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Dynamical upper bounds for Hausdorff dimension of invariant sets

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Abstract. We study the Hausdorff dimension of invariant sets for expanding maps and that of hyperbolic sets on unstable manifolds. Upper bounds for the Hausdorff dimension are given in terms of topological pressure, or topological entropy and Lyapunov exponents.

1. *Introduction*

In the past few years, there has been an increasing interest in searching for relations between dimension and topological pressure [**6, 9, 16**]. The topological pressure may give a more accurate estimate to the dimension of invariant sets than that given by other dynamical quantities. Bowen [**2**] was the first to express the Hausdorff dimension of an invariant set as a solution of an equation involving topological pressure. Ruelle [**18**] refined Bowen's method and proved the following result. Assume that f is a $C^{1+\gamma}$ conformal expanding map, Λ is an isolated compact invariant set and $f|_\Lambda$ is topologically mixing, then the Hausdorff dimension of Λ, $\dim_H \Lambda$, is given by the unique solution α of the equation

$$P(f|_\Lambda, -\alpha \log \|D_x f\|) = 0, \tag{1}$$

where $P(f|_\Lambda, \cdot)$ is the topological pressure functional. The smoothness $C^{1+\gamma}$ was recently relaxed to C^1 [**8**].

At about the same time, McCluskey and Manning [**14**] studied a similar problem for hyperbolic sets. Let f be a C^1 Axiom A diffeomorphism of a compact surface M, and let Λ be a basic set with splitting $T_x M = E^s(x) \oplus E^u(x)$, $x \in \Lambda$. Use $W^u(x)$ to denote the unstable manifolds. Then for every $\bar{x} \in \Lambda$, $\dim_H(W^u(\bar{x}) \cap \Lambda)$ is the unique number α satisfying

$$P(f|_\Lambda, -\alpha \log \|D_x f|E^u(x)\|) = 0. \tag{2}$$

The above two theorems were obtained separately. However, since in the hyperbolic case the restriction of f on $W^u(x)$, $f|_{W^u(x)}$, is expanding, they are certainly closely related. We may say that both results are essentially one-dimensional. The first goal of this paper is to generalize them to multi-dimensional systems. The invariant sets for expanding maps and the hyperbolic sets are treated in a parallel manner. Our results will

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also be expressed in terms of equations involving topological pressure. The equations are like (1) and (2), but with the following modifications. First, an expanding map f, or a restriction of a diffeomorphism $f|_{W^u(x)}$, usually has different expanding rates in different directions. These rates are given by the singular values of the derivative $D_x f$, or $D_x f|E^u(x)$, respectively. Therefore, we use these singular values to construct functions superseding the norms in (1) and (2). Second, the derivatives of f^n, for all $n \in \mathbf{Z}^+$, have to be considered rather than just $D_x f$. So our new equation involves the limit of a sequence of topological pressures, one for each f^n. We show that the unique solution of the equation is an upper bound for the Hausdorff dimension.

For expanding maps, this upper bound is equivalent to that in Falconer [**6**]. However, our method and result are different. The proof in [**6**] requires a careful estimate of the shape of arbitrarily high iterates of small balls. Therefore a bounded distortion condition $\|(D_x f)^{-1}\|^2 \|D_x f\| < 1$ is necessary. Our trick, however, is to calculate the Hausdorff measure at each iteration (see §5). The distortion of a ball after more than one iteration is irrelevant to the proof. So our result is valid for all C^1 expanding maps. Furthermore, this method works equally well for the dimension of a hyperbolic set on unstable manifolds.

Based on the topological pressure method, we also give upper bounds for the Hausdorff dimension in terms of Lyapunov exponents and topological entropy. These bounds are constructed in a similar way to the Lyapunov dimension [**1, 12, 13**]. Some inequalities of this type have been obtained before [**7, 13**].

According to the way they are produced, all the above upper bounds may be called *dynamical quantities*. It is natural to ask how close they are to the dimension. We point out that the bounds by topological pressure are sharp estimates of the Hausdorff dimension. For some large classes of invariant sets, they agree with the dimension prevalently. In contrast, the Lyapunov exponents and topological entropy can only provide rough bounds.

This paper is organized as follows. Basic definitions and notations are given in §2. In §3, we describe the construction of two upper bounds for Hausdorff dimension for expanding maps. In §4 we study the similar problems for hyperbolic sets. §5 is devoted to proving a basic inequality on the change of Hausdorff measure after one iteration, which is used in §6 to complete the proof of all theorems.

2. *Definitions and notations*

We state the definitions of Hausdorff dimension and topological pressure in a general setting.

Let X be a compact metric space. Use $B(x, r)$ to denote the open ball centered at $x \in X$ with radius r. Fix $\alpha > 0$. For $Y \subset X$ and $\rho > 0$ define

$$\mathcal{H}^\alpha_\rho(Y) = \inf \left\{ \sum_j r_j^\alpha : \bigcup_j B(x_j, r_j) \supset Y, x_j \in X \text{and } r_j \le \rho \right\},$$

and

$$\mathcal{H}^\alpha(Y) = \lim_{\rho \to 0} \mathcal{H}^\alpha_\rho(Y).$$

$\mathcal{H}^\alpha$ is called the α-dimensional *Hausdorff (ball) measure*. The *Hausdorff dimension* of

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Y is given by

$$\dim_H Y = \inf\{\alpha : \mathcal{H}^\alpha(Y) = 0\}.$$

See [**5**] for detailed discussions.

Let $f : X \to X$ be a continuous map. For $x \in X$ and $r > 0$, denote

$$B_n(x, r) = \{y \in X : f^i y \in B(f^i x, r), \text{for all } i = 0, \ldots, n-1\}.$$

If ϕ is a real continuous function on X and $n \in \mathbf{Z}^+$ let

$$(S_n\phi)(x) = \sum_{i=0}^{n-1} \phi(f^i x)$$

and for $\rho > 0$, set

$$Q(\phi, n, \rho) = \inf\left\{\sum_j \exp\{(S_n\phi)(x_j)\} : \bigcup_j B_n(x_j, \rho) \supset X\right\}.$$

Then the *topological pressure* of ϕ is given by

$$P(f, \phi) = \lim_{\rho \to 0} \limsup_{n \to \infty} \frac{1}{n} \log Q(\phi, n, \rho).$$

In this paper, we only consider the expanding and the hyperbolic systems, which are both *expansive* [**20**, Definition 5.1]. So there is some $\rho_0 > 0$ such that for any $\rho < \rho_0$ and any continuous function ϕ

$$P(f, \phi) = \limsup_{n \to \infty} \frac{1}{n} \log Q(\phi, n, \rho) \tag{3}$$

[**20**, Chapter 9]. In this case, $P(f, \phi)$ is always finite.

In particular, the topological pressure of the constant zero function, $P(f, 0)$, is called the topological entropy of f, denoted by $h_{\text{top}}(f)$.

Now we define the logarithmic singular value function $g^\alpha(L)$ for any expanding linear map L. Let V and V' be two real inner product spaces with the same dimension m. A linear map $L : V \to V'$ is said to be expanding if $\|Lv\| > \|v\|$ for all $v \in V$ and $v \neq 0$. We write the logarithms of the singular values of L (a singular value of L is the positive square root of an eigenvalue of L^*L, which must all be real and non-negative) in decreasing order:

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m,$$

where $\lambda_m > 0$ because of the expansion. Then for any $\alpha \in [0, m]$ set

$$g^\alpha(L) = \sum_{i=m-[\alpha]+1}^{m} \lambda_i + (\alpha - [\alpha])\lambda_{m-[\alpha]},$$

where $[\alpha]$ is the largest integer $\leq \alpha$. $g^\alpha(L)$ is continuous and strictly increasing in α. $g^0(L) = 0$ and $g^m(L) = \sum_{i=1}^m \lambda_i = \log|\operatorname{Jac}(L)|$, where $\operatorname{Jac}(L)$ is the Jacobean of L.

g^α has the following superadditive property. If $L : V \to V'$ and $L' : V' \to V''$ are two linear expanding maps, where $\dim V = \dim V' = \dim V'' = m$, then for any $\alpha \in [0, m]$

$$g^\alpha(L'L) \geq g^\alpha(L) + g^\alpha(L'). \tag{4}$$

See [**4, 13**] for a proof.

