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Singularities, incompleteness and the Lorentzian distance function

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Abstract. A space–time (M, g) is singular if it is inextendible and contains an inextendible nonspacelike geodesic which is incomplete. In this paper nonspacelike incompleteness is studied using the Lorentzian distance $d(p, q)$. A compact subset K of M causally disconnects two divergent sequences $\{p_n\}$ and $\{q_n\}$ if $0 < d(p_n, q_n) < \infty$ for all n and all nonspacelike curves from p_n to q_n meet K. It is shown that no space–time (M, g) can satisfy all of the following three conditions: (1) (M, g) is chronological, (2) each inextendible nonspacelike geodesic contains a pair of conjugate points and (3) there exist two divergent sequences $\{p_n\}$ and $\{q_n\}$ which are causally disconnected by a compact set K. This particular result extends a theorem of Hawking and Penrose. It also implies that if (M, g) satisfies conditions (1) and (3), then there is a C^0-neighbourhood of g in the space of metrics conformal to g such that any metric in this neighbourhood which satisfies the generic condition and the strong energy condition is nonspacelike incomplete.

1. *Introduction.* The purpose of this paper is to study new methods for showing a space–time is nonspacelike incomplete. Here we will say that a space–time is *nonspacelike incomplete* if it has at least one inextendible nonspacelike geodesic which is incomplete. An incomplete geodesic is one which cannot be extended to all positive and negative values of an affine parameter.

Hawking (8), Hawking and Penrose (9), Hawking and Ellis (10), Tipler (16, 17, 18), and others have studied space–times which are nonspacelike incomplete and inextendible. Such space–times are said to be singular. Nonspacelike incompleteness implies the space–time is either extendible or has a singularity. However, such a singularity is not necessarily a curvature singularity.

The distance function of a positive definite riemannian manifold has been very useful in the development of riemannian geometry. It is therefore to be expected that the distance function on a Lorentzian manifold has important mathematical applications. Also the Lorentzian distance function should have physical applications since it measures the supremum of possible times it takes to go from one event to an event in its causal future, compare (9, 10). Instead of the familiar triangle inequality of riemannian geometry, the Lorentzian distance function satisfies the reverse triangle inequality $d(p, q) \geqslant d(p, r) + d(r, q)$ whenever r is in the causal future of p and q is in the causal future of r. The distance between two points may be infinite. However, if the space–time (M, g) satisfies the finite distance condition $d(p, q) < \infty$ for all $p, q \in M$, then any bounded conformal change preserves this condition. The unifying theme of

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this paper is Lorentzian distance. We will use this distance to define cut points and more importantly to construct maximal geodesics in order to obtain nonspacelike incompleteness.

In this paper we will define a future-directed nonspacelike geodesic γ to be *maximal* if it has the property that $d(\gamma(t_1), \gamma(t_2))$ is equal to the length of γ from $\gamma(t_1)$ to $\gamma(t_2)$ whenever $t_1 < t_2$ are in the domain of γ. Thus a maximal geodesic maximizes the distance between any two of its points. Given a point p on a future directed nonspacelike geodesic, the geodesic is maximizing from p up to and including the cut point. Beyond the cut point the geodesic is no longer maximizing. Our definition of maximal geodesic is analogous to the riemannian definition of minimizing. Hawking and Ellis (10) use a different concept of maximality which they apply, however, just to timelike geodesics. In their approach a timelike geodesic from p is maximal up to the first conjugate point. Since the cut point comes before or at the first conjugate point, our definition is more restrictive for timelike geodesics. On the other hand, our approach applies to null geodesics as well. Our definitions of null cut points and maximal null geodesics are easily shown to be invariant under conformal changes. Investigation of null cut points thus leads to consideration of the space of metrics on M conformal to given metric g.

All nonspacelike geodesics in a nonspacelike complete space–time which satisfies the generic condition and the strong energy condition must have conjugate points ((9), p. 539). Since a maximal nonspacelike geodesic cannot contain conjugate points in its interior, such a geodesic in a space–time which satisfies the generic condition and the strong energy condition implies incompleteness. Consequently, we will be interested in constructing maximal geodesics.

A new ingredient in our study of incompleteness is the concept of a causally disconnecting set. A compact set K is said to *causally disconnect* two divergent sequences $\{p_n\}$ and $\{q_n\}$ if $0 < d(p_n, q_n) < \infty$ for all n and each nonspacelike curve from p_n to q_n meets K. The causally disconnecting set K plays a role similar to that of the future horismos of a trapped set (9, 10). However, space–times which do not contain trapped sets may contain causally disconnecting sets. In particular, any globally hyperbolic space–time with a regular spacelike boundary point contains a causally disconnecting set. Even if a space–time has no spacelike boundary points, it may still admit a causally disconnecting set. Minkowski space–time, Schwarzschild space–time, Einstein static space–time, de Sitter space–time, Reissner–Nordström space–times, the steady-state universe and the Friedmann cosmological models all contain causally disconnecting sets. In each of these space–times a causally disconnecting set with associated divergent sequences $\{p_n\}$ and $\{q_n\}$ may be found such that

$$0 < d(p_n, q_n) < \infty$$

for each n. A general construction utilizing such sequences then yields the existence of maximal nonspacelike geodesics in these space–times.

Generalizing a theorem of Hawking and Penrose ((9), p. 538), we prove that no space-time can satisfy all of the following three conditions: (1) (M, g) is chronological, (2) every inextendible nonspacelike geodesic contains a pair of conjugate points, and

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(3) there exist two divergent sequences $\{p_n\}$ and $\{q_n\}$ which are causally disconnected by a compact set. Our result implies the Hawking and Penrose theorem because a trapped set yields a future (or past) horismos which may be used as a causally disconnecting set. Our theorem implies that a chronological space–time satisfying the generic condition and strong energy condition is nonspacelike incomplete whenever there exist two divergent sequences $\{p_n\}$ and $\{q_n\}$ which are causally disconnected by a compact set. Furthermore, there is a C^0-neighbourhood of the metric g in the space of metrics on M conformal to g such that any metric in this neighbourhood satisfying the generic condition and the strong energy condition is nonspacelike incomplete.

Now suppose that the original metric satisfies the stronger curvature condition $\mathrm{Ric}\,(g)\,(v,v) > 0$ for all nonspacelike v in addition to the above causality and causally disconnecting assumptions. There is then a C^2-neighbourhood of g in the space of Lorentzian metrics for M conformal to g such that *all* metrics in this neighbourhood are nonspacelike incomplete. Thus nonspacelike incomplete metrics for a given space–time are not necessarily isolated in the space of metrics conformal to a given metric.

We have given a weaker version of the main result assuming that (M,g) satisfies the finite distance condition in (3), théorème 8.

2. *Limit curves and upper semi-continuity of length.* Let (M,g) be an arbitrary connected space–time of $\dim M \geqslant 3$. This means M is a smooth manifold with a countable basis, a smooth Lorentz metric g of signature $(-,+,\ldots,+)$ and a time orientation. Let $\pi: TM \to M$ denote the tangent bundle of M. Recall that a nonzero tangent vector $v \in TM$ is said to be *timelike* (resp. *nonspacelike, null, spacelike*) according to whether $g(v,v) < 0$ (resp. $\leqslant 0$, $= 0$, > 0). We will use the standard notations $p \ll q$ if there is a future-directed timelike curve from p to q and $p \leqslant q$ if there is a future-directed nonspacelike curve from p to q. The chronological future of p is defined by $I^+(p) = \{q | p \ll q\}$ and the causal future of p is defined by $J^+(p) = \{q | p \leqslant q\}$. The sets $I^-(p)$ and $J^-(p)$ are defined dually.

