Combinatorics, Probability and Computing (2001) 10, 543–555. © 2001 Cambridge University Press
DOI: 10.1017/10.1017/S0963548301004837 Printed in the United Kingdom

On the Edge Distribution of a Graph

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Received 5 June 2000; revised 1 February 2001

We investigate a graph function which is related to the local density, the maximal cut and the least eigenvalue of a graph. In particular it enables us to prove the following assertions.

Let $p \geqslant 3$ be an integer, $c \in (0, 1/2)$ and G be a K_p-free graph on n vertices with $e \leqslant cn^2$ edges. There exists a positive constant $\alpha = \alpha(c, p)$ such that:

(a) some $\lfloor n/2 \rfloor$-subset of $V(G)$ induces at most $\left(\frac{c}{4} - \alpha\right) n^2$ edges (this answers a question of Paul Erdős);
(b) G can be made bipartite by the omission of at most $\left(\frac{c}{2} - \alpha\right) n^2$ edges.

1. Notation and conventions

We consider only simple finite, undirected graphs and use the standard graph terminology and notation of [1]. All graphs are assumed to be defined on the vertex set $\{1, 2, \ldots, n\}$. We let $V(G)$ denote the set of vertices of G, but simply write V instead of $V(G)$ when no ambiguity arises. Given a vertex i, N_i denotes the set of its neighbours, d_i denotes its degree and t_i denotes the number of triangles containing it. Given a vertex i, we let t_i'' denote the number of edges (j, k) such that $j \neq i$ and $k \neq i$ and neither j nor k are adjacent to i.

We let $G(n)$ denote a graph on n vertices and let $G(n, e)$ denote a graph on n vertices with e edges. For a graph G and $X \subset V(G)$, $G[X]$ denotes the subgraph of G induced by the set of vertices X. If there is no ambiguity, we write $e(X)$ instead of $e(G[X])$. K_p and $\overline{K}_p$ denote the complete graph and the empty graph on p vertices, respectively.

For a graph G, T_3 is the number of triangles of G and T_3'' is the number of induced subgraphs of G of order 3 and size 1.

2. Introduction

In the present paper we investigate a graph function which describes, in some sense, the edge distribution of a graph. First, it is helpful to better locate the topic within the current state of the art.

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Recall the following popular problems of Paul Erdős and his collaborators.

Problem 2.1. *Find the minimal c such that any triangle-free graph on n vertices contains a set of $\lfloor n/2 \rfloor$ vertices which spans at most $cn^2 + o\left(n^2\right)$ edges.*

This problem was stated in [5], and the best bounds of c so far seem to be those found by Krivelevich in [6].

Problem 2.2. *Find the minimal c such that any triangle-free graph on n vertices can be made bipartite by the omission of $cn^2 + o\left(n^2\right)$ edges.*

Problem 2.2 appears in [4] and is discussed by Krivelevich in [6]. Both these problems look quite similar but their relationship is by no means obvious. The question of their interdependence is treated in some detail by Krivelevich in [6], but only for regular graphs.

We believe that both problems stem from a common ancestor problem and hope that our results show that they can at least be considered from a common viewpoint. A similar approach was adopted by Brandt in [2], though he looks rather for a common descendant of both problems.

On the other hand, our results are close to the research on quasi-random graphs initiated by Rödl [8], Thomason [9] and Chung, Graham and Wilson [3]. In fact, some results obtained by our approach have alternative proofs in the general vein of quasi-random consideration, as noted by the referee, but not all of them, and not fully. We hope that our function gives a common viewpoint of the local density, the maximal cut and the least eigenvalue of a graph, to name but a few important graph characteristics.

3. Main results

Let $G = G(n,e)$. Consider the function

$$\Phi(G,k) = \min\left\{\frac{e(U)}{k} + \frac{e\left(V\setminus U\right)}{n-k} - \frac{e}{n}\right\},$$

where the minimum is taken over all proper subsets $U \subset V$ such that $|U| = k$.

Let

$$\varphi(G) = \min \Phi(G,k)$$

where k runs from 1 to $n-1$.

It is evident that $\Phi(G,k)$ and $\varphi(G)$ depend on the uniformity of edge distribution of G. Indeed, if G is quasi-random in the sense of Chung, Graham and Wilson [3], then $\Phi(G,k) = o(n)$ for every k $(1 \leqslant k \leqslant n-1)$ and consequently $\varphi(G) = o(n)$. On the other hand, the graph consisting of two disjoint complete graphs on n vertices shows that the reverse implication does not always hold.

Our main result is contained in the following theorem.