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3. *Upper bounds of Hausdorff dimension for expanding maps*

We study expanding maps in this section. Let M be a C^∞ Riemannian manifold, $\dim M = m$. Let U be an open subset of M and let $f : U \to M$ be a C^1 map. Suppose $\Lambda \subset U$ is a compact invariant set on which f is expanding, that is, $f\Lambda = \Lambda$ and there is $k > 1$ such that for all $x \in \Lambda$ and $v \in T_x M$,

$$\|D_x f v\| \geq k\|v\|,$$

where $\|\cdot\|$ is the norm induced by an adapted Riemannian metric. We do not assume that Λ is isolated or $f|_\Lambda$ has any transitivity property.

If $x \in \Lambda$, then $D_x f : T_x M \to T_{fx} M$ is a linear expanding map. Denote the logarithms of the singular values of $D_x f$ by

$$\lambda_1(x, f) \geq \cdots \geq \lambda_m(x, f)$$

and for $\alpha \in [0, m]$, write

$$g^\alpha(x, f) \equiv g^\alpha(D_x f) = \sum_{i=m-[\alpha]+1}^{m} \lambda_i(x, f) + (\alpha - [\alpha])\lambda_{m-[\alpha]}(x, f).$$

Since f is C^1, the functions $x \to \lambda_i(x, f)$ and $x \to g^\alpha(x, f)$ are all continuous.

For each $\alpha \in [0, m]$, consider $P_1(\alpha) \equiv P(f|_\Lambda, -g^\alpha(\cdot, f))$. It is obvious from the definition of topological pressure that $P_1(\alpha)$ is continuous in α and is strictly decreasing. $P_1(0) = h_{\text{top}}(f|_\Lambda) \geq 0$. The sign of $P_1(m) = P(f|_\Lambda, -\log|\operatorname{Jac}(D_x f)|)$ is, in general, unknown. However, when f is C^2 and Λ is an isolated invariant set, $P_1(m)$ is non-positive. The following is our main lemma, which will be proved in §6.

LEMMA 1. *Set*

$$D_1 = \max\{\alpha \in [0, m] : P_1(\alpha) \geq 0\}.$$

Then

$$\dim_H \Lambda \leq D_1.$$

The definition of D_1 gives that if $P(f|_\Lambda, -\log|\operatorname{Jac}(D_x f)|) \leq 0$, then D_1 is the unique number α satisfying

$$P_1(\alpha) = 0;$$

otherwise, $D_1 = m$. If the map f is conformal, then $g^\alpha(x, f) = \alpha \log\|D_x f\|$. By [**2, 18**], D_1 actually equals $\dim_H \Lambda$ under mild additional assumptions. When f is not conformal, however, D_1 is only a rough bound for $\dim_H \Lambda$. Sharper bounds can be obtained by considering the derivatives of all iterates of f.

$f\Lambda = \Lambda$ implies $f^n\Lambda = \Lambda$. f^n is also expanding on Λ. Let the logarithms of the singular values of $D_x f^n$ be

$$\lambda_1(x, f^n) \geq \cdots \geq \lambda_m(x, f^n)$$

and set

$$g^\alpha(x, f^n) = \sum_{i=m-[\alpha]+1}^{m} \lambda_i(x, f^n) + (\alpha - [\alpha])\lambda_{m-[\alpha]}(x, f^n).$$

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By (4), the functions $g^\alpha(\cdot, f^n)$ satisfy

$$g^\alpha(x, f^{n+l}) \geq g^\alpha(x, f^n) + g^\alpha(f^n x, f^l). \tag{5}$$

Define a sequence of functions $P_n : [0, m] \to \mathbf{R}$ as follows:

$$P_n(\alpha) = P\left(f|_\Lambda, -\frac{1}{n} g^\alpha(\cdot, f^n)\right).$$

We show the following lemma in §6.

LEMMA 2. *For every $\alpha \in [0, m]$, the following limit exists*

$$\lim_{n\to\infty} P_n(\alpha) = \inf_{n\in\mathbf{Z}^+} P_n(\alpha).$$

Set $P^*(\alpha) = \lim_{n\to\infty} P_n(\alpha)$. Then P^* is continuous and strictly decreasing on $[0, m]$. $P^*(0) = h_{\text{top}}(f|_\Lambda) \geq 0$, and $P^*(m) = P(f|_\Lambda, -\log|\operatorname{Jac}(D_x f)|)$.

From Lemma 1 we deduce that the zero of P^* is an upper bound for the Hausdorff dimension.

THEOREM 1. *Let*

$$\mathcal{D}(f, \Lambda) = \max\{\alpha \in [0, m] : P^*(\alpha) \geq 0\}.$$

Then

$$\dim_H \Lambda \leq \mathcal{D}(f, \Lambda).$$

The proof will also be given in §6. The number $\mathcal{D}(f, \Lambda)$, if less than m, is equivalent to the upper bound in [**6**]. The author proves $\dim_H \Lambda \leq \mathcal{D}(f, \Lambda)$ (actually he proves the inequality for box dimension, which automatically implies that for Hausdorff dimension) under the condition $\|(D_x f)^{-1}\|^2 \|D_x f\| < 1$, which he needs to control the distortion of small balls after many iterations (see [**6** Lemma 5.2]). In contrast, a basic technique of our proof is to estimate the Hausdorff measure of fA for small sets A, where only one iteration is applied (Lemma 3). Therefore, we are able to ignore the complicated distortion problem caused by multiple iterations and so the bounded distortion condition is not needed. However, without assuming such a condition, we do not have the inequality for box dimension. Also, the subadditive thermodynamic formalism in [**6**] does not apply, so the definition of $\mathcal{D}(f, \Lambda)$ is slightly different.

Using Lyapunov exponents and topological entropy, we can give another upper bound for $\dim_H \Lambda$. Let $\mathcal{E}(\Lambda, f)$ be the set of f-invariant ergodic measures with support in Λ. If $\mu \in \mathcal{E}(\Lambda, f)$, by expansion all the Lyapunov exponents of μ are positive. We order them as usual:

$$\lambda_1(\mu) \geq \cdots \geq \lambda_m(\mu)$$

and write

$$g^\alpha(\mu) = \sum_{i=m-[\alpha]+1}^{m} \lambda_i(\mu) + (\alpha - [\alpha])\lambda_{m-[\alpha]}(\mu).$$

By the multiplicative ergodic theorem we have

$$\lim_{n\to\infty} \frac{1}{n} \lambda_i(x, f^n) = \lambda_i(\mu), \quad \mu\text{-a.e. and in } L^1(\mu)$$

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and

$$\lim_{n\to\infty}\frac{1}{n}g^{\alpha}(x,\,f^{n})=g^{\alpha}(\mu),\quad \mu\text{-a.e. and in } L^{1}(\mu).$$

For each $\mu\in\mathcal{E}(\Lambda,f)$, we define a number $d_{\Lambda}(\mu)$ by its Lyapunov exponents and topological entropy $h_{\mathrm{top}}(f|_{\Lambda})$:

$$d_{\Lambda}(\mu)=\max\{\alpha\in[0,m]:g^{\alpha}(\mu)\le h_{\mathrm{top}}(f|_{\Lambda})\}.$$

THEOREM 2.

$$\dim_{H}\Lambda\le\sup_{\mu\in\mathcal{E}(\Lambda,f)}d_{\Lambda}(\mu).$$

The definition of $d_{\Lambda}(\mu)$ is motivated by the Lyapunov dimension [**1, 12, 13**]. If $\sum_{i=1}^{m}\lambda_{i}(\mu)>h_{\mathrm{top}}(f|_{\Lambda})$, one can rewrite

$$d_{\Lambda}(\mu)=l+\frac{h_{\mathrm{top}}(f|_{\Lambda})-\lambda_{m}(\mu)-\cdots-\lambda_{m-l+1}(\mu)}{\lambda_{m-l}(\mu)},\tag{6}$$

where $l=\max\{i:\lambda_{m}(\mu)+\cdots+\lambda_{m-i+1}(\mu)\le h_{\mathrm{top}}(f|_{\Lambda})\}$. Let us compare (6) with the definition of Lyapunov dimension. Suppose ν is an ergodic measure of any differentiable map f with Lyapunov exponents $\lambda_{1}\ge\cdots\ge\lambda_{u}>0\ge\lambda_{u+1}\ge\cdots\ge\lambda_{m}$, then the Lyapunov dimension

