = D_m \frac{\partial^2 w}{\partial x^2} - d_m w + u(t,r,x), \tag{1.4}$$

where $u(t,r,x)$ is called the maturation rate and it can be obtained by Fourier transforms from (1.1) and the boundary condition (1.3), with a formula given by

$$u(t,r,x) = \epsilon \int_{-\infty}^{\infty} b(w(t-r,y)) f_\alpha(x-y)dy,$$

where

$$\epsilon = \exp\left[-\int_0^r d_I(a)da\right], \quad \alpha = \int_0^r D_I(a)da,$$

and

$$f_\alpha(x) = \frac{\exp^{(-x^2/4\alpha)}}{\sqrt{4\pi\alpha}}.$$

The functions $D_I(a)$ and $d_I(a)$ given above are the diffusion and death rates of the immature population, respectively.

As such, a non-local reaction diffusion equation with delay can be obtained:

$$\frac{\partial w}{\partial t} = D_m \frac{\partial^2 w}{\partial x^2} - d_m w + \epsilon \int_{-\infty}^{\infty} b(w(t-r,y)) f_\alpha(x-y)dy.$$

During the past decade, there have been some further studies on this model. In [14], Mei and So investigated the stability of travelling wave solutions in the case of Nicholson's blow-flies birth function. Liang and Wu in [12] investigated the existence of travelling wave solutions for different birth functions. In [12], Liang, So, Zhang and Zou considered the above model on a bounded domain, where the diffusion and death rates are constants. They also investigated the behaviour of the solution using numerical simulations.

Thieme and Zhao in [25] considered the following general stage-structure model

$$\begin{cases} \partial_t u + \partial_a u = d_I(a)\Delta_x u - \mu_I(a)u, & 0 < a < r, \quad x \in \mathbb{R}^n \\ u(t,0,x) = f(u_m(t,x)), & t \geqslant -r, x \in \mathbb{R}^n, \\ \partial_t u_m = D_m \Delta_x u_m - d_m g(u_m) + u(t,r,x), & t > 0, x \in \mathbb{R}^n, \end{cases}$$

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where u_m and u are the population density of the mature and the immature populations, $f(u_m)$ and $g(u_m)$ are the birth and death functions, $D_I(a)$ and $\mu_I(a)$ are the diffusion and death rates of the immature population, and D_m and d_m are age independent diffusion and death rates of the mature population. They investigated the existence of travelling wave solutions of this model, when the spatial domain is the whole real line $\mathbb{R}$. When the spatial domain is bounded, the above model was studied by Xu and Zhao in [28] and by Jin and Zhao in [11] where they investigated the existence of steady-state solutions.

Also, with a slight modification, these types of models have been recently studied by many mathematicians. For example, Al-Omari and Gourley [1], Gourley and Kuang in [4], Gourley and So in [5], Gourley, So, and Wu in [6] and Ou and Wu in [18]. Overall, all these papers study the models with a crucial assumption that *the diffusion and death rates of the mature population are constants* so that equation (1.4) can be easily derived. Actually, all models above seem to fail if $D_m(a)$ and $d_m(a)$ are not constants, because *equation (1.4) is not valid anymore if both* $D_m(a)$ *and* $d_m(a)$ *are age-dependent*.

Biologically, it is more realistic to include the age effects in the mathematical models during the whole life of the species. For example, women in the age between 15–40 years have higher birth rate and lower death rate. Another example is the predation of animals; the predation could be heavier on some certain age groups of mature population, which makes the death rate depend on the age. In epidemiology, it is also important to consider the death rate of mature population to be age-dependent. Indeed, the disease may have different infection rates for different age groups of mature population. An example of this is the sexually-transited diseases (STDs), which spread through mature individuals; most AIDS cases occur in young adults. This causes a variation in the death rate amongst the different age of the mature individuals. Therefore, it is natural and more realistic to consider the diffusion and death rates to be age-dependent during all the life time.

In this paper, we aim to investigate the populations dynamics for this case (age-dependent in the whole life). Therefore, we consider the model (1.1)–(1.3) with age-dependent diffusion and death rates during all life time, and with a spatial domain being unbounded as well as bounded. For the unbounded domain case, we derive a model in terms of an integral equation. We show that this integral equation admits a unique (up to translation) monotone travelling wave solution connecting 0 and u^*, for all $c > c^*$. For the bounded domain case, we derive two new models in terms of integral equations. Under a proper condition, we show that the first integral equation admits a positive steady-state solution, whilst the second integral equation admits a non-negative and not identically zero steady-state solution. We show that the positive (resp. non-negative and not identically zero) steady-state solution is stable provided that λ_0 (the principal eigenvalue of the linear integral equation) is positive.

The paper is organised as follows. We start in Section 2 by establishing the model when the spatial domain is unbounded. In Section 3, we obtain the global existence of the solution. In Section 4, we investigate the existence of travelling wave solutions. In Section 5, a model when the spatial domain is bounded is derived. Also, we investigate the existence of steady-state solutions when the birth function is linear. In Section 6, we investigate the existence of the steady-state solutions and their global stability. Section 7 is devoted to numerical simulations and Section 8 is devoted to discussing the main results.

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2 The derivation of the new model

To derive the model, we consider (1.1)–(1.3). We fix s, $0 < s < A_l$, and define a function v by $v(s,a,x) = u(s+a,a,x)$. Then we obtain

$$\begin{aligned}\frac{\partial v}{\partial a} &= \left[\frac{\partial u}{\partial t}(t,a,x) + \frac{\partial u}{\partial a}(t,a,x)\right]_{t=a+s} \\ &= D(a)\frac{\partial^2 u}{\partial x^2}(a+s,a,x) - d(a)u(a+s,a,x) \\ &= D(a)\frac{\partial^2 v}{\partial x^2}(s,a,x) - d(a)v(s,a,x). \end{aligned} \tag{2.1}$$

By applying Fourier transforms to the equation

$$\frac{\partial v}{\partial a} = D(a)\frac{\partial^2 v}{\partial x^2}(s,a,x) - d(a)v(s,a,x), \tag{2.2}$$

we get

$$\frac{d}{da}\overline{V}(s,a,\omega) = -\left[D(a)\omega^2\overline{V}(s,a,\omega) + d(a)\overline{V}(s,a,\omega)\right].$$

Therefore,

$$\overline{V}(s,a,\omega) = K(s,\omega)\,\exp\left[-\int_0^a \left(D(\xi)\omega^2 + d(\xi)\right)d\xi\right] = \beta(a)K(s,\omega)e^{-\alpha(a)\omega^2},$$

where

$$\alpha(a) = \int_0^a D(\xi)d\xi, \quad \beta(a) = \exp\left[-\int_0^a d(\xi)d\xi\right],$$

and $K(s,\omega) = \mathscr{F}\{b(w(s,x)\}$. By taking the inverse transformation, we obtain

$$\begin{aligned} v(s,a,x) &= \mathscr{F}^{-1}\{\overline{V}(s,a,\omega)\} \\ &= \beta(a)\left[\mathscr{F}^{-1}\{K(s,\omega)\} * \mathscr{F}^{-1}\{e^{\alpha(a)\omega^2}\}\right] \\ &= \frac{\beta(a)}{\sqrt{4\pi\alpha(a)}}\int_{-\infty}^{\infty} b(w(s,y))\,e^{-\frac{(x-y)^2}{4\alpha(a)}}dy, \end{aligned}$$

where for the last equality, we have made use of the relation

$$\mathscr{F}^{-1}\left\{e^{-\alpha(a)\,\omega^2}\right\} = \frac{1}{\sqrt{4\pi\alpha(a)}}\,e^{\frac{-x^2}{4\alpha(a)}}.$$

Thus, it follows that

$$u(t,a,x) = v(t-a,a,x) = \frac{\beta(a)}{\sqrt{4\pi\alpha(a)}}\int_{-\infty}^{\infty} b(w(t-a,y))\,e^{-\frac{(x-y)^2}{4\alpha(a)}}dy. \tag{2.3}$$

Then we substitute (2.3) into (1.2) to get

$$w(t,x) = \int_r^{A_l}\int_{-\infty}^{\infty} b\left(w(t-s,y)\right)\frac{e^{-(x-y)^2/4\alpha(s)}}{\sqrt{4\pi\alpha(s)}}\,\beta(s)\,dy\,ds, \tag{2.4}$$

which is an integral equation.

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Remark 2.1 The integral equation (2.3) is only valid for $t \geqslant a$. As such, (2.4) is technically true for $t \geqslant A_l$. However, we will concentrate on the long-time behaviour of (2.4), and it is meaningful to study this equation for all $t \geqslant 0$.

3 Global existence of the solution

By changing the variables in (2.4), we get

$$\begin{aligned} w(t,x) &= \int_r^{A_l} \int_{-\infty}^{\infty} b\,(w(t-s,x-y))\,\frac{e^{-y^2/4\alpha(s)}}{\sqrt{4\pi\alpha(s)}}\,\beta(s)\,dy\,ds \\ &= \int_0^{\infty} \int_{-\infty}^{\infty} b\,(w(t-s,x-y))\,K(s,y)\,dy\,ds\,, \end{aligned} \tag{3.1}$$

where

$$K(s,x) = \begin{cases} 0\,, & 0 \leqslant s < r \text{ or } s > A_l,\ x \in \mathbb{R}, \\ \dfrac{\beta(s)}{\sqrt{4\pi\alpha(s)}}\ \exp\{-x^2/4\alpha(s)\}\,, & r \leqslant s \leqslant A_l,\ x \in \mathbb{R}\,, \end{cases} \tag{3.2}$$

$$\beta(s) = \exp\left\{-\int_0^s d(\xi)d\xi\right\},$$

and

$$\alpha(s) = \int_0^s D(\xi)d\xi.$$

Hence, (3.1) can be written in the following form:

$$w(t,x) = \int_0^{\infty} \int_{-\infty}^{\infty} F\,(w(t-s,x-y),s,y)\,dy\,ds\,. \tag{3.3}$$

In the following sections, we shall study the above integral equation and investigate the existence as well as the uniqueness of travelling waves. In fact, we will consider three kinds of birth function $b(.)$. First, as an illustration, we will consider the Nicholson's blow-flies birth function

$$b_0(u) = f(u) = pue^{-au^q},$$

where p, q and a are positive constants.

To start, we may first choose $q = 1$. In this case, for simplicity, we may assume $a = p = 1$ after some scalings of the variables and parameters. Now, $F(u,s,y) = f(u)K(s,y)$, where $K(s,y)$ is the kernel given in (3.2) and $f(u) = ue^{-u}$. We now use the properties of two functions $f(u)$, $K(s,u)$ and some results in [25] to prove the existence as well as the uniqueness of the travelling wave solution of (3.1) for each $c > c^*$, where c^* is the minimal wave speed.

Now, to obtain the global existence of solutions of (3.1), we introduce the known results in [25]. Suppose that the following conditions (A) are assumed:

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(A1) $\int_0^\infty \int_{-\infty}^\infty K(s,y)dx\ dy < \infty$; in this case define $k^* = \int_0^\infty \int_{-\infty}^\infty K(s,y)dx\ dy$.

