Emmental squares are tasty

TONY CRILLY, STAN DOLAN and COLIN R. FLETCHER

1. *Introduction*

A shape is called *equable* if its area and perimeter are numerically equal relative to some given system of units. For example, if a square is equable, then its side, a, must satisfy $4a = a^2$. So there is only one equable square, and it has side 4.

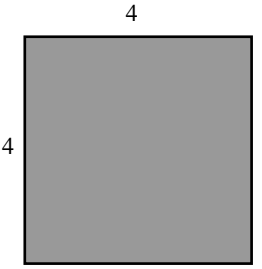


FIGURE 1

It is easy to investigate this idea for other shapes. Though not connected with this problem, the work of Imre Lakatos suggested a generalisation to us. Lakatos showed that Euler's classical formula $V + F = E - 2$ for polyhedra could be extended when the notion of tunnels was introduced [1].

In Lakatosian spirit we shall consider squares with square holes, which we term Emmental squares. The number of square holes will be called the order of the Emmental square. These holes must lie strictly inside the containing square and not touch each other. We shall assume throughout that all dimensions have integer values.

An example of an Emmental square of order 3 is shown below. It has side 9 and its three holes have sides 2, 2 and 3.

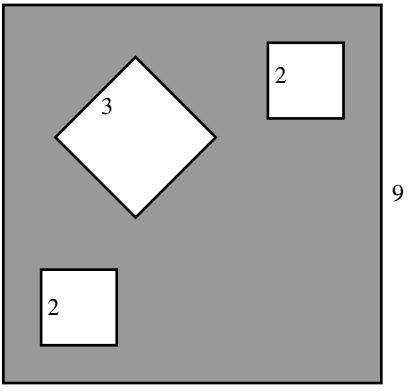


FIGURE 2

https://doi.org/10.1017/mag.2018.3 Published online by Cambridge University Press

The example above is easily seen to be equable. Its area and its perimeter, including the internal perimeters of the 3 holes, are both equal to 64.

Emmental squares prove to be a surprisingly rich source of mathematics. It should be noted that the notion of *equable shapes* is predicated on the choice of units. If a shape is equable with one choice of units it will cease to be equable if those units are changed. We shall assume the units of measurement are fixed at the outset.

In this article we shall concentrate on finding various families of equable Emmental squares and this work offers links with both

- a result of Erdős on maximising the perimeters of squares drawn without touching in a fixed square;
- the problem of dissecting a square into a number of smaller squares which has attracted considerable attention in the mathematical literature, most famously by Henry Dudeney, [2], and by the Cambridge mathematicians W. T. Tutte, C. A. B. Smith, A. H. Stone and R. L. Brooks, as described in [3].

At the end of this article we shall suggest some further ideas on Emmental squares. These include problems which require solutions of Pell's equation and the finding of integer points on elliptic curves. Emmental squares are indeed tasty!

2. *The basic Diophantine equation*

In dealing with integer dimensions the equable condition leads us to a Diophantine equation. For an equable Emmental square of order n with side a this basic equation is:

$$a^2 - \sum_{n \text{ holes}} x_i^2 = 4a + \sum_{n \text{ holes}} 4x_i$$

$$\Rightarrow \qquad a(a-4) = \sum_{n \text{ holes}} x_i(x_i+4) \tag{1}$$

where the i th hole has side length x_i.

As an example of the solution of this equation, consider the case of a square of side a with only one hole, of side length x. Equation (1) gives

$$a(a-4) = x(x+4)$$

and hence

$$a^2 - 4a + 4 = x^2 + 4x + 4$$

which gives the solution $x = a - 4$.

This means that, for a given square of side $a > 4$ we simply cut out a square of side 4 units less to give an equable Emmental square. Therefore

https://doi.org/10.1017/mag.2018.3 Published online by Cambridge University Press

there are infinitely many equable Emmental squares, starting with $a = 5, 6, 7, \ldots$.

There is a two-stage process involved here. In addition to solving equation (1) we have to check that the solution is physically possible. Each algebraic solution of (1) has to be checked to show the square holes can be cut inside the enclosing square and be non-touching.

The diagram below shows that a serious complication can arise:

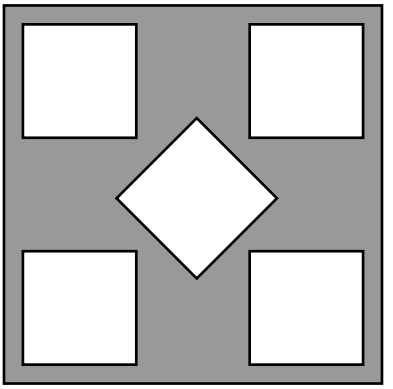


FIGURE 3

In this example, an Emmental square of side 3 with 5 unit square holes can only be formed if at least one hole is at a skew angle. For the purposes of this article we shall need a result concerning square holes at skew angles.

