Developing quadrature themes

JOHN D. MAHONY

Introduction

Readers will most likely be aware of the works of Johann Sebastian Bach in the field of music, particularly of his Goldberg variations and the changes that can be rung, wherein aesthetically appealing alterations to structure produce a raft of colourful sounding themes. Similarly in the field of quadrature it is possible to revisit and re-establish well-known formulae by developing variations on the theme of a three-point interpolating quadratic formed to represent a function that is to be integrated.

There are of course very-well established, benchmark approaches to quadrature and these involve the use of Cote's formula and finite difference methods that can be found in most seminal texts on numerical integration (see [1, 2, 3, 4]), particularly in relation to equally spaced abscissa points. For example, these methods are used to establish at an undergraduate level numerical integration formulae such as 'Simpson's rule', the 'Three-eighths rule' and 'Weddle's formulae', to name but a few. In the case of Simpson's rule, the interpolating quadratic approach for equal intervals is often presented to students at sixth form level and it is the purpose here to show that this approach can be employed also to establish other well-known formulae by judiciously combining results of quadratic interpolation across unequally spaced intervals. It is likely that such an approach is known, but the author is not yet aware of the literature containing it, the library being filled somewhat with the classical texts. No attempt will be made to dwell on the important aspect of quantifying before use the magnitude of accuracies, which is inherent in the more esoteric approaches. In this respect, the approach here is somewhat simplistic but it should serve nonetheless to provide the younger mathematician with a very quick avenue to well-known results and it might possibly enable them to develop quadrature formulae of their own. In the final analysis, the proof of the pudding is in the eating and so the results of using the formulae developed hereunder to evaluate a test integral numerically will also be presented.

The approximating quadratic

With reference to Figure 1 below, a typical interval of integration is assumed to be of width $2h$ and within the interval there is a point of subdivision at the abscissa point $x = \lambda h$ which does not necessarily divide the interval into two equal parts.

For this interval the integration is expressed symbolically in the form

$$I = \int_0^{2h} y\,dx$$

and the function to be integrated, $y\big(= f(x)\big)$ say, is approximated by a

https://doi.org/10.1017/mag.2021.113 Published online by Cambridge University Press

quadratic of the form

$$y = f(x) = Ax^2 + Bx + C$$

passing through the three points $(0, y_0)$, $(\lambda h, y_1)$ and $(2h, y_2)$.

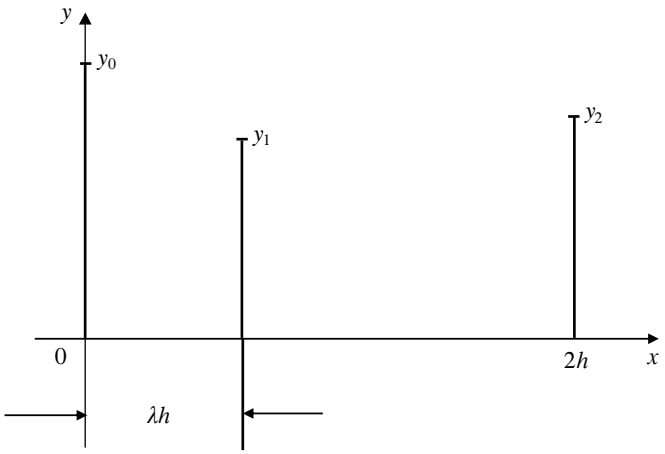


FIGURE 1: The integration schematic for three unequally spaced abscissa points

In order to ascertain suitable values for the coefficients A, B and C, the quadratic is forced to pass through the above three points, and the intermediate point, which can be varied, will be referred to hereafter as the hinge point of the interpolation. Usually the abscissae are taken to be equispaced but for the purpose of this exercise they are not. Thus, by adopting the prescription for the quadratic through these points, it can be established in a straightforward manner that the associated values are

$$A = \frac{2(y_1 - y_0) - \lambda(y_2 - y_0)}{2\lambda(\lambda - 2)h^2}$$

$$B = \frac{4(y_1 - y_0) - \lambda^2(y_2 - y_0)}{2\lambda(2 - \lambda)h}$$

and $$C = y_0.$$

These values can be substituted into the above expression for the integrand and the integrations can be carried out explicitly to produce the intermediate result

$$I = \frac{h}{3}(ay_0 + 4by_1 + cy_2) \tag{1}$$

where

$$a = \frac{3\lambda - 2}{\lambda},\ b = \frac{1}{\lambda(2 - \lambda)} \text{ and } c = \frac{4 - 3\lambda}{2 - \lambda}.$$