The space–time is said to be *chronological* if $p \notin I^+(p)$ for all p. It is *strongly causal* if each point has arbitrarily small neighbourhoods such that no nonspacelike curve which leaves one of these neighbourhoods ever returns, see (10), p. 192. A strongly causal space–time such that $J^+(p) \cap J^-(q)$ is compact for all $p, q \in M$ is called *globally hyperbolic*.

Let $\{\gamma_n\}$ be a sequence of nonspacelike curves. Two different types of convergence for such a sequence have been useful in General Relativity, cf. (10, 15). The first type of convergence involves the notion of a limit curve while the second uses the C^0-topology on curves. We will show that for strongly causal space–times these two types of convergence are closely related.

Hawking and Ellis (10) call a curve γ a *limit curve of the sequence* $\{\gamma_n\}$ if there is a subsequence $\{\gamma_m\}$ such that, for all $p \in \gamma$, each neighbourhood of p intersects all but a finite number of curves of the subsequence. This subsequence $\{\gamma_m\}$ will be said to *distinguish* the limit curve γ. Equivalently, the curve γ must lie in the set $\liminf\{\gamma_m\}$. This first form of convergence is defined even if the given curves do not have endpoints.

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Assume now that γ and all curves of the sequence $\{\gamma_n\}$ are defined on the same compact interval $[a, b]$. We will say that the sequence $\{\gamma_n\}$ *converges* to γ *in the* C^0*-topology* on curves if $\gamma_n(a) \to \gamma(a)$, $\gamma_n(b) \to \gamma(b)$ and if for each open set V containing γ there is an N such that $\gamma_n \subset V$ for all $n \geqslant N$. For an arbitrary space–time, convergence in the limit curve sense does *not* imply convergence in the C^0-topology.

In the next Lemma we will let S denote the open subset of M consisting of all points where M is strongly causal.

LEMMA 2·1. *Let* $\{\gamma_n\}$ *be a sequence of nonspacelike curves defined on* $[a, b]$. *Suppose that* $\gamma: [a, b] \to S$ *is a nonspacelike limit curve of the sequence. If* $\gamma_n(a) \to \gamma(a)$ *and* $\gamma_n(b) \to \gamma(b)$ *then there is a subsequence* $\{\gamma_m\}$ *of* $\{\gamma_n\}$ *which converges to* γ *in the* C^0*-topology.*

Proof. Assume without loss of generality that γ and $\{\gamma_n\}$ are all future-directed curves. Let V be an open set with $\gamma \subset V \subset S$. Cover the compact image of γ with convex normal neighbourhoods $W_1, \ldots, W_k$ such that each $W_i \subset V$ and no nonspacelike curve which leaves W_i returns to W_i. There exists a subdivision $a = t_0 < t_1 < \ldots < t_j = b$ of $[a, b]$ such that for all $0 \leqslant i \leqslant j-1$ each pair $\gamma(t_i)$, $\gamma(t_{i+1})$ lies in some W_h. Here $h = h(i)$ and $1 \leqslant h(i) \leqslant k$ for all i. Let $\{\gamma_m\}$ be a subsequence which distinguishes γ as a limit curve. For each m, let $p(0, m) = \gamma_m(a)$ and $p(j, m) = \gamma_m(b)$. Furthermore, for each fixed i with $0 < i < j$ choose $p(i, m) \in \gamma_m$ such that $p(i, m)$ converges to $\gamma(t_i)$. Since $\gamma(t_{i+1})$ lies in the causal future of $\gamma(t_i)$ and the set S is strongly causal, the point $p(i+1, m)$ lies in the causal future of $p(i, m)$ for all m larger than some N_1. Also there is some N_2 such that $p(i, m)$ and $p(i+1, m)$ lie in $W_{h(i)}$ for all $0 \leqslant i \leqslant j-1$ and $m \geqslant N_2$. Let $N = \max(N_1, N_2)$. The portion of γ_m joining $p(i, m)$ to $p(i+1, m)$ must lie entirely in $W_{h(i)}$ for $m \geqslant N$ because no nonspacelike curve can leave W_h and return. It follows that $\gamma_m \subset W_1 \cup \ldots \cup W_k \subset V$ for $m \geqslant N$. This establishes the result.

Let γ be a nonspacelike curve which lies in the strongly causal subset S and is defined on $[a, b]$. Choose C to be any compact set with $\gamma \subset \mathrm{Int}(C) \subset C \subset S$. Let the nonspacelike curves which lie in C be given the C^0-topology. It is known ((15), p. 54) that length is an upper semi-continuous function on this class of curves. Consequently, if $L(\gamma)$ denotes the length of γ we have the following result.

Remark 2·2. Let $\gamma: [a, b] \to S$ be a nonspacelike curve. If the sequence of nonspacelike curves $\{\gamma_n\}$ converges to γ in the C^0-topology, then $L(\gamma) \geqslant \limsup L(\gamma_n)$.

Using Lemma 2·1 and Remark 2·2 we obtain the next Lemma.

LEMMA 2·3. *Let* $\gamma: [a, b] \to S$ *be a given nonspacelike curve. Suppose* $\{\gamma_n\}$ *is a sequence of nonspacelike curves defined on* $[a, b]$ *such that* $\gamma_n(a) \to \gamma(a)$, $\gamma_n(b) \to \gamma(b)$ *and that* $\lim L(\gamma_n) = c$. *If* γ *is a limit curve of* $\{\gamma_n\}$, *then* $L(\gamma) \geqslant c$.

3. *Maximal geodesics and distance.* The Lorentzian distance function

$$d: M \times M \to R \cup \{\infty\}$$

is defined as follows. Given $p \in M$, set $d(p, q) = 0$ for $q \in M - J^+(p)$ and for $q \in J^+(p)$ let $d(p, q)$ be the supremum of lengths of all future-directed nonspacelike curves from p to q.

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This distance function satisfies the reverse triangle inequality

$$d(p,q) \geqslant d(p,r)+d(r,q)$$

whenever $p \leqslant r \leqslant q$. Also the distance function is lower semi-continuous where it is finite ((10), p. 215). In addition, if $p_n \to p$, $q_n \to q$ and $d(p,q) = \infty$, then

$$\lim d(p_n, q_n) = \infty.$$

For globally hyperbolic spaces, this function is continuous and finite valued ((10), p. 215). In more general space–times, however, the distance function may fail to be upper semi-continuous and thus fail to be continuous. In this paper, we will hence not assume that (M,g) has a continuous distance function. However, the lower semi-continuity of distance will be utilized.

If there is a closed future-directed timelike curve from p to itself, then we may traverse this curve an arbitrarily large number of times and find that $d(p,p) = \infty$. Therefore, if d is finite on the diagonal of $M \times M$ the space–time has no closed timelike curves and must be chronological. It is not hard to show that for strongly causal space–times the distance function is finite and continuous on a neighbourhood of the diagonal of $M \times M$.

Let γ be a future-directed nonspacelike curve in a causal space–time. Let $p = \gamma(s)$ and $q = \gamma(t)$ with $s < t$ and $p \neq q$. It is convenient to introduce the notational

CONVENTION 3·1. *Let $\gamma[p,q]$ denote the restriction of γ to the interval $[s,t]$.*

We will use a notion of maximal geodesic segment which is different from the Jacobi field-conjugate point approach used in Hawking and Ellis (10). Our definition is motivated by the concept of minimal (i.e. distance-realizing) geodesic segment in riemannian geometry.