3. *A result of Erdős*

The required result is believed to have been first articulated by Paul Erdős in 1932. In a 1975 paper coauthored by Erdős, [4], it was noted that this result was set in a high school examination paper in Hungary:

Lemma (Erdős)

If an $a \times a$ square strictly contains a $b \times b$ square and a non-touching $c \times c$ square, then

$$b + c < a.$$

A proof of this is given by one of us in [5]. Using this lemma we are in a position to explore equable Emmental squares with n holes. If we specify n, how many equable Emmental squares can be constructed and identified?

We shall find that in the case of n holes, there are infinitely many equable Emmental squares for all natural numbers except $n = 2$ and $n = 3$. These two cases prove to be exceptional, as we shall see later.

https://doi.org/10.1017/mag.2018.3 Published online by Cambridge University Press

4. *Some infinite families of equable Emmental squares*

(i) $n = 1$

For the case $n = 1$ we have already seen that there is a family of equable Emmental squares, as illustrated below.

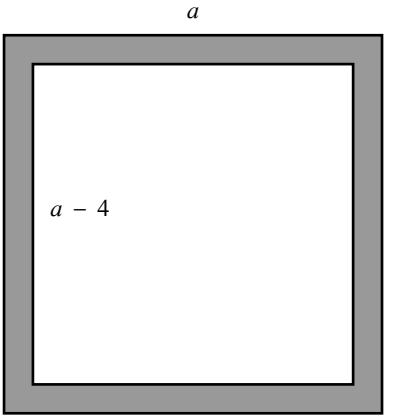


FIGURE 4

Once the hole has been removed, the area of each member of this family is $8a - 16$.

An important point to note is that, for large values of a, the area of the Emmental square becomes negligible in comparison with a^2 the area of the original square. For large a, the Emmental square therefore becomes effectively just a square border.

(ii) $n = 5$

Writing $a = 2b$ for some parameter b, there is a family of equable Emmental squares, with

- 1 hole of side 2,
- 4 holes of side $b - 3$.

These Emmental squares satisfy (1) since

$$a(a - 4) = \sum_{5 \text{ holes}} x_i(x_i + 4) = 4(b - 3)(b + 1) + 12 = 4b^2 - 8b.$$

https://doi.org/10.1017/mag.2018.3 Published online by Cambridge University Press

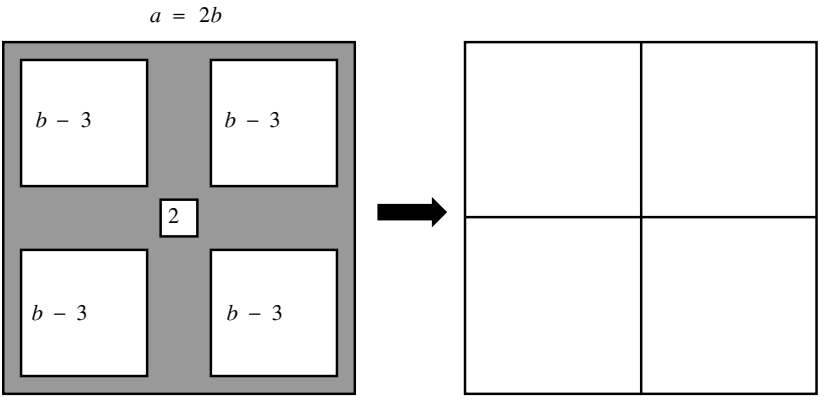


FIGURE 5

In the case when a is large, one of the holes becomes relatively negligible and the Emmental squares tend to a patchwork of 4 squares (Figure 5).

In general, the area of an equable Emmental square (with the n holes removed) is bounded above, and since $x_i < a$ applied to (1), we have

$$a^2 - \sum_{n \text{ holes}} x_i^2 = 4a + \sum_{n \text{ holes}} 4x_i < 4(n+1)a$$

and so

$$\frac{a^2 - \sum_{n \text{ holes}} x_i^2}{a^2} < \frac{4(n+1)}{a}.$$

For large values of a, the area of the Emmental square becomes negligible in comparison with a^2 the area of the enclosing square. So, for any infinite family of solutions, as the sides of the Emmental squares increase in size, the equable Emmental squares will tend to a simple patchwork of square outlines.