(A2) $0 \leqslant F(u,s,x) \leqslant u\,K(s,x)$, $\forall u, s \geqslant 0$, $x \in \mathbb{R}$.

(A3) For every compact interval I in $(0,\infty)$, there exists an $\epsilon > 0$ such that $F(u,s,x) \geqslant \epsilon\,K(s,x)$, $\forall u \in I$, $s > 0$ and $x \in \mathbb{R}$.

(A4) For every $\epsilon > 0$, there exists a $\delta > 0$ such that

$$F(u,s,x) \geqslant (1-\epsilon)\,u\,K(s,x)\,,$$

for all $u \in [0,\delta]$, $s \geqslant 0$, and $x \in \mathbb{R}$.

(A5) For every $w > 0$, there exists a $\Lambda > 0$ such that

$$|F(u,s,x) - F(v,s,x)| \leqslant \Lambda\,|u-v|\,K(s,x)\,,$$

for each $u, v \in [0,w]$, $s \geqslant 0$ and $x \in \mathbb{R}$.

Then the following proposition from [25] holds.

Proposition 3.1 *[25] Let (A) hold. Then, for every Borel measurable, non-negative and bounded function $u_0(t,x)$, there exists a unique measurable solution $u : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}_+$ of the nonlinear equation $u(t,x) = u_0(t,x) + \int_0^t \int_{-\infty}^\infty F(u(t-s,x-y),s,y)\,dy\,ds$ and u is bounded on $[0,r] \times \mathbb{R}$, for every $r > 0$. Furthermore, the following results hold:*

a) The solution u is bounded if there exist $c_1, c_2 > 0$ such that $c_1 k^* < 1$ and $F(u,s,x) \leqslant (c_1 + c_2 u)\,K(s,x)$, for all $u, s \geqslant 0$ and $x \in \mathbb{R}$.

b) If $r > 0$ and $\lim_{|x|\to\infty} u_0(x,t) = 0$ uniformly for $t \in [0,r]$, then the same holds for u.

Remark 3.1 The kernel given in (3.2) satisfies $\int_0^\infty \int_{-\infty}^\infty K(s,x)\,dx\,ds < \infty$.

In fact,

$$\begin{aligned} k^* := \int_0^\infty \int_{-\infty}^\infty K(s,x)\,ds\,dx &= \int_r^{A_l} \int_{-\infty}^\infty \frac{\beta(s)}{\sqrt{4\pi\alpha(s)}}\, e^{\frac{-x^2}{4\alpha(s)}}\,ds\,dx \\ &= \int_r^{A_l} \beta(s)\,ds < \infty. \end{aligned}$$

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Now, we turn to prove the existence of a unique solution of the integral equation (3.1) for the given birth function $b(u) = ue^{-u}$. Let t_0 be fixed and let $w(s,x) = \phi(s,x)$, $s \in [t_0 - A_l, t_0]$, $x \in \mathbb{R}$, be the initial data of (3.1). Then we have the following theorem:

Theorem 3.1 *Let $b(u) = ue^{-u}$. Then equation (3.1) has a unique bounded solution for $t \geqslant t_0$ with the initial data given by $w(s,x) = \phi(s,x)$, $s \in [t_0 - A_l, t_0]$, $x \in \mathbb{R}$.*

Proof. To prove this result, we consider (3.3). First, we remark that (3.3) is the limiting equation of

$$u(t,x) = u_0(t,x) + \int_0^t \int_{-\infty}^{\infty} F(u(t-s, x-y), s, y)\, dy\, ds.$$

Therefore, by the above proposition, we need to check the validity of the condition (A).
(A1): In fact (A1) holds by the assumptions and by Remark (3.1).
(A2): Since $0 < e^{-u} \leqslant 1$ for every $u \geqslant 0$, we have $0 < u\,e^{-u} \leqslant u$. Thus, $0 < u\,e^{-au}K(s,x) \leqslant uK(s,x)$, $\forall u, s \geqslant 0$ and $x \in \mathbb{R}$.
(A3): Let $I = [a,b]$ be a compact interval in $(0,\infty)$. We have $f'(u) = e^{-u}(1-u)$. Hence, f takes its maximum at $u = 1$, since $f(u)$ is increasing on the interval $[0,1]$ and decreasing on the interval $[1,\infty]$. If $b \leqslant 1$, let $\epsilon = f(a)$; if $a \geqslant 1$, let $\epsilon = f(b)$; and if $a < 1$ and $b > 1$, let $\epsilon = \min\{f(b), f(c)\}$. Hence, $f(u) = ue^{-au} \geqslant \epsilon$, for all $u \in I$. Therefore, $F(u,x,s) = f(u)K(s,x) \geqslant \epsilon K(s,x)$, for all $u \in I$, $s \geqslant 0$, and $x \in \mathbb{R}$.
(A4): Notice that $F(u,s,x) \geqslant 0$, for all $u, s \geqslant 0$, and $x \in \mathbb{R}$. Thus, the inequality in (A4) holds for every $\epsilon \geqslant 1$. Now, let $\epsilon < 1$ and $\delta = \ln(\frac{1}{1-\epsilon}) > 0$. Since e^{-u} is decreasing on $[0,\delta]$, we get the result directly.
(A5): $f(u)$ is continuous and its derivative exists for all $u \geqslant 0$. Therefore, the inequality in (A5) holds by applying the Mean Value Theorem.

Since $f(u) = u\,e^{-u}$ takes its maximum value at $u = 1$, we have $u\,e^{-u} \leqslant \frac{1}{e}$, for $u \geqslant 0$. Therefore, $u\,e^{-u} \leqslant \frac{1}{e} + c_2 u$, where $c_2 > 0$ and satisfies $c_2 k^* < 1$. Indeed such c_2 exists since k^* is finite in our assumption. This completes the proof. □

The global existence of solutions of (3.1) for other birth functions in the sections below can be obtained by a similar approach. We will omit the details.

4 Existence and uniqueness of travelling wave solutions

4.1 Case 1: $b(u) = ue^{-u}$

At the beginning of this section, we introduce the following assumptions:

(B) Let $K : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}_+$ be a Borel measurable function such that

(B1) $k^* := \int_0^{\infty} \int_{-\infty}^{\infty} K(s,y)dyds \in (1,\infty)$.

(B2) $\exists\ \lambda_0$, such that $\int_0^{\infty} \int_{-\infty}^{\infty} e^{\lambda_0 y} K(s,y)dy\, ds < \infty$.

(B3) $\exists\ \sigma_2 > \sigma_1 > 0, \rho > 0$ such that $K(s,x) > 0$, for $s \in (\sigma_1, \sigma_2)$ and $x \in [0,\rho]$.

(B4) $K(x,y)$ is isotropic in x, i.e., $K(s,x) = K(s,-x)$.

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Remark 4.1 It is easy to see that the kernel function $K(s,x)$, given in equation (3.2), satisfies the conditions (B2), (B3) and (B4). Moreover, it satisfies the condition (B1) if $1 < k^* = \int_r^{A_l} \beta(s)\,ds$ holds.

Definition 4.1 (Travelling Wave Solutions). A solution $w(x,t)$ is called a travelling wave solution of the integral equation in (3.3), if $w(x,t) = W(x+ct)$ for some function W. The constant c is called the wave speed and the solution $W(.)$ is called the travelling wave front solution or the travelling wave profile.

Now, to prove our main theorem, we introduce the conditions (C) [25]:

(C) Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be a continuous function such that

(C1) $f(0) = 0$ and there exists solution u^* (> 0) of equation $u = k^* f(u)$ such that $k^* f(u) > u$, for $u \in (0, u^*)$.

(C2) $f(u)$ is differentiable at $u = 0$, $f'(0) = 1$, and $f(u) \leqslant u$ for any $u \in [0, u^*]$.

Then we have the following two theorems:

Theorem 4.1 *[25] Let $F(u,s,x) = f(u)K(s,x)$, and assume that the conditions (B) and (C) hold. Suppose that f is increasing on $[0,u^*]$, and $f(u) \geqslant u - au^2$, $\forall u \in [0,u^*]$, for some $a > 0$. Then, there is a constant $c^* > 0$ so that for any $c > c^*$, equation*

$$u(t,x) = \int_0^\infty \int_{-\infty}^\infty F\left(u(t-s, x-y), s, y\right) dy\, ds$$

has a monotone travelling wave solution connecting 0 and u^* with speed c.

Theorem 4.2 *[25] Let $F(u,s,x) = f(u)K(s,x)$ and assume that the conditions (B) and (C) hold. Suppose that f is increasing on $[0,u^*]$, and*

$$|f(u) - f(v)| \leqslant |u - v|,\ \forall u, v \in [0, u^*]\ .$$

Then for $c > c^*$, equation

$$u(t,x) = \int_0^\infty \int_{-\infty}^\infty F\left(u(t-s, x-y), s, y\right) dy\, ds$$

admits at most one (up to translation) increasing travelling wave solution $u(x + ct)$ connecting 0 and u^*.

Remark 4.2 The constant c^* is called the minimum wave speed, and it is defined by

$$c^* := \inf\left\{c \geqslant 0 : \mathscr{K}(c,\lambda) < 1,\ \text{for some } \lambda > 0\right\},$$

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where

$$\mathcal{K}(c,\lambda) = \int_r^{A_l}\int_{-\infty}^{\infty} \frac{\beta(s)}{\sqrt{4\pi\alpha(s)}} \exp\{-y^2/4\alpha(s)\}\, e^{-\lambda(y+cs)} dy ds$$
$$= \int_r^{A_l} \beta(s)\exp\{\alpha(s)\lambda^2 - cs\lambda\} ds.$$

The minimum wave speed c^* can be determined uniquely by solving the system

$$\begin{cases} \mathcal{K}(c,\lambda) = 1 \\ \frac{d}{d\lambda}\mathcal{K}(c,\lambda) = 0, \end{cases}$$

for c^* and λ^*; see Lemma 2.2 [23].

For similar results to Theorem 4.1 and 4.2, one can see [2, 3]. Now, we can directly apply these two theorems to study the existence and uniqueness of a monotone travelling wave solution for our equation in the case $b(u) = ue^{-u}$. Therefore, we obtain the following theorem.

Theorem 4.3 *There is a constant $c^* > 0$ so that for $c > c^*$, equation (3.1) has a unique (up to translation) monotone travelling solution connecting $u = 0$ and $u^* = \ln(k^*)$, provided that $1 < k^* \leqslant e$.*

Proof. Existence: To prove the existence, we apply Theorem 4.1. Notice that conditions (B) hold by the assumption.

Now, we check the satisfaction of conditions (C). We notice that $f(u) = u\ e^{-u}$ is continuous with $f(0) = 0$ and $u^* = \ln(k^*)$ is a solution of $u = k^*\ f(u)$. Moreover, for every $u \in (0, u^*)$, we have

$$u < \ln(k^*) \Leftrightarrow e^u < k^* \Leftrightarrow \frac{1}{k^*} < e^{-u} \Leftrightarrow 1 < k^*\ e^{-u} \Leftrightarrow u < k^*\ u\ e^{-u} = k^*\ f(u)\ .$$

Therefore, condition (C1) holds. Next, for condition (C2), it is obvious that f is differentiable at $u = 0$ with $f'(0) = 1$ and $f(u) = u\ e^{-u} \leqslant u$ for all $u \in [0, u^*]$, since $e^{-u} \leqslant 1$ for $u \in [0, u^*]$. Moreover, f is increasing provided that $k^* \leqslant e$, which implies that $f(u)$ is increasing in $u \in [0, u^*]$. Also, if we let $g(u) = e^{-u} + u - 1$, then $g'(u) \geqslant 0$ for all $u \geqslant 0$. This implies that $f(u) \geqslant u - u^2$, for all $u \in [0, u^*]$. This completes our proof for the existence part.

