Mathematika 56 (2010) 26–34 © University College London
doi:10.1112/S0025579309000369

INFINITE PATHS WITH NO SMALL ANGLES

IMRE BÁRÁNY AND ATTILA PÓR

Abstract. It is shown here that given a discrete (and infinite) set of points in the plane, it is possible to arrange them on a polygonal path so that every angle on the polygonal path is at least 9°. This has been known to hold for finite sets (with 20°). The main result holds for discrete sets in higher dimensions as well, with a smaller bound on the angle.

§1. *Introduction and the main result.* A set $X \subset \mathbb{R}^2$ is *discrete* by definition if every disk contains only finitely many elements of X. Of course, X is finite or countable. An ordering of the points of X is either $x_1, x_2, \ldots$ (a one-way infinite sequence) or $\ldots, x_{-1}, x_0, x_1, x_2, \ldots$ (a two-way infinite sequence) or $x_1, x_2, \ldots, x_n$ (when X is finite). Such an ordering is identified with a polygonal path P on X: its edges are the segments connecting x_i to x_{i+1}. The angle of P at x_i is just $\angle x_{i-1}x_ix_{i+1}$. The path is called α-good if all of its angles are at least α where $\alpha > 0$. In answer to a question of Fekete [3] from 1992 (see also [4]) and of Dumitrescu [2] from 2005, we proved in [1] the following result.

THEOREM 1. *If X is a finite set in the plane, then there is an α-good path on X with $\alpha = \pi/9$.*

The aim of this paper is to extend the above result of [1] to infinite, discrete sets $X \subset \mathbb{R}^2$. The condition of discreteness is quite natural. For instance, when X is the set of rational points on the x-axis, the ordering is either increasing or decreasing but it is unclear how to define angles along this path. Even worse, it is equally unclear what the definition of a path or an angle could be when X is the image of the rational points on the Peano curve. The following is our main result.

THEOREM 2. *Assume that $0 < \alpha < \pi/18$ and that X is a discrete set in the plane. Then there exists an α-good path on the points of X.*

Here one cannot guarantee that the path is one- or two-way infinite. The example showing this is when X is the set of positive integer points, and integer points, respectively, on the x-axis. The next example is interesting as it highlights the difficulties of finding an α-good path. Let $q \in \mathbb{R}$ be large and define $x_n = (q^{3n+1}, 0) \in \mathbb{R}^2$, $y_n = (0, q^{3n+2}) \in \mathbb{R}^2$, $z_n = (-q^{3n+3}, 0) \in \mathbb{R}^2$, and $X = \bigcup_0^\infty \{x_n, y_n, z_n\}$. Every pair of points in X determines a segment that is either

Received 30 January 2009.
MSC (2000): 52C10 (primary), 52C30 (secondary).

https://doi.org/10.1112/S0025579309000369 Published online by Cambridge University Press

almost vertical or almost horizontal. In view of Theorem 1 there is a good path on every finite subset of X. But how to extend such a path to an infinite one? What is an α-good path on X? How many α-good paths are there on X?

We observe that Kynčl [5] has recently improved the bound in Theorem 1 from $\pi/9$ to $\pi/6$, which is actually the best possible value of α. The details are not yet available, but most likely his result combined with our proof would imply that Theorem 2 holds for every $\alpha < \pi/12$.

§2. *Auxiliary lemmas.* The proof of Theorem 2 consists of several steps. We now introduce some notation and terminology and state the two main lemmas needed for the proof. For a point $z \in \mathbb{R}^2$, $|z|$ denotes its distance from the origin and $\overline{z}$ denotes the unit vector $z/|z|$ (assuming that $z \neq 0$). So $\overline{z} \in S^1$ where S^1 is the unit circle, so it can be thought of as a direction or angle. It will be convenient to use the notation $\overline{z} \in I$, meaning that I is an arc on S^1. Such an arc is just $I = (\beta, \gamma)$ where β, γ are angles and (β, γ) means the anticlockwise arc from β to γ. Given distinct points $u, v \in \mathbb{R}^2$ we let $\overline{uv}$ denote the unit vector $(v-u)/|v-u|$.