Definition 3·2. A nonspacelike curve γ from p to q, with $p \leqslant q$ and $p \neq q$ is said to be *maximal* if $L(\gamma) = d(p,q)$.

The reverse triangle inequality implies that, if x and y are points along the maximal curve γ with $p \leqslant x \leqslant y \leqslant q$, then $d(x,y) = L(\gamma[x,y])$. A curve defined on an open interval of the real axis is called *maximal* if it is maximal between any pair of points.

It is convenient to state the space–time analogue of the well-known riemannian result that the radial geodesics emanating from the centre p of a normal neighbourhood are the unique curves that realize the distance from p. This is almost established in Penrose's result on arc length ((15), p. 53) but is not stated in precisely the form we need.

PROPOSITION 3·3. *Let U be a convex normal neighbourhood centred at $p \in M$. For $q \in J^+(p)$, let $\overline{pq}$ denote the unique (up to parametrization) nonspacelike geodesic segment from p to q in U. If γ is a future-directed nonspacelike curve in U from p to q and*

$$L(\gamma) = d(p,q),$$

then γ coincides with $\overline{pq}$ up to parametrization.

We obtain the following corollary.

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COROLLARY 3·4. *Let γ be a future-directed nonspacelike curve joining the distinct points p and q. If γ is maximal, then γ is a geodesic segment.*

To illustrate the usefulness of our Definition 3·2 of 'maximal' and of Corollary 3·4, we now give a simple proof of a basic result in elementary causality theory.

LEMMA 3·5. *If $p \leqslant q$, but $p \ll q$ is false, there is a null geodesic from p to q.*

Proof. The causality assumptions on p and q imply that $d(p, q) = 0$. Now let c be a future-directed nonspacelike curve from p to q. By definition of distance, $d(p, q) \geqslant L(c)$, whence $0 = L(c) = d(p, q)$. Corollary 3·4 now implies that c is a null geodesic.

In strongly causal space–times the nonspacelike geodesics can be characterized as the nonspacelike curves which are locally maximal. In other words, a nonspacelike curve γ in a strongly causal space–time is a geodesic if and only if each $p \in \gamma$ has a neighbourhood such that any (connected) portion of γ contained in this neighbourhood is maximal.

Globally hyperbolic subsets have been used in the proofs of various singularity theorems (cf. (**8**, **9**, **10**)), in order to construct nonspacelike geodesics which are maximal according to our definition. Instead of globally hyperbolic subsets we use causally disconnecting sets and a finite-distance assumption in obtaining our results. We will say that the compact set K *causally disconnects* two points p, q with $p \leqslant q$ if each nonspacelike curve from p to q intersects K.

We will adopt the following convention.

CONVENTION 3·6. *A nonspacelike curve γ is said to be* inextendible *if it is both past and future inextendible.*

In Lemma 3·8 we will use sequences of curves which are almost maximal to construct maximal geodesics. The following elementary property of almost maximal nonspacelike curves will be used.

LEMMA 3·7. *Let $\gamma: [0, 1] \to M$ be a nonspacelike curve from p to q, with $L(\gamma) < \infty$, and $L(\gamma) \geqslant d(p, q) - \epsilon$. For any two distinct points x, y along γ with $p \leqslant x \leqslant y \leqslant q$, the inequality $L(\gamma[x, y]) \geqslant d(x, y) - \epsilon$ holds.*

Proof. Assume that $L(\gamma[x, y]) < d(x, y) - \epsilon$. Then

$$\begin{aligned} L(\gamma) &= L(\gamma[p, x]) + L(\gamma[x, y]) + L(\gamma[y, q]) \\ &\leqslant d(p, x) + L(\gamma[x, y]) + d(y, q) \\ &< d(p, x) + d(x, y) - \epsilon + d(y, q) \\ &\leqslant d(p, q) - \epsilon. \end{aligned}$$

Now we are ready to prove Lemma 3·8.

LEMMA 3·8. *Let (M, g) be a strongly causal space–time. Assume p and q are causally disconnected by the compact set K and $0 < d(p, q) < \infty$. Then at least one of the following holds:*

(1) *There exists a maximal timelike geodesic from p to q.*

(2) *There exists a future-directed maximal nonspacelike geodesic which starts at p, intersects K and is future inextendible.*

(3) *There exists a future-directed maximal nonspacelike geodesic which ends at q, intersects K and is past inextendible.*

(4) *There exists a maximal nonspacelike geodesic which intersects K and is inextendible.*

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Proof. Let γ_n be a sequence of future-directed timelike curves from p to q such that $d(p,q)-1/n \leqslant L(\gamma_n) \leqslant d(p,q)$. Extend each γ_n to an inextendible nonspacelike curve. By the definition of causally disconnecting set, each γ_n must intersect K. Let

$$r_n \in \gamma_n \cap K.$$

Since K is compact, there must be a subsequence of $\{r_n\}$ which converges to some $r \in K$. Using lemma 6·2·1 of Hawking and Ellis (**10**), p. 185, we see that there must be an inextendible nonspacelike curve γ through r which is a limit curve. There are four cases to consider: (1) γ contains both p and q, (2) γ contains p but not q, (3) γ contains q but not p, and (4) γ contains neither p nor q. The proof that γ is maximal on the set indicated in the Lemma is essentially the same in all four cases so we only consider case (2). In view of Corollary 3·4 we may show that γ is a maximal geodesic by establishing that $d(x,y) = L(\gamma[x,y])$ for any two distinct points x, y lying along γ with $p \leqslant x \leqslant y$. We must have $x, y \in \bar{J}^-(q)$ because $q \in \gamma_n$ for all n and $q \notin \gamma$.

Let $\{\gamma_m\}$ be a subsequence of $\{\gamma_n\}$ which distinguishes γ. Choose $x_m, y_m \in \gamma_m$ with $p \leqslant x_m \leqslant y_m$, $x_m \to x$ and $y_m \to y$. Then $d(p,q) \geqslant \liminf d(x_m, y_m) \geqslant d(x,y)$, this last inequality by lower semi-continuity of distance. Hence $d(x,y)$ is finite. By Lemma 2·1 we may assume the subsequence $\gamma_m[x_m, y_m]$ converges to $\gamma[x,y]$ in the C^0-topology by passing to a subsequence of $\{\gamma_m\}$ if necessary. We now apply upper semi-continuity of length to obtain

$$L(\gamma[x,y]) \geqslant \limsup L(\gamma_m[x_m, y_m]).$$

On the other hand,

$$d(x_m, y_m) - 1/m \leqslant L(\gamma_m[x_m, y_m]) \leqslant d(x_m, y_m)$$

by Lemma 3·7. Thus we obtain

$$L(\gamma[x,y]) \geqslant \limsup d(x_m, y_m) \geqslant d(x,y).$$

But $d(x,y) \geqslant L(\gamma[x,y])$ by definition of the distance function. Hence

$$d(x,y) = L(\gamma[x,y])$$

as required.

It is a basic result of the calculus of variations that a geodesic in a riemannian space fails to be distance realizing beyond a conjugate point. The space–time analogue of this result has been given in (**10**). We now restate this result in terms of our definition of a maximal geodesic segment (Definition 3·2) for use in the sequel.

LEMMA 3·9. *If γ is a maximal nonspacelike geodesic from p to q, then γ does not contain a pair of conjugate points between p and q.*

This Lemma implies that a maximal nonspacelike geodesic defined on an open interval is free of conjugate points.