Theorem 3.1. *Let $p \geqslant 3$ be an integer, $c > 0$ be real, and $G = G(n,e)$ be a K_p-free graph*

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with $e \geqslant cn^2$. *There exists a constant* $\beta = \beta(c, p)$ *such that*

$$\Phi\left(G, \lfloor n/2 \rfloor\right) \leqslant -\beta n$$

holds for sufficiently large n. In particular, G can be made bipartite by the omission of at most $\frac{e}{2} - \beta \frac{n^2}{2}$ *edges.*

One particular application of Theorem 3.1 is the solution of the following problem, which is mentioned in [1, p. 363, problem 25] as unsolved.

Problem 3.2 (Paul Erdős). *Let* $c > 0$. *Suppose* $G = G(n, \lfloor cn^2 \rfloor)$ *is such that* $e(G[W]) \geqslant c\left(n/2\right)^2 (1 + o(1))$ *for every* $W \subset V$, $|W| = \lfloor n/2 \rfloor$. *Then, for every fixed k and sufficiently large n, the graph G contains a* K_k.

It is interesting to note that this problem, being a difficult one in the early 1980s, looks more or less routine now, given the speed with which V. Rödl in a private letter and the referee in his report proposed different, though similar, solutions. Their proofs are short yet self-contained, thus revealing a totally new perception of such problems. As an illustration of Theorem 3.1, let us show in brief how it implies Problem 3.2.

Assume the assertion of Problem 3.2 does not hold. Then there exist an integer $p \geqslant 3$ and a real $c > 0$ such that, for any $\varepsilon > 0$ and n_0, there exists a K_p-free graph $G = G\left(n, \lfloor cn^2 \rfloor\right)$ of order $n > n_0$ with the following property: for any $W \subset V(G)$ with $|W| = \lfloor n/2 \rfloor$, $e(G[W]) > \left(c/4 - \varepsilon\right) n^2 \geqslant e(G)/4 - \varepsilon n^2$. By Theorem 3.1, for sufficiently large n we have

$$\Phi\left(G, \lfloor n/2 \rfloor\right) \leqslant -\beta n;$$

thus, by the definition of Φ, there exists some $U \subset V$ with $|U| = \lfloor n/2 \rfloor$ and

$$\frac{e(U)}{\lfloor n/2 \rfloor} + \frac{e\left(V \backslash U\right)}{\lceil n/2 \rceil} - \frac{e}{n} \leqslant -\beta n.$$

Let us consider the case of even n, since the case of odd n is easily derived from it when n is sufficiently large. Without loss of generality we may assume that $e\left(V \backslash U\right) \geqslant e(U)$. Therefore we have

$$e(U) \leqslant \frac{1}{4} e - \frac{\beta}{4} n^2,$$

and, since this contradicts our assumption, the assertion of Problem 3.2 does hold.

All subsequent proofs are gathered in Section 4 in the hope that the mainstream of the paper might be better followed.

3.1. Properties of $\Phi(G, k)$ and $\varphi(G)$

The functions $\Phi(G, k)$ and $\varphi(G)$ have many interesting properties, some of which are outlined in the following lemmas.

Lemma 3.3. *For any* $G = G(n, e)$ *and any* k $(1 \leqslant k < n)$,

$$-\frac{k(n-k)}{n} \leqslant \Phi(G, k) \leqslant -\frac{e}{(n-1)n}.$$

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Note that Lemma 3.3 holds without any constraints on G, but for that reason it does not yield any effective upper estimate of $\Phi(G,k)$.

By Lemma 3.3, we obviously have

$$-\frac{1}{n}\left\lfloor\frac{n^2}{4}\right\rfloor \leqslant \varphi(G) \leqslant -\frac{e}{(n-1)n}.$$

These inequalities are not surprising in view of the following claim.

Claim 3.4. *For any graph G, the least eigenvalue $\lambda(G)$ of the adjacency matrix of G satisfies*

$$\lambda(G) \leqslant 2\varphi(G).$$

We do not prove this assertion here because it would distract from the flow of our arguments. A proof can be found in [7].

The following recursive inequality is a key element in our argument.

Lemma 3.5. *Let G be a graph of order n. For any integer k $\left(1 \leqslant k \leqslant \frac{n}{2} - 1\right)$, we have*

$$\frac{k+1}{n-k-1}\Phi(G,k+1) \leqslant \frac{k}{n-k}\Phi(G,k).$$

Note that, by Lemma 3.5, we have

$$\Phi\left(G, \left\lfloor\frac{n}{2}\right\rfloor\right) \leqslant \frac{k\lceil n/2\rceil}{(n-k)\lfloor n/2\rfloor}\Phi(G,k),$$

and, since $\Phi(G,k) \leqslant 0$, we immediately obtain the following corollary.