$$\mathrm{Lyap\,dim}(\nu)=l+\frac{\lambda_{1}+\cdots+\lambda_{u}+\lambda_{u+1}+\cdots+\lambda_{l}}{|\lambda_{l+1}|},$$

where $l=\max\{i:\lambda_{1}+\cdots+\lambda_{i}\ge 0\}$. When f is a diffeomorphism and ν is absolutely continuous with respect to the Lebesgue measure (or only absolutely continuous on unstable manifolds), by Pesin's formula [**15**] we have

$$\mathrm{Lyap\,dim}(\nu)=l+\frac{h_{\nu}(f)+\lambda_{u+1}+\cdots+\lambda_{l}}{|\lambda_{l+1}|},$$

where $h_{\nu}(f)$ is the metric entropy and $l=\max\{i:-\lambda_{u+1}-\cdots-\lambda_{i}\le h_{\nu}(f)\}$. This expression is similar to (6). In the definition of $d_{\Lambda}(\mu)$, however, we use $h_{\mathrm{top}}(f|_{\Lambda})$ instead of $h_{\mu}(f)$ in order to prove Theorem 2. Notice that $\mathrm{Lyap\,dim}(\mu)$ does not apply to expanding systems, since all exponents of μ are positive. So $d_{\Lambda}(\mu)$ may be considered as a reasonable substitute. Thus, Theorem 2 is a supplement to the following inequality of Ledrappier [**13**]

$$\dim_{H}\Lambda\le\sup_{\nu\in\mathcal{E}(\Lambda,f)}\mathrm{Lyap\,dim}(\nu),$$

where Λ is a compact invariant set of any differentiable map f.

A construction similar to $d_{\Lambda}(\mu)$ also appears in [**7**], where, instead of $g^{\alpha}(\mu)$, a function consisting of uniform Lyapunov exponents is used.

The question arises of how good are these upper bounds. Whereas it is easy to show by examples that $\sup_{\mu\in\mathcal{E}(\Lambda,f)}d_{\Lambda}(\mu)$ is a rough bound for $\dim_{H}\Lambda$, $\mathcal{D}(f,\Lambda)$ often has a surprising property which seems to indicate that it is very sharp. (It can be shown that $\mathcal{D}(f,\Lambda)\le\sup_{\mu\in\mathcal{E}(\Lambda,f)}d_{\Lambda}(\mu)$, see §6). For some large classes of self-affine sets, the equality $\dim_{H}\Lambda=\mathcal{D}(f,\Lambda)$ holds almost surely. We describe two such classes.

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Let there be l non-singular linear contractions $T_j : \mathbf{R}^m \to \mathbf{R}^m$, $\|T_j\| < 1$, $j = 1, \ldots, l$. Given any $b_1, \ldots, b_l \in \mathbf{R}^m$ we define l affine maps $t_j(x) = T_j x + b_j$. Let $B(T_1, \ldots, T_l)$ be the set of $(b_1, \ldots, b_l) \in \mathbf{R}^{m \times l}$ satisfying the property that there is a compact set $K \subset \mathbf{R}^m$ with $\operatorname{int} K \neq \emptyset$, where $\operatorname{int} K$ is the interior of K, such that the sets $K_j \equiv t_j K$ are disjoint and are all contained in $\operatorname{int} K$. Then $B(T_1, \ldots, T_l)$ is an open set. Let us only consider such $T_1, \ldots, T_l$ that $B(T_1, \ldots, T_l) \neq \emptyset$. Fix $(b_1, \ldots, b_l) \in B(T_1, \ldots, T_l)$. Denote $K_{j_1 \ldots j_n} = t_{j_1} \circ \cdots \circ t_{j_n}(K)$, $j_i = 1, \ldots, l$. Then $K_{j_1 \ldots j_n j_{n+1}} \subset \operatorname{int} K_{j_1 \ldots j_n}$, and if $j'_{n+1} \neq j_{n+1}$ then $K_{j_1 \ldots j_n j_{n+1}} \cap K_{j_1 \ldots j_n j'_{n+1}} = \emptyset$. Put $\Lambda = \bigcap_{n=0}^{\infty} \bigcup_{j_1, \ldots, j_n = 1}^{l} K_{j_1 \ldots j_n}$. Λ is called a self-affine set. See [**5**] for more about this construction. Now define a C^1 expanding map f on the open set $U = \bigcup_{j=1}^{l} \operatorname{int} K_j$ in a natural way: $f|_{\operatorname{int} K_j} \equiv t_j^{-1}$. It is easy to check that $f\Lambda = \Lambda$.

The following cases have been studied.

(1) [**4, 6**] If $\|T_j\| < 1/3$, $j = 1, \ldots, l$, then for Lebesgue almost all $(b_1, \ldots, b_l) \in B(T_1, \ldots, T_l)$, $\dim_H \Lambda = \mathcal{D}(f, \Lambda)$.

(2) Consider two affine contractions on $\mathbf{R}^2$ with the same linear part

$$T_1 = T_2 = \begin{pmatrix} \chi_1 & 0 \\ 0 & \chi_2 \end{pmatrix}, \quad 0 < \chi_2 \leq \chi_1, \quad \text{and} \quad \chi_2 \leq \tfrac{1}{2},$$

which map the unit square $R = [0, 1] \times [0, 1]$ into two disjoint rectangles inside R. (Notice that each rectangle has a horizontal side of length χ_1 and a vertical side of length χ_2.) Then:

(a) [**17**] if $\chi_1 \leq \frac{1}{2}$, then $\dim_H \Lambda = \mathcal{D}(f, \Lambda) = -\log 2 / \log \chi_1$, except that $b_1 - b_2$ happens to be a vertical vector;

(b) [**17, 19**] for Lebesgue almost every $\chi_1 \in (\frac{1}{2}, 1)$, $\dim_H \Lambda = \mathcal{D}(f, \Lambda) = 1 - \log 2\chi_1 / \log \chi_2$, except that $b_1 - b_2$ happens to be a vertical vector.

These facts strongly suggest that $\dim_H \Lambda = \mathcal{D}(f, \Lambda)$ should hold prevalently. (Another class of self-affine sets satisfying this equality was recently studied in [**11**]. We thank the referee for pointing this out to us.) For general nonlinear maps, Lebesgue measure is obviously not appropriate to describe the prevalence. To formulate the right concept and to prove that the equality $\dim_H \Lambda = \mathcal{D}(f, \Lambda)$ is indeed prevalent should be an interesting problem. The situation here is parallel to the Kaplan–Yorke conjecture on the Hausdorff dimension of invariant measures [**1, 12**].

4. *Hausdorff dimension of hyperbolic sets on unstable manifolds*

The technique in studying the Hausdorff dimension of hyperbolic sets on unstable manifolds is similar. Let M be a compact Riemannian manifold and let f be a C^1 diffeomorphism on M. A compact set Λ is called a *hyperbolic set* for f, if $f\Lambda = \Lambda$ and there is a continuous splitting $T_x M = E^s(x) + E^u(x)$ on Λ satisfying:

(1) $(D_x f)E^s(x) = E^s(fx)$, $(D_x f)E^u(x) = E^u(fx)$, $x \in \Lambda$;

(2) there is a constant $k > 1$, such that

$$\|D_x f | E^s(x)\| \leq k^{-1}, \quad \|D_x f^{-1} | E^u(x)\| \leq k^{-1}.$$

$f|_\Lambda$ need not be transitive. However, for simplicity we assume that the splitting has a constant dimension $\dim E^s(x) = s$ and $\dim E^u(x) = u$.

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For $x \in \Lambda$, denote by $W^u(x)$ the global unstable manifold containing x. $W^u(x)$ is a C^1 injective immersion of a Euclidean space $\mathbf{R}^u$ into M. Let d^u be the induced metric on $W^u(x)$ and let $B^u(x, r)$ be the set $\{y \in W^u(x) : d^u(x, y) < r\}$. We study the Hausdorff dimension of the set $W^u(x) \cap \Lambda$.