From now on we assume that $X \subset \mathbb{R}^2$ is infinite and discrete and $\alpha \in (0, \pi/18)$. We assume, without loss of generality, that the origin, to be denoted by 0, is not contained in X and also that $|x|$ and $|y|$ are different for each pair $x, y \in X$, $x \neq y$.

Fix $\beta \in (0, \pi/18)$ and define K to be the cone consisting of vectors z with $\overline{z} \in [-\beta, \beta]$. As usual, let $-K$ be the reflection of K with respect to 0 and $K^* = K \cup (-K)$ be the corresponding double cone. Here is the cone lemma, an auxiliary result needed for Theorem 2.

LEMMA 1. *If $X \setminus K^*$ is finite, then there is an α-good path on X.*

The same conclusion holds, of course, if $X \setminus K_0^*$ is finite where K_0^* is a rotated copy (around the origin) of K^*. We now slightly reformulate the cone lemma. Let $\triangle = \triangle(X)$ denote the set of limit directions in X, that is, $z \in \triangle(X)$ if and only if there is a sequence of distinct elements $z_1, z_2, \ldots$ of X with $\lim \overline{z_n} = z$. Clearly $\triangle(X) \subset S^1$ is closed. When I is an arc on S^1 we define $I^* = I \cup (-I)$. Here is the cone lemma in a slightly different form, more suitable for our purposes.

LEMMA 2. *Assume that $\triangle(X) \subset I^*$ for some open arc $I \subset S^1$ of length $\pi/9$. Then there is an α-good path on X.*

It will suffice to prove Lemma 1 because of the following.

CLAIM 1. *Lemma 1 implies Lemma 2.*

Proof. Assume that the conditions of Lemma 2 hold. Since I is open and $\triangle(X)$ is closed, there is a closed arc $J \subset I$ with $\triangle(X) \subset J^*$. Let K_0 be the cone hull of J; then K_0 is a cone with half angle $\beta \in (0, \pi/18)$ and $X \setminus K_0^*$ is finite, so Lemma 1 applies. □

https://doi.org/10.1112/S0025579309000369 Published online by Cambridge University Press

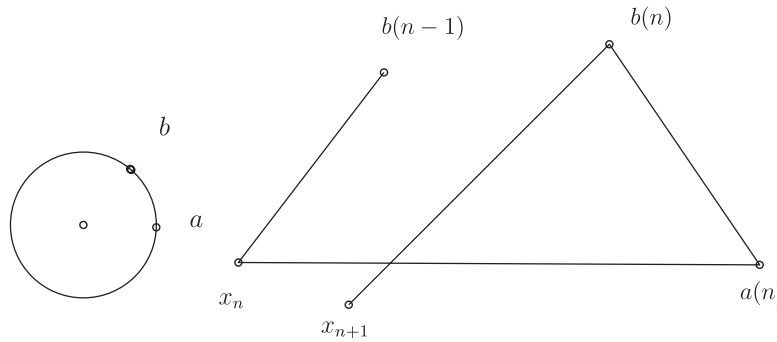


Figure 1: The construction of $a(n)$, $b(n)$, x_{n+1}.

Now we come to the second auxiliary lemma. A point $a \in X$ is called *sharp* if $\angle 0ab < \pi/18$ for every $b \in X$ with $|b| < |a|$. Set $\gamma = \pi/9$.

LEMMA 3. *Assume that all but finitely many elements of X are sharp. Then there is a γ-good path on X.*

For our purposes, an α-good path on X would do as well. But, as we will see later, the proof gives a γ-good path on X.

§3. *Proof of Theorem 2.* A pair $a, b \in X$ is said to be *fat* if all angles of the triangle $0ab$ are at least $\pi/18$. The proof of the following result is simple.

PROPOSITION 1. *If X contains infinitely many fat pairs, then there is an α-good path on X.*

Proof. We choose a sequence of (distinct) fat pairs, a_k, b_k, from X with $\lim \overline{a_k} = a \in S^1$ and $\lim \overline{b_k} = b \in S^1$. This is clearly possible, and $\angle a0b \geq \pi/18$. Also, $\angle a0b \leq \pi - 2\pi/18$ since the angles at a and b of the triangle $a0b$ are at least $\pi/18$.