4. *Conformal changes.* Two Lorentz metrics g and g' on M are said to be *conformal* if there is a smooth function $\Omega: M \to \mathbb{R}$ with $\Omega > 0$ such that $g' = \Omega^2 g$. Let $C(M,g)$ denote the set of all Lorentz metrics for M conformal to g. If $g' \in C(M,g)$, then a curve γ is timelike (resp. null, spacelike) for (M,g') if and only if it is timelike (resp. null, spacelike) for (M,g). Thus conformal metrics induce the same causal structure on M.

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Conversely, two Lorentz metrics which induce the same causal structure on M are known to be conformal ((10), p. 180).

Relevant to this paper are the C^r-topologies on $C(M,g)$ for $r = 0$ and $r = 2$. Let $\{B_i\}$ be a fixed countable collection of coordinate neighbourhoods covering M such that each B_i has compact closure and such that any compact subset of M meets only finitely many of these neighbourhoods. Let $\delta: M \to (0,\infty)$ be a continuous function. Then $|g_2 - g_1|_r < \delta$ if, for each $p \in M$, all of the corresponding coefficients and derivatives up to order r of the two metric tensors are $\delta(p)$-close at p when calculated in the fixed coordinates of each of the corresponding coordinate charts B_i containing p. The sets $\{g_2 \in C(M,g) \mid |g_2 - g_1|_r < \delta\}$ with $g_1 \in C(M,g)$ and $\delta: M \to (0,\infty)$ arbitrary form a basis for the C^r topology on $C(M,g)$.

For each $g_1 \in C(M,g)$ let $B(g_1)$ consist of all $g_2 \in C(M,g)$ such that $g_2 = \Omega^2 g_1$ for some smooth bounded function $\Omega: M \to \mathbb{R}$. Notice that $g_2 \in B(g_1)$ does not imply $g_1 \in B(g_2)$. If M is noncompact, the set $B(g_1)$ is a proper subset of $C(M,g)$.

LEMMA 4·1. *For each nonnegative integer r, there is a C^r-open neighbourhood $U(g_1)$ of g_1 in $C(M,g)$ such that $U(g_1) \subset B(g_1)$.*

Proof. Let $\delta: M \to (0,\infty)$ be a continuous function such that for each chart B_i of the fixed covering of M we have $\delta(p)$ less than the maximum of the absolute values of the components of $g_1(p)$ calculated in the coordinates of B_i for all $p \in B_i$. If α is the maximum of the absolute values of the components of $g_1(p)$ with respect to some B_i, then

$$\alpha \geqslant \delta(p) > |g_2 - g_1|_0 = |\Omega^2 g_1 - g_1|_0 = |\Omega^2 - 1|\,|g_1|_0 \geqslant |\Omega^2 - 1|\,\alpha$$

whenever $|g_2 - g_1|_r < \delta$. It follows that $|\Omega^2 g_1 - g_1|_r < \delta$ implies $0 < \Omega^2 < 2$. This establishes the Lemma.

Fix $g_2 \in B(g_1)$. Then $g_2 = \Omega^2 g_1$ for some $\Omega: M \to \mathbb{R}$ and by definition of $B(g_1)$ there is a constant $\beta > 0$ such that $0 < \Omega < \beta$. Let d_i denote the distance function induced by g_i on M for $i = 1, 2$.

LEMMA 4·2. *If $g_2 = \Omega^2 g_1$ with $0 < \Omega < \beta$, then $d_2(p,q) \leqslant \beta d_1(p,q)$ for all $p, q \in M$.*

Proof. Given a future-directed timelike curve γ from p to q, let $L_i(\gamma)$ denote the length of γ with respect to g_i for $i = 1, 2$. Then

$$L_2(\gamma) = \int \sqrt{[-\Omega^2 g_1(\dot\gamma, \dot\gamma)]}\,dt < \beta \int \sqrt{[-g_1(\dot\gamma, \dot\gamma)]}\,dt = \beta L_1(\gamma),$$

using the law of the mean. The result now follows from the definition of distance.

An immediate consequence of Lemmas 4·1 and 4·2 is the following

COROLLARY 4·3. *Let (M, g_1) satisfy the condition $d_1(p,q) < \infty$ for all $p, q \in M$. Then there is a C^0-neighbourhood $U(g_1)$ of g_1 in $C(M,g)$ such that for any $g_2 \in U(g_1)$ and all $p, q \in M$, the condition $d_2(p,q) < \infty$ holds.*

Let Ric (g) denote the Ricci curvature tensor of g.

LEMMA 4·4. *Let (M,g) satisfy* Ric $(g)(v,v) > 0$ *for all nonspacelike $v \in TM$. There is a C^2-neighbourhood $U(g)$ of g in $C(M,g)$ such that* Ric $(g_1)(v,v) > 0$ *for all $g_1 \in U(g)$ and all nonspacelike $v \in TM$.*

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Proof. Fix an auxiliary riemannian metric h for M. Then

$$S_i = \{v \in TM \mid \pi(v) \in \bar{B}_i, \quad h(v,v) = 1 \quad \text{and} \quad g(v,v) \leqslant 0\}$$

is a compact subset of TM for each of the fixed coordinate charts B_i. To check that $\mathrm{Ric}\,(g_1) > 0$ for all nonspacelike tangent vectors it suffices to check that

$$\mathrm{Ric}\,(g_1)(v,v) > 0 \quad \text{for any} \quad v \in \cup_i S_i.$$

Utilizing the compactness of the sets S_i, it is clear that the lemma is implied by the pointwise C^2-dependence of $\mathrm{Ric}\,(g)$ on g and the definition of the C^2-topology for $C(M,g)$.

5. *The null cut locus.* Let $\gamma: [0,a) \to M$ be a future-directed null geodesic with $p = \gamma(0)$. Set $t_0 = \sup\{t \in [0,a) | d(p,\gamma(t)) = 0\}$. The cylinder $S^1 \times \mathbb{R}$ with the metric $ds^2 = d\theta\, dt$ shows that $t_0 = 0$ and $t_0 = \infty$ are possible. If $0 < t_0 < a$, then we will say that $\gamma(t_0)$ is the *future null cut point* of p on γ. Past null cut points are defined dually.

The geometrical significance of t_0 is related to the maximality or lack of maximality of $\gamma|[0,t)$ for $t < a$. The geodesic γ is maximizing from p up to and including the null cut point $\gamma(t_0)$, i.e. $L(\gamma|[0,t]) = d(p,\gamma(t))$ for all $t \leqslant t_0$. Thus there is no timelike curve joining p to $\gamma(t)$ for any t with $t \leqslant t_0$. In contrast, the geodesic γ is no longer maximizing beyond the cut point $\gamma(t_0)$. In fact, each point $\gamma(t)$ for $t_0 < t < a$ may be joined to p by a timelike curve.

The null cut locus of p, denoted by $C_\Lambda(p)$, consists of all future and past null cut points of p. The null cut locus may be empty. For instance, the null cut locus of each point in Minkowski space–time is empty because any pair of points which lie on a common null geodesic are zero distance apart.

Let R and Ric denote the curvature tensor and Ricci curvature tensor of g respectively. An inextendible nonspacelike geodesic γ with tangent vector W satisfies the *generic condition* ((10), p. 101) if there is some point of γ at which

$$W^c W^d W_{[a} R_{b]cd[e} W_{f]} \neq 0.$$

The space–time (M,g) satisfies the *generic condition* if each of its inextendible nonspacelike geodesics satisfies this condition. In particular, if $\mathrm{Ric}\,(v,v) > 0$ for all nonspacelike vectors v, then (M,g) satisfies the generic condition.