By the definition of hyperbolic sets, for each $n \in \mathbf{Z}^+$, $D_x f^n | E^u(x)$ is an expanding linear map from $E^u(x)$ onto $E^u(f^n x)$. Denote by

$$\lambda_1(x, f^n) \geq \cdots \geq \lambda_u(x, f^n) \quad (\geq n \log k)$$

the logarithms of the singular values of $D_x f^n | E^u(x)$, and for $\alpha \in [0, u]$ set

$$g^\alpha(x, f^n) = \sum_{i=u-[\alpha]+1}^{u} \lambda_i(x, f^n) + (\alpha - [\alpha])\lambda_{u-[\alpha]}(x, f^n).$$

(The notations $\lambda_i(x, f^n)$ and $g^\alpha(x, f^n)$ are used in both expanding and hyperbolic situations. Their meaning should be clear from the context.) It follows from the continuity of the splitting that $\lambda_i(x, f^n)$ and $g^\alpha(x, f^n)$ are continuous in x. Consider the topological pressure functions

$$P_n^u(\alpha) = P\left(f|_\Lambda, -\frac{1}{n} g^\alpha(\cdot, f^n)\right), \quad \alpha \in [0, u].$$

Since the functions $g^\alpha(\cdot, f^n)$ also satisfy the superadditive condition (5), we can show, as in Lemma 2, that the limit function

$$P^{u*}(\alpha) \equiv \lim_{n\to\infty} P_n^u(\alpha) = \inf_{n\in\mathbf{Z}^+} P_n^u(\alpha)$$

exists. Furthermore, P^{u*} is continuous and strictly decreasing in $\alpha \in [0, u]$: $P^{u*}(0) = P_n^u(0) = h_{\text{top}}(f|_\Lambda) \geq 0$ and $P^{u*}(u) = P_n^u(u) = P(f|_\Lambda, -\log|\operatorname{Jac}(D_x f|E^u(x))|)$. If f is C^2 and Λ is a hyperbolic set with the local product structure, then it is known that $P(f|_\Lambda, -\log|\operatorname{Jac}(D_x f|E^u(x))|) \leq 0$. So the equation

$$P^{u*}(\alpha) = 0$$

has a unique solution $\alpha \in [0, u]$. In general, however, the sign of $P(f|_\Lambda, -\log|\operatorname{Jac}(D_x f|E^u(x))|)$ is not clear. Parallel to Theorem 1, we have the following result.

THEOREM 3. *Let*

$$\mathcal{D}^u(f, \Lambda) = \max\{\alpha \in [0, u] : P^{u*}(\alpha) \geq 0\}.$$

Then for every $x \in \Lambda$,

$$\dim_H(W^u(x) \cap \Lambda) \leq \mathcal{D}^u(f, \Lambda).$$

In the proof of Theorem 3 (§6), we will calculate the Hausdorff dimension with the induced metric d^u. However, since $W^u(x)$ is a C^1 submanifold, using the original Riemannian metric to compute the dimension would yield the same value.

In particular, when $\dim E^u(x) = 1$, it is easy to check that $P^{u*}(\alpha) = P_n^u(\alpha) = P(f|_\Lambda, -\alpha \log\|D_x f|E^u(x)\|)$. So Theorem 3 generalizes [**14**, Theorem 1] (stated in §1) in one direction.

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Again as in §3, we derive another upper bound for $\dim_H(W^u(x)\cap\Lambda)$ by Lyapunov exponents and topological entropy. If $\mu\in\mathcal{E}(\Lambda, f)$, then μ has exactly u positive Lyapunov exponents. Write them in the usual order

$$\lambda_1(\mu)\geq\cdots\geq\lambda_u(\mu)$$

and let

$$g^\alpha(\mu)=\sum_{i=u-[\alpha]+1}^{u}\lambda_i(\mu)+(\alpha-[\alpha])\lambda_{u-[\alpha]}(\mu).$$

Then define for each $\mu\in\mathcal{E}(\Lambda, f)$

$$d_\Lambda^u(\mu)=\max\{\alpha\in[0,u]: g^\alpha(\mu)\leq h_{\text{top}}(f|_\Lambda)\}.$$

This number takes the following form if $\sum_{i=1}^{u}\lambda_i(\mu)>h_{\text{top}}(f|_\Lambda)$:

$$d_\Lambda^u(\mu)=l+\frac{h_{\text{top}}(f|_\Lambda)-\lambda_u(\mu)-\cdots-\lambda_{u-l+1}(\mu)}{\lambda_{u-l}(\mu)},$$

where $l=\max\{i:\lambda_u(\mu)+\cdots+\lambda_{u-i+1}(\mu)\leq h_{\text{top}}(f|_\Lambda)\}$.

THEOREM 4. *For any $x\in\Lambda$,*

$$\dim_H(W^u(x)\cap\Lambda)\leq\sup_{\mu\in\mathcal{E}(\Lambda,f)}d_\Lambda^u(\mu).$$

It can be shown that $\mathcal{D}^u(f,\Lambda)\leq\sup_{\mu\in\mathcal{E}(\Lambda,f)}d_\Lambda^u(\mu)$. In view of the argument at the end of last section, it seems reasonable to believe that $\mathcal{D}^u(f,\Lambda)$ is a close bound for the dimension, while $\sup_{\mu\in\mathcal{E}(\Lambda,f)}d_\Lambda^u(\mu)$ is only a coarse estimate.

When Λ is a basic set of an Axiom A surface diffeomorphism, it is shown in [**14**] that $\dim_H\Lambda=\dim_H(W^s(x)\cap\Lambda)+\dim_H(W^u(x)\cap\Lambda)$, where $W^s(x)$ is a stable manifold. So a formula for $\dim_H\Lambda$ directly follows from their calculation on $\dim_H(W^u(x)\cap\Lambda)$. In the higher dimensional case, however, the relation between $\dim_H\Lambda$, $\dim_H(W^s(x)\cap\Lambda)$ and $\dim_H(W^u(x)\cap\Lambda)$ is not clear and so our results can not immediately provide upper bounds for $\dim_H\Lambda$. We mention that some upper bounds similar to $\sup_{\mu\in\mathcal{E}(\Lambda,f)}d_\Lambda^u(\mu)$ have been obtained for $\dim_H\Lambda$ [**7, 13**].

5. *Change of Hausdorff measure under iteration*

A basic tool used to prove the theorems is the following lemma, which estimates the Hausdorff measure of the image of a small set under an expanding map.

LEMMA 3. *Let $U\subset\mathbf{R}^m$ be an open set and let $f: U\to\mathbf{R}^m$ be a C^1 map. Assume that, at a point $x\in U$, $D_xf:\mathbf{R}^m\to\mathbf{R}^m$ is an expanding linear map. Then for every $\alpha\in[0,m]$, there exists $\rho_0>0$, such that $B(x,\rho_0)\subset U$ and for any set $A\subset B(x,\rho_0)$ we have*

$$\mathcal{H}_{b\rho}^\alpha(A)\leq C\mathcal{H}_\rho^\alpha(fA) \tag{7}$$

for all $\rho<\rho_0$, where

$$b = 2\sqrt{m}\exp\{-\lambda_{m-[\alpha]}(x,f)\}, \tag{8}$$

$$C = 2^\alpha m^{\alpha/2}\exp\{-g^\alpha(x,f)\}. \tag{9}$$

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Proof. We follow the ideas in [**3**]. Let ϵ be a small positive number with $(1+\epsilon)e^{\epsilon} < 2$. Since $D_y f : \mathbf{R}^m \to \mathbf{R}^m$ is continuous in y, we can choose ρ_0 satisfying the following conditions:

(1) f maps $B(x, 2\rho_0)$ diffeomorphically onto $fB(x, 2\rho_0)$ and $\|fz - fy\| > \|z - y\|$, for all $y, z \in B(x, 2\rho_0)$ with $y \neq z$;

(2) $|\lambda_i(y, f) - \lambda_i(x, f)| < \epsilon$, for all $y \in B(x, \rho_0)$ and $i = 1, \ldots, m$;

(3) $\|z - y - (D_y f)^{-1}(fz - fy)\| < \epsilon\|z - y\|$, for all $y \in B(x, \rho_0)$ and $z \in B(y, \rho_0)$.