We will construct an α-good path P on X of the form

$$x_1, a(1), b(1), x_2, a(2), b(2), x_3, \ldots$$

satisfying the condition

$$\text{for every } n, \text{ each } x \in X \text{ with } |x| < |x_n| \text{ appears before } x_n \text{ on } P. \tag{1}$$

Here $a(n), b(n)$ are fat pairs from the sequence a_k, b_k. The construction is quite straightforward (see Figure 1). Evidently, x_1 is the shortest element of X. Assume that $P_n = x_1, a(1), b(1), x_2, \ldots, b(n-1), x_n$ has been constructed satisfying condition (1) and, further, that $\overline{x_n b(n)} \approx b$, meaning that $\overline{x_n b(n)}$ and b are less than $(\pi/18 - \alpha)/2$ apart on S^1. Clearly x_{n+1} has to be the shortest vector in X missing from P_n. Choose $a(n), b(n)$ from the sequence of fat pairs so far from x_n and x_{n+1} that $\overline{x_n a(n)} \approx a$ and $\overline{x_{n+1} b(n)} \approx b$ (with the same meaning of $\approx$ as before). It is not hard to see now that $P_{n+1} = P_n, a(n), b(n), x_{n+1}$ is an α-good path. □

https://doi.org/10.1112/S0025579309000369 Published online by Cambridge University Press

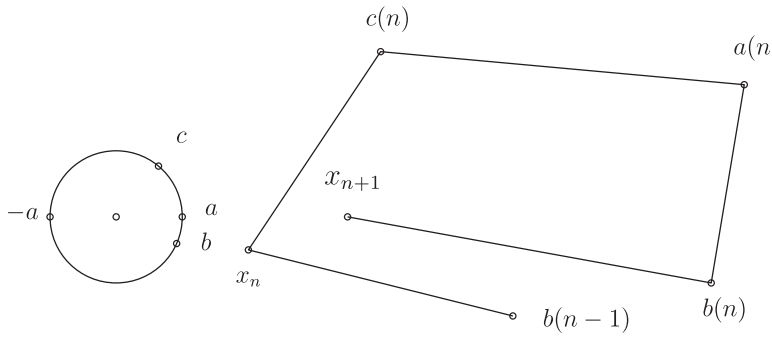


Figure 2: The construction of $a(n)$, $b(n)$, $c(n)$, x_{n+1}.

Next we call a pair $a, b \in X$ *balanced* if $\angle 0ab \geq \pi/18$, $\angle 0ba \geq \pi/18$ and $\angle a0b < \pi/18$.

PROPOSITION 2. *If X contains infinitely many balanced pairs, then there is an α-good path on X.*

Proof. We again choose a sequence of (distinct) balanced pairs a_k, b_k from X with $\lim \overline{a_k} = a \in S^1$ and $\lim \overline{b_k} = b \in S^1$. This is clearly possible, $\angle a0b \leq \pi/18$, and $a, b \in \triangle(X)$.

For $z \in S^1$, let I_z be the open arc of S^1 of length $\pi/9$, centered at z.

Assume that there exists $c \in \triangle(X)$ with $c \notin I_{-a} \cup I_b$. Let $c_k \in X$ be a sequence with $\overline{c_k} \to c$ and $|c_k| \to \infty$. We will construct an α-good path P on X of the form

$$x_1, c(1), a(1), b(1), x_2, c(2), a(2), b(2), x_3, \ldots$$

satisfying condition (1) where $a(n), b(n)$ are pairs from the sequence a_k, b_k and $c(n)$ is a subsequence of c_k (see Figure 2). The construction is similar to the previous one. We start with x_1, the shortest element in X. Assume that the path $P_n = x_1, c(1), a(1), b(1), \ldots, \underline{b(n-1)}, x_n$ has been constructed and satisfies condition (1), and, further, that $\overline{x(n)b(n-1)} \approx b$. Again, x_{n+1} has to be the shortest vector in X missing from P_n. Choose $c(n)$ so that $\overline{x_n c(n)} \approx c$ and then the pair $a(n), b(n)$ so far away from $c(n)$ and x_{n+1} that $\overline{a(n)c(n)} \approx -a$ and $\overline{x_{n+1}b(n)} \approx b$. It is clear that $P_{n+1} = P_n, c(n), a(n), b(n), x_{n+1}$ is an α-good path.