Corollary 3.6. *Let k be an integer with $0 < k \leqslant \left\lfloor\frac{n}{2}\right\rfloor$. Then*

$$\Phi\left(G, \left\lfloor\frac{n}{2}\right\rfloor\right) \leqslant \frac{k}{n-k}\Phi(G,k).$$

By the definition of $\varphi(G)$, we trivially have $\varphi(G) \leqslant \Phi\left(G, \lfloor n/2\rfloor\right)$. However, if we know some upper bound of $\varphi(G)$, we may effectively estimate $\Phi\left(G, \lfloor n/2\rfloor\right)$ from above as well. More precisely, the following lemma holds.

Lemma 3.7. *Let $\varphi(G) \leqslant -\gamma n$ where $\gamma \leqslant 1/4$. Then*

$$\Phi\left(G, \left\lfloor\frac{n}{2}\right\rfloor\right) \leqslant -\frac{1-\sqrt{1-4\gamma}}{1+\sqrt{1-4\gamma}}\gamma n.$$

The following lemma relates $\varphi(G)$ and $\varphi\left(G'\right)$ of an induced subgraph $G' \subset G$. Unlike the simple inequality for the least eigenvalue $\lambda(G) \leqslant \lambda\left(G'\right)$, a similar inequality for $\varphi(G)$ apparently does not hold.

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Lemma 3.8. *Let G be a graph of order n and $G' \subset G$ be an induced subgraph of G of order $m < n$. Then*

$$n\varphi(G) + \frac{e(G)}{n+1} \leqslant m\varphi(G') + \frac{e(G')}{m+1}.$$

In some cases better inequalities for $\varphi(G)$ of induced subgraphs are possible. We state the following lemma, which is not needed for the main argument but is rather instructive anyway.

Lemma 3.9. *Let G be a graph of order $2n$ and $G' \subset G$ be an induced subgraph of G of order $2m < 2n$. Then $n\Phi(G,n) \leqslant m\Phi(G',m)$.*

3.2. Main theorem and proof-supporting lemmas

In order to prove Theorem 3.1 we have to loosen its assertion in the following way.

Theorem 3.10. *For any integer $p \geqslant 3$ and real $c > 0$ there exists a constant $\alpha = \alpha(c,p)$ such that, for any K_p-free graph $G = G(n,e)$ with $e \geqslant cn^2$, the inequality*

$$\varphi(G) \leqslant -\alpha n$$

holds for sufficiently large n.

Although Theorem 3.1 seems stronger than Theorem 3.10, by Lemma 3.7, we have

$$\Phi\left(G, \left\lfloor \frac{n}{2} \right\rfloor\right) \leqslant -\frac{1-\sqrt{1-4\alpha}}{1+\sqrt{1-4\alpha}}\alpha n,$$

and we readily obtain the assertion of Theorem 3.1 from Theorem 3.10. However, Theorem 3.10 is better suited for proof, and this was the reason for introducing it together with the function $\varphi(G)$.

Before proceeding to the proof of Theorem 3.10 we shall investigate the function $\varphi(G)$ for triangle-free graphs G, since in that case the results are more explicit and, also, we obtain a base for an inductive proof of Theorem 3.10.

For a proof of the following simple lemma see [1, p. 302].

Lemma 3.11. *For any $G = G(n,e)$,*

$$3T_3 = T_3'' - ne + \sum_{i=1}^{n} d_i^2. \tag{3.1}$$

The following lemma is an immediate consequence of Lemma 3.11 and the definition of $\varphi(G)$.

Lemma 3.12. *For any $G = G(n,e)$ without isolated vertices,*

$$\varphi(G)\left(n^2 - 2e\right) \leqslant n\sum_{i=1}^{n}\frac{t_i}{d_i} - \frac{1}{2}\sum_{i=1}^{n} d_i^2 \leqslant n\sum_{i=1}^{n}\frac{t_i}{d_i} - \frac{2e^2}{n}. \tag{3.2}$$

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In the case of triangle-free graphs we may omit the density constraint of G, and prove the following more general result.

Lemma 3.13. *For any triangle-free $G = G(n,e)$, the following inequality holds:*

$$\varphi(G) \leqslant -\frac{\sum_{i=1}^{n} d_i^2}{2\left(n^2-2e\right)}.$$

Hence, it is not hard to obtain by Lemma 3.13 an explicit upper bound of $\varphi(G)$ which holds for any dense triangle-free graph. Indeed, in view of

$$\varphi(G) \leqslant -\frac{\sum_{i=1}^{n} d_i^2}{2\left(n^2-2e\right)} \leqslant \frac{-2e^2}{n\left(n^2-2e\right)} \leqslant \frac{-2c^2}{(1-2c)}n,$$

we obtain the following result.