Condition (3) holds because $(D_y f)^{-1}$ is also continuous in y. Assume $A \subset B(x, \rho_0)$. Fix any $\rho < \rho_0$. Let $a = \mathcal{H}^{\alpha}_{\rho}(fA)$ (a is obviously finite). Then for any $\eta > 0$, there are finitely many balls $B(z_j, r_j)$, $z_j \in fB(x, \rho_0)$, $r_j \leq \rho$, such that $\bigcup_j B(z_j, r_j) \supset fA$ and $\sum_j r_j^{\alpha} < a + \eta$. Let $B'_j = \{y \in B(x, \rho_0) : fy \in B(z_j, r_j)\}$. Then $\bigcup_j B'_j \supset A$. By (1) and (3) $B'_j \subset y_j + (1+\epsilon)(D_{y_j} f)^{-1}B(0, r_j)$, where $y_j \in B(x, \rho_0)$ and $f(y_j) = z_j$. The definition of the singular values implies that $(D_{y_j} f)^{-1}B(0, r_j)$ is an ellipsoid which has principle axes of lengths $2r_j \exp\{-\lambda_m(y_j, f)\} \geq \cdots \geq 2r_j \exp\{-\lambda_1(y_j, f)\}$. Therefore, B'_j is contained in a rectangular box with sides $2(1+\epsilon)r_j \exp\{-\lambda_m(y_j, f)\}, \ldots, 2(1+\epsilon)r_j \exp\{-\lambda_1(y_j, f)\}$, which can be covered by $\exp\{-\sum_{i=m-[\alpha]+1}^{m} \lambda_i(y_j, f)\} \exp\{[\alpha]\lambda_{m-[\alpha]}(y_j, f)\}$ balls with radius $\sqrt{m}(1+\epsilon)r_j \exp\{-\lambda_{m-[\alpha]}(y_j, f)\}$. By (2) $\exp\{-\lambda_{m-[\alpha]}(y_j, f)\} \leq e^{\epsilon} \exp\{-\lambda_{m-[\alpha]}(x, f)\}$. Summing up over all j and using the assumption $(1+\epsilon)e^{\epsilon} < 2$ we get

$$\begin{aligned}
\mathcal{H}^{\alpha}_{b\rho}(A) &\leq \sum_j \exp\left\{-\sum_{i=m-[\alpha]+1}^{m} \lambda_i(y_j, f)\right\} \exp\{[\alpha]\lambda_{m-[\alpha]}(y_j, f)\} \\
&\quad \cdot m^{\alpha/2}(1+\epsilon)^{\alpha} r_j^{\alpha} \exp\{-\alpha\lambda_{m-[\alpha]}(y_j, f)\} \\
&= (1+\epsilon)^{\alpha} m^{\alpha/2} \sum_j r_j^{\alpha} \exp\{-g^{\alpha}(y_j, f)\} \\
&\leq (1+\epsilon)^{\alpha} \exp\{\alpha\epsilon\} m^{\alpha/2} \sum_j r_j^{\alpha} \exp\{-g^{\alpha}(x, f)\} \\
&\leq 2^{\alpha} m^{\alpha/2} \exp\{-g^{\alpha}(x, f)\} \sum_j r_j^{\alpha} \\
&\leq C(a + \eta).
\end{aligned}$$

Since $\eta > 0$ is arbitrary, this proves the lemma. □

By a routine procedure we can extend the inequality to any compact set.

COROLLARY 1. *Let M be a Riemannian manifold, $\dim M = m$, and let $f : M \to M$ be a C^1 map. Assume $X \subset M$ is compact and f is expanding on X (X need not be f invariant). Fix $\alpha \in [0, m]$. Then for any $b_0 > 2\sqrt{m}$ and $C_0 > 2^{\alpha} m^{\alpha/2}$, there is $\rho_0 > 0$, such that for all $x \in X$ and all $A \subset B(x, \rho_0)$ (7) holds for every $\rho < \rho_0$, where b and C (depending on x) are given by*

$$b = b_0 \exp\{-\lambda_{m-[\alpha]}(x, f)\}, \tag{10}$$

$$C = C_0 \exp\{-g^{\alpha}(x, f)\}. \tag{11}$$

Proof. Since $X \cup fX$ is compact, there is $\rho_1 > 0$, such that for any $x \in X \cup fX$, $\exp_x$ maps $\widetilde{B}_x(0, \rho_1)$, the ρ_1-ball at 0 in $T_x M$, diffeomorphically onto the ball $B(x, \rho_1) \subset M$. Choose

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$\rho_2 < \rho_1$ so that for every $x \in X$, $fB(x, \rho_2) \subset B(fx, \rho_1)$. Then $\widetilde{f}_x \equiv \exp_{fx}^{-1} \circ f \circ \exp_x$ is defined on $\widetilde{B}_x(0, \rho_2)$. Obviously, $D_0\widetilde{f}_x$ can be identified with $D_x f$. Again by compactness, both $\exp_x$ and $\exp_{fx}^{-1}$ uniformly C^1 approach isometries on $\widetilde{B}_x(0, \rho)$ and $B(fx, \rho)$, as $\rho \to 0$. So there is $\rho_0 < \rho_2$, such that at every $x \in X$, $\widetilde{f}_x$ satisfies conditions (1)–(3) in the proof of Lemma 3. Applying Lemma 3 to $\widetilde{f}_x$, for every $\widetilde{A} \subset \widetilde{B}_x(0, \rho_0)$ we have

$$\mathcal{H}^{\alpha}_{b\rho}(\widetilde{A}) \le C\mathcal{H}^{\alpha}_{\rho}(\widetilde{f}_x(\widetilde{A})) \tag{12}$$

for all $\rho < \rho_0$, where b and C are given by (8) and (9). Set $A = \exp_x(\widetilde{A})$. Then $\widetilde{f}_x(\widetilde{A}) = \exp_{fx}^{-1} \circ f(A)$. Reducing ρ_0 one more time if necessary, we may make all $\exp_x$ and $\exp_{fx}^{-1}$ so close to an isometry that (12) can be replaced by (7) with slightly larger b and C, which are pre-assigned by (10) and (11). This completes the proof. □

Next we show that a similar statement holds on unstable manifolds of a hyperbolic set. It is needed in the proof of Theorem 3. Remember that if a set is contained in a piece of unstable manifold, its Hausdorff measure is calculated under the induced metric.

COROLLARY 2. *Let Λ be a hyperbolic set of a C^1 diffeomorphism f. All notations are as in §4. Fix $\alpha \in [0, u]$. Then for any $b_0 > 2\sqrt{u}$ and $C_0 > 2^{\alpha}u^{\alpha/2}$, there is $\rho_0 > 0$, such that for all $x \in \Lambda$ and all $A \subset B^u(x, \rho_0)$, (7) holds for every $\rho < \rho_0$, where b and C (depending on x) are given by*

$$b = b_0 \exp\{-\lambda_{u-[\alpha]}(x, f)\}, \tag{13}$$

$$C = C_0 \exp\{-g^{\alpha}(x, f)\}. \tag{14}$$

Proof. Choose $\rho_1 > 0$, so that for every $x \in \Lambda$, $\exp_x$ maps the ball $\widetilde{B}_x(0, \rho_1)$ diffeomorphically onto $B(x, \rho_1)$. From standard properties of unstable manifolds [**10**], there is $\rho_2 < \rho_1$ such that for every $x \in \Lambda$, $B^u(x, \rho_2) \subset fB^u(f^{-1}x, \rho_2) \subset B(x, \rho_1)$, and $\exp_x^{-1}(fB^u(f^{-1}x, \rho_2))$ is the graph of a C^1 function from an open set of $E^u(x)$ to $E^s(x)$. Use p_x to denote the projection of T_xM onto $E^u(x)$ along $E^s(x)$. Let q_x be the inverse map of p_x restricted to $\exp_x^{-1}(fB^u(f^{-1}x, \rho_2))$. Then q_x is a C^1 diffeomorphism and D_0q_x may be identified with the identity map on $E^u(x)$. Set $\widetilde{f}_x \equiv q_{fx}^{-1} \circ \exp_{fx}^{-1} \circ f \circ \exp_x \circ q_x$. Then $D_0\widetilde{f}_x = D_x f|E^u(x)$. As $\rho \to 0$, all $\exp_x$ and q_x uniformly C^1 approach isometries on $\widetilde{B}_x(0, \rho)$ and $\widetilde{B}_x(0, \rho) \cap E^u(x)$, respectively. Thus there is $\rho_0 < \rho_2$ such that Lemma 3 applies to $\widetilde{f}_x$ uniformly for all $x \in \Lambda$. So if $x \in \Lambda$ and $\widetilde{A} \subset \widetilde{B}_x(0, \rho_0) \cap E^u(x)$, then

$$\mathcal{H}^{\alpha}_{b\rho}(\widetilde{A}) \le C\mathcal{H}^{\alpha}_{\rho}(\widetilde{f}_x(\widetilde{A})) \tag{15}$$

for all $\rho < \rho_0$, where b and C are as in (8) and (9) (with m replaced by u). Now set $A = \exp_x \circ q_x(\widetilde{A})$. Then $\widetilde{f}_x(\widetilde{A}) = q_{fx}^{-1} \circ \exp_{fx}^{-1} \circ f(A)$. Further reducing ρ_0 to make all $\exp_x$ and q_x closer to isometries, we can replace (15) by (7) with b and C satisfying (13) and (14), completing the proof. □

6. *Proof of the theorems*

First we use Corollary 1 to prove Lemma 1. For simplicity we will assume that f is defined and is C^1 on the whole of M.