Having established such a 'Lego' building block, so to speak, the exercise now will focus on showing how selected values for the parameter λ can be used and varied to re-establish well known quadrature formulae, without recourse to the usual higher-flown methods.

https://doi.org/10.1017/mag.2021.113 Published online by Cambridge University Press

Varying the values of the parameter λ

The λ-values of interest here will be those shown in Table 1 below, but it will become apparent that other values could be considered to secure yet more integration formulae, if required. Also shown are the names of well-known integration formulae which will be developed here from the associated parameter values.

Number of equal strip widths	Rule/formula designation	Associated values of the parameter λ	Corresponding ordinate values
2	Simpson	0, 1, 2	y_0, y_1, y_2
3	Three-eighths	0, 2/3, 4/3, 2	y_0, y_1, y_2, y_3
4	Milne [2]	0, 1/2, 1, 3/2, 2	y_0, y_1, y_2, y_3, y_4
5	—	0, 2/5, 4/5, 6/5, 8/5, 2	$y_0, y_1, y_2, y_3, y_4, y_5$
6	Weddle	0, 1/3, 2/3, 1, 4/3, 5/3, 2	$y_0, y_1, y_2, y_3, y_4, y_5, y_6$

TABLE 1: Parameter values and associated integration formulae names

For each entry in the Table, the end point values for λ are 0, 2 and the hinge point values are the values between. In what follows, an interpolating quadratic will be forced to pass through the end points and each hinge point in turn. The ordinate names, corresponding to the abscissae (hinge and end points) in the third column, are shown in the last column of the Table.

With reference to future use of (1), it will be convenient to display also the values of the above parameters a, $4b$ and c due to these tabulated λ-values. They are shown in Table 2.

$\boldsymbol{\lambda}$**-values**	$\boldsymbol{a}$	$\boldsymbol{4b}$	$\boldsymbol{c}$
1/3	−3	36/5	9/5
2/5	−2	25/4	7/4
1/2	−1	16/3	5/3
2/3	0	9/2	3/2
4/5	1/2	25/6	4/3
1	1	4	1
6/5	4/3	25/6	1/2
4/3	3/2	9/2	0
3/2	5/3	16/3	−1
8/5	7/4	25/4	−2
5/3	9/5	36/5	−3

TABLE 2: Values of a, $4b$ and c due to associated λ-values

https://doi.org/10.1017/mag.2021.113 Published online by Cambridge University Press

Not surprisingly, there is a skew symmetry of data in the table. Having established the preliminaries, it will be appropriate to address now the integration formulae that can be extracted using the proposed approach.

Two equal intervals

In the case when the hinge point is at $\lambda = 1$ the spacing between abscissae in Figure 1 is equal and, for associated data from both Tables, the interpolating quadratic of (1) through the three points where $\lambda = 0, 1, 2$ reduces simply to Simpson's rule, i.e.

$$I = \frac{h}{3}(y_0 + 4y_1 + y_2). \tag{2}$$

Three equal intervals

With reference to Table 1, the required hinge point values are $\lambda = 2/3$ and $\lambda = 4/3$, and the ordinate values at these points are y_1 and y_2, respectively. For the interpolating quadratic through the end points and the hinge point at $\lambda = 2/3$, values of a, $4b$ and c can be extracted from Table 2 to produce a result from (1) in the form

$$\frac{3I}{h} = \frac{9y_1}{2} + \frac{3y_3}{2}.$$

Similarly, when the hinge point is at $\lambda = 4/3$, the exercise will produce the result

$$\frac{3I}{h} = \frac{3y_0}{2} + \frac{9y_2}{2}.$$

Each of these two expressions represents the area under the curve $y = f(x)$ in the interval $(0, 2h)$. Accordingly, so too must their weighted sum. For example, multiply the first expression by α, say, and the second similarly by β, and add to obtain the result

$$I = \left(\frac{h}{2(\alpha + \beta)}\right)(\beta y_0 + 3\alpha y_1 + 3\beta y_2 + \alpha y_3).$$