Uniqueness: In fact, we would like to prove

$$|f(u) - f(v)| \leqslant |u - v|,\ \forall u, v \in [0, u^*]\ .$$

To do this, we notice that $f(u)$ is continuous and differentiable on $[0, u^*]$. So, it suffices to show that $|f'(u)| \leqslant 1$. Actually, $f'(u) = (1-u)e^{-u}$, and $f''(u) = (u-2)e^{-u}$. So, $f'(u)$ is decreasing on $[0, 2]$. However, $1 < k^* \leqslant e$ implies that $0 < \ln(k^*) \leqslant 1$, and so $f'(u)$ is decreasing and positive on $[0, u^*] \subseteq [0, 1]$. Therefore, $0 \leqslant f'(u^*) \leqslant f'(u) \leqslant f'(0) = 1$, $\forall u \in [0, u^*]$. Hence, the inequality holds by applying the Mean Value Theorem. □

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Theorem 4.4 *[25] Let conditions (A) and (B) hold. Then for each $c \in (0, c^*)$, there exists no positive travelling wave solution of the equation*

$$u(t,x) = \int_0^\infty \int_{-\infty}^\infty F\left(u(t-s, x-y), s, y\right) dy\, ds$$

with speed c.

Remark 4.3 By applying the above theorem, (3.1) with the birth function $b(u) = ue^{-u}$ has no positive travelling wave solution for speed $c \in (0, c^*)$.

4.2 Case 2: $b_1(u) = pue^{-au^q}$.

Now, we will consider other cases of birth function for the integral equation (3.1). To prove the existence and uniqueness of a travelling wave solution, we state first the following theorem.

Theorem 4.5 *[25] Let (A2) and (B) hold. Assume that $F(.,s,x)$ is increasing in $[0,u^*]$ for each $(s,x) \in \mathbb{R}_+ \times \mathbb{R}$, and $F(u,s,x) \geqslant (u - bu^\sigma)K(s,x), \forall u \in [0,\delta]$, $(s,x) \in \mathbb{R}_+ \times \mathbb{R}$ for appropriate $\delta \in (0,u^*]$, $\sigma > 1$ and $b > 0$. Then for each $c > c^*$, there exists a monotone travelling wave solution of the equation*

$$u(t,x) = \int_0^\infty \int_{-\infty}^\infty F\left(u(t-s, x-y), s, y\right) dy\, ds$$

connecting 0 and u^* with speed c.

Consider the birth function $b(u) = p\,u\,e^{-au^q}$, where a, p and $q > 0$. In condition (C_2), we require $f'(0) = 1$. Therefore, we denote $f(u) = u\,e^{-au^q}$. The constant coefficient p in the birth function can be included into the kernel function $K(s,x)$. After this setting, for $b(u) = p\,u\,e^{-au^q}$, we can also prove the following theorem, which gives the existence and uniqueness (up to translation) of travelling wave solution of (3.1).

Theorem 4.6 *There exists a constant $c^* > 0$ so that equation (3.1) has a unique monotone travelling wave solutions with speed $c > c^*$, connecting $u = 0$ and $u^* = \left[\frac{1}{a}\ln(pk^*)\right]^{1/q}$, provided that $1 < pk^* \leqslant e^{1/q}$. On the other hand, for $c \in (0, c^*)$, equation (3.1) has no positive travelling waves.*

Proof. We notice that $F(u,s,x)$ is increasing in u on the interval $[0,u^*]$, since $f_1(u)$ is increasing on $[0,u^*]$. Furthermore, $F(u,s,x) \geqslant (u - bu^\sigma)\,K(s,x)$, $\forall u \in [0,\delta]$, $(s,x) \in \mathbb{R}_+ \times \mathbb{R}$ and for any $\delta \in (0,u^*]$, where $\sigma = q+1$ and $b = a$. Moreover, conditions (A2) and (B) hold. Thus, by Theorem 4.5, there exists a monotone travelling wave solution of (3.1). The uniqueness, up to translation, of the monotone travelling wave solution can be obtained by applying Theorem 4.2. We remark here that the inequality in Theorem 4.2 holds by the Mean Value Theorem. Indeed, we have $|f_1'(u)| \leqslant 1$. Therefore, (3.1) admits a unique, up to translation, monotone travelling wave solution for $c > c^*$. The nonexistence of a

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positive travelling wave solution for $c \in (0, c^*)$ follows directly from Theorem 4.4. This completes the proof. □

4.3 Case 3: $b_2(u) = \frac{p\,u}{1+au^q}$

In this subsection, we consider the birth function $b(u) = \frac{p\,u}{1+au^q}$, where a, p and $q > 0$. In fact we let $f(u) = \frac{u}{1+au^q}$, so that $f(0) = 0$, $f'(0) = 1$. The coefficient p is again put into the kernel function k. Similarly we have the following theorem for this case.

Theorem 4.7 *There is a constant $c^* > 0$ so that for each $c > c^*$, (3.1) admits a unique, up to translation, increasing travelling wave solution connecting $u = 0$ and $u^* = [\frac{pk^*-1}{a}]^{1/q}$, provided that $1 < pk^* < \infty$ if $q \leqslant 1$, or $1 < pk^* < \frac{q}{q-1}$ if $q > 1$. On the other hand, for $c \in (0, c^*)$, equation (3.1) has no positive travelling waves.*

Proof. Again the proof comes by applying Theorem 4.5 and Theorem 4.2. □

4.4 Case 4: $b_3(u) = pu\left(1 - \frac{u^q}{k^q}\right)$

Now we Consider

$$b(u) = \begin{cases} pu\left(1 - \dfrac{u^q}{k^q}\right), & 0 \leqslant u \leqslant k \\ 0, & u > k, \end{cases}$$

and we let

$$f(u) = \begin{cases} u\left(1 - \dfrac{u^q}{k^q}\right), & 0 \leqslant u \leqslant k \\ 0, & u > k. \end{cases}$$

For this case, we have the following theorem:

Theorem 4.8 *There is a constant $c^* > 0$, so that for $c > c^*$, (3.1) has a unique, up to translation, monotone travelling solution connecting $u = 0$ and $u^* = K(1 - \frac{1}{pk^*})^{1/q}$, where $1 < pk^* \leqslant \frac{q}{1+q}$. On the other hand, for $c \in (0, c^*)$, (3.1) has no positive travelling waves.*

Proof. Again the proof comes by applying Theorem 4.5 and Theorem 4.2. □

Remark 4.4 We consider the above birth functions due to their wide use in the field of mathematical biology, especially, for the cases $q = 1$ in $b_1(u)$, $q = 2$ in $b_2(u)$ and $q = 1$ in $b_3(u)$, see also [26].

Remark 4.5 If we assume the diffusion and death rates of the mature population are age-independent (i.e., $D(a)$ and $d(a)$, $\forall a \in [r, A_l]$, are constants), then $w(t, x)$ given in (2.4) solves the reaction diffusion equation (1.4). In addition, it is easy to see that conditions (A) and (B) hold. This particular case has been studied in [13,22], and our results actually cover theirs.

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Remark 4.6 The study of the stability of the travelling wave solutions is much more interesting and challenging, and we leave it for future studies.

5 Age-structured model on bounded domains

In this section, we consider the model in (5.1) in the case of a bounded domain. In the first section, we derive the model, and in the second section, we prove the existence of a eigen-solution when the birth function is linear.

5.1 The derivation of the model over a bounded domain

Let $u(t, a, x)$ denote the density of population of the species at time $t \geqslant 0$, age $a \geqslant 0$ and location $x \in \Omega = [0, \pi]$. We consider the model

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial a} = D(a)\frac{\partial^2 u}{\partial x^2} - d(a)u, \tag{5.1}$$

where $D(a)$ and $d(a)$ are the diffusion and death rates, respectively, at age a. Let $r \geqslant 0$ be the maturation time for the species. Then the total matured population density at time t and location x is given by

$$w(t, x) = \int_r^{A_l} u(t, a, x)da. \tag{5.2}$$

Since only the mature can reproduce, we have

$$u(t, 0, x) = b(w(t, x)), \tag{5.3}$$

where $b(.)$ is the birth function.

For this case (the bounded domain case), we supply the above problem with two kinds of boundary conditions and thus derive two new models. To be more precise, we consider Neumann boundary conditions and Dirichlet boundary conditions. First, we consider Neumann boundary conditions. The derivation of the model for the Dirichlet case is similar and we omit the details.

Now, we consider the problem in the equations (5.1)–(5.3) subject to the Neumann boundary conditions

$$\frac{\partial}{\partial x}u(t, a, 0) = 0 \text{ and } \frac{\partial}{\partial x}u(t, a, \pi) = 0, \quad t \geqslant 0, \quad a \geqslant 0. \tag{5.4}$$

We fix $s > 0$ and define a function $v(s, a, x) = u(s + a, a, x)$. Then we obtain

$$\begin{aligned}\frac{\partial v}{\partial a} &= \left[\frac{\partial u}{\partial t}(t, a, x) + \frac{\partial u}{\partial a}(t, a, x)\right]_{t=a+s} \\ &= D(a)\frac{\partial^2 u}{\partial x^2}(a + s, a, x) - d(a)u(a + s, a, x) \\ &= D(a)\frac{\partial^2 v}{\partial x^2}(s, a, x) - d(a)v(s, a, x).\end{aligned}$$

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We re-arrange the above equation with Neumann boundary conditions as

$$\frac{\partial v}{\partial a} = D(a)\frac{\partial^2 v}{\partial x^2}(s,a,x) - d(a)v(s,a,x), \tag{5.5}$$

$$\frac{\partial}{\partial x}v(s,a,0) = 0, \quad \text{and} \quad \frac{\partial}{\partial x}v(s,a,\pi) = 0. \tag{5.6}$$

To solve the previous boundary value problem, we use separation of variables. Therefore, we let $v(s,a,x) = A(a)X(x)$ and then substitute it into (5.5) to get

$$A'(a)X(x) = D(a)A(a)X''(x) - d(a)A(a)X(x). \tag{5.7}$$

The corresponding eigenvalue problem for the boundary value problem (5.5)–(5.6) is

$$\frac{d^2X}{dx^2} + \lambda^2 X = 0, \quad 0 < x < \pi, \tag{5.8}$$

$$X'(0) = 0, \quad X'(\pi) = 0. \tag{5.9}$$

The solutions of (5.8) and (5.9) are

$$X_n(x) = \cos nx, \quad n = 0, 1, 2, \ldots,$$

and their corresponding eigenvalues are

$$\lambda_n = n, \quad n = 0, 1, 2 \ldots.$$

Moreover, the solutions of the differential equation

$$A'(a) + \left(d(a) + n^2 D(a)\right) A(a) = 0 \tag{5.10}$$

are given by

$$A_n(a) = c_n(s)\beta(a)\exp\left\{-n^2\alpha(a)\right\}, \quad n = 0, 1, 2, \ldots, \tag{5.11}$$

where

$$\alpha(a) = \int_0^a D(\xi)d\xi, \quad \beta(a) = \exp\left\{-\int_0^a d(\xi)d\xi\right\}.$$