Corollary 3.14. *For any real $c \in (0,1/2)$ and every triangle-free $G = G(n,e)$ with $e \geqslant cn^2$,*

$$\varphi(G) \leqslant \frac{-2c^2}{(1-2c)}n.$$

4. Proofs

Proof of Lemma 3.3. Let us first prove the left inequality. Let

$$\Phi(G,k) = \frac{e(U)}{k} + \frac{e\left(V\backslash U\right)}{n-k} - \frac{e}{n}$$

for some U with $|U| = k$. We obviously have $e = e(U) + e\left(V\backslash U\right) + e\left(U, V\backslash U\right)$, and thus

$$\begin{aligned}\Phi(G,k) &= \frac{e(U)}{k} + \frac{e\left(V\backslash U\right)}{n-k} - \frac{e(U) + e\left(V\backslash U\right) + e\left(U, V\backslash U\right)}{n}\\ &\geqslant e(U)\left(\frac{1}{k}-\frac{1}{n}\right) + e\left(V\backslash U\right)\left(\frac{1}{n-k}-\frac{1}{n}\right) - \frac{k(n-k)}{n}\\ &\geqslant -\frac{k(n-k)}{n}.\end{aligned}$$

Let us now prove the right inequality. Consider separately the case $k = 1$. We obviously have

$$\begin{aligned}n\Phi(G,1) &\leqslant \sum_{|U|=1}\left(\frac{e(U)}{1} + \frac{e\left(V\backslash U\right)}{n-1} - \frac{e}{n}\right)\\ &= \sum_{i=1}^{n}\left(\frac{e-d_i}{n-1} - \frac{e}{n}\right) = \frac{-e}{n-1} < 0.\end{aligned}$$

Similarly, for $k \geqslant 2$, we have

$$\binom{n}{k}\Phi(G,k) \leqslant \sum_{|U|=k}\left(\frac{e(U)}{k} + \frac{e\left(V\backslash U\right)}{n-k} - \frac{e}{n}\right).$$

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Any edge of G is contained in $\binom{n-2}{k-2}$ sets of size k and in the complement of $\binom{n-2}{k}$ sets of size k. Thus,

$$\begin{aligned}\binom{n}{k}\Phi(G,k) &\leqslant \left(\frac{1}{k}\binom{n-2}{k-2}+\frac{1}{n-k}\binom{n-2}{k}-\frac{1}{n}\binom{n}{k}\right)e\\ &= \frac{1}{k}\binom{n-2}{k-2}\left(1+\frac{n-k-1}{k-1}-\frac{n-1}{(k-1)}\right)e\\ &= -\frac{1}{k(k-1)}\binom{n-2}{k-2}e<0,\end{aligned}$$

and we are done. □

Proof of Lemma 3.5. Let $U \subset V$ be a proper subset with $|U| = k$, and

$$\Phi(G,k) = \frac{e(U)}{k}+\frac{e\left(V\backslash U\right)}{n-k}-\frac{e}{n}.$$

Consider all proper subsets of V which are obtained by removing a single vertex from $V\backslash U$ and adding it to U. For every $v \in V\backslash U$ we have, by the minimality of $\Phi(G,k+1)$,

$$\Phi(G,k+1) \leqslant \frac{e(U+v)}{k+1}+\frac{e\left(V\backslash U-v\right)}{n-k-1}-\frac{e}{n}.$$

Let us sum this inequality for all $v \in V\backslash U$. We obtain

$$\begin{aligned}&(n-k)\Phi(G,k+1)\\ &\leqslant \sum_{v\in V\backslash U}\frac{e(U+v)}{k+1}+\sum_{v\in V\backslash U}\frac{e\left(V\backslash U-v\right)}{n-k-1}-\frac{n-k}{n}e\\ &= \frac{n-k}{k+1}e(U)+\frac{1}{k+1}e\left(U,V\backslash U\right)+\frac{n-k-2}{n-k-1}e\left(V\backslash U\right)-\frac{n-k}{n}e\\ &= \frac{n-k-1}{k+1}e(U)+\left(\frac{n-k-2}{n-k-1}-\frac{1}{k+1}\right)e\left(V\backslash U\right)-\left(\frac{n-k}{n}-\frac{1}{k+1}\right)e.\end{aligned}$$