A geodesic γ with affine parameter t is said to be *complete* if it can be extended to arbitrarily large positive and negative values of t (cf. Wolf (19)). A space–time is called *nonspacelike* (resp. *timelike*, *null*, *spacelike*) *complete* if all inextendible nonspacelike (resp. timelike, null, spacelike) geodesics are complete. Examples show that timelike, null and spacelike completeness are all logically inequivalent, compare (1, 5, 11). In the positive definite case, Nomizu and Ozeki (13) have shown that any riemannian manifold may be made geodesically complete by a conformal change. The corresponding result is not true for all space–times. However, it has been shown (2) that for a large class of space–times, including the strongly causal space–times, there is a conformal factor which makes the space-time nonspacelike complete. However, the construction of this conformal factor given in (2) will in general cause the Ricci curvature to assume negative values on some null vectors.

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A space-time (M, g) is said to be *globally conformally diffeomorphic* to an open subset U of the space–time (M', g') if there is a diffeomorphism $f: M \to U$ of M onto U such that $f^*(g'|U) = \Omega^2 g$ for some $\Omega: M \to (0, \infty)$.

PROPOSITION 5·1. *Let (M, g) be a space–time such that all inextendible null geodesics satisfy the generic condition and such that* $\mathrm{Ric}(v, v) \geqslant 0$ *for all null vectors. If (M, g) is globally conformally diffeomorphic to an open subset of a space–time (M', g') which has no null cut points, then* all *null geodesics of (M, g) are incomplete.*

Proof. Proposition 4·4·5 of Hawking and Ellis (10), p. 101, shows that, if the Ricci curvature is nonnegative on all null vectors, then each complete null geodesic which satisfies the generic condition has conjugate points. Since maximal geodesics do not contain conjugate points, we need only show that all null geodesics of (M, g) are maximal to establish their incompleteness. Assume that γ is a future-directed null geodesic from p to q which is not maximal. Then there is a timelike curve from p to q. Since conformal diffeomorphisms take null geodesics to null geodesics and timelike curves to timelike curves, the image of γ must be a nonmaximal null geodesic in (M', g'). This implies that (M', g') has a null cut point and hence yields a contradiction.

Consider the Einstein static universe which is the cylinder $x^2 + y^2 + z^2 + w^2 = 1$ embedded in $\mathbb{R}^5$ with the metric induced from the Minkowski metric

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 + dw^2.$$

The Einstein static universe is thus $S^3 \times \mathbb{R}$ with a (Lorentzian) product metric. The geodesics and null cut points are easy to determine in this space–time. The null cut locus of the point (t, x, y, z, w) merely consists of the two points $(t \pm \pi, -x, -y, -z, -w)$. Consequently, the subset $M' = \{(t, x, y, z, w) | 0 < t < \pi, x^2 + y^2 + z^2 + w^2 = 1\}$ has the property that $C_N(p) \cap M' = \varnothing$ for all $p \in M'$. Proposition 5·1 then has the following

COROLLARY 5·2. *Let (M, g) be a space–time such that all null geodesics satisfy the generic condition and such that* $\mathrm{Ric}(v, v) \geqslant 0$ *for all null vectors. If (M, g) is globally conformally diffeomorphic to some portion of the subset M' of the Einstein static universe, then* all *null geodesics of (M, g) are incomplete.*

Since Minkowski space–time may be conformally embedded in M', the above result remains valid if we replace M' with Minkowski space–time.

Friedmann space–times are used in cosmology as models of the universe. In these spaces it is assumed that the universe is filled with a perfect fluid having zero pressure and positive energy density. We assume the cosmological constant Λ is zero for these models. These space–times are then Robertson–Walker spaces (10) with $p = \Lambda = 0$. It is possible to choose coordinates for these space-times such that the metric has the form $ds^2 = -dt^2 + S^2(t)\, d\sigma^2$, where $d\sigma^2$ is the metric of a space of constant curvature. These space–times may be conformally embedded in the subset M' of the Einstein static universe defined above, compare (10, 14). Furthermore, $\mathrm{Ric}(g)(v, v) > 0$ for all nonspacelike vectors v in a Friedmann model (M, g). Using Lemma 4·4 we see that there is a C^2-neighbourhood $U(g)$ of g in $C(M, g)$ such that $\mathrm{Ric}(g')(v, v) > 0$ for all $g' \in U(g)$. Since $\mathrm{Ric}(g')(v, v) > 0$ implies the generic condition (9), p. 540, Corollary 5·2 yields

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COROLLARY 5·3. *Let* (M,g) *be a Friedmann cosmological model. There is a* C^2*-neighbourhood* $U(g)$ *of* g *in* $C(M,g)$ *such that* every *null geodesic in* (M,g') *is incomplete for* all $g' \in U(g)$.

We now define timelike cut points. Let $\gamma: [0,a) \to M$ be a future-directed timelike geodesic with $p = \gamma(0)$. Set $t_0 = \sup\{t \in [0,a) | d(p,\gamma(t)) = L(\gamma|[0,t])\}$. If (M,g) is strongly causal, then $0 < t_0 \leqslant a$. Minkowski space–time shows $t_0 = a = \infty$ is possible. If $0 < t_0 < a$, then $\gamma(t_0)$ is said to be the future timelike cut point of p on γ.

By definition, the geodesic γ is maximizing from p to $\gamma(t)$ when $t < t_0$. The lower semi-continuity of distance implies γ is also maximizing from p up to the cut point $\gamma(t_0)$. Unlike the null cut locus, however, the timelike cut locus is *not* invariant under conformal changes. In a subsequent paper we will make a more detailed investigation of the timelike and null cut loci in strongly causal space–times.

6. *Causally disconnecting sets and incompleteness.* Let $\{p_n\}$ be a sequence of points in (M,g). Borrowing from the traditional phraseology of riemannian geometry, we will say that $\{p_n\}$ is a *divergent sequence of points* if for every compact subset C of M there is some integer $N = N(C)$ such that $p_n \notin C$ for all $n \geqslant N$.

We extend our previous notion of a compact disconnecting set for two points to pairs of divergent sequences.

Definition 6·1. Let two divergent sequences $\{p_n\}$ and $\{q_n\}$ be given such that

$$0 < d(p_n, q_n) < \infty$$

for each n. The compact set K *causally disconnects* the two sequences if for each n all nonspacelike curves from p_n to q_n intersect K. We will say that a space–time (M,g) admitting such a compact K causally disconnecting two sequences is *causally disconnected*.

We will say that the causally disconnecting set K has *associated* sequences $\{p_n\}$ and $\{q_n\}$. Notice that, while q_n must lie in the chronological future of p_n, the point q_n is not necessarily causally related to any p_k or q_k for $k \neq n$. Also the compact set K may be quite different from a Cauchy surface. Instead of requiring that *all* future-directed inextendible nonspacelike curves intersect K, we only require that nonspacelike curves passing through both p_n and q_n must meet K.

We make two observations on the stability of the causal disconnection condition. First, Lemmas 4·1 and 4·2 imply that, if (M,g) is causally disconnected, then there is a C^0-neighbourhood $U(g)$ of g in $C(M,g)$ such that, for all metrics $g' \in U(g)$, the space–time (M,g') is causally disconnected. Thus causal disconnection is a C^0-open property. Secondly, if (M,g) is globally hyperbolic and causally disconnected, then (M,g') is causally disconnected for *all* metrics $g' \in C(M,g)$. This assertion holds since all metrics $g' \in C(M,g)$ are globally hyperbolic and hence all satisfy the finite distance condition $d(g')(p,q) < \infty$ for all $p,q \in M$.

We will need the following lemma which has been given in (9), p. 536, and (12), p. 41.