Thus, we have

$$v(s,a,x) = \sum_{n=0}^{\infty} c_n(s)\beta(a)\exp\left\{-n^2\alpha(a)\right\}\cos nx. \tag{5.12}$$

To evaluate the coefficients $c_n(s)$ in the above equation, we use the initial condition $v(s,0,x) = b(w(s,x))$. Let $a = 0$ in (5.12), then we have

$$b(w(s,x)) = v(s,0,x) = \sum_{n=0}^{\infty} c_n(s)\cos nx.$$

Therefore, it follows that

$$c_0 = \frac{1}{\pi}\int_0^{\pi} b(w(s,x))dx,$$

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and

$$c_n = \frac{2}{\pi}\int_0^{\pi} b(w(s,x))\cos nx\,dx, \quad n = 1,2,3,\ldots.$$

On the other hand,

$$\begin{aligned} u(t,a,x) &= v(t-a,a,x) \\ &= \beta(a)\sum_{n=0}^{\infty} c_n(t-a)\exp\left\{-n^2\alpha(a)\right\}\cos nx \\ &= \frac{\beta(a)}{\pi}\int_0^{\pi} b(w(t-a,y))\left(1+2\sum_{n=1}^{\infty} e^{-n^2\alpha(a)}\cos nx\,\cos ny\right)dy \\ &= \frac{\beta(a)}{\pi}\int_0^{\pi} b(w(t-a,y)) \\ &\quad\times\left\{1+\sum_{n=1}^{\infty}\left[\cos n(x-y)+\cos n(x+y)\right]e^{-n^2\alpha(a)}\right\}dy \\ &= \int_0^{\pi} b(w(t-a,y))\overline{K}_1(a,x,y)dy, \end{aligned}$$

where

$$\begin{aligned} \overline{K}_1(a,x,y) &= \frac{\beta(a)}{\pi}\left(1+2\sum_{n=1}^{\infty} e^{-n^2\alpha(a)}\cos nx\,\cos ny\right) \\ &= \frac{\beta(a)}{\pi}\left\{1+\sum_{n=1}^{\infty}\left[\cos n(x-y)+\cos n(x+y)\right]e^{-n^2\alpha(a)}\right\}. \end{aligned}$$

If we substitute this into (5.2), we get the following integral equation:

$$w(t,x) = \int_r^{A_l}\int_0^{\pi} b(w(t-a,y))\overline{K}_1(a,x,y)dyda, \quad t \geqslant A_l. \tag{5.13}$$

Now, we consider (5.1) subject to Dirichlet boundary conditions. The Dirichlet boundary conditions are given as

$$u(t,a,0) = 0, \quad u(t,a,\pi) = 0, \quad t \geqslant 0, \quad a \geqslant 0. \tag{5.14}$$

Similarly, we apply the technique of separation of variables to the following boundary value problem

$$\frac{\partial v}{\partial a} = D(a)\frac{\partial^2 v}{\partial x^2}(s,a,x) - d(a)v(s,a,x), \tag{5.15}$$

subject to the boundary conditions

$$v(s,a,0) = 0, \quad v(s,a,\pi) = 0, \tag{5.16}$$

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to obtain

$$\begin{aligned} u(t,a,x) &= v(t-a,a,x) \\ &= \beta(a)\sum_{n=1}^{\infty} c_n(t-a)\exp\left\{-n^2\alpha(a)\right\}\sin nx \\ &= \frac{\beta(a)}{\pi}\int_0^{\pi} b(w(t-a,y))\left(\sum_{n=1}^{\infty}\sin nx\,\sin ny\right)dy \\ &= \frac{2\beta(a)}{\pi}\int_0^{\pi} b(w(t-a,y)) \\ &\quad\times\sum_{n=1}^{\infty}\left[\cos n(x-y)-\cos n(x+y)\right]e^{-n^2\alpha(a)}dy \\ &= \int_0^{\pi} b(w(t-a,y))\overline{K}_2(a,x,y)dy, \end{aligned}$$

where

$$\begin{aligned} \overline{K}_2(a,x,y) &= \frac{2\beta(a)}{\pi}\sum_{n=1}^{\infty} e^{-n^2\alpha(a)}\sin nx\,\sin ny \\ &= \frac{\beta(a)}{\pi}\sum_{n=1}^{\infty}\left[\cos n(x-y)-\cos n(x+y)\right]e^{-n^2\alpha(a)}. \end{aligned}$$

If we substitute it into (5.2), we get the following integral equation:

$$w(t,x) = \int_r^{A_l}\int_0^{\pi} b(w(t-a,y))\overline{K}_2(a,x,y)dyda. \tag{5.17}$$

5.2 The existence of steady-state solution for a linear birth function

In this section, we show that the integral equations (5.13) and (5.17) admit a positive (non-negative and not identically zero) temporal-spatial eigen-solution with a form $e^{\lambda t}W(x)$, when the birth function is linear, i.e., $b(w) = pw,\ p > 0$. To do this, we assume that $w(t,x) = e^{\lambda t}W(x)$ and we substitute it into (5.13) and (5.17), we get

$$W(x) = \int_r^{A_l}\int_0^{\pi} e^{-\lambda a}W(y)\overline{K}_1(a,x,y)dyda \tag{5.18}$$

and

$$W(x) = \int_r^{A_l}\int_0^{\pi} e^{-\lambda a}W(y)\overline{K}_2(a,x,y)dyda, \tag{5.19}$$

respectively. Now, we define

$$K_1(x,y) = \frac{p}{\pi}\left(\Gamma_0(\lambda) + 2\sum_{n=1}^{\infty}\Gamma_n(\lambda)\cos nx\cos ny\right), \tag{5.20}$$

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where

$$\Gamma_n(\lambda) = \int_r^{A_l} e^{-\left(\lambda a+\gamma(a)+n^2\alpha(a)\right)} da, \quad \gamma(a) = \int_0^a d(\xi)d\xi\ ; \quad n = 0, 1, 2, \ldots. \tag{5.21}$$

Similarly, we define

$$K_2(x, y) = \frac{2p}{\pi} \sum_{n=1}^{\infty} \Gamma_n(\lambda) \sin nx \sin ny, \tag{5.22}$$

where $\Gamma_n(\lambda)$, $n = 1, 2, \ldots$, is given in (5.21). If we interchange the order of the integrals in (5.18) and (5.19), then we get

$$W(x) = \int_0^{\pi} W(y)K_1(x, y)dy, \tag{5.23}$$

and

$$W(x) = \int_0^{\pi} W(y)K_2(x, y)dy. \tag{5.24}$$

Let $W(x) = 1, x \in [0, \pi]$, in (5.23). Then by using the fact that $\{\cos nx\}_{n=0}^{\infty}$ are orthogonal functions, we have the following theorem:

Theorem 5.1 *The integral equation (5.18) admits a principal eigenvalue λ_0 with a corresponding eigenfunction $W(x) = 1$. The eigenvalue λ_0 can be determined by solving the following equation:*

$$p\Gamma_0(\lambda) = 1,$$

where

$$\Gamma_0(\lambda) = \int_r^{A_l} \exp\{-(\lambda a + \gamma(a))\}\, da,$$

and

$$\gamma(a) = \int_0^a d(\xi)d\xi.$$

Similarly, if we let $W(x) = \sin x, x \in [0, \pi]$, in (5.24), then we have the following theorem (here, we again use the fact that $\{\sin nx\}_{n=1}^{\infty}$ are orthogonal functions).

Theorem 5.2 *The integral equation (5.19) admits a principal eigenvalue λ_0 with a corresponding eigenfunction $W(x) = \sin x, x \in [0, \pi]$. The eigenvalue λ_0 can be determined by solving the equation*

$$p\Gamma_1(\lambda) = 1,$$

where

$$\Gamma_1(\lambda) = \int_r^{A_l} \exp\{-(\lambda a + \gamma(a) + \alpha(a))\}\, da.$$

The functions $\gamma(a)$ and $\alpha(a)$ are given in the following formulas:

$$\gamma(a) = \int_0^a d(\xi)d\xi \quad \textit{and} \quad \alpha(a) = \int_0^a D(\xi)d\xi.$$

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6 Existence of steady-state solutions and their stability

In this section, we consider a nonlinear birth function $b(w)$, e.g., $b(w) = pwe^{-sw^q}$, where p, q and s are positive constants. To see the existence of steady-state solution for this case, we assume $w(t, x)$ is free of the time variable t, i.e., $w(t, x) = w(x)$. We substitute this into (5.13) and (5.17) to get

$$w(x) = \int_r^{A_l} \int_0^{\pi} b(w(y))\overline{K}_1(a, x, y)dyda,$$

and

$$w(x) = \int_r^{A_l} \int_0^{\pi} b(w(y))\overline{K}_2(a, x, y)dyda.$$

For a particular case, if we substitute $b(w(x)) = pf(w),\ p > 0$, where $f(w)$ is a nonlinear function, into the above equations, we get

$$w(x) = \int_r^{A_l} \int_0^{\pi} pf(w(y))\overline{K}_1(a, x, y)dyda,$$

and

$$w(x) = \int_r^{A_l} \int_0^{\pi} pf(w(y)\overline{K}_2(a, x, y)dyda.$$

By integrating the above two equations with respect to the variable a, we have

$$w(x) = \int_0^{\pi} f(w(y))K_1^*(x, y)dy, \tag{6.1}$$

and

$$w(x) = \int_0^{\pi} f(w(y))K_2^*(x, y)dy. \tag{6.2}$$

The kernels $K_1^*(x, y)$ and $K_2^*(x, y)$ are given by

$$K_1^*(x, y) = \frac{p}{\pi}\left(c_0 + 2\sum_{n=1}^{\infty} c_n \cos\, nx \cos\, ny\right),$$

and

$$K_2^*(x, y) = \frac{2p}{\pi}\sum_{n=1}^{\infty} c_n \sin\, nx \sin\, ny,$$

where

$$c_n = \int_r^{A_l} e^{-\left(\gamma(a)+n^2\alpha(a)\right)}da,\ \ \gamma(a) = \int_0^a d(\xi)d\xi\,,\ \ n = 0, 1, 2, \ldots.$$

The constants $c_n,\ n = 1, 2, 3, \ldots$, satisfy the following inequality:

$$0 \leqslant c_n \leqslant \frac{e^{-(m_d+n^2m_D)r}}{m_d + n^2m_D}. \tag{6.3}$$

Therefore, the above two series converge uniformly and absolutely. Hence, $K_1^*(x, y)$ and $K_2^*(x, y)$ are continuous in the variables x and y.