The same argument works, exchanging the roles of a and b, when there is $c \in \triangle(X)$ with $c \notin I_{-b} \cup I_a$. Thus we can assume that there is no $c \in \triangle(X)$ with $c \notin I_{-a} \cup I_b$ or $c \notin I_{-b} \cup I_a$. This means that, with $I = I_a \cap I_b$, $\triangle(X) \subset I^*$. Now the cone lemma (Lemma 2) can be applied since I is an open interval of length at most $\pi/9$. □

Thus we are left with the case when there are only finitely many fat pairs and finitely many balanced pairs in X. Choose r so large that all fat and balanced pairs in X are inside D_r, the disk of radius r centered at 0. We claim then that every $x \in X \setminus D_r$ is sharp.

https://doi.org/10.1112/S0025579309000369 Published online by Cambridge University Press

Indeed, consider $x \in X \setminus D_r$ and assume that $z \in X$ with $|z| < |x|$. Then, of course, $\angle 0zx > \angle 0xz$. If $\angle 0xz \geq \pi/18$, then the pair x, z is either fat (since $\angle x0z \geq \pi/18$) or balanced (if $\angle x0z < \pi/18$). But both cases are excluded as x is outside D_r. Thus $\angle 0xz < \pi/18$ and x is sharp.

A direct application of Lemma 3 completes the proof of the theorem. □

§4. *Proof of the cone lemma.* We need a stronger version of Theorem 1 which is proved in [**1**]. To state it we require two additional definitions.

Given a path $z_1, z_2, \ldots, z_n$ the directions $\overline{z_2z_1}$ and $\overline{z_{n-1}z_n}$ are called the *end directions* of the path. We call a subset R of S^1 a *restriction* if it is the disjoint union of two closed arcs $R_1, R_2 \subset S^1$ such that both have length 4γ and their distance from each other (along the unit circle) is larger than 2γ. (Recall that $\gamma = \pi/9$.) We call the path $z_1, \ldots, z_n$ *R-avoiding* if the two end directions are not in the same R_i $(i = 1, 2)$ and the path is γ-good.

THEOREM 3. *Let X be a finite set of points in the plane. For every restriction R there is an R-avoiding path on all the points of X.*

We now begin the proof of the cone lemma. Call a pair $a, b \in X$ *steep* if the angle between the x-axis and the line through a and b is at least 2γ.

If there is no steep pair in X, then ordering the points of X by increasing first component gives an α-good path on X, even with $\alpha = 5\pi/9$.

We let C be the cone consisting of all $z \in \mathbb{R}^2$ with $\overline{z} \in [-\pi/18, \pi/18]$, and set $C^* = C \cup (-C)$. Since $\beta < \pi/18$, the cone K lies in the interior of the cone C. One more piece of notation: z^1 denotes the first coordinate of $z \in \mathbb{R}^2$.

Assume next that there are only finitely many steep pairs in X. For $t_i > 0$ define the strip $T_i = \{x \in \mathbb{R}^2 : |x^1| \leq t_i\}$. Choose t_1 so large that T_1 contains all steep pairs and the set $X \setminus K^*$ as well. Next choose $t_2 \in \mathbb{R}$ so large that $X \setminus T_2 \subset x + C^*$ for every $x \in X \cap T_1$. Such a t_2 exists because $K \subset C$.

Set $R_1 = [-2\gamma, 2\gamma]$, $R_2 = [\pi - 2\gamma, \pi + 2\gamma]$. Then $R = R_1 \cup R_2 \subset S^1$ is a restriction, so by Theorem 3, there is an R-avoiding path, $P = x_1, x_2, \ldots, x_n$ on $X \cap T_2$ (even with $\alpha = \pi/9$). One end direction of P is not in R_1 and the other one is not in R_2. For the sake of simplicity assume that $\overline{x_2x_1} \notin R_1$ and $\overline{x_{n-1}x_n} \notin R_2$.

Let $x_{n+1}, x_{n+2}, \ldots$ and $x_0, x_{-1}, x_{-2}, \ldots$ respectively be the points of $(X \setminus T_2) \cap K$ in increasing order and the points of $(X \setminus T_2) \cap (-K)$ in decreasing order.