LEMMA 6·2. *If* (M,g) *is a chronological space–time such that every inextendible null geodesic contains a pair of conjugate points, then* (M,g) *is strongly causal.*

We now prove our main result.

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THEOREM 6·3. *No space–time (M, g) can satisfy all of the following three conditions:*

(1) *(M, g) is chronological.*

(2) *Every inextendible nonspacelike geodesic contains a pair of conjugate points.*

(3) *There exist two divergent sequences $\{p_n\}$ and $\{q_n\}$ which are causally disconnected by a compact set K.*

Proof. Conditions (1) and (2) imply by Lemma 6·2 that we may assume (M, g) is strongly causal. Lemma 3·9 implies that we need only show the existence of an inextendible maximal nonspacelike geodesic in order to find a contradiction to the assumed existence of (M, g).

Apply Lemma 3·8 to each pair p_n, q_n. If case (4) ever occurs, then we have an inextendible nonspacelike geodesic which is maximal. Therefore, assume at least one of cases (1), (2) or (3) occurs for an infinite number of n. Since all three cases are essentially the same we will only consider the situation where case (1) holds for infinitely many n. Let $\{m\}$ be a subsequence of $\{n\}$ such that for each m there is an inextendible timelike geodesic γ_m which contains p_m and q_m and is maximal between p_m and q_m. Each γ_m intersects K in at least one point r_m. Let v_m be a future-pointing tangent vector to γ_m at r_m. Assume without loss of generality that each v_m has unit length with respect to an auxiliary positive definite riemannian metric on M. Since K is compact, the sphere bundle $S(K)$ over K with respect to the auxiliary riemannian metric is compact. Thus we may find a subsequence $\{k\}$ of $\{m\}$ such that $\{v_k\}$ converges to some $v \in S(K)$. Let $\gamma(v)$ denote the inextendible nonspacelike geodesic with tangent vector v at $\pi(v) \in K$. The $\{\gamma_k\}$ have $\gamma(v)$ as a limit curve by construction. We must show $\gamma(v)$ is maximal. Let $x, y \in \gamma(v)$ be distinct points with $x \leqslant y$. Since the geodesic segment $\gamma(v)[x, y]$ from x to y is compact, there is some compact set C with $\gamma(v)[x, y] \subset \operatorname{Int}(C)$. Choose points $x_k, y_k \in \gamma_k$ such that $x_k \to x$ and $y_k \to y$. Then $\{\gamma_k[x_k, y_k]\}$ converges to $\gamma(v)[x, y]$ in the C^0-topology on curves. Since the sequences $\{p_k\}$ and $\{q_k\}$ diverge, there is some N such that $p_k \notin C$, $q_k \notin C$ and $p_k \leqslant x_k \leqslant y_k \leqslant q_k$ for all $k \geqslant N$. Now

$$d(x_k, y_k) = L(\gamma_k[x_k, y_k])$$

since γ_k is maximal from p_k to q_k. Thus using the upper semi-continuity of length and the definition of Lorentzian distance we find that

$$d(x, y) \geqslant L(\gamma(v)[x, y]) \geqslant \limsup L(\gamma_k[x_k, y_k]) = \limsup d(x_k, y_k).$$

On the other hand, the lower semi-continuity of distance implies that

$$d(x, y) \leqslant \liminf d(x_k, y_k).$$

Thus $d(x, y) = L(\gamma(v)[x, y])$. This shows $\gamma(v)$ is maximal and hence (M, g) is nonspacelike incomplete.

Notice that the actual construction of the maximal nonspacelike geodesic in the proof of Theorem 6·3 was only dependent on conditions (1) and (2) to the extent that these conditions imply (M, g) is strongly causal. Consequently, we obtain the following proposition.

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PROPOSITION 6·4. *Let (M,g) be a strongly causal space–time. Assume $\{p_n\}$ and $\{q_n\}$ are two divergent sequences of points which are causally disconnected by a compact set K. Then (M,g) contains a maximal inextendible nonspacelike geodesic which intersects K.*

We will now indicate how our Theorem 6·3 generalizes a result of Hawking and Penrose (9), p. 538. We must first recall some definitions. The *future horismos* of a set S is defined to be $E^+(S) = J^+(S) - I^+(S)$ and the past horismos is defined dually. The set S is said to be *achronal* if for each pair of distinct points neither is in the chronological future of the other. An achronal set S is called *future* (resp. *past*) *trapped* if $E^+(S)$ (resp. $E^-(S)$) is compact.

Hawking and Penrose have proved that no space–time can satisfy conditions (1) and (2) of our Theorem 6·3 as well as the condition (3′) that there is a future (or past) trapped set $S \subset M$. In their paper, they give arguments which, with very slight modification, show that if S is a future (or past) trapped set, then there is an inextendible timelike curve γ which lies in the Cauchy development of the set $E^+(S)$ (resp. $E^-(S)$), compare (9), p. 545. Thus one may choose sequences $\{p_n\}$ and $\{q_n\}$ on γ which diverge and are causally disconnected by the compact set $E^+(S)$ (resp. $E^-(S)$). Since $J^+(p_n) \cap J^-(q_n)$ is always compact in this case, we have $d(p_n, q_n) < \infty$ for each n.

In other words, our result implies the Hawking and Penrose Theorem because a trapped set yields a horismos which is a causally disconnecting set.

7. *The non-isolation of nonspacelike incomplete spacetimes in $C(M,g)$.* Recall from Section 4 that $C(M,g)$ denotes the set of all Lorentzian metrics for M conformal to a given metric g for M. In this section we use Theorem 6·3 and the results of Section 4 on the C^r topologies for $C(M,g)$ to study the nonisolation of nonspacelike incomplete metrics in $C(M,g)$. Our results also yield alternative interpretations of Theorem 6·3.

The following energy condition has been crucial to the Hawking–Penrose singularity theory. It is discussed in Hawking and Ellis (10), p. 95, where it is called the timelike convergence condition for (M,g).

STRONG ENERGY CONDITION. $\mathrm{Ric}(g)(v,v) \geqslant 0$ *for all nonspacelike vectors* $v \in TM$.

Recall that (M,g) satisfies the generic condition if each inextendible nonspacelike geodesic γ with tangent vector W has at least one point where $W^c W^d W_{[a} R_{b]cd[e} W_{f]} \neq 0$.

We first consider C^0-neighbourhoods in $C(M,g)$ of a chronological causally disconnected spacetime (M,g) which satisfies the generic and the strong energy conditions.

THEOREM 7·1. *Let (M,g) be a chronological, causally disconnected spacetime which satisfies the generic and strong energy conditions. Then (M,g) is nonspacelike incomplete and moreover there is a C^0-neighbourhood $U(g)$ of g in $C(M,g)$ such that every $g_1 \in U(g)$ which satisfies the generic and strong energy conditions is nonspacelike incomplete.*

The proof of Theorem 7·1 is immediate from the following result and will thus be omitted.

THEOREM 7·2. *Let (M,g) be as in Theorem 7·1. Suppose $g_1 = \Omega^2 g$ for a bounded positive function $\Omega: M \to \mathbb{R}$ and g_1 satisfies the generic and strong energy conditions. Then (M,g_1) is nonspacelike incomplete.*

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Proof. Theorem 6·3 implies that (M, g) itself is nonspacelike incomplete because every complete nonspacelike geodesic in a space-time which satisfies the generic and strong energy conditions has conjugate points. The fact that (M, g_1) is incomplete then follows from Lemma 4·2.

Making a stronger curvature assumption on (M, g), we obtain a C^2 non-isolation theorem.