Hence, we have

$$\begin{aligned}&(n-k)\Phi(G,k+1)-\frac{(n-k-1)k}{k+1}\Phi(G,k)\\ &\leqslant \frac{n-k-1}{k+1}e(U)+\left(\frac{n-k-2}{n-k-1}-\frac{1}{k+1}\right)e\left(V\backslash U\right)-\left(\frac{n-k}{n}-\frac{1}{k+1}\right)e\\ &-\frac{(n-k-1)k}{k+1}\left(\frac{e(U)}{k}+\frac{e\left(V\backslash U\right)}{n-k}-\frac{e}{n}\right)\\ &= \left(\frac{n-k-2}{n-k-1}-\frac{1}{k+1}-\frac{(n-k-1)k}{(n-k)(k+1)}\right)e\left(V\backslash U\right)\\ &= -\frac{n}{(n-k-1)(k+1)(n-k)}e\left(V\backslash U\right)\leqslant 0.\end{aligned}$$

Therefore,

$$(n-k)\Phi(G,k+1) \leqslant \frac{(n-k-1)k}{k+1}\Phi(G,k)$$

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and thus

$$\frac{k+1}{n-k-1}\Phi(G,k+1) \leqslant \frac{k}{n-k}\Phi(G,k). \qquad \square$$

Proof of Lemma 3.7. Let

$$\varphi(G) = \frac{e(U)}{k} + \frac{e\left(V\backslash U\right)}{n-k} - \frac{e}{n}.$$

We obviously have $e = e(U) + e\left(V\backslash U\right) + e\left(U, V\backslash U\right)$, and hence

$$\left(\frac{1}{k} - \frac{1}{n}\right) e(U) + \left(\frac{1}{n-k} - \frac{1}{n}\right) e\left(V\backslash U\right) + \gamma n \leqslant \frac{e\left(U, V\backslash U\right)}{n} \leqslant \frac{k(n-k)}{n}.$$

Therefore, $k^2 - nk + \gamma n^2 \leqslant 0$, and thus

$$k \geqslant \frac{1-\sqrt{1-4\gamma}}{2} n.$$

By Corollary 3.6,

$$\begin{aligned}\Phi\left(G, \left\lfloor \frac{n}{2} \right\rfloor\right) &\leqslant \frac{k}{(n-k)}\Phi(G,k) \leqslant \frac{1-\sqrt{1-4\gamma}}{1+\sqrt{1-4\gamma}}\Phi(G,k)\\ &\leqslant -\frac{1-\sqrt{1-4\gamma}}{1+\sqrt{1-4\gamma}}\gamma n. \qquad \square\end{aligned}$$

Proof of Lemma 3.8. Obviously, it suffices to prove the assertion only for $m = n-1$. Let G' be of order $n-1$ and size e, and let $U \subset V = V\left(G'\right)$ be a proper subset of $V\left(G'\right)$ with $|U| = k$ and

$$\varphi\left(G'\right) = \frac{e(U)}{k} + \frac{e\left(V\backslash U\right)}{n-k-1} - \frac{e}{n-1}.$$

For brevity, let e_1 denote $e(U)$ and let e_2 denote $e\left(V\backslash U\right)$. Thus the above equality can be rewritten as

$$\varphi\left(G'\right) = \frac{e_1}{k} + \frac{e_2}{n-k-1} - \frac{e}{n-1}. \tag{4.1}$$

Now, let v be the vertex of G, which does not belong to $V\left(G'\right)$. Let

$$d' = |N_v \cap U|,\ d'' = |N_v\backslash U|,\ W_1 = U \cup \{v\},\ W_2 = \left(V\backslash U\right) \cup \{v\}.$$

We have, by the minimality of $\varphi(G)$,

$$\varphi(G) \leqslant \frac{e(G[W_1])}{k+1} + \frac{e\left(G\left[V(G)\backslash W_1\right]\right)}{n-k-1} - \frac{e(G)}{n} \tag{4.2}$$

and

$$\varphi(G) \leqslant \frac{e\left(G\left[V(G)\backslash W_2\right]\right)}{k} + \frac{e(G[W_2])}{n-k} - \frac{e(G)}{n}. \tag{4.3}$$

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On the other hand, we manifestly have

$$\begin{aligned} e(G[W_1]) &= e_1 + d', \\ e\left(G\left[V(G)\setminus W_1\right]\right) &= e_2, \\ e(G[W_2]) &= e_2 + d'', \\ e\left(G\left[V(G)\setminus W_2\right]\right) &= e_1, \\ e(G) &= e + d' + d''. \end{aligned} \tag{4.4}$$

Hence, replacing $e(G[W_1])$ and $e\left(G\left[V(G)\setminus W_1\right]\right)$ in (4.2), we obtain

$$\begin{aligned} \varphi(G) &\leqslant \frac{e_1 + d'}{k+1} + \frac{e_2}{n-k-1} - \frac{e + d' + d''}{n} \\ &= \frac{e_1}{k+1} + \frac{e_2}{n-k-1} - \frac{e}{n} + \frac{n-k-1}{(k+1)n}d' - \frac{1}{n}d'', \end{aligned}$$

and thus

$$(k+1)\varphi(G) \leqslant e_1 + \frac{(k+1)e_2}{n-k-1} - \frac{(k+1)e}{n} + \frac{n-k-1}{n}d' - \frac{k+1}{n}d''.$$