CLAIM 2. *The path $\ldots, x_{-1}, x_0, x_1, \ldots, x_n, x_{n+1}, \ldots$ is α-good on X.*

Proof. We only have to check $\angle x_{n-1}x_nx_{n+1} \geq \alpha$ and $\angle x_0x_1x_2 \geq \alpha$. By symmetry it suffices to check the latter. Either x_1, x_2 is a steep pair (see Figure 3), in which case $x_1, x_2 \in T_1$ and $\angle x_0x_1x_2 \geq 2\gamma - \pi/18 = \pi/6 > \alpha$ because $x_0^1 < 0$, $x_1 \in X \cap T_1$, and $x_0 \in X \setminus T_2 \subset x_1 - C$ imply $\overline{x_0x_1} \in -C$; or x_1, x_2 is not a steep pair, in which case $\overline{x_2x_1} \in R_2$ because $\overline{x_2x_1} \notin R_1$. But then $\angle x_0x_1x_2 \geq \pi - \gamma - \pi/18 = 13\pi/18 > \alpha$. □

https://doi.org/10.1112/S0025579309000369 Published online by Cambridge University Press

So we are left with the case when there are infinitely many steep pairs. We first construct an α-good path on X under the extra condition that $X \subset K$, and explain how to extend the argument for the general case later.

Let Π be the set of steep pairs in X. We will use them to create U-turns on the α-good path to be constructed.

We recursively define numbers $t_0 = 0 < t_1 < t_2 < \cdots$ and pairs $\{a_i, b_i\}$ and $\{c_i, d_i\}$ in Π (all of them distinct points of X) satisfying conditions (Ai) and (Ci) below. We set $T_i = \{z \in \mathbb{R}^2 \mid 0 \le z^1 \le t_i\}$. The conditions are as follows.

(Ai) $\{a_i, b_i\} \in \Pi$, $a_i, b_i \in T_i \backslash T_{i-1}$ and $X \backslash T_i \subset (a_i + C) \cap (b_i + C)$.
(Ci) $\{c_i, d_i\} \in \Pi$, $c_i, d_i \in T_{i+1} \backslash T_i$ and $X \cap T_i \subset (c_i - C) \cap (d_i - C)$.

The recursive definition starts with choosing a steep pair a_1, b_1 and then t_1 so large that (A1) is satisfied. This is possible since the angle of K is smaller than that of C. Assume that $t_i, a_i, b_i, c_{i-1}, d_{i-1}$ have been defined for all $i = 1, 2, \ldots, k$ (except c_{-1}, d_{-1}, which are not needed) and satisfy all conditions. Then we choose a steep pair, c_k, d_k, outside T_k satisfying (Ck). Next we choose another steep pair a_{k+1}, b_{k+1} outside T_k (both distinct from c_k, d_k). Finally, we fix t_{k+1} so large that condition (A$k+1$) holds. This is clearly possible.

The construction of the α-good path is now easy (see Figure 4). We add a dummy point $a_0 = (-1, 0)$. Theorem 1 guarantees the existence of a γ-good path Q_i on the finite set

$$((X \cap (T_i \backslash T_{i-1})) \backslash \{a_i, b_i, c_{i-1}, d_{i-1}\}) \cup \{a_{i-1}, c_i\}.$$

Neither a_{i-1} nor c_i is an interior point of Q_i because of condition (A$i-1$) and (Ci). In the case of a_0 this follows from $X \subset K$. Thus a_{i-1} and c_i are the endpoints of Q_i with the end direction at a_{i-1} in C and at c_i in $-C$. It follows now that the path $Q_1, d_1, b_1, Q_2, d_2, b_2, Q_3, d_3, \ldots$ is α-good on $X \cup \{a_0\}$. Deleting the dummy vertex from it gives an α-good path on X. This completes the proof when $X \subset K$.

In the general case we proceed as follows. If there are infinitely many steep pairs both in K and $-K$, then we choose a steep pair $a_1, b_1 \in K$ and another $a_{-1}, b_{-1} \in -K$ and then fix t_1 so large that $T_1 = \{z \in \mathbb{R}^2 : |z^1| \le t_1\}$ contains all $Z \backslash K^*$ and, further, the conditions

$$(X \backslash T_1) \cap K \subset (a_1 + C) \cap (b_1 + C)$$
$$(X \backslash T_1) \cap (-K) \subset (a_{-1} - C) \cap (b_{-1} - C)$$

are satisfied. This is clearly possible. We then proceed the same way as before, but moving in two directions.