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Proof of Lemma 1. Assume $D_1 < m$, otherwise the assertion is obvious. Then $P_1(D_1) = 0$. Take any $d > D_1$. Since $P_1(\alpha)$ is strictly decreasing, $\epsilon \equiv -\frac{1}{2}P_1(d) = -\frac{1}{2}P(f|_\Lambda, -g^d(\cdot, f))$ is a positive number. Choose $b_0 > 2\sqrt{m}$ and $C_0 > 2^d m^{d/2}$. Since $\|D_x f v\| \geq k\|v\|$ for all $x \in \Lambda$ and $v \in T_x M$, $\|D_x f^n v\| \geq k^n \|v\|$. This implies that $\lambda_i(x, f^n) \geq n \log k$ for all $i = 1, \ldots, m$ and $n \in \mathbf{Z}^+$.

Pick $N \in \mathbf{Z}^+$ satisfying the inequalities

$$\beta \equiv b_0 k^{-N} < 1, \tag{16}$$

$$C_0 \exp\{-N\epsilon\} < 1. \tag{17}$$

Applying Corollary 1 to the map f^N, we can find $\rho_0 > 0$ such that for all $x \in \Lambda$ and $A \subset B(x, \rho_0)$

$$\mathcal{H}^d_{\beta\rho}(A) \leq C_0 \exp\{-g^d(x, f^N)\}\mathcal{H}^d_\rho(f^N A) \tag{18}$$

for all $\rho < \rho_0$. Also assume ρ_0 is so small that the topological pressure can be calculated by (3) for any $\rho < \rho_0$. Fix such a ρ. By the definition of ϵ,

$$\limsup_{n\to\infty} \frac{1}{n} \log Q(-g^d(\cdot, f), n, \rho) = -2\epsilon.$$

Hence there exists $n_0 \in \mathbf{Z}^+$ such that $n > n_0$ implies

$$Q(-g^d(\cdot, f), nN, \rho) < \exp\{-nN\epsilon\}.$$

So there are points $\{x_j\} \subset \Lambda$, $\bigcup_j B_{nN}(x_j, \rho) \supset \Lambda$, such that

$$\sum_j \exp\left\{-\sum_{i=0}^{nN-1} g^d(f^i x_j, f)\right\} < \exp\{-nN\epsilon\}.$$

Since $B_{iN}(x, \rho) \subset B(x, \rho_0)$ and $f^N B_{iN}(x, \rho) \subset B_{(i-1)N}(f^N x, \rho)$ for all $i = 1, \ldots, n$ and all $x \in \Lambda$, applying (18) $n-1$ times to $B_{nN}(x_j, \rho)$ yields

$$\begin{aligned}
\mathcal{H}^d_{\beta^n\rho}(B_{nN}(x_j, \rho)) &\leq C_0^{n-1} \exp\left\{-\sum_{i=0}^{n-2} g^d(f^{iN} x_j, f^N)\right\} \mathcal{H}^d_{\beta\rho}(B_N(f^{(n-1)N} x_j, \rho)) \\
&\leq C_0^{n-1} C_1 C_2 \exp\left\{-\sum_{i=0}^{n-1} g^d(f^{iN} x_j, f^N)\right\},
\end{aligned} \tag{19}$$

where $C_1 = \sup_{x\in\Lambda} \mathcal{H}^d_{\beta\rho}(B(x, \rho))$ and $C_2 = \sup_{x\in\Lambda} \exp\{g^d(x, f^N)\}$. By (5),

$$\sum_{i=0}^{n-1} g^d(f^{iN} x_j, f^N) \geq \sum_{i=0}^{n-1} \sum_{l=0}^{N-1} g^d(f^{iN+l} x_j, f) = \sum_{i=0}^{nN-1} g^d(f^i x_j, f).$$

Summing up (19) over j we have

$$\begin{aligned}
\mathcal{H}^d_{\beta^n\rho}(\Lambda) &\leq \sum_j \mathcal{H}^d_{\beta^n\rho}(B_{nN}(x_j, \rho)) \\
&\leq C_1 C_2 C_0^{n-1} \sum_j \exp\left\{-\sum_{i=0}^{nN-1} g^d(f^i x_j, f)\right\} \\
&< C_1 C_2 C_0^{n-1} \exp\{-nN\epsilon\}.
\end{aligned}$$

By (16) and (17), letting $n \to \infty$ we get $\mathcal{H}^d(\Lambda) = 0$. Therefore $d \geq \dim_H \Lambda$. Since d can be made arbitrarily close to D_1, this proves $\dim_H \Lambda \leq D_1$. □

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Proof of Lemma 2. All results here easily follow from the standard properties of topological pressure.

By (5) and [**20**, Theorem 9.7],

$$\begin{aligned} P_{n+l}(\alpha) &= P\left(f|_\Lambda, -\frac{1}{n+l}g^\alpha(x, f^{n+l})\right) \\ &\le P\left(f|_\Lambda, \frac{n}{n+l}\left(-\frac{1}{n}g^\alpha(x, f^n)\right) + \frac{l}{n+l}\left(-\frac{1}{l}g^\alpha(f^n x, f^l)\right)\right) \\ &\le \frac{n}{n+l}P\left(f|_\Lambda, -\frac{1}{n}g^\alpha(x, f^n)\right) + \frac{l}{n+l}P\left(f|_\Lambda, -\frac{1}{l}g^\alpha(f^n x, f^l)\right) \\ &= \frac{n}{n+l}P_n(\alpha) + \frac{l}{n+l}P_l(\alpha). \end{aligned}$$

This, together with [**20**, Theorem 4.9], proves the existence of the limit

$$P^*(\alpha) = \lim_{n\to\infty} P_n(\alpha) = \inf_{n\in\mathbf{Z}^+} P_n(\alpha).$$

For each n, $P_n(0) = h_{\text{top}}(f|_\Lambda)$ and

$$\begin{aligned} P_n(m) &= P\left(f|_\Lambda, -\frac{1}{n}\log|\operatorname{Jac}(D_x f^n)|\right) \\ &= P\left(f|_\Lambda, -\frac{1}{n}\sum_{i=0}^{n-1}\log|\operatorname{Jac}(D_{f^i x} f)|\right) \\ &= P(f|_\Lambda, -\log|\operatorname{Jac}(D_x f)|). \end{aligned}$$

So $P^*(0) = h_{\text{top}}(f|_\Lambda)$ and $P^*(m) = P(f|_\Lambda, -\log|\operatorname{Jac}(D_x f)|)$. Set $K = \sup_{x\in\Lambda}\|D_x f\|$. Then for any $0 \le \alpha_1 < \alpha_2 \le m$,

$$n\log k \le \frac{g^{\alpha_2}(x, f^n) - g^{\alpha_1}(x, f^n)}{\alpha_2 - \alpha_1} \le n\log K.$$

By [**20**, Theorem 9.7],

$$-\log K \le \frac{P_n(\alpha_2) - P_n(\alpha_1)}{\alpha_2 - \alpha_1} \le -\log k,$$

which implies that P^* is continuous and strictly decreasing in α. □

Theorem 1 follows from Lemma 1 and some general arguments about topological pressure.