Similarly, replacing $e(G[W_2])$ and $e\left(G\left[V(G)\setminus W_2\right]\right)$ in (4.3), we obtain

$$(n-k)\varphi(G) \leqslant \frac{(n-k)e_1}{k} + e_2 - \frac{e(n-k)}{n} + \frac{k}{n}d'' - \frac{n-k}{n}d'.$$

Summing the last two inequalities, we have

$$(n+1)\varphi(G) \leqslant \frac{ne_1}{k} + \frac{ne_2}{n-k-1} - \frac{(n+1)e}{n} - \frac{d' + d''}{n},$$

and therefore

$$\begin{aligned} n\varphi(G) &\leqslant \frac{n^2}{n+1}\left(\frac{e_1}{k} + \frac{e_2}{n-k-1}\right) - e - \frac{d' + d''}{n+1} \\ &= \left(n - 1 + \frac{1}{n+1}\right)\left(\frac{e_1}{k} + \frac{e_2}{n-k-1}\right) - e - \frac{d' + d''}{n+1}. \end{aligned}$$

Hence, by (4.1),

$$n\varphi(G) \leqslant (n-1)\varphi\left(G'\right) + \frac{1}{n+1}\left(\frac{e_1}{k} + \frac{e_2}{n-k-1}\right) - \frac{d' + d''}{n+1}.$$

Since by Lemma 3.3 we have

$$\varphi\left(G'\right) \leqslant -\frac{e\left(G'\right)}{(n-2)(n-1)},$$

therefore

$$\frac{1}{n+1}\left(\frac{e_1}{k} + \frac{e_2}{n-k-1}\right) \leqslant \frac{1}{n+1}\frac{(n-3)e\left(G'\right)}{(n-2)(n-1)} < \frac{e\left(G'\right)}{n(n+1)}.$$

Consequently,

$$n\varphi(G) \leqslant (n-1)\varphi\left(G'\right) + \frac{e\left(G'\right)}{(n+1)n} - \frac{d' + d''}{n+1}.$$

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Now, by (4.4) we obtain

$$\begin{aligned} n\varphi(G) &\leqslant (n-1)\,\varphi\left(G'\right) + \frac{e\left(G'\right)}{(n+1)\,n} - \left(e(G) - e\left(G'\right)\right)\frac{1}{n+1} \\ &= (n-1)\,\varphi\left(G'\right) + \frac{e\left(G'\right)}{n} - \frac{e(G)}{n+1}, \end{aligned}$$

and we are done. □

Proof of Lemma 3.9. Obviously it suffices to prove the assertion only for $m = n-1$. Let G' be of order $2(n-1)$, and let $U \subset V\left(G'\right)$ be such that

$$\Phi\left(G', n-1\right) = \frac{e\left(G'[U]\right)}{n-1} + \frac{e\left(\left[V\left(G'\right)\setminus U\right]\right)}{n-1} - \frac{e\left(G'\right)}{2(n-1)}.$$

Let 1 and 2 be the two vertices of G that do not belong to $V\left(G'\right)$. Consider the sets $W_1 = U \cup \{1\}$ and $W_2 = U \cup \{2\}$. Both are of order n, so we have

$$\begin{aligned} 2\Phi(G,n) \leqslant &\frac{e(G[W_1])}{n} + \frac{e\left(\left[V(G)\setminus W_1\right]\right)}{n} - \frac{e(G)}{2n} \\ &+ \frac{e(G[W_2])}{n} + \frac{e\left(\left[V(G)\setminus W_2\right]\right)}{n} - \frac{e(G)}{2n}. \end{aligned}$$

Let

$$\begin{aligned} &d_{1,1} = U \cap N_1,\ d_{1,2} = U \cap N_2, \\ &d_{2,1} = \left(V\left(G'\right)\setminus U\right) \cap N_1,\ d_{2,2} = \left(V\left(G'\right)\setminus U\right) \cap N_2. \end{aligned}$$

Obviously, we have

$$\begin{aligned} e(G[W_1]) &= e\left(G'[U]\right) + d_{1,1}, \\ e(G[W_2]) &= e\left(G'[U]\right) + d_{1,2}, \\ e\left(G\left[V(G)\setminus W_1\right]\right) &= e\left(G'\left[V\left(G'\right)\setminus U\right]\right) + d_{2,1}, \\ e\left(G\left[V(G)\setminus W_2\right]\right) &= e\left(G'\left[V\left(G'\right)\setminus U\right]\right) + d_{2,2}. \end{aligned}$$