If, finally, there are infinitely many steep pairs in K yet only finitely many in $-K$, then an obvious combination of the previous methods produces an α-good path on X. The details are straightforward and therefore omitted. □

Remark. The bound $\alpha < \pi/18$ comes from this part of the proof. Namely, Theorem 1 gives the γ-good path Q_i; its endpoints are forced to be a_{i-1} and c_i only when the angle of K is less than $\pi/9$.

https://doi.org/10.1112/S0025579309000369 Published online by Cambridge University Press

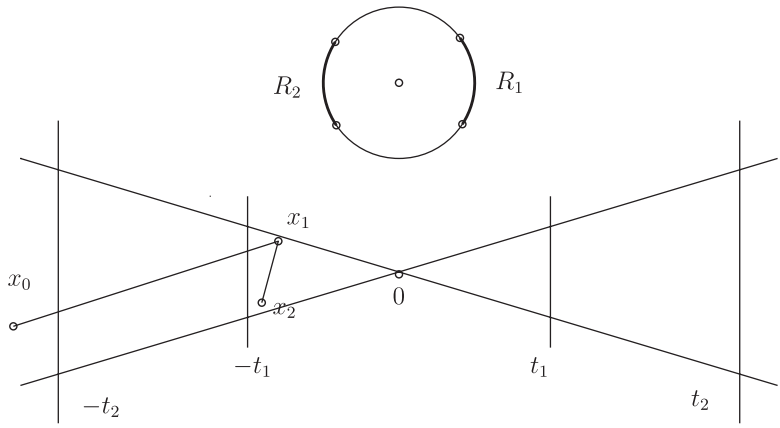


Figure 3: The angle $\angle x_0x_1x_2$.

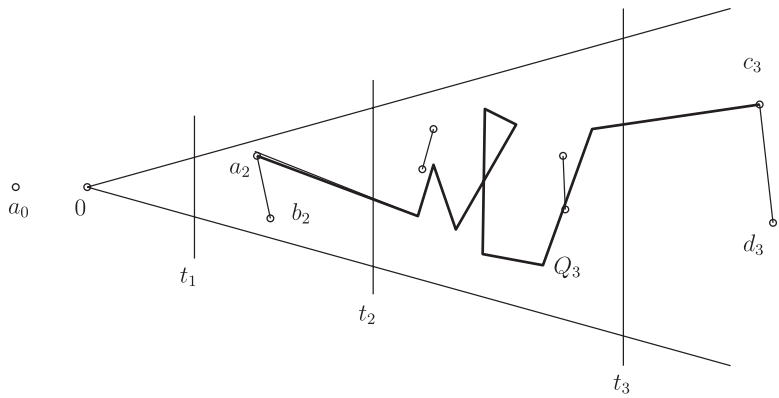


Figure 4: The construction of Q_3.

§5. *Proof of Lemma 3.* Let $r > 0$ be so large that $X_0 = X \cap D_r$ contains all non-sharp points of X. Order the elements of $X \setminus X_0$ by increasing distance from the origin, so if $x_1, x_2, \ldots$ is this order, then $|x_n| < |x_{n+1}|$ for all $n \in \mathbb{N}$. Set, further, $X(n) = X_0 \cup \{x_1, \ldots, x_n\}$ and fix a γ-good path, P_n, on $X(n)$.

CLAIM 3. *For every $n \in \mathbb{N}$, x_n is an endpoint of every γ-good path on $X(n)$.*

Proof. Assume to the contrary that x_n is an interior point of such a path. Then the two neighbors of x_n (a, b, say) are in $X(n)$ and $\angle 0x_na$ and $\angle 0x_nb$ are both smaller than $\pi/18$ and therefore $\angle ax_nb < \pi/9$, which is a contradiction. □

For every $1 \le n \le k$ we define, by backward induction on n, a γ-good path $P_k[n]$ on $X(n)$ as follows. Set $P_k[k] = P_k$. If $P_k[n]$ has been defined and $n > 1$, then, by the previous claim, x_n is an end vertex of $P_k[n]$. Delete this end vertex from $P_k[n]$ to get $P_k[n-1]$.

https://doi.org/10.1112/S0025579309000369 Published online by Cambridge University Press

Let L be an infinite subset of the natural numbers and $n \in \mathbb{N}$. There exists an infinite subset L' of L such that for every $j, k \in L'$ the two paths $P_j[n]$ and $P_k[n]$ are the same. Indeed, partition L by paths on $X(n)$, that is, for every $k \in L$, $k \geq n$, the element k is put into the class $P_k[n]$. Since there are finitely many paths on $X(n)$, one of the classes L' is infinite.