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Remark 6.1 *Using the parabolic maximum principle, we can show that* $K_1^*(x,y)$ *is positive on* $[0,\pi]\times[0,\pi]$, *and* $K_2^*(x,y)$ *is positive on* $(0,\pi)\times(0,\pi)$.

The nonlinear integral equations in (6.1) and (6.2) can be written in abstract forms as follows:

$$w(x) = N_1(w)(x) := \int_0^\pi f(w(y))K_1^*(x,y)dy, \tag{6.4}$$

and

$$w(x) = N_2(w)(x) := \int_0^\pi f(w(y))K_2^*(x,y)dy. \tag{6.5}$$

Remark 6.2 We remark that $\int_0^\pi K_1^*(x,y)dy = pk^*$, where $k^* := c_0 = \int_r^{A_l} e^{-\gamma(a)}da$. Therefore, if $pk^* > 1$, then (6.4) admits a positive homogeneous steady-state solution w^*, where w^* is the unique solution of the algebraic equation $w^* = pk^*f(w^*)$.

Now, we aim to show that (6.5) admits a unique non-negative and non-zero steady-state solution when $f(w)$ is a monotone increasing function and $\lambda_0 > 0$. We also want to prove the global stability of the positive homogeneous steady-state solution w^* of (6.4). For this purpose, we use the approach of infinite dynamical systems as well as the persistence theory. For a basic setup, we consider the ordered Banach space $X = C_0^1([0,\pi])$, the space of all continuous functions that vanish at the boundaries and have a continuous first derivative with a norm defined by

$$\|\phi(x)\|_X = \max_{x\in[0,\pi]}|\phi(x)| + \max_{x\in[0,\pi]}|\phi'(x)|. \tag{6.6}$$

Let

$$X_+ = \{\phi(x)\in X \mid \ \phi(x)\geqslant 0\}$$

be its positive cone. By this consideration of the Banach space X, the positive cone X_+ has a non-empty interior ($Int(X_+) \neq \phi$). Therefore, we can define a strongly positive relation on X_+. We also consider the space $Y = C([t_0 - A_l, t_0], X)$, where $t_0 \geqslant A_l$ is fixed, with its ordered positive cone $Y_+ = C([t_0 - A_l, t_0], X_+)$. For convenience, we identify each $\phi \in Y_+$ as a function from $[t_0 - A_l, t_0]\times[0,\pi]$ to $\mathbb{R}$ as follows: $\phi(s,x) = \phi(s)(x)$. For any function $y(.) : [t_0 - A_l, b) \to X$, where $b > t_0$, define $y_t \in Y$ by $y_t(s) = y(t+s)$, $\forall t \in [t_0, b)$. For given $\beta > 0$, we define the positive cone $\Sigma_\beta := \{\phi(x)\in X_+ \mid \ \phi(x)\leqslant\beta\}$ and the function space $Z_\beta = C([t_0 - A_l, t_0], \Sigma_\beta)$. By using the method of steps [27], a unique solution $w(t,x,\phi)$ of the integral equation

$$\begin{cases} w(t,x) = \displaystyle\int_r^{A_l}\int_0^\pi f(w(t-a,y))\overline{K_2}(a,x,y)dyda & \\ w(t,0) = w(t,\pi) = 0, & t\geqslant t_0 \\ w(s,x) = \phi(s,x)\geqslant 0, & t_0 - A_l \leqslant s\leqslant t_0,\ x\in[0,\pi] \end{cases} \tag{6.7}$$

globally exists for any $\phi \in Y_+$, provided that $f(w)$ is a Lipschitz function. Therefore, we can define the semiflow $\Phi(t) : Y_+ \to Y_+$, by $(\Phi(t)\phi)(s,x) = w(t+s,x,\phi)$, $\forall s \in [t_0 - A_l, t_0]$, $x \in$

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$[0,\pi]$. Moreover, the semiflow $\Phi(t) : Y_+ \to Y_+$ is compact for $\forall t > t_0$. The concept of a semiflow can be found in the book [30, p. 8], see also [20, p. 2].

To go further for our analysis, we need the following assumptions on the function $f(w)$:

(H) Assume that

(H1) $f \in C^1\left(\mathbb{R}^+, \mathbb{R}^+\right)$, $f(0) = 0$, $f'(0) = 1 > 0$ and $f(w) \leqslant w, \forall w \geqslant 0$.

(H2) f is bounded and Lipschitz function, $\forall w \geqslant 0$.

(H3) f is a sublinear function, in the sense $f(\sigma w) \geqslant \sigma f(w), \forall w \geqslant 0$ and $\sigma \in (0,1)$.

To prove the main results in this section, we first prove a sequence of lemmas. We start with the following lemma.

Lemma 6.1 *Assume that (H1) and (H2) hold. Then for any $\phi \in Y_+$, the semiflow solution $w(t,x,\phi)$ is uniformly bounded. Furthermore, the semiflow $\Phi(t) : Y_+ \to Y_+$ admits a connected global attractor on Y_+, which attracts every bounded set in Y_+.*

Proof. Let $\phi \in Y_+$. Since $f(w)$ and $\overline{K}_2(a,x,y)$ are bounded and non-negative, we have

$$\begin{aligned} w(t,x,\phi) &= \int_r^{A_l} \int_0^{\pi} f(w(t-a,x))\overline{K_2}(a,x,y)dyda \\ &\leqslant \|f\|_\infty \int_0^{\pi} \int_r^{A_l} \overline{K_2}(a,x,y)dady \\ &\leqslant \|f\|_\infty k^* := M, \end{aligned}$$

where $k^* = \max_{x,y\in[0,\pi]} \int_r^{A_l} \pi\overline{K_2}(a,x,y)da$. Hence, $\limsup_{t\to\infty} w(t,x,\phi) \leqslant M$, $\forall \phi \in Y_+$, $x \in [0,\pi]$. Therefore, the semiflow $\Phi(t) : Y_+ \to Y_+$ is point dissipative. Since $\Phi(t) : Y_+ \to Y_+$ is compact and point dissipative, then, by Theorem 3.4.8 [8], the semiflow $\Phi(t) : Y_+ \to Y_+$ admits a connected global attractor on Y_+, which attracts every bounded set in Y_+. □

Remark 6.3 From the above lemma, it is easy to see that $\Phi(t)Z_M \subseteq Z_M$, and the omega limit set $\omega(\phi) \subseteq Z_M, \forall \phi \in Y_+$.

Re-consider the linear eigenvalue problem

$$\begin{aligned} w(x) &= \int_r^{A_l} \int_0^{\pi} pe^{-\lambda a} w(y)\overline{K_2}(a,x,y)dyda \\ &= \int_0^{\pi} w(y)K_2(\lambda,x,y)dy, \end{aligned} \tag{6.8}$$

and recall that its principal eigenvalue λ_0 exists. The following lemma shows that $\Phi(t) : Y_+ \to Y_+$ is uniformly persistent if $\lambda_0 > 0$. A semiflow $\Phi(t) : \mathbb{X} \to \mathbb{X}$ is called uniformly persistent with respect to a set $\mathbb{X}_0$ in $\mathbb{X}$, if there exists a positive constant δ_0 such that

$$\liminf_{t\to\infty} dist(\Phi(t)\phi, \partial\mathbb{X}_0) \geqslant \delta_0, \ \forall \phi \in \mathbb{X}_0,$$

where $\partial\mathbb{X}_0 = \mathbb{X} \setminus \mathbb{X}_0$.

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Lemma 6.2 *Assume that (H1) and (H2) hold, and assume* $\lambda_0 > 0$. *Then the semiflow* $\Phi(t) : Y_+ \to Y_+$ *is uniformly persistent with respect to* $Y_0 := \{\phi \in Y_+ \mid \phi \not\equiv 0\}$.

Proof. According to Theorem 4.6 in [24], it suffices to show that zero is a uniform weak repeller for Y_0, i.e., there exists a $\delta_0 > 0$ such that

$$\limsup_{t\to\infty} \|\Phi(t)\phi\|_X \geqslant \delta_0, \quad \forall \phi \in Y_0.$$

The proof will be done by contradiction. Let us consider the eigenvalue problem (6.8). Since this equation admits a principal eigenvalue λ_0, and by the assumption $\lambda_0 > 0$, there exists a sufficiently small number $\epsilon > 0$, such that

$$w(x) = (1-\epsilon)\int_0^\pi w(y)K_2(\lambda, x, y)dy$$

admits a positive eigenvalue λ_ϵ with a corresponding eigenfunction $\phi_\epsilon(.)$. Moreover, recall that $f'(0) = 1 > 0$. Therefore, there exists δ_ϵ such that

$$f(w) > (1-\epsilon)w, \quad \forall w \in (0, \delta_\epsilon).$$

Let $\delta_0 = \delta_\epsilon$. Assume there exists $\phi_0 \in Y_0$ such that

$$\limsup_{t\to\infty} \|\Phi(t)\phi_0\|_X < \delta_0.$$

Then there exists a $t' > t_0$ such that

$$\|w(t, ., \phi_0)\|_\infty \leqslant \|w(t, ., \phi_0)\|_X < \delta_0, \quad \forall t \geqslant t' - t_0.$$

Therefore, $w(t, x, \phi_0) \in (0, \delta_\epsilon)$ and so $w(t, x)$ satisfies

$$f(w(t, x)) > (1-\epsilon)w(t, x), \ t \geqslant t', x \in [0, \pi]. \tag{6.9}$$

Since $w(t, x, \phi_0) > 0$, $\forall t > t_0, \forall x \in (0, \pi)$, there exists $\eta > 0$ such that $w(t' + s, x, \phi_0) \geqslant \eta w_\epsilon(s, x), s \in [t_0 - A_l, t_0]$ and $x \in [0, \pi]$, where $w_\epsilon(t, x)$ is the solution of the following linear problem:

$$\begin{cases} w(t, x) = (1-\epsilon)\displaystyle\int_0^{A_l}\int_0^\pi w(t-a, y)\overline{K}_2(a, x, y)dyda, & \\ \phi_0(t, 0) = \phi_0(t, \pi) = 0, & t > t' \\ w(s, x) = \phi_0(s, x), & s \in [t_0 - A_l, t_0], \ x \in [0, \pi]. \end{cases}$$

Using the Inequality (6.9) and the comparison principle, e.g., Lemma 3.2 in [24], we have

$$w(t, x, \phi_0) \geqslant \eta w_\epsilon(t - t') = \eta w(x)e^{\lambda_\epsilon(t-t')},$$

which is unbounded when $t \to \infty$. This is a contradiction. □

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Remark 6.4 As mentioned before, the positive cone X_+ has a non-empty interior. Therefore, by the standard maximum principle, we can generate a strongly positive relation on Y_+, i.e., $\Phi(t)\left(Y_+ \setminus \{0\}\right) \subseteq Int(Y_+)$, $\forall t > t_0$. Moreover, if we assume that $f(w)$ is increasing on $[0, M]$, then, by the standard comparison principle, the semiflow $\Phi(t) : Z_M \to Z_M$ is strongly monotone.

Now, we turn to prove the main results in this section.