On the other hand,

$$e(G) \leqslant e\left(G'\right) + d_{1,1} + d_{1,2} + d_{2,1} + d_{2,2}.$$

Therefore,

$$\begin{aligned} 2n\Phi(G,n) &\leqslant 2e\left(G'[U]\right) + 2e\left(G'\left[V\left(G'\right)\setminus U\right]\right) - e\left(G'\right) \\ &= 2(n-1)\,\Phi\left(G', n-1\right), \end{aligned}$$

and we are done. □

Proof of Lemma 3.12. For every $i \in V(G)$, by the definition of $\varphi(G)$ applied to $U = N_i$, we have

$$\varphi(G)(n-d_i) \leqslant \frac{t_i}{d_i}(n-d_i) + t_i'' - \frac{e}{n}(n-d_i).$$

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Therefore, summing over all $i \in V(G)$ we obtain, by Lemma 3.11,

$$\begin{aligned}\varphi(G)\left(n^2-2e\right) &\leqslant n\sum_{i=1}^{n}\frac{t_i}{d_i}-3T_3+T''-\frac{e}{n}\left(n^2-2e\right)\\ &= n\sum_{i=1}^{n}\frac{t_i}{d_i}+ne-\sum_{i=1}^{n}d_i^2-\frac{e}{n}\left(n^2-2e\right)\\ &\leqslant n\sum_{i=1}^{n}\frac{t_i}{d_i}-\frac{1}{2}\sum_{i=1}^{n}d_i^2,\end{aligned}$$

and this proves the left inequality of Lemma 3.12. The right inequality of Lemma 3.12 follows from the left one. □

Proof of Lemma 3.13. Let us assume first that G does not contain isolated vertices, that is, $d_i > 0$ for every $i \in V$. Since G is triangle-free, $t_i = 0$ for every $i \in V$, and by (3.2) we have

$$\varphi(G)\left(n^2-2e\right) \leqslant -\frac{1}{2}\sum_{i=1}^{n}d_i^2.$$

Thus, our assertion is proved for G without isolated vertices. If G contains isolated vertices we may remove them and obtain a graph G' of smaller order m without any isolated vertices. We have then, by Lemma 3.8,

$$n\varphi(G)+\frac{e}{n+1} \leqslant m\varphi\left(G'\right)+\frac{e}{m+1} \leqslant -m\frac{\sum_{i=1}^{n}d_i^2}{2\left(m^2-2e\right)}+\frac{e}{m+1}.$$

Hence,

$$\varphi(G) \leqslant -m\frac{\sum_{i=1}^{n}d_i^2}{2n\left(m^2-2e\right)}+\frac{e(n-m)}{n(n+1)(m+1)}.$$

To complete the proof we have to show that

$$-m\frac{\sum_{i=1}^{n}d_i^2}{2n\left(m^2-2e\right)}+\frac{e(n-m)}{n(n+1)(m+1)} \leqslant -\frac{\sum_{i=1}^{n}d_i^2}{2\left(n^2-2e\right)},$$

or, equivalently,

$$\begin{aligned}\frac{e(n-m)}{(n+1)(m+1)} &\leqslant \frac{1}{2}\sum_{i=1}^{n}d_i^2\left(\frac{m}{\left(m^2-2e\right)}-\frac{n}{\left(n^2-2e\right)}\right)\\ &= \frac{1}{2}\sum_{i=1}^{n}d_i^2\left(\frac{(n-m)(mn+2e)}{\left(m^2-2e\right)\left(n^2-2e\right)}\right).\end{aligned} \tag{4.5}$$

We shall show that, in fact, we have

$$\frac{1}{(n+1)(m+1)} < \frac{2e(mn+2e)}{m\left(m^2-2e\right)\left(n^2-2e\right)} \tag{4.6}$$

and obviously (4.6) implies (4.5) in view of $\sum_{i=1}^{n}d_i^2 \geqslant 4e^2/m$.

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One easily verifies that the functions $x/\left(n^2-x\right)$ and $(mn+x)/\left(m^2-x\right)$ are increasing with respect to x in the interval $m \leqslant x \leqslant m(m-1)$. Since $m \leqslant 2e \leqslant m(m-1)$, we obtain

$$\begin{aligned}\frac{2e(mn+2e)}{m\left(m^2-2e\right)\left(n^2-2e\right)} &\geqslant \frac{m(mn+m)}{m\left(m^2-m\right)\left(n^2-m\right)} = \frac{(n+1)}{(m-1)\left(n^2-m\right)} \\ &> \frac{n+1}{(m+1)\left(n^2-1\right)} > \frac{1}{(n+1)(m+1)},\end{aligned}$$

and the proof is completed. □

Proof of Theorem 3.10. We shall use induction with respect to p. The basic idea of the proof is quite straightforward. If $t_i = o\left(n^2\right)$ for every $i \in V$, then we derive the assertion by (3.2). If $t_i = c'n^2$ for some $i \in V$ then, by the induction hypothesis, $\varphi(G[N_i]) < -c''d_i$ and hence $\varphi(G) < -c'''n$.