Next we define, by induction, a chain of infinite sets $L_1 \supset L_2 \supset L_3 \supset \cdots$ with the property that, for every $j, k \in L_n$, the two paths $P_j[n]$ and $P_k[n]$ are the same. Start with $L_0 = \mathbb{N}$. Let $n \geq 1$ and assume that $L = L_{n-1}$ has been defined. The previous argument gives a suitable infinite $L' \subset L$, and we set $L_n = L'$. The sets form an infinite chain $L_0 = \mathbb{N}, L_1, L_2, \ldots$ with each L_n infinite and containing L_{n+1}, and, further, for $i, j \in L_n$ with $n \geq 1$, the condition $P_j[n] = P_k[n]$ is satisfied.

For $n \in \mathbb{N}$ let Q_n be the path $P_k[n]$ for some $k \in L_n$. For $n < m$, Q_n is a subpath of Q_m by construction. Define the infinite path Q as the union of the paths Q_n. The path Q is an infinite γ-good path on X. □

Remark. In the example of §1 there are neither fat nor balanced pairs, and the conditions of Lemma 2 do not hold. So in our proof, the α-good path on X is found via the above procedure. The argument in Claim 3 can be used to show that all α-good paths on X are of the following form. The order of the x_ns and z_ns is $\ldots, z_2, z_1, x_1, x_2, x_3, \ldots$ and y_1 is either between x_1 and x_2 or between x_1 and z_1 and, for $n \geq 2$, y_n is either between x_n and x_{n+1} or between z_{n-1} and z_n. It is easy to see that each such path is indeed α-good.

§6. *Higher dimensions.* In the paper [**1**] we proved the higher-dimension analogue of Theorem 1 in the following form.

THEOREM 4. *For every $d \geq 2$ there is a positive α_d such that for every finite set of points $X \subset \mathbb{R}^d$ there exists an α_d-good path on X.*

The actual value of α_d is $\pi/42$ (for $d > 2$); see [**1**]. The proof of Theorem 2 goes through in higher dimensions without any real difficulty, and gives the following result.

THEOREM 5. *For every $d \geq 2$ for every discrete set of points $X \subset \mathbb{R}^d$ and every $\alpha \in (0, \pi/84)$ there exists a α-good path on X.*

Acknowledgements. Partial support from Hungarian National Foundation Grants No. 060427 and 062321 is acknowledged. This work was also supported by the Discrete and Convex Geometry project, MTKD-CT-2005-014333, of the European Community.

References

1. I. Bárány, A. Pór and P. Valtr, Paths without small angles. *SIAM J. Discrete Math.* (2009), to appear.
2. A. Dumitrescu, 2005, personal communication.
3. S. Fekete, Geometry and the traveling salesman problem, *PhD Thesis*, Department of Combinatorics and Optimization, The University of Waterloo, 1992.
4. S. Fekete and G. J. Woeginger, Angle-restricted tours in the plane. *Comput. Geom.* **8** (1997), 195–218.
5. J. Kynčl, 2009, personal communication.

https://doi.org/10.1112/S0025579309000369 Published online by Cambridge University Press

Imre Bárány,
Rényi Institute of Mathematics,
Hungarian Academy of Sciences,
P.O. Box 127, 1364 Budapest,
Hungary
and
Department of Mathematics,
University College London,
Gower Street, London WC1E 6BT,
U.K.
E-mail: barany@renyi.hu

Attila Pór,
Department of Mathematics,
Western Kentucky University,
Bowling Green, KY 42101,
U.S.A.
E-mail: apor@math.wku.edu

https://doi.org/10.1112/S0025579309000369 Published online by Cambridge University Press