Theorem 6.1 *Let (H1) and (H2) hold, and assume that $\lambda_0 < 0$, then for any $\phi \in Y_+$, the unique solution $w(t, x, \phi)$ of (6.7) satisfies $\lim_{t\to\infty} \|w(t, ., \phi)\|_X = 0, \forall \phi \in Y_+$.*

Proof. Since $f(w) \leqslant w$, then (6.7) is dominated by the following linear equation:

$$\begin{cases} v(t,x) = p\displaystyle\int_r^{A_l}\int_0^{\pi} v(t-a,y)\overline{K_2}(a,x,y)dyda \\ v(t,0) = v(t,\pi) = 0, \qquad\qquad t \geqslant t_0, \\ v(s,x) = \phi(s,x) \geqslant 0, \qquad\qquad t_0 - A_l \leqslant s \leqslant t_0,\ x \in [0,\pi]. \end{cases} \tag{6.10}$$

Therefore, $w(t, ., \phi) \leqslant v(t, ., \phi), \forall t \geqslant t_0 - A_l$, $\forall \phi \in Y_+$. Since $\lambda_0 < 0$, by the properties of the solution $v(t, ., \phi)$, we have $v(t, ., \phi) \to 0$, as $t \to \infty$. Hence, $\lim_{t\to\infty} \|w(t, ., \phi)\|_X = 0, \forall \phi \in Y_+$. □

Theorem 6.2 *Let (H1)–(H3) hold, and $\lambda_0 > 0$. Moreover, assume that f is increasing on $[0, M]$ and f is strictly sublinear on $[0, M]$. Then there exists a unique steady-state solution $\phi^* \in Y_+ \setminus \{0\}$ of 6.7. Furthermore, ϕ^* is globally stable, i.e., $\omega(\phi) = \phi^*$, $\forall \phi \in Y_+ \setminus \{0\}$.*

Proof. Consider the semiflow $\Phi(t) : Z_M \to Z_M$. Then $\Phi(t) : Z_M \to Z_M$ is compact $\forall t > t_0$, point dissipative, and uniformly persistent with respect to $Z_M \cap Y_0$. Hence, by Theorem 1.3.7 [30], $\Phi(t) : Z_M \cap Y_0 \to Z_M \cap Y_0$ has a global attractor A_0 in $Z_M \cap Y_0$, which attracts every solution in $Z_M \cap Y_0$, and there exists at least one steady-state solution ϕ^* in A_0 ($\phi^* \neq 0$). Furthermore, the semiflow $\Phi(t) : Z_M \to Z_M$ is strongly monotone (Remark 6.4), and $\Phi(t) : Z_M \to Z_M$ is strictly sublinear, since f is strictly sublinear on $[0, M]$. Hence, by Lemma 1 [29], the steady-state solution ϕ^* is unique, and A_0 contains only ϕ^*. Moreover, by Remark 6.3, the omega limit set $\omega(\phi) \in Z_M$, $\forall \phi \in Y_+$. Also, by Remark 6.4, $\Phi(t)(Y_+ \setminus \{0\}) \subseteq Int(Y_+)$. Therefore, every non-zero orbit is attracted to Z_M, and hence to A_0. Since A_0 contains only ϕ^*, every non-zero orbit converges to ϕ^*, by the Hirsch Attractivity Theorem (*If an attractor K contains only one equilibrium p, then every trajectory attracted to K converges to p*; Theorem 3.3 [9]), i.e., $\omega(\phi) = \phi^*, \forall \phi \in Y_+ \setminus \{0\}$. This completes the proof. □

Now, we consider the case of Neumann's boundary conditions. As mentioned before, (6.4) admits a positive homogeneous steady-state solution w^*, when $pk^* > 1$ (see Remark 6.2). Moreover, w^* is unique if $f(w)$ is a sublinear function. To prove the stability of this homogeneous steady-state solution, we follow the same argument above with a slightly change in the function space. In fact, we consider the ordered Banach space $X = C([0, \pi])$ with the supremum norm. We also consider its positive cone $X_+ := \{\phi \in X \| \phi \geqslant 0\}$. Let

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$Y := C([t_0 - A_l, t_0], X)$ with the positive cone $Y := C([t_0 - A_l, t_0], X_+)$. Since X_+ has a non-empty interior, the semiflow $\Phi(t) : Y_+ \to Y_+$ is strongly positive (i.e., $\Phi(t)Y_+ \setminus \{0\} \subseteq Int(Y_+)$). Now, we consider the following problem:

$$\begin{cases} w(t,x) = \displaystyle\int_r^{A_l}\int_0^{\pi} f(w(t-a,y))\overline{K_1}(a,x,y)dyda, \\ w_x(t,0) = w_x(t,\pi) = 0, \qquad t \geqslant t_0, \\ w(s,x) = \phi(s,x) \geqslant 0, \qquad t_0 - A_l \leqslant s \leqslant t_0,\ x \in [0,\pi]. \end{cases} \tag{6.11}$$

For this equation, we have the following observation:

Remark 6.5 Let (H1) and (H2) hold. Then for any $\phi \in Y_+$, the unique solution $w(t,x,\phi)$ of (6.11) satisfies

$$\begin{aligned} w(t,x,\phi) &= \int_0^{\pi}\int_r^{A_l} f(w(t-a,y))\overline{K_1}(a,x,y)dady \\ &\leqslant k^* \|f\|_\infty := M, \end{aligned}$$

where the last inequality comes from the fact that $f(w)$ is non-negative and $\overline{K_1}$ is positive. Here, $k^* = \int_r^{A_l} \exp\{-\gamma(a)\}da$.

In this case, we have the following results:

Theorem 6.3 *Let (H1) and (H2) hold, and assume that $\lambda_0 < 0$. Then for any $\phi \in Y_+$ the unique solution $w(t,x,\phi)$ of (6.11) satisfies $\lim_{t\to\infty} \|w(t,.,\phi)\|_\infty = 0$.*

Proof. The proof is similar to the proof of Theorem 6.1. □

Theorem 6.4 *Let (H1)–(H3) hold, and $\lambda_0 > 0$. Moreover, assume that f is increasing on $[0,M]$ and f is strictly sublinear on $[0,M]$. Then the positive homogeneous steady-state solution w^* satisfies $lim_{t\to\infty} \|w(t,.,\phi) - w^*\|_\infty = 0$ for any non-zero $\phi \in Y_+$.*

Proof. First, we remark that $0 < w^* = k^* f(w^*) \leqslant k^* \|f\|_\infty = M$. Hence, $w^* \in [0,M]$. By the strict sublinearity of f over $[0,M]$, the homogeneous steady-state solution w^* is unique. The proof of the global stability of w^* is similar to the proof of Theorem 6.2, and therefore, we omit the details. □

Finally, we present two examples. In these examples, we consider the diffusion and death rates to be age independent. Let $D(a) = D > 0$ and $d(a) = d > 0, \forall a \in [0, A_l]$. We consider the birth function to be the Nicholson's blow-flies birth function, i.e., $f(w) = pwe^{-aw}$, $a, p > 0$. Then, $f(w)$ is increasing on $[0, 1/a]$, and it takes its maximum at $w = 1/a$ and the maximum value of $f(w)$ is $\frac{p}{a}e^{-1}$.

Example 6.1 *In this example, we consider the case of Neumann boundary conditions. To simplify and approximate the calculations, we may assume A_l to be large (i.e., $A_l \gg 1$).*

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For this case, the homogeneous steady-state solution is given by $w^* = \frac{1}{a}\ln(pk^*)$, *and* w^* *is positive if* $pk^* > 1$. *Since* $A_l \gg 1$, $k^* \sim e^{-dr}/d$. *The birth function* $f(w)$ *is monotone on* $[0, M]$ *if* $M \leqslant \frac{1}{a}$ *(equivalently* $\frac{pk^*}{a}e^{-1} \leqslant \frac{1}{a}$*). This implies that* f *is monotone on* $[0, M]$ *if* $pk^* \leqslant e$. *Therefore,* w^* *exists and unique if* $1 < pk^* \leqslant e$. *To study the global stability of* w^*, *we analyse the corresponding characteristic equation* $p\Gamma_0(\lambda) = 1$. *Simple calculations show that* $\Gamma_0(\lambda) \sim \frac{e^{-(\lambda+d)r}}{\lambda+d}$, *when* $A_l \gg 1$. *Therefore, the equation* $p\Gamma_0(\lambda) = 1$ *reduces to an approximate equation* $e^{-\lambda r} = \frac{1}{pk^*}(\lambda + d) := g(\lambda)$. *This equation has a unique solution* λ_0, *since the slope of the line* $g(\lambda)$ *is* $\frac{1}{pk^*} > 0$. *Moreover,* $g(\lambda)$ *intersects* $\lambda-$*axis at* $\lambda = -d < 0$ *and the* $y-$*intercept of* $g(\lambda)$ *is* $\frac{d}{pk^*}$. *Therefore, we have the following conclusions:*

(*i*) *if* $\frac{d}{pk^*} > 1$ *(equivalently,* $d > pk^*$*), then* $\lambda_0 < 0$, *and hence, the zero solution is stable.*

(*ii*) *if* $\frac{d}{pk^*} < 1$ *(equivalently,* $d < pk^*$*), then* $\lambda_0 > 0$, *and hence,* w^* *is globally stable.*

Example 6.2 *In this example, we consider the case of Dirichlet boundary conditions. We assume that* A_l *is large, and we assume* $pk^* \leqslant e$, *where* $k^* = \max_{x,y\in[0,\pi]} \int_r^{A_l} \pi\overline{K_2}(a, x, y)da$. *Under these assumptions, 6.7 admits a unique non-negative and non-zero steady-state solution* ϕ^*. *To study the global stability of* ϕ^*, *we analyse the corresponding characteristic equation* $p\Gamma_1(\lambda) = 1$. *Indeed, we have* $\Gamma_1(\lambda) \sim \frac{e^{-(\lambda+d+D)r}}{\lambda+d+D}$, *when* $A_l \gg 1$. *Therefore, the characteristic equation is approximated by* $p\Gamma_1(\lambda) \sim p\frac{e^{-(\lambda+d+D)r}}{\lambda+d+D} = 1$, *or* $e^{-\lambda r} = \frac{1}{pe^{-(d+D)r}}(\lambda + d + D) := g(\lambda)$. *The slope of the line* $g(\lambda)$ *is* $\frac{1}{pe^{-(d+D)r}} (> 0)$. *Hence, there exists a unique solution* λ_0 *solves the equation* $g(\lambda) = e^{-\lambda r}$. *Moreover,* $g(\lambda)$ *intersects* $\lambda-$*axis at* $\lambda = -(d + D) < 0$ *and the* $y-$*intercept of* $g(\lambda)$ *is* $\frac{d+D}{pe^{-(d+D)r}}$. *Therefore, we have the following conclusions:*

(*i*) *if* $\frac{d+D}{pe^{-(d+D)r}} > 1$ *(equivalently,* $d+D > pe^{-(d+D)r}$*), then* $\lambda_0 < 0$, *and hence, the zero solution is stable.*