THEOREM 7·3. *Let (M, g) be a chronological, causally disconnected space–time which satisfies the curvature condition* $\mathrm{Ric}(g)(v, v) > 0$ *for all nonspacelike $v \in TM$. Then there is a C^2-neighbourhood $U(g)$ of g in $C(M, g)$ such that* all *metrics in $U(g)$ (including g) are nonspacelike incomplete.*

Proof. The curvature condition $\mathrm{Ric}(g')(v, v) > 0$ for all nonspacelike v implies that the generic and strong energy conditions hold for (M, g'). The theorem is thus implied by Lemmas 4·1, 4·4 and Theorem 7·1.

Using theorem 6·1 of (4) and Theorem 7·3 we obtain the following nonisolation result.

THEOREM 7·4. *Let (M, g) be a globally hyperbolic, causally disconnected space–time satisfying the strong energy condition. Suppose that (M, g) admits a smooth globally hyperbolic time function $h: M \to \mathbb{R}$ such that some Cauchy hypersurface $h^{-1}(t)$ satisfies the curvature condition* $\mathrm{Ric}(g)(v, v) > 0$ *for all nonspacelike tangent vectors v with*

$$\pi(v) \in h^{-1}(t).$$

Then there is an open set of nonspacelike incomplete metrics for M in $C(M, g)$.

Proof. Choose a C^0-neighbourhood $U(g)$ of g in $C(M, g)$ so that all metrics $g' \in U(g)$ are causally disconnected. Now by theorem 6·1 of (4) we may find a metric $g_1 \in U(g)$ with $\mathrm{Ric}(g_1)(v, v) > 0$ for all nonspacelike tangent vectors v. Apply Theorem 7·3 to g_1 to get a C^2-neighbourhood of g_1 in $C(M, g_1) = C(M, g)$ of nonspacelike incomplete metrics.

Remark 7·5. The requirement that (M, g) be causally disconnected in Theorem 7·4 is automatically satisfied if the Cauchy hypersurface $h^{-1}(t)$ is compact.

Theorem 1 of Gromoll and Meyer ((7), p. 80) implies that a noncompact manifold having more than one end does not admit a complete riemannian metric with everywhere positive Ricci curvature. Reinterpreting our Theorem 7·1 in this spirit yields the following Lorentzian analogue to theorem 1 of Gromoll and Meyer. In the Lorentzian case, the concept of causal disconnexion plays the role of the topological concept of end in the riemannian case.

COROLLARY 7·6. *Let (M, g) be a chronological, causally disconnected space–time. Then there is a C^0-neighbourhood U of g in $C(M, g)$ such that no nonspacelike complete metric in U can have everywhere positive nonspacelike Ricci curvature.*

Proof. Choose a C^0-neighbourhood U of g such that all metrics in U are causally disconnected and apply Theorem 7·3 to U.

If (M, g) is globally hyperbolic, then (M, g) satisfies the finite distance condition

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$d(p,q) < \infty$ for all $p,q \in M$ (cf. (10), p. 215). Thus if (M,g) is globally hyperbolic, all metrics in $C(M,g)$ satisfy the finite distance condition. Consequently, if (M,g) is a globally hyperbolic, causally disconnected space–time, then (M,g') is causally disconnected for *all* $g' \in C(M,g)$. We thus have a complete analogue of theorem 1 of Gromoll and Meyer for globally hyperbolic, causally disconnected space–times.

COROLLARY 7·7. *Let (M,g) be a globally hyperbolic, causally disconnected space–time. Then no nonspacelike complete metric in $C(M,g)$ can have everywhere positive nonspacelike Ricci curvature.*

8. *Boundaries.* Geroch, Kronheimer and Penrose (6) have defined an ideal boundary for space–times using past and future indecomposable sets. Let this completion M^* be written $M \cup \Delta$, where Δ is the causal boundary (compare (10), p. 220). This completion has a Hausdorff topology and the original topology on M agrees with the topology induced on M as a subset of M^*.

Consider an ideal point $\bar{p}$ at which the boundary is differentiable. Let $U^*(\bar{p})$ be a neighbourhood of $\bar{p}$ in M^*. Denote by (U,g) the metric g restricted to the set

$$U = U^*(\bar{p}) \cap M.$$

A *conformal representation* of $U^*(\bar{p})$ will be a space–time (M',g') and a homeomorphic embedding $f\colon U^* \to M'$ such that:

(1) f is a smooth map which embeds U^* as a manifold with smooth boundary ∂U in M'.

(2) There is a smooth function $\Omega\colon U \to \mathbb{R}$ such that $\Omega > 0$ and $\Omega^2 g = f^*(g')$.

Definition 8·1. An ideal point $\bar{p}$ will be called a *regular spacelike* (resp. *null*, *timelike*) boundary point if there is a conformal representation of some open neighbourhood $U^*(\bar{p})$ such that the corresponding boundary ∂U is spacelike (resp. null, timelike).

A nonspacelike curve of M which tends in M^* to a boundary point $\bar{p}$ is said to have $\bar{p}$ as an *endpoint*. If $\bar{p}$ is a regular spacelike boundary point, then it is not hard to show there must exist a compact subset K of M such that all inextendible nonspacelike curves with one endpoint at $\bar{p}$ must intersect K. In fact, K may be chosen as an achronal set which is the closure of a spacelike hypersurface. Also K may be chosen to be a subset of any given neighbourhood $U^*(\bar{p})$ of $\bar{p}$. A regular spacelike boundary point $\bar{p}$ and one choice of K are illustrated in Fig. 1.

Let us call the condition that $d(p,q) < \infty$ for all $p, q \in M$ the *finite distance condition*. This condition has been defined in Beem and Ehrlich (3). Notice that we are not assuming that the distances are bounded, only that, for each fixed pair of points, they are finite. Also the upper semi-continuity of distance is not assumed in this definition. The finite distance condition is satisfied for all globally hyperbolic space–times and a number of other space–times. In particular, this condition holds for Schwarzschild space–time, Minkowski space–time, Einstein static space–time, de Sitter space–time and the Friedmann cosmological models. The naturality of the finite distance condition may be justified by stating the following result. The proof will be given in a subsequent paper.

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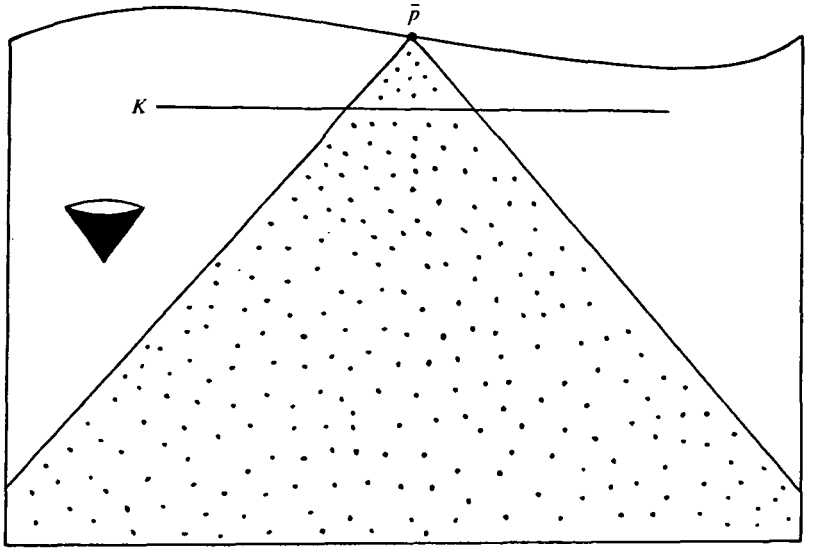


Fig. 1. All inextendible nonspacelike curves which have the regular spacelike boundary point $\bar{p}$ as one endpoint must intersect the compact set K.