Let us now develop that idea with greater care. By Corollary 3.14, the assertion of the theorem holds for $p = 3$ with $\alpha(c,3) = 2c^2/(1-2c)$. Assume $p > 3$ and let the assertion hold for any p' such that $3 \leqslant p' < p$. We confine ourselves to the case of G without isolated vertices, since the general case easily follows from it. We have, by Lemma 3.12,

$$\varphi(G) \leqslant \frac{n}{n^2-2e}\sum_{i=1}^{n}\frac{t_i}{d_i} - \frac{2e^2}{n\left(n^2-2e\right)} \leqslant \frac{n}{n^2-2e}\sum_{i=1}^{n}\frac{t_i}{d_i} - \frac{2c^2}{(1-2c)}n.$$

On the other hand, by Turán's theorem on K_p-free graphs, we have $e \leqslant \frac{p-2}{2(p-1)}n^2$, and hence

$$\varphi(G) \leqslant \frac{p-1}{n}\sum_{i=1}^{n}\frac{t_i}{d_i} - \frac{2c^2}{(1-2c)}n.$$

Since G is K_p-free, for every $i \in V(G)$ the graph $G(N_i)$ is K_{p-1}-free and hence, by Turán's theorem, we have

$$\frac{t_i}{d_i} \leqslant \sqrt{\frac{p-3}{2(p-2)}t_i}.$$

Therefore, we obtain

$$\varphi(G) \leqslant \frac{1}{n}\sqrt{\frac{(p-1)^2(p-3)}{2(p-2)}}\sum_{i=1}^{n}\sqrt{t_i} - \frac{2c^2}{(1-2c)}n. \tag{4.7}$$

For brevity, set

$$\beta = \sqrt{\frac{(p-1)^2(p-3)}{2(p-2)}},\ \gamma = \frac{c^2}{(1-2c)} \text{ and } \delta = \left(\frac{\gamma}{\beta}\right)^2. \tag{4.8}$$

Assume that $t_i \geqslant \delta n^2$ for some vertex i and let $m = d_i$. Consider the graph $G' = G[N_i]$. This graph is K_{p-1}-free with $e\left(G'\right) = t_i \geqslant \delta n^2 > \delta m^2$. Thus, by the induction hypothesis,

$$\varphi\left(G'\right) \leqslant -\alpha(\delta, p-1)m$$

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holds for sufficiently large m. Since $\binom{m}{2} \geqslant t_i \geqslant \delta n^2$, we obtain $m^2 > 2\delta n^2$ and thus

$$\varphi\left(G'\right) < -\alpha\left(\delta, p-1\right)\sqrt{2\delta}n.$$

Now, by Lemma 3.8, we have

$$n\varphi\left(G\right) + \frac{e}{n+1} \leqslant -\alpha\left(\delta, p-1\right)\sqrt{2\delta n^2} + \frac{t_i}{m+1}.$$

Hence, for sufficiently large n,

$$\begin{aligned}\varphi\left(G\right) &\leqslant -\alpha\left(\delta, p-1\right)\sqrt{2\delta}n + \frac{t_i}{\left(m+1\right)n} \\ &< -\alpha\left(\delta, p-1\right)\sqrt{2\delta}n + \frac{1}{2},\end{aligned}$$

and thus

$$\varphi\left(G\right) < -\alpha\left(\delta, p-1\right)\sqrt{\delta}n$$

holds for sufficiently large n. Therefore, we have bounded $\varphi\left(G\right)$ in the desired way provided $t_i \geqslant \delta n^2$ for some vertex i.

Now, let $t_i < \delta n^2$ for every vertex $i \in V\left(G\right)$. By (4.7) and (4.8), we have

$$\varphi\left(G\right) \leqslant \frac{1}{n}\beta\sum_{i=1}^{n}\sqrt{t_i} - 2\gamma n \leqslant \beta\sqrt{\delta}n - 2\gamma n = -\gamma n.$$

It suffices to let

$$\alpha\left(c, p\right) = \min\left\{\gamma, \alpha\left(\delta, p-1\right)\delta\right\}$$

to complete the induction step and the proof of Theorem 3.10 as well. □

Acknowledgements

The author is indebted to the referee for his numerous and valuable notes and suggestions, to V. Rödl for his comments on Problem 3.2 and especially to B. Bollobás for his kind attention and support.

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