Proof of Theorem 1. Without loss of generality, we assume $P(f|_\Lambda, -\log|\operatorname{Jac}(D_x f)|) \le 0$. For $n \ge 2$, consider the equation

$$\widetilde{P}_n(\alpha) \equiv P(f^n|_\Lambda, -g^\alpha(\cdot, f^n)) = 0.$$

Since $\widetilde{P}_n(0) = h_{\text{top}}(f^n|_\Lambda) = nh_{\text{top}}(f|_\Lambda) \ge 0$ and $\widetilde{P}_n(m) = nP(f|_\Lambda, -\log|\operatorname{Jac}(D_x f)|) \le 0$, the equation has a unique solution, which we denote by D_n. Applying Lemma 1 to the expanding map f^n yields $\dim_H \Lambda \le D_n$. So $\dim_H \Lambda \le \inf_{n\in\mathbf{Z}^+} D_n$. The proof of Theorem 1 will be completed if the following inequality is verified

$$\inf_{n\in\mathbf{Z}^+} D_n \le \mathcal{D}(f, \Lambda). \tag{20}$$

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For small $\epsilon > 0$, let $d = \inf_{n\in\mathbf{Z}^+} D_n - \epsilon$. As in the proof of Lemma 2, for any $\alpha_1 < \alpha_2$,

$$\widetilde{P}_n(\alpha_2) - \widetilde{P}_n(\alpha_1) \le -n \log k \cdot (\alpha_2 - \alpha_1).$$

Substituting $\alpha_1 = d$ and $\alpha_2 = D_n$, yields

$$\widetilde{P}_n(d) \ge n \log k \cdot (D_n - d) \ge n \log k \cdot \epsilon.$$

Fix any $N \in \mathbf{Z}^+$. We show $P_N(d) > 0$.

Set $C = \max\{g^d(x, f^N) : x \in \Lambda\}$. Let $l \in \mathbf{Z}^+$ be so big that $lN \log k \cdot \epsilon > C$. For each $j = 0, 1, \ldots, N-1$, using (5) we have

$$\begin{aligned}\sum_{i=0}^{l-1} g^d(f^{iN+j}x, f^N) &\le (g^d(x, f^j) + \sum_{i=0}^{l-2} g^d(f^{iN+j}x, f^N) \\ &\quad + g^d(f^{(l-1)N+j}x, f^{N-j})) + C \\ &\le g^d(x, f^{lN}) + C.\end{aligned}$$

Summing up over j, we get

$$\frac{1}{N}\sum_{i=0}^{lN-1} g^d(f^i x, f^N) \le g^d(x, f^{lN}) + C. \tag{21}$$

Now by [**20**, Theorem 9.8(i) and Theorem 9.7]

$$\begin{aligned}P_N(d) &= P\left(f|_\Lambda, -\frac{1}{N}g^d(x, f^N)\right) \\ &= \frac{1}{lN}P\left(f^{lN}|_\Lambda, -\frac{1}{N}\sum_{i=0}^{lN-1} g^d(f^i x, f^N)\right) \\ &\ge \frac{1}{lN}P(f^{lN}|_\Lambda, -g^d(x, f^{lN}) - C) \\ &= \frac{1}{lN}(\widetilde{P}_{lN}(d) - C) \\ &\ge \frac{1}{lN}(lN \log k \cdot \epsilon - C)\end{aligned}$$

which is > 0 by the definition of l.

Since $P_N(d) > 0$ for all N, $P^*(d) \ge 0$ and so $d \le \mathcal{D}(f, \Lambda)$. As $\epsilon = \inf_{n\in\mathbf{Z}^+} D_n - d$ can be arbitrarily small, we prove (20), completing the proof of Theorem 1. □

Note that a little more effort can actually show $\mathcal{D}(f, \Lambda) = \inf_{n\in\mathbf{Z}^+} D_n$.

One more lemma is required to prove Theorem 2. The lemma is similar to Proposition 2 in [**13**]. Mimic the definition of $d_\Lambda(\mu)$; for any $x \in \Lambda$ and $n \in \mathbf{Z}^+$ we set

$$d_\Lambda(x, f^n) = \max\{\alpha \in [0, m] : g^\alpha(x, f^n) \le h_{\text{top}}(f^n|_\Lambda)\}.$$

Notice that $h_{\text{top}}(f^n|_\Lambda) = nh_{\text{top}}(f|_\Lambda)$.

LEMMA 4.

$$\inf_{n\in\mathbf{Z}^+}\sup_{x\in\Lambda} d_\Lambda(x, f^n) = \sup_{\mu\in\mathcal{E}(\Lambda,f)} d_\Lambda(\mu).$$

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Proof. Let $d = \inf_{n\in\mathbf{Z}^+} \sup_{x\in\Lambda} d_\Lambda(x, f^n)$. We first show $d \le \sup_{\mu\in\mathcal{E}(\Lambda,f)} d_\Lambda(\mu)$. Let us denote $h = h_{\mathrm{top}}(f|_\Lambda)$ for simplicity. Since every $d_\Lambda(x, f^n)$ is uniformly continuous on Λ, there is a sequence $\{x_n\} \subset \Lambda$ such that $d_\Lambda(x_n, f^n) = \sup_{x\in\Lambda} d_\Lambda(x, f^n) \ge d$ for all $n \in \mathbf{Z}^+$. This implies $g^d(x_n, f^n) \le nh$. Now fix $N \in \mathbf{Z}^+$. If $n > N$ and $0 \le j < N$, we have by (5)

$$\begin{aligned}
&\sum_{i=0}^{[n/N]-k(j)} g^d(f^{iN+j}x_n, f^N) \\
&\le\ g^d(x_n, f^j) + \sum_{i=0}^{[n/N]-k(j)} g^d(f^{iN+j}x_n, f^N) + g^d(f^{l(j)}x_n, f^{n-l(j)}) \\
&\le\ g^d(x_n, f^n) \\
&\le\ nh,
\end{aligned}$$

where $k(j) = 1$, if $0 \le j \le n - N[n/N]$; $k(j) = 2$, if $n - N[n/N] < j < N$, and $l(j) = N([n/N] - k(j) + 1) + j$. Summing up over j yields

$$\sum_{i=0}^{n-N} g^d(f^i x_n, f^N) \le nNh.$$

Denoting $\nu_n = \frac{1}{n}\sum_{i=0}^{n-1} \delta_{f^i x_n}$, where δ_x is the Dirac measure at x, we rewrite the inequality as

$$\int_\Lambda g^d(x, f^N)\, d\nu_n(x) \le Nh + \frac{N}{n}C, \tag{22}$$

where $C = \sup_{x\in\Lambda} g^d(x, f^N)$. Let ν be a vague limit point of $\{\nu_n\}$. Then ν is an invariant probability measure on Λ (see [**20**, Theorem 6.9]) and from (22)

$$\int_\Lambda g^d(x, f^N)\, d\nu(x) \le Nh$$

or

$$\int_\Lambda \frac{1}{N} g^d(x, f^N)\, d\nu(x) \le h. \tag{23}$$

Let $\nu = \int_\Lambda \mu_x\, d\nu(x)$ be an ergodic decomposition of ν, where $\mu_x \in \mathcal{E}(\Lambda, f)$, ν-a.e. $x \in \Lambda$. Then $\lim_{N\to\infty} \frac{1}{N} g^d(x, f^N) = g^d(\mu_x)$, ν-a.e. and in $L^1(\nu)$. Thus by (23)

$$\int_\Lambda g^d(\mu_x)\, d\nu(x) \le h.$$

Therefore $g^d(\mu_x) \le h$ for some $\mu_x \in \mathcal{E}(\Lambda, f)$, i.e. $d \le d_\Lambda(\mu_x)$.

Now we prove the converse. Obviously, $d \ge \sup_{x\in\Lambda} \inf_n d_\Lambda(x, f^n)$. For any $\mu \in \mathcal{E}(\Lambda, f)$, there is $x \in \Lambda$ such that $\lim_{n\to\infty} \frac{1}{n} g^\alpha(x, f^n) = g^\alpha(\mu)$, all $\alpha \in [0, m]$. So $\lim_{n\to\infty} d_\Lambda(x, f^n) = d_\Lambda(\mu)$. By (5), $d_\Lambda(x, f^{n+l}) \le \sup\{d_\Lambda(x, f^n), d_\Lambda(f^n x, f^l)\}$, for all $n, l \in \mathbf{Z}^+$. Applying this inequality repeatedly we obtain $\lim_{n\to\infty} d_\Lambda(x, f^n) \le \sup_{n\in\mathbf{Z}^+} \inf_{l\in\mathbf{Z}^+} d_\Lambda(f^n x, f^l) \le d$, which proves $d_\Lambda(\mu) \le d$. □

Proof of Theorem 2. By Lemma 4 and the fact that $\dim_H \Lambda \le \inf_{n\in\mathbf{Z}^+} D_n$ (see the proof of Theorem 1), we only need to show that for each n

$$D_n \le \sup_{x\in\Lambda} d_\Lambda(x, f^n).$$

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Let $d = \sup_{x\in\Lambda} d_\Lambda(x, f^n)$. Then $g^d(x, f^n) \geq nh$ for all $x \in \Lambda$. So $\widetilde{P}_n(d) = P(f^n|_\Lambda, -g^d(\cdot, f^n)) \leq P(f^n|_\Lambda, -nh) = P(f^n|_\Lambda, 0) - nh = nh - nh = 0$. By the definition of D_n, we get $d \geq D_n$, as desired. □

As mentioned before, $\mathcal{D}(f, \Lambda) = \inf_{n\in\mathbf{Z}^+} D_n$. Therefore we have just shown that $\mathcal{D}(f, \Lambda) \leq \sup_{\mu\in\mathcal{E}(\Lambda, f)} d_\Lambda(\mu)$.