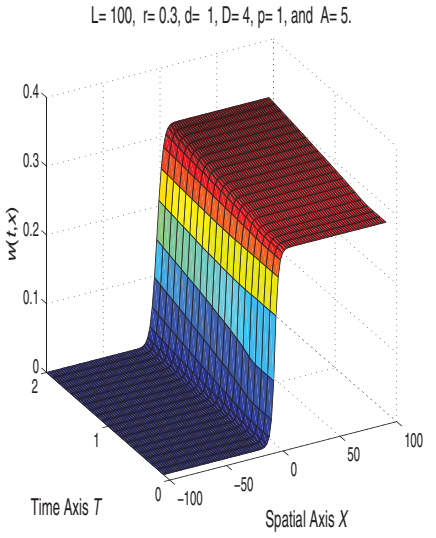
(*ii*) *if* $\frac{d+D}{pe^{-(d+D)r}} < 1$ *(equivalently,* $d + D < pe^{-(d+D)r}$*), then* $\lambda_0 > 0$, *and hence,* ϕ^* *is globally stable.* □

7 Numerical simulations

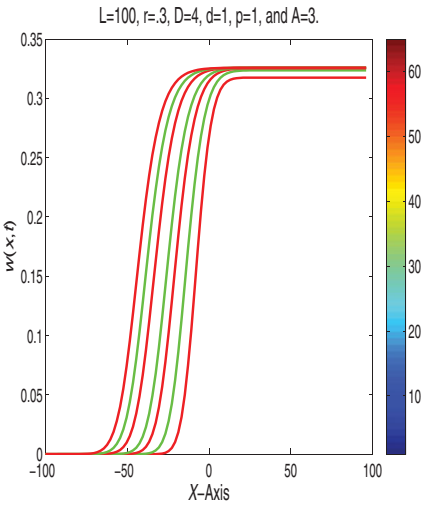
In this section, we present numerical simulations to investigate the long time behaviour of the solution $w(t, x)$. In the simulations, we consider the Nichelson's blow-flies birth function $f(u) = pue^{-u}$. For the infinite domain case, we present a numerical simulation to show that the integral equation 3.1 admits a monotone travelling wave solution. For the finite case, we present numerical simulations for the Dirichlet boundary value problems as well as the Neumann boundary value problems. These simulations show that the solution $w(t, x)$ either converges to the zero solution or to a non-negative and non-zero solution (resp. positive solution).

In the numerical simulations, we apply the composite Simpson's rule to evaluate the integrals in (1.2) and (5.2). To evaluate the solution $u(t, a, x)$ at the mesh points $a_i, i = 1, 2, \ldots, N$, we solve the reaction diffusion equations given in (2.2), (5.6) and (5.15), by using the method of lines.

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(a) Travelling wave solution. The values of the parameters are $L = 100,\ r = 0.3,\ A = 3,\ p = 1,\ D = 4$ and $d = 1$.



(b) A two dimensional graph shows the travelling wave solution. The values of the parameters are $L = 100,\ r = 0.3,\ A = 3,\ p = 1,\ D = 4$ and $d = 1$.

FIGURE 1. (Colour online) Travelling wave fronts for the constant case.

7.1 Numerical simulations for the unbounded domain case

In this section, we present numerical simulations for the infinite domain case. In fact, we distinguish two cases: the constant case (death and diffusion rates are age independent) and the age dependent case.

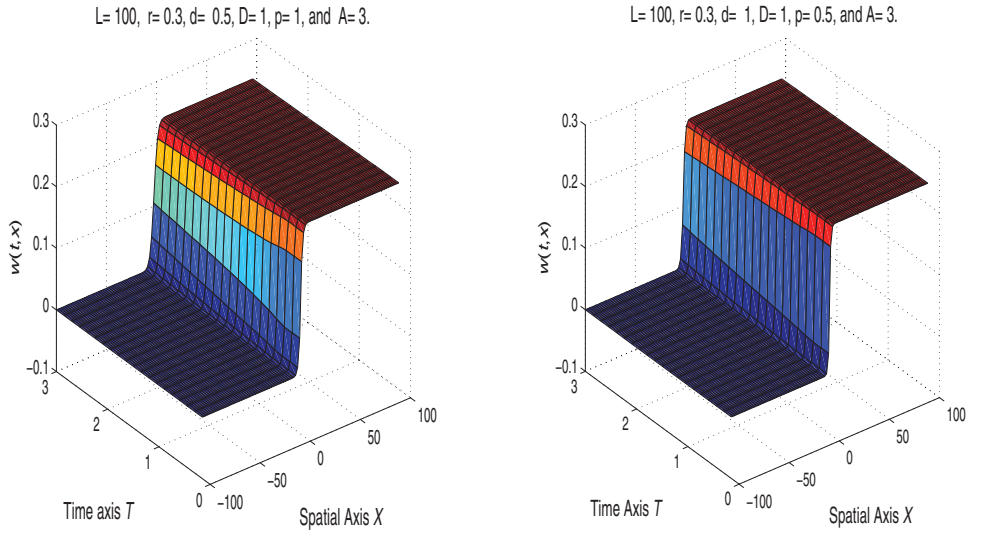
Case 1: In this case, we consider the diffusion and death rates to be constants. This numerical simulation shows that (3.1) admits a monotone travelling wave solution. We apply our technique for the values $L = 100$, $r = 0.3$, $A = 3$, $p = 1$, $D = 4$ and $d = 1$ (see Figure 1).

Case 2: To investigate the effect of age dependence on the solution, we add an exponential variation to the diffusion and death rates. Indeed, we consider $D(a) = D + e^{-a^2}$ and $d(a) = d + e^{-a^2}$. In this case, we choose $L = 100$, $r = 0.3$, $A = 3$, $p = 0.5$, $D = 1$, and $d = 1$ (see Figure 2(a)). Also, we consider $\alpha(a) = e^{-Da^2}$ and $\beta(a) = e^{-da^2}$ (i.e., the diffusion and death rates are linear) with the parameters $L = 100$, $r = 0.3$, $A = 3$, $p = 0.5$, $D = 1$ and $d = 1$ (see Figure 2(b)).

7.2 Numerical simulations for the bounded domain case

In this section, we present numerical simulations for the finite domain case. Similarly, we distinguish two cases: the constant case (death and diffusion rates are constant) and the age dependent case. In each case we consider the Dirichlet boundary conditions and the Neumann boundary conditions.

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(a) Travelling wave solutions when $D(a) = D + e^{-a^2}$ and $d(a) = d + e^{-a^2}$. The values of the parameters are $L = 100,\ r = 0.3,\ A = 3,\ p = 1,\ D = 1$ and $d = 0.5$.

(b) Travelling wave solutions when $\alpha(a) = e^{-Da^2}$ and $\beta(a) = e^{-da^2}$. The values of the parameters are $L = 100,\ r = 0.3,\ A = 3,\ p = 0.5,\ D = 1$ and $d = 1$.

FIGURE 2. (Colour online) Travelling wave fronts with age dependence.

Case 1: We consider the diffusion and death rates to be constants. The numerical simulation of this case is given in Figure 3. In fact, this simulation shows that the solution converges to the zero solution or to a non-negative and non-zero solution (resp. positive solution).

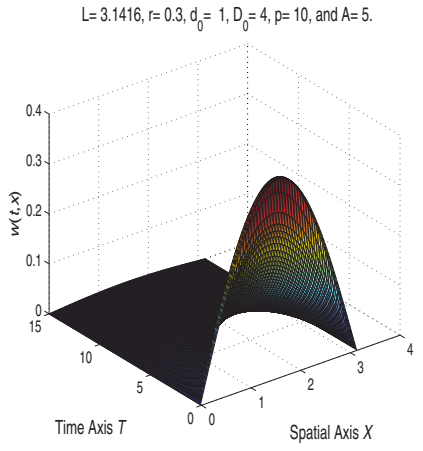
Case 2: The age dependence case. In this case we consider $D(a) = D + e^{-a^2}$ and $d(a) = d + e^{-a^2}$. The simulation for this case is given in Figure 4. Also, we consider $\alpha(a) = e^{-Da^2}$ and $\beta(a) = e^{-da^2}$. The simulation for this case is given in Figure 5.

Remark 7.1 In the case of age dependence, we considered the death rate $d(a) = d\ a$. In fact, this linear death rate is a rough attempt to represent a human population death rate. Indeed, the original formula is given by $d(a) = \frac{100}{A_l}(\frac{a}{A_l})$, where A_l is the life limit (see [10]).

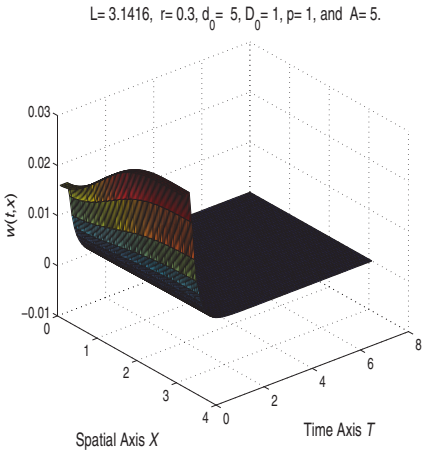
8 Conclusion

In this paper, we considered the age-dependent model (1.1)–(1.3), when the spatial domain is either unbounded or bounded. For the unbounded domain case, we derived a model in terms of an integral equation (3.1). In Section 4, we showed that this integral equation admits a unique (up to translation) monotone travelling wave solution, which connecting 0 and u^*, for all $c > c^*$, for different choices of birth functions, provided that the birth functions are monotone on $[0, u^*]$. For the case that the diffusion and death rates of the mature population are age independent (i.e., $D_m(a) = D$ and $d_m(a) = d$ are constants), it is easy to see that conditions (A) and (B) hold for our choice of the birth functions;

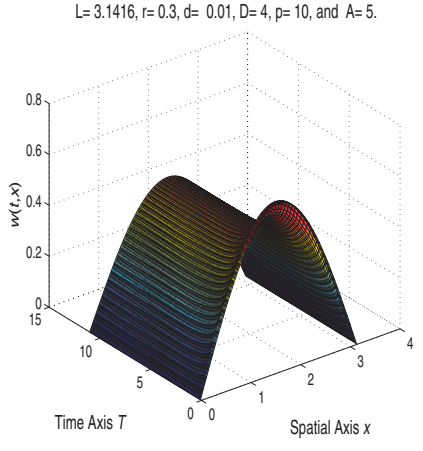
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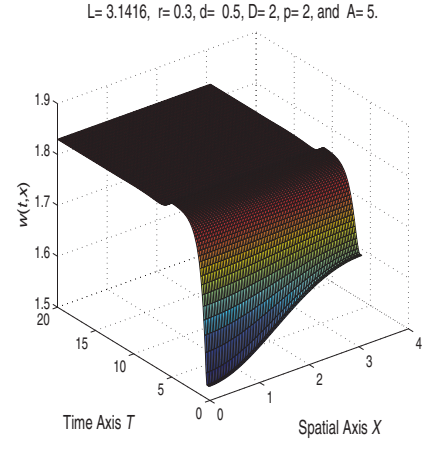
(a) Dirichlet boundary conditions. The values of the parameters are $L = \pi$, $r = 0.3$, $A = 5$, $p = 1$, $D = 1$, and $d = 5$. The initial condition is $\phi(s,x) = \sin x$, $0 \le x \le \pi$; $0 \le s \le 0.3$. In this case, the solution $w(t,x)$ converges to the zero solution for large time t.