Writing $\beta/\alpha = \mu$, this may be rewritten in the form

$$I = \left(\frac{h}{2(1 + \mu)}\right)(\mu y_0 + 3y_1 + 3\mu y_2 + y_3). \tag{3}$$

This weighted expression involves four equally-spaced ordinate values and the distance between respective abscissae can be denoted by h'. Then $h' = 2h/3$ and if μ is unity the result reduces to the well-known Three-eighths rule [2], i.e.

$$I = \frac{3h'}{8}(y_0 + 3y_1 + 3y_2 + y_3).$$

The result has been obtained without reference to either Cote's formula or finite difference methods by judiciously combining results for abscissae that

https://doi.org/10.1017/mag.2021.113 Published online by Cambridge University Press

are not uniformly spaced. The interested reader might care to examine other formulae from the above expression by selecting a value for μ that is different from unity. As is well-known, it is possible also to obtain this result using an interpolating cubic to pass through four ordinate points whose abscissa are equally spaced.

Four equal intervals

The hinge point values required to produce rules for integration over four intervals are shown in the Table 1 and the corresponding values for a, $4b$ and c are shown in Table 2. Specifically, at the hinge point $\lambda = 1/2$, the result from (1) can be written in the form

$$\frac{3I}{h} = -y_0 + \frac{16y_1}{3} + \frac{5y_4}{3}.$$

Similarly, for the next hinge point value ($\lambda = 1$) it can be shown that

$$\frac{3I}{h} = y_0 + 4y_2 + y_4.$$

Finally, for the remaining hinge point at $\lambda (= 3/2)$, it is possible to obtain similarly the result

$$\frac{3I}{h} = \frac{5y_0}{3} + \frac{16y_3}{3} - y_4.$$

As before, in the three-interval case, it will prove to be of interest to examine a weighted summation of these three expressions. Specifically, in this instance a weighted symmetric summation will be chosen by multiplying the first and third expressions by α and then adding them to the second expression multiplied by β to produce the symmetric result

$$I = \frac{h}{18\alpha + 9\beta}\big((2\alpha + 3\beta)y_0 + 16\alpha y_1 + 12\beta y_2 + 16\alpha y_3 + (2\alpha + 3\beta)y_4\big).$$

Again, introducing the above parameter μ, this expression reduces to

$$I = \frac{h}{18 + 9\mu}\big((2 + 3\mu)y_0 + 16y_1 + 12\mu y_2 + 16y_3 + (2 + 3\mu)y_4\big). \quad (4)$$

It is possible now to envisage a raft of expressions for different values of the parameter μ. For example, if $\mu = 2/3$ and h is replaced by $2h'$, it is a simple matter to show that Simpson's rule for four intervals is captured. Alternatively, if $\mu = -2/3$, the expression reduces to

$$I = \frac{4h'}{3}(2y_1 - y_2 + 2y_3)$$

which is also a known result; it serves as an approximation to the numerical value of the integral over the specified range that avoids tabulation of the integrand at the end points of the interval of integration. Again, a judicious combination of results for unequally spaced abscissae has resulted in an integration rule where ordinate values are obtained at uniformly spaced abscissae.

https://doi.org/10.1017/mag.2021.113 Published online by Cambridge University Press