THEOREM 8·2. *The strongly causal space–time (M, g) is globally hyperbolic if and only if (M, g') satisfies the finite distance condition for all $g' \in C(M, g)$.*

We now establish a corollary to Theorem 7·1.

COROLLARY 8·3. *Let (M, g) be a chronological space–time which satisfies the generic condition, the strong energy condition and the finite distance condition. If (M, g) has a regular spacelike boundary point, then (M, g) is nonspacelike incomplete.*

Proof. Let $\bar{p}$ be a regular spacelike boundary point. Choose K to be a compact set such that any inextendible nonspacelike curve which has $\bar{p}$ as one endpoint must meet K. Let $\gamma: (-\infty, \infty) \to M$ be a future-directed timelike curve which is inextendible with $\bar{p}$ as one end point and $\gamma(0) \in K$. Let $p_n = \gamma(-n)$ and $q_n = \gamma(n)$. For sufficiently large n the points p_n and q_n are causally disconnected by K. The corollary now follows from Theorem 7·1.

Using the fact that the above set K may be chosen arbitrarily close to $\bar{p}$ in the topology on M^* and the fact that a limit of maximal curves is maximal in a strongly causal space-time, we may establish the following corollary.

COROLLARY 8·4. *Let (M, g) be a strongly causal space–time with a regular spacelike boundary point. Assume that (M, g) satisfies the generic condition, the strong energy condition and the finite distance condition. For each regular spacelike boundary point $\bar{p}$ of (M, g) there is an incomplete nonspacelike geodesic which is inextendible and has $\bar{p}$ as endpoint.*

9. *The existence of causally disconnecting sets.* In this last section we comment briefly on the existence of causally disconnecting sets. In this section K will always denote a compact causally disconnecting set such that the associated divergent sequences $\{p_n\}$ and $\{q_n\}$ satisfy $0 < d(p_n, q_n) < \infty$ for each n.

First, in any space-time having a regular spacelike boundary point and satisfying

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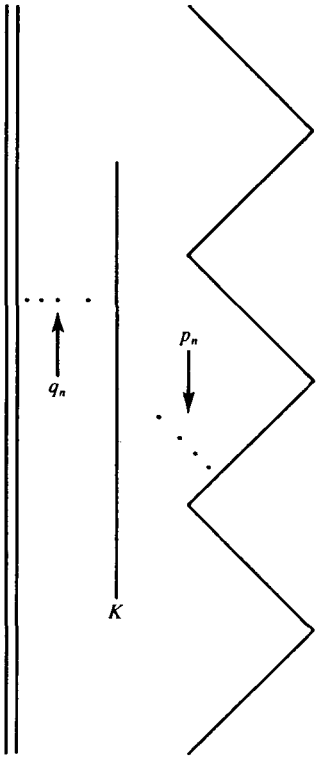


Fig. 2. The Penrose diagram for a Reissner–Nordström space–time with $e^2 = m^2$. The sequences $\{p_n\}$ and $\{q_n\}$ are causally disconnected by the compact set K.

the finite distance condition, it is always possible to find a compact K with associated sequences. Consequently, the extended Schwarzschild space–times (10) and the Friedmann models all contain such a K with associated $\{p_n\}$ and $\{q_n\}$.

In many space–times that do not have spacelike boundary points it is possible to find causally disconnecting sets. In particular, if (M, g) is Minkowski space–time and K is any compact subset with nonempty interior, then there exist divergent sequences $\{p_n\}$ and $\{q_n\}$ which are causally disconnected by K and which satisfy $0 < d(p_n, q_n) < \infty$. Finally, we mention that in all of the extended Reissner–Nordström space–times (10) there exist compact causally disconnecting sets with associated $\{p_n\}$ and $\{q_n\}$ satisfying $0 < d(p_n, q_n) < \infty$ for each n. In fact, we may choose $\{p_n\}$ and $\{q_n\}$ such that $J^+(p_n) \cap J^-(q_n)$ is always compact (see Fig. 2).

One may easily construct examples of nonchronological space-times which have causally disconnecting sets where the causality violation takes place on a proper subset. On the other hand, a number of space–times which are not causal do not have causally disconnecting sets. In particular, a *totally vicious* space–time (18) satisfies $I^+(p) \cap I^-(p) = M$ for all $p \in M$ and cannot contain a causally disconnecting set because $d(p, q) = \infty$ for all $p, q \in M$. Naturally, there are physical objections to space-times which have causality violation. The totally vicious space–times are the space–times which have complete causality violation.

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REFERENCES

(1) BEEM, J. K. Some examples of incomplete space–times. *General Relativity and Gravitation* **7** (1976), 501–509.
(2) BEEM, J. K. Conformal changes and geodesic completeness. *Comm. Math. Phys.* **49** (1976), 179–186.
(3) BEEM, J. K. and EHRLICH, P. E. Distance lorentzienne finie et géodésiques f-causales incomplètes. *C. R. Acad. Sci. Paris Sér.* A **285** (1977), 1129–1131.
(4) BEEM, J. K. and ERHLICH, P. E. Conformal deformations, Ricci curvature and energy conditions on globally hyperbolic spacetimes. *Math. Proc. Cambridge Philos. Soc.* **84** (1978), 159–175.
(5) GEROCH, R. P. What is a singularity in general relativity? *Ann. Physics* **48** (1968), 526–540.
(6) GEROCH, R. P., KRONHEIMER, E. H. and PENROSE, R. Ideal points in space–time. *Proc. Roy. Soc. London Ser.* A **327** (1972), 545–567.
(7) GROMOLL, D. and MEYER, W. On complete open manifolds of positive curvature. *Ann. of Math.* **90** (1969), 75–90.
(8) HAWKING, S. W. The occurrence of singularities in cosmology III. Causality and singularities. *Proc. Roy. Soc. London Ser.* A **300** (1967), 187–201.
(9) HAWKING, S. W. and PENROSE, R. The singularities of gravitational collapse and cosmology. *Proc. Roy. Soc. London Ser.* A **314** (1970), 529–548.
(10) HAWKING, S. W. and ELLIS, G. F. R. *The large-scale structure of space–time* (Cambridge University Press, 1973).
(11) KUNDT, W. Note on the completeness of space–times. *Z. Physik* **172** (1963), 488–489.
(12) LERNER, D. Techniques of topology and differential geometry in general relativity. *Springer Lect. Notes Phys.* **14** (1972), 1–44.
(13) NOMIZU, K. and OZEKI, H. The existence of complete riemannian metrics. *Proc. Amer. Math. Soc.* **12** (1961), 889–891.
(14) PENROSE, R. Structure of space–time. In *Battelle Rencontres*; 1967 *Lectures in Mathematics and Physics*, ed. de Witt and Wheeler (New York, W. A. Benjamin), 121–235.
(15) PENROSE, R. Techniques of differential topology in relativity. *Regional Conference Series in Applied Math.* 7 (SIAM, Philadelphia, 1972).
(16) TIPLER, F. Causally symmetric space–times. *J. Math. Phys.* **18** (1977), 1568–1573.
(17) TIPLER, F. Singularities in universes with negative cosmological constant. *Astrophys. J.* **209** (1976), 12–15.
(18) TIPLER, F. Singularities and causality violation. *Ann. Physics* **108** (1977), 1–36.
(19) WOLF, J. A. *Spaces of constant curvature* (New York, McGraw-Hill, 1967).

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