(b) Neumann boundary conditions. The values of the parameters are $L = \pi$, $r = 0.3$, $A = 3$, $p = 0.5$, $D = 1$, and $d = 1$. The initial condition is $\phi(s,x) = 1 - \cos x$, $0 \le x \le \pi$; $0 \le s \le 0.3$. In this case, the solution $w(t,x)$ converges to the zero solution for large time t.



(c) Dirichlet boundary conditions. The values of the parameters are $L = \pi$, $r = 0.3$, $A = 5$, $p = 10$, $D = 4$ and $d = 0.01$. The initial condition is $\phi(s,x) = \sin x$, $0 \le x \le \pi$; $0 \le s \le 0.3$. In this case, the solution $w(t,x)$ converges to a non-negative and not identically zero solution for large time t.

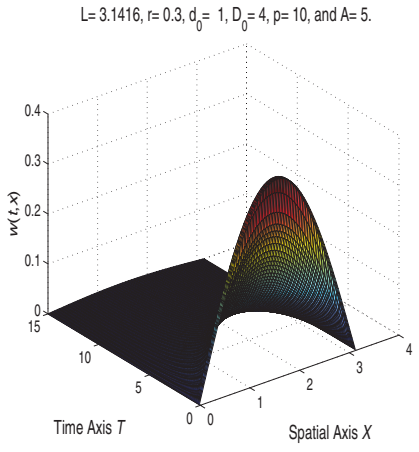


(d) Neumann boundary conditions.The values of the parameters are $L = \pi$, $r = 0.3$, $A = 5$, $p = 2$, $D = 2$ and $d = 0.5$. The initial condition is $\phi(s,x) = 1 - \cos x$, $0 \le x \le \pi$; $0 \le s \le 0.3$. In this case, the solution $w(t,x)$ converges to a positive solution for large time t.

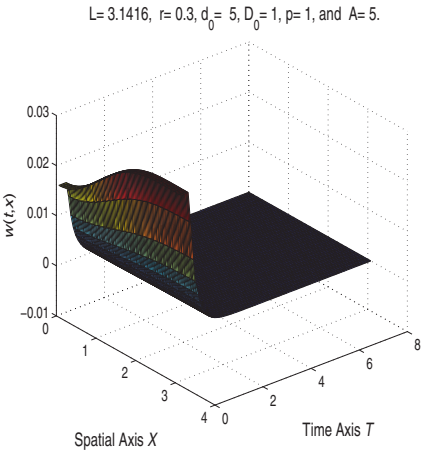
FIGURE 3. (Colour online) The behaviour of the mature population $w(t,x)$ for large time t.

and hence, our results hold for this case. The existence and uniqueness of a monotone travelling wave solution for this case have been studied in [13,22], and our results actually cover their results. Also, we remark that (3.1) admits a monotone travelling wave solution

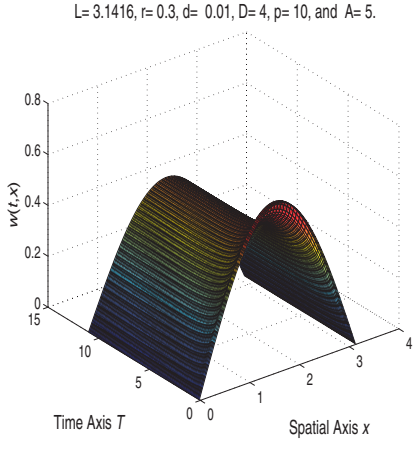
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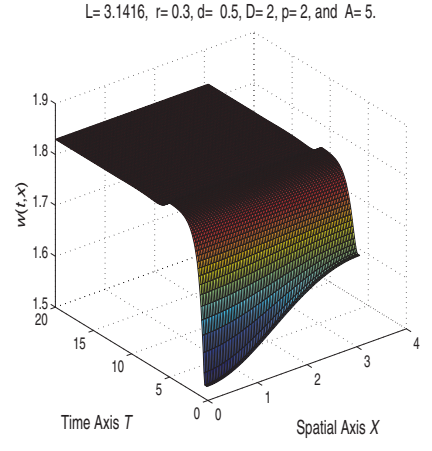
(a) Dirichlet boundary conditions. The values of the parameters are $L = \pi,\ r = 0.3,\ A = 5,\ p = 1,\ D = 1$ and $d = 5$. The initial condition is $\phi(s,x) = \sin x,\ 0 \le x \le \pi;\ 0 \le s \le 0.3$. In this case, the solution $w(t,x)$ converges to the zero solution for large time t.



(b) Neumann boundary conditions. The values of the parameters are $L = \pi,\ r = 0.3,\ A = 3,\ p = 0.5,\ D = 1$ and $d = 1$. The initial condition is $\phi(s,x) = 1 - \cos x,\ 0 \le x \le \pi;\ 0 \le s \le 0.3$. In this case, the solution $w(t,x)$ converges to the zero solution for large time t.



(c) Dirichlet boundary conditions. The values of the parameters are $L = \pi,\ r = 0.3,\ A = 5,\ p = 10,\ D = 4$ and $d = 0.01$. The initial condition is $\phi(s,x) = \sin x,\ 0 \le x \le \pi;\ 0 \le s \le 0.3$. In this case, the solution $w(t,x)$ converges to a non-negative and not identically zero solution for large time t.



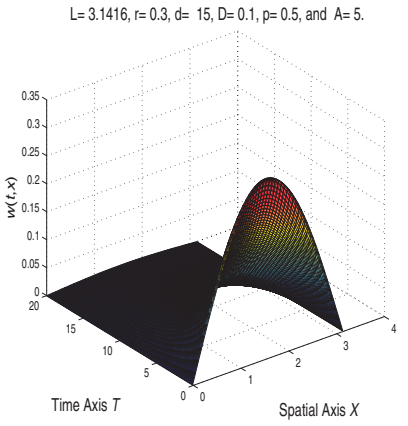
(d) Neumann boundary conditions.The values of the parameters are $L = \pi,\ r = 0.3,\ A = 5,\ p = 2,\ D = 2$ and $d = 0.5$. The initial condition is $\phi(s,x) = 1 - \cos x,\ 0 \le x \le \pi;\ 0 \le s \le 0.3$. In this case, the solution $w(t,x)$ converges to a positive solution for large time t.

FIGURE 4. (Colour online) The behaviour of the mature population $w(t,x)$ for large time t.

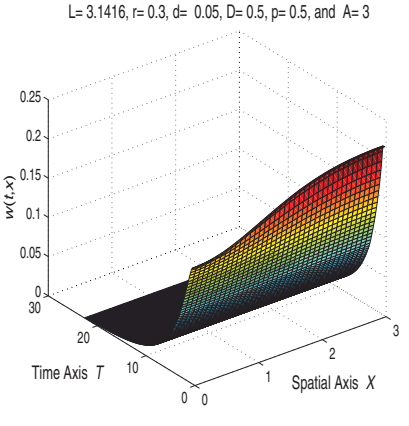
(U, c^*) (i.e., for $c = c^*$) connecting 0 and u^*, provided that the birth functions are monotone on $[0, u^*]$ (we guarantee this by applying Theorem 3.4 [25]).

For the bounded domain case, we derived two new models in terms of integral equations [(5.13) and (5.17)]. In Section 5, we gave conditions under which (5.13) [resp. (5.17)] admits

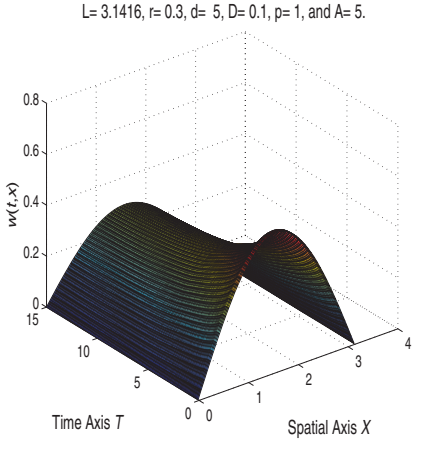
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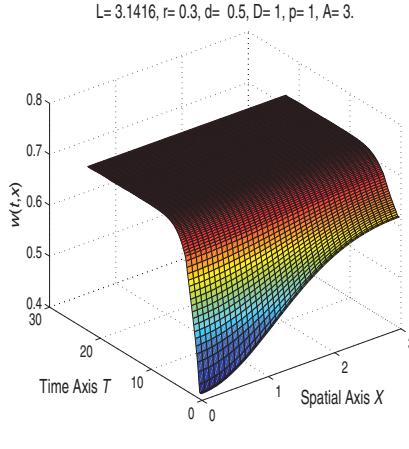
(a) Dirichlet boundary conditions. The values of the parameters are $L = \pi,\ r = 0.3,\ A = 3,\ p = 5,\ D = 2$ and $d = 1$. The initial condition is $\phi(s,x) = \sin x,\ 0 \le x \le \pi;\ 0 \le s \le 0.3$. In this case, the solution $w(t,x)$ converges to the zero solution for large time t.



(b) Neumann boundary conditions. The values of the parameters are $L = \pi,\ r = 0.3,\ A = 3,\ p = 1,\ D = 1$ and $d = 2$. The initial condition is $\phi(s,x) = 1 - \cos x,\ 0 \le x \le \pi;\ 0 \le s \le 0.3$. In this case, the solution $w(t,x)$ converges to the zero solution for large time t.



(c) Dirichlet boundary conditions. The values of the parameters are $L = \pi,\ r = 0.3,\ A = 5,\ p = 1,\ D = 0.1$ and $d = 5$. The initial condition is $\phi(s,x) = \sin x,\ 0 \le x \le \pi;\ 0 \le s \le 0.3$. In this case, the solution $w(t,x)$ converges to a non-negative and not identically zero solution for large time t.



(d) Neumann boundary conditions. The values of the parameters are $L = \pi,\ r = 0.3,\ A = 5,\ p = 1,\ D = 1$ and $d = 0.5$. The initial condition is $\phi(s,x) = 1 - \cos x,\ 0 \le x \le \pi;\ 0 \le s \le 0.3$. In this case, the solution $w(t,x)$ converges to a positive solution for large time t.

FIGURE 5. (Colour online) The behaviour of the mature population $w(t,x)$ for large time t.

a positive (resp. a non-negative and not identically zero) steady-state solution for linear birth functions. Since the stability of the positive (resp. non-negative and not identically zero) steady-state solution is determined by the sign of the principal eigenvalue λ_0 of the corresponding linear integral equation, we showed in Section 5 that (5.13) [resp. (5.17)]

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admits a unique real principal eigenvalue λ_0. In Section 6, for nonlinear birth function, we showed that the positive (resp. non-negative and not identically zero) steady-state solution exists and is stable provided that $\lambda_0 > 0$, whilst the zero solution is stable when $\lambda_0 < 0$. The particular case $D_m(a) = D$ and $d_m(a) = d$ have been studied in [28], and our results cover those results for this case. Finally, to support our mathematical results, we presented in Section 7 numerical simulations for each case.

Acknowledgement

The authors thank the referees' valuable comments, which greatly improved the exposition of the paper.

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