We can therefore base any search for families of equable Emmental squares on specific dissections of a square into a number of smaller squares.

https://doi.org/10.1017/mag.2018.3 Published online by Cambridge University Press

(iii) $n = 2k + 2, k \geqslant 1$ (i.e. all even n greater than 2)

We shall consider Emmental squares which will tend to a patchwork of $2k + 2$ holes, as shown for $k = 4$ (Figure 6):

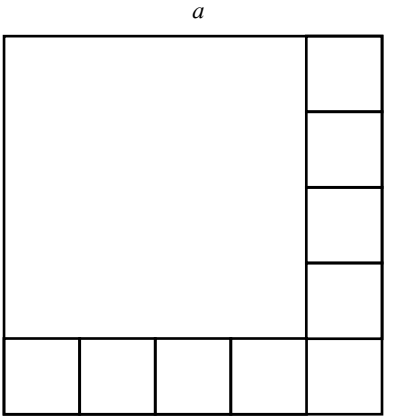


FIGURE 6

A little experimentation with precise sizes for the holes suggests the following arrangement of squares, for some parameter b:

- 1 hole of side kb;
- k holes of side $b - 4$;
- $k + 1$ holes of side b.

Then

$$\begin{aligned} a(a - 4) &= \sum_{n \text{ holes}} x_i(x_i + 4) \\ &= kb(kb + 4) + k(b - 4)b + (k + 1)b(b + 4) \\ &= \left[(k + 1)b + 4\right](k + 1)b. \end{aligned}$$

Putting $a = (k + 1)b + 4$ therefore gives an infinite family of equable Emmental squares for all even numbers n greater than 2.

(iv) $n = 2k + 3, k \geqslant 2$ (i.e. all odd n greater than 5)

In the previous example, we can simply replace two holes of side $b - 4$ and two of side b by two holes of side $b - 3$, two of side $b - 1$ and one of side 2.

This works because of the identity

$$2(b - 4)b + 2b(b + 4) \equiv 2(b - 3)(b + 1) + 2(b - 1)(b + 3) + 12.$$

This replacement does not alter $\sum_{n \text{ holes}} x_i(x_i + 4)$ and so the Emmental square which results is still equable.

https://doi.org/10.1017/mag.2018.3 Published online by Cambridge University Press

Cases (i), (ii), (iii) and (iv) cover all positive integer values of n except for 2 and 3 and so we have proved

Proposition 1: There are infinite families of equable Emmental squares of all orders n except possibly for $n = 2$ and $n = 3$.

5. *The cases* $n = 2$ *and* $n = 3$

The idea of relating the asymptotic behaviour of families of equable Emmental squares to dissections of squares will now be used to prove that there are no such infinite families for $n = 2$ or 3. We shall again make use of Erdős's lemma.

For $n = 2$, the basic equation (1) can be written as

$$(a + x)(a - x - 4) = y(y + 4) \tag{2}$$

and for $n = 3$

$$(a + x)(a - x - 4) = y(y + 4) + z(z + 4). \tag{3}$$

In both cases

$$x < a \Rightarrow 2x(a - x - 4) \leqslant 2y(y + 4)$$

and so by Erdős's lemma,

$$x(y - 4) < y(y + 4)$$

from which it follows that

$$\frac{y - 4}{y + 4} < \frac{y}{x}. \tag{4}$$

A square has the geometrical property that *it cannot be dissected into two or three smaller squares*, and it is this geometrical property which is key. For either of $n = 2$ or 3, suppose there were infinitely many possibilities for a. Then, for large values of a, all but one hole would become relatively negligible and therefore $\frac{y}{x}$ can be made arbitrarily small. However, for $y \geqslant 5$ equation (4) implies $\frac{y}{x} > \frac{1}{9}$.

Therefore there must be infinitely many values of a with $y \leqslant 4$. However, from (3) such solutions have $a + x \leqslant 64$ and are therefore finite in number. This contradiction proves the following result.

Proposition 2: There are only finitely many equable Emmental squares of orders $n = 2$ and $n = 3$.

It only remains to find all the solutions for $n = 2$ and 3 and this will be done in the next two sections.

https://doi.org/10.1017/mag.2018.3 Published online by Cambridge University Press

6. *Equable Emmental squares of order* 2

We shall assume that the holes are of sides x, y with $y \leqslant x$. For $n = 2$ holes, the basic equation (1) is

$$a^2 = 4a + x^2 + 4x + y^2 + 4y. \tag{5}$$

By Erdős's lemma, $x + y < a$ and we can introduce a slack variable $u > 0$ with $x + y + u = a$.