Five equal intervals

The values for λ that are required to produce a rule for this scenario can be extracted from Table 1 and the corresponding values for a, $4b$ and c can be drawn from Table 2. As before, the corresponding ordinate values at the various λ-points are also shown in the table. Then the interpolating quadratics through the end points and each of the various hinge points in turn can be obtained as described above. For the hinge point where $\lambda = 2/5$, it is possible to obtain the result

$$\frac{3I}{h} = -2y_0 + \frac{25y_1}{4} + \frac{7y_5}{4}.$$

Similarly, for the hinge point $\lambda = 4/5$,

$$\frac{3I}{h} = \frac{y_0}{2} + \frac{25y_2}{6} + \frac{4y_5}{3}.$$

Also, for the hinge point where $\lambda = 6/5$,

$$\frac{3I}{h} = \frac{4y_0}{3} + \frac{25y_3}{6} + \frac{y_5}{2}.$$

Finally, at the hinge point $\lambda = 8/5$

$$\frac{3I}{h} = \frac{7y_0}{4} + \frac{25fy_4}{4} - 2y_5.$$

Again, a symmetric, weighted summation of these four expressions can be formed in the usual way. Specifically, multiply the first and last of these by α, say, and multiply the second and third by β to obtain, on addition, the result

$$I = \frac{5h'}{144(\alpha+\beta)}\left((22\beta-3\alpha)y_0 + 75\alpha y_1 + 50\beta y_2 + 50\beta y_3 + 75\alpha y_4 + (22\beta-3\alpha)y_5\right)$$

where now $h = 5h'/2$. As before, make the change $\beta/\alpha = \mu$ to rewrite this expression in the form

$$I = \frac{5h'}{144(1+\mu)}\left((22\mu-3)y_0 + 75y_1 + 50\mu y_2 + 50\mu y_3 + 75y_4 + (22\mu-3)y_5\right). \quad (5)$$

Again, in the usual fashion, one can experiment with different values of the parameter μ. For example with $\mu = 3/22$ it is possible to obtain an integration rule that does not involve function evaluation at the end points of integration.

Six equal intervals (*Weddle's formula*)

This is a well-known integration formula that can be found in various texts [2], or by googling the words. The procedure to secure a formula is pretty much the same as that described above for the various cases and involves first the determination of the interpolating quadratics through the various hinge points shown in Table 1. It is left as an exercise for the reader to fathom that a symmetric weighted summation of the ensuing quadratics

https://doi.org/10.1017/mag.2021.113 Published online by Cambridge University Press

(with weighting coefficients α, β and γ) can be developed to produce the result

$$I = \frac{h}{30a + 12\beta + 3\gamma}\big((\gamma + 3\beta - 6\alpha)y_0 + 36\alpha y_1 + 9\beta y_2 + 4\gamma y_3 + 9\beta y_4 + 36\alpha y_5 + (\gamma + 3\beta - 6\alpha)y_6\big).$$

As before, effect the parameter change $\beta/\alpha = \mu$ and also the change $\gamma/\alpha = \nu$ to rewrite this expression in the form

$$I = \frac{h}{30 + 12\mu + 3\nu}\big((\nu + 3\mu - 6)fy_0 + 36y_1 + 9\mu y_2 + 4\nu y_3 + 9\mu y_4 + 36y_5 + (\nu + 3\mu - 6)y_6\big). \quad (6)$$

If now the weighting coefficients are chosen so that $\mu = 4/5$ and $\nu = 54/5$, Weddle's rule is captured:

$$I = \frac{3h'}{10}\big(y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + y_6\big),$$

where now $h' = h/3$. It is of course possible to consider other values for these weighting coefficients to obtain different-looking expressions. For example, if $\mu = 2$ and $\nu = 9$, it is possible to capture the expression for Simpson's rule across six equally spaced intervals. Further, if $\mu = 4$ and $\nu = 6$, it is possible to capture similarly the Three-eighths rule applied over the six intervals viewed as consecutive tranches of three equally spaced intervals. The result due to Weddle can be related also to a combination of Simpson's rule and the Three-eighths rule. To this end, denote the numerical integrations corresponding to Weddle's formula, Simpson's rule and the Three-eighths rule by W, S and T respectively. Then it may be verified that $W = (9S - 4T)5$.