Now we turn to the proof of the results for hyperbolic sets. All properties of (un)stable manifolds used here are standard, and can be found in [**10**]. The key step is the following dimension estimate.

LEMMA 5. *Set*

$$D_1^u = \max\{\alpha \in [0, u] : P_1^u(\alpha) \geq 0\}.$$

Then for any $x \in \Lambda$,

$$\dim_H(W^u(x) \cap \Lambda) \leq D_1^u.$$

Proof. Fix $\bar{x} \in \Lambda$. Let W be any compact piece of $W^u(\bar{x})$. We show

$$\dim_H(W \cap \Lambda) \leq D_1^u. \tag{24}$$

This obviously implies Lemma 5.

Without loss of generality, assume $D_1^u < u$, i.e. $P(f|_\Lambda, -\log|\operatorname{Jac}(D_x f|E^u(x))|) < 0$ and $P_1^u(D_1^u) = 0$. Take any $d > D_1^u$ and set $\epsilon = -\frac{1}{2}P_1^u(d)$. Since P_1^u is strictly decreasing, $\epsilon > 0$. Choose $b_0 > 2\sqrt{u}$, $C_0 > 2^d u^{d/2}$, $\eta > 0$ and $N \in \mathbf{Z}^+$ satisfying

$$\beta \equiv b_0 k^{-N} < 1, \tag{25}$$

$$C_0 e^\eta \exp\{-N\epsilon\} < 1. \tag{26}$$

Consider Λ as a hyperbolic set for f^N. Clearly, $\|D_x f^N|E^s(x)\| \leq k^{-N}$ and $\|D_x f^{-N}|E^u(x)\| \leq k^{-N}$. By Corollary 2, there is $\rho' > 0$, such that for all $x \in \Lambda$, $A \subset B^u(x, \rho')$ and $\rho < \rho'$ we have

$$\mathcal{H}^d_{\beta\rho}(A) \leq C_0 \exp\{-g^d(x, f^N)\}\mathcal{H}^d_\rho(f^N A). \tag{27}$$

Choose small positive numbers r, ρ_0 and ρ_1 (in this order) with $\rho_0 \ll r$, $\rho_0 \leq \rho'$ and $\rho_1 < \rho_0$, so that the following hold:

(1) if $x \in \Lambda$ and $y \in B(x, \rho_1) \cap \Lambda$, then $B^u(y, r) \cap B(x, \rho_1) \subset B^u(y, \rho_0)$;
(2) if $x \in \Lambda$ and $y \in W \cap B(x, \rho_1) \cap \Lambda$, then $W \cap B(x, \rho_1) \subset B^u(y, \rho_0)$;
(3) for any $x \in \Lambda$, $f^N B^u(x, \rho_0) \subset B^u(f^N x, r)$;
(4) if $x \in \Lambda$ and $y \in B(x, \rho_1) \cap \Lambda$, then $|g^d(x, f^N) - g^d(y, f^N)| < \eta$;
(5) for any $\rho < \rho_1$ and any continuous function ϕ

$$P(f|_\Lambda, \phi) = \limsup_{n\to\infty} \frac{1}{n} \log Q(\phi, n, \rho).$$

Here, (1) is due to some well-known facts about the unstable manifolds. (2) follows from the compactness of W. It roughly means that W intersects any $B(x, \rho_1)$ at most once. (3)–(5) are obviously true for all small ρ_0 and ρ_1.

Fix $\rho < \rho_1$. By the definition of ϵ,

$$\limsup_{n\to\infty} \frac{1}{n} \log Q(-g^d(\cdot, f), n, \rho) = -2\epsilon.$$

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So for all large n,

$$Q(-g^d(\cdot, f), nN, \rho) < \exp\{-nN\epsilon\}.$$

Thus, there are points $\{x_j\} \subset \Lambda$, $\cup_j B_{nN}(x_j, \rho) \supset \Lambda$, such that

$$\sum_j \exp\left\{-\sum_{i=0}^{nN-1} g^d(f^i x_j, f)\right\} < \exp\{-nN\epsilon\}. \tag{28}$$

The sets $\{B_{nN}(x_j, \rho)\}$ cover $W \cap \Lambda$. Let $A_j = W \cap \Lambda \cap B_{nN}(x_j, \rho)$, and consider the iterates $f^{iN}A_j$, $i = 0, 1, \ldots, n$, if $A_j \neq \emptyset$.

By (2), $A_j \subset W \cap B(x_j, \rho) \subset B^u(y_j, \rho_0)$, for some $y_j \in A_j$. We use induction to show

$$f^{iN}A_j \subset B^u(f^{iN}y_j, \rho_0) \quad i = 0, 1, \ldots, n-1. \tag{29}$$

Assume we already have $f^{(i-1)N}A_j \subset B^u(f^{(i-1)N}y_j, \rho_0)$. Then by (3), $f^{iN}A_j \subset B^u(f^{iN}y_j, r)$. Moreover, $f^{iN}A_j \subset B(f^{iN}x_j, \rho)$ by the fact that $A_j \subset B_{nN}(x_j, \rho)$. So by (1), $f^{iN}A_j \subset B^u(f^{iN}y_j, r) \cap B(f^{iN}x_j, \rho) \subset B^u(f^{iN}y_j, \rho_0)$, proving (29).

(4) implies

$$|g^d(f^{iN}y_j, f^N) - g^d(f^{iN}x_j, f^N)| < \eta.$$

From this and applying (27) n-times, yields

$$\begin{aligned}
\mathcal{H}^d_{\beta^n\rho}(A_j) &\leq C_0^n \exp\left\{-\sum_{i=0}^{n-1} g^d(f^{iN}y_j, f^N)\right\}\mathcal{H}^d_\rho(f^{nN}A_j) \\
&\leq C_1 C_0^n \exp\{n\eta\} \exp\left\{-\sum_{i=0}^{n-1} g^d(f^{iN}x_j, f^N)\right\},
\end{aligned}$$

where $C_1 = \sup\{\mathcal{H}^d_\rho(B^u(x, r)) : x \in \Lambda\}$, which is clearly finite. By the superadditivity (5)

$$\sum_{i=0}^{n-1} g^d(f^{iN}x_j, f^N) \geq \sum_{i=0}^{nN-1} g^d(f^i x_j, f).$$

Since $\cup_j A_j \supset W \cap \Lambda$, we get

$$\begin{aligned}
\mathcal{H}^d_{\beta^n\rho}(W \cap \Lambda) &\leq \sum_j \mathcal{H}^d_{\beta^n\rho}(A_j) \\
&\leq C_1 C_0^n \exp\{n\eta\} \sum_j \exp\left\{-\sum_{i=0}^{nN-1} g^d(f^i x_j, f)\right\} \\
&< C_1 C_0^n \exp\{n\eta\} \exp\{-nN\epsilon\},
\end{aligned}$$

where the last inequality is from (28). Using (25) and (26) and letting $n \to \infty$, yields

$$\mathcal{H}^d(W \cap \Lambda) = 0.$$

This gives $\dim_H(W \cap \Lambda) \leq d$. Taking $d \to D_1^u$ proves (24). □

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Note that we have given an alternative proof for half of [**14**, Theorem 1]. Again, because we only calculate the change of Hausdorff measure for each iteration, the use of Markov partitions and the discussion on the structure of n-cylinders are avoided.

Let us make a final comment on completing the proof of Theorems 3 and 4. Following Lemma 5, we may just repeat the proofs of Theorem 1, Lemma 4 and Theorem 2, almost word for word. All we have to use are general properties of topological pressure and Lyapunov exponents, and the superadditivity of the functions $g^{\alpha}(\cdot, f^n)$, which are shared by both expanding and hyperbolic systems. Thus, the detailed proofs are omitted.

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