Eliminating a from (5), we have

$$2xy = (4 - u)(2x + 2y + u).$$

Then u must be even and $u < 4$. The only possibility is $u = 2$ and then

$$xy = 2x + 2y + 2$$

can be factorised as $(x - 2)(y - 2) = 6$ and so

$$\{x - 2, y - 2\} = \{6, 1\} \text{ or } \{3, 2\}.$$

We obtain the surprising result (in comparison with all the solutions for other values of n) that there are only two equable Emmental squares with two holes.

a	x	y	Area = Perimeter
11	5	4	80
13	8	3	96

7. *Equable Emmental squares of order* 3

In this case we shall assume that the 3 holes are of sides x, y, z with $z \leqslant y \leqslant x$. Then (1) is

$$a^2 = 4a + x^2 + 4x + y^2 + 4y + z^2 + 4z. \tag{6}$$

Again, by Erdős's lemma, we can introduce a slack variable $u > 0$ with

$$x + y + u = a.$$

Eliminating a from (6), we have

$$2xy = (4 - u)(2x + 2y + u) + z^2 + 4z.$$

As $z \leqslant y \leqslant x$ we have $xy \geqslant z^2$ so we obtain

$$xy \leqslant (5 - u)4x + (4 - u)u.$$

Therefore $u \leqslant 4$ and, also, $xy \leqslant 16x + 4$ and so $y \leqslant 16$.

There are now various ways to solve (6). A routine way is to simply check each possibility given by $z \leqslant y \leqslant 16$ and for each of $u = 1, 2, 3, 4$. This shows that there are precisely 17 equable Emmental squares of order 3:

https://doi.org/10.1017/mag.2018.3 Published online by Cambridge University Press

u	a	x	y	z	Area = Perimeter
1	17	10	6	3	144
1	21	10	10	7	192
1	21	14	6	5	184
1	21	16	4	1	168
1	29	14	14	13	280
1	29	24	4	3	240
2	13	6	5	2	104
2	14	8	4	2	112
2	18	8	8	6	160
2	19	11	6	6	168
2	19	13	4	4	160
2	19	14	3	2	152
2	21	11	8	8	192
3	11	6	2	1	80
3	15	6	6	5	128
4	9	3	2	2	64
4	12	4	4	4	96

Final remarks

We have seen that finding equable Emmental squares of order n and side a depends upon solving the equation

$$a(a-4) = \sum_{n \text{ holes}} x_i(x_i+1). \tag{1}$$

In this article we have considered the problem of finding families of solutions for given values n. However, other problems suggest themselves and produce an interesting array of types of solution. The following are just two ideas which interested readers might wish to explore.

Finding solutions with a given number of identical square holes

For holes of the same side x, (1) becomes

$$a(a-4) = nx(x+4).$$

This can be solved by conversion into a Pell-type equation.

https://doi.org/10.1017/mag.2018.3 Published online by Cambridge University Press

Finding solutions with n holes of consecutive sides $1, 2, \ldots, n$

In this case, (1) becomes

$$6a(a - 4) = n(n + 1)(2n + 13).$$

This equation can be solved by considering integer points on an elliptic curve. There are precisely 6 such equable Emmental squares; the largest of which is a monster square with side 147900 and with 4031 holes.

Acknowledgement

We would like to thank the anonymous referee for making useful suggestions on the original submission.

References

1. I. Lakatos, *Proofs and refutations: the logic of mathematical discovery*, Cambridge University Press (1976).
2. J. H. Conway, 'Mrs. Perkins' Quilt', *Mathematical Proceedings of the Cambridge Philosophical Society*, **60** (1964), pp. 363-368.
3. W. T. Tutte, Squaring the square, *Scientific American* **199** (5) (November 1958) pp. 136-142.
4. P. Erdős, R. L. Graham, On packing squares with equal squares. *Journal of Combinatorial Theory*, Series A, **19** (1975), pp. 119-123.
5. S. Dolan, Thoughts on a conjecture of Erdős, *Math. Gaz.* **101** (November, 2017) pp. 449-457.

10.1017/mag.2018.3

TONY CRILLY
10 Lemsford Road, St Albans AL1 3PB
e-mail: *t.crilly@btinternet.com*
STAN DOLAN
126A Harpenden Road, St Albans AL3 6BZ
e-mail: *stan@standolan.co.uk*
COLIN R. FLETCHER
Atalaya, Lon Glanfred, Llandre, Aberystwyth SY24 5BY
e-mail: *Atalaya1@btinternet.com*

https://doi.org/10.1017/mag.2018.3 Published online by Cambridge University Press