Discussion

As with all numerical integration formulae, their relative strengths can be assessed via a numerical exercise. To this end it is appropriate to display the results of a numerical exercise involving all of the above integration formulae applied to a well-known integral. In this instance 'Ahmed's Integral' (see [5]) will be chosen as the test integral. It is

$$I = \int_0^1 \frac{\tan\sqrt{2 + x^2}}{(1 + x^2)\sqrt{2 + x^2}}\,dx = \frac{5\pi^2}{96}.$$

The integration formulae given by equations (2) – (5) were computed in various cases and a Table of accuracies for different numbers of integration strips was drawn up, as shown below:

https://doi.org/10.1017/mag.2021.113 Published online by Cambridge University Press

Equation	Number of strips							
μ-value	**3**	**4**	**5**	**6**	**8**	**9**	**10**	**12**
(2) –	–	1.E–03	–	7.E–05	2.E–05	–	6.E–06	3.E–06
(3) **1**	1.E–01	–	–	4.E–04	–	3.E–05	–	7.E–06
(4) **–2 / 3**	–	3.E–01	–	–	1.E–03	–	–	6.E–05
(4) **2**	–	1.E–01	–	–	4.E–04	–	–	2.E–05
(4) **2 / 9**	–	6.E–02	–	–	2.E–04	–	–	1.E–05
(5) **1**	–	–	1.E–02	–	–	–	3.E–05	–
(5) **3 / 2**	–	–	4.E–02	–	–	–	1.E–04	–
(5) **3 / 22**	–	–	2.E–01	–	–	–	7.E–04	–

TABLE 3: Absolute % accuracies, i.e. $\left|(1 - \mathit{Approximate}/\mathit{Exact}) \times 100\right|$

In this Table, the left-hand column shows the equation number from the text (in brackets), and underneath is a typical value chosen for the parameter μ, where appropriate. Clearly, and perhaps not surprisingly, Simpson's rule (the top entry in the Table) is the most attractive, but the other options also converge albeit at a slower rate. It is of course possible to consider other test integrals and the reader might care, like the author, to do so and see if the story changes significantly. It is also of interest to write down in full the integral expressions from the Table for the various values of the parameter μ. For example, in the case of the fifth data entry in the Table pertaining to (4) when $\mu = 2/9$, the relevant form for integration over four equally-spaced intervals is

$$I = \frac{4h'}{15}\left(y_0 + 6y_1 + y_2 + 6y_3 + y_4\right)$$

which is not cumbersome to the eye. However, this is not always the case as the reader might appreciate by considering for example the integral expression for integration over five equally spaced intervals arising from the sixth data entry in the Table.

https://doi.org/10.1017/mag.2021.113 Published online by Cambridge University Press

Data pertaining to Weddle's rule (see (6) above) is not contained in the Table and the exploitation of the two parameters there (μ and ν) to obtain different looking integral expressions for integration over six equally spaced intervals is left as an exercise for the reader.

Conclusion

An exercise has been carried out to show how some of the standard numerical integration formulae that an early undergraduate might be exposed to can be obtained by judicious combination of quadratics interpolated through three points whose abscissae are unequally spaced. In addition to securing the well-known numerical integration formulae it is possible to obtain others, seemingly at will. Convergence rates for the latter can be seen to be somewhat slower than the former, at least in the case of a specified test integral. The author has considered other test integrals and a similar picture emerges. Moreover, attention here has focused on a symmetric weighted combination of outputs obtained from the interpolating quadratics because it leads to a simplicity in the sums. It is left as an exercise for the reader to determine the effect of other types of weightings.

Acknowledgement

The author is pleased to acknowledge input from a referee concerning up to date references.

References

1. Carl-Erik Froberg, *Introduction to numerical analysis*, Addison-Wesley Publishing Company (1968).
2. W. E. Milne, *Numerical solutions of differential equations*, Wiley, (1953).
3. Josef Stoer and Roland Bulirsch, *Introduction to numerical analysis*, Springer (1980).
4. Dan Zwillinger, *CRC Standard mathematical tables and formulas (advances in applied mathematics)*, Chapman and Hall (2018).
5. *Wolfram MathWorld*, Ahmed's integral, accessed April 2021 at http://mathworld.wolfram.com/AhmedsIntegral.html

JOHN D. MAHONY
5 Bluewater View, Mt. Pleasant, Christchurch 8081, New Zealand
e-mail: *johndmahony@gmail.com*

10.1017/mag.2021.113
© The Mathematical Association 2021

https://doi.org/10.1017/mag.2021.113 Published online by Cambridge University Press