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Embedding theorems for pro-p groups

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Baumslag [**Bm1**] and Remeslennikov [**R1**] independently showed that the abstract group with finite presentation

$$\langle a, x, y \mid a^x = aa^y, [x, y] = 1 = [a, a^y] \rangle.$$

is metabelian; the derived subgroup is free abelian of countably infinite rank. Both generalized this example to show that every finitely generated metabelian group embeds in a finitely presented metabelian group. Thomson [**T**] generalized this result still further to show that every finitely generated group of upper triangular matrices over an abstract ring A lies inside a finitely presented group of upper triangular matrices over some extension ring of A. For a valuable survey of these results, see [**St**].

Baumslag proved analogous theorems for Lie algebras in [**Bm2**]. In this paper we shall prove some analogous results for pro-p groups.

Theorem A. *Suppose that $p > 2$ and let k be $\mathbb{Z}_p$ or $\mathbb{F}_p$. Let Q be the free abelian pro-p group on the set $\{x_j, y_j \mid 1 \leqslant j \leqslant d\}$ and let A be a free module of finite rank over the formal power series algebra $k[\![X_1, X_2, \ldots, X_d]\!]$. Suppose that Q acts on A by $a.x_j = a(1 + X_j)$ and $a.y_j = a(1 + 2X_j)$. Then the split extension $Q \ltimes A$ is finitely presented. In the case that $d = 1$ and A has rank 1, the group has presentation*

$$\langle a, x, y \mid a^{2x} = aa^y, [x, y] = 1 = [a, a^y] \rangle.$$

We remark that the abstract group given by this presentation is not metabelian or residually finite-p (see Proposition 2·1). We need to produce a separate example for the case $p = 2$.

Theorem B. *Suppose that $p = 2$ and let k be $\mathbb{Z}_p$ or $\mathbb{F}_p$. Let Q be the free abelian pro-p group on the set $\{x_j, y_j \mid 1 \leqslant j \leqslant d\}$ and let A be a free module of finite rank over the formal power series algebra $k[\![X_1, X_2, \ldots, X_d]\!]$. Suppose that Q acts on A by $a.x_j = a(1 + X_j)$ and $a.y_j = a(1 + X_j + X_j^2)$. Then the split extension $Q \ltimes A$ is finitely presented. In the case that $d = 1$ and A has rank 1, the group has presentation*

$$\langle a, x, y \mid aa^x a^{x^2} = a^y, [x, y] = 1 = [a, a^x] \rangle.$$

We use these examples to prove the following embedding theorem, which shows that the subgroup structure of a finitely presented metabelian pro-p group is in general complicated.

Theorem C. *Every finitely generated metabelian pro-p group can be embedded in a finitely presented metabelian pro-p group.*

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We shall adapt this theorem to a class of linear pro-p groups. Let A be a ring and P and S be subsets of A. We write $\mathrm{T}_n(P,S)$ for the set of all upper triangular $n \times n$ matrices of form

$$\begin{pmatrix} p_1 & s_{12} & s_{13} & \cdots & s_{1n} \\ & p_2 & s_{23} & \cdots & s_{2n} \\ & & p_3 & \cdots & s_{3n} \\ & & & \ddots & \vdots \\ & & & & p_n \end{pmatrix}$$

such that $p_i \in P$ and $s_{ij} \in S$.

THEOREM D. *Let A be a commutative pro-p ring with (Jacobson) radical R. Then the set $\mathrm{T}_n(1+R,A)$ is a pro-p group and any finitely generated subgroup lies in a finitely presented subgroup.*

In the course of proving Theorem D we collect some useful results about pro-p rings in general. This includes the following intriguing example of a pro-p ring. Let S be a topological ring. We say that S is *topologically* (*right*) *Noetherian* if and only if every ascending chain of closed right ideals is eventually constant. We say that S is *weakly* (*right*) *Noetherian* if and only if every closed right ideal is topologically finitely generated.

PROPOSITION E. *Let I be a set of distinct positive integers. Let C be the Cartesian product of the finite fields $\{\mathbb{F}_{p^i} \mid i \in I\}$. Then C has radical zero, is generated as a topological ring by a single element and is weakly Noetherian. Moreover, C is topologically Noetherian if and only if I is a finite set.*

In Section 1 we establish some notation and terminology. The proofs of Theorems A and B appear in Section 2. We prove Theorem C in Section 3. Theorem D and Proposition E appear in Section 4.

1. *Notation and terminology*

Throughout this account, p will denote a rational prime. We write $\mathbb{F}_p$ for the field with p elements, $\mathbb{Z}_p$ for the ring of p-adic integers and $\mathbb{Q}_p$ for the field of p-adic numbers.

Recall that a *profinite group* is a topological group given by the inverse limit of a system of finite groups, each equipped with the discrete topology. As topological spaces, profinite groups are compact and totally disconnected (hence Hausdorff). A *pro-p group* is a profinite group such that each finite group in the inverse limit is a finite-p group. See [**DDMS**] for basic facts about profinite and pro-p groups.

By convention, when we refer to a subgroup of a profinite group we mean a closed subgroup, unless otherwise indicated. Similarly, a homomorphism of profinite groups is always assumed to be continuous. This convention extends to our notation, where we shall use notation normally used for abstract group algebras, tensor products and so forth for their topological analogues.

If x and y are elements of a group, we write $[x,y]$ for the commutator $x^{-1}y^{-1}xy$ and x^y for the conjugate $y^{-1}xy$.

If H is an abstract group then the *pro-p completion* of H is a pro-p group $\hat{H}$ together with a group homomorphism $i\colon H \to \hat{H}$ which satisfies the following universal

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property: given any pro-p group G and a group homomorphism $f\colon H \to G$, there is a unique (continuous) homomorphism $g\colon \hat{H} \to G$ such that $f = ig$. We may construct $\hat{H}$ as $\varprojlim H/N$, where the inverse limit is taken over all normal subgroups N of H such that the quotient H/N is a finite-p group. Note that H is residually finite-p if and only if i is a monomorphism.

Let G be a pro-p group. We say that G is *finitely generated* if and only if there is a finite set $X = \{x_1, x_2, \ldots, x_n\}$ and an epimorphism π from the free pro-p group $G(X)$ on X to G. Moreover, G is *finitely presented* if and only if in addition the kernel of π is finitely generated as a (closed) normal subgroup of $G(X)$. A *finite presentation* of G is such an epimorphism together with a set of relations $\{f_1, f_2, \ldots, f_r\}$ which generate $\ker \pi$. We write such a presentation as

$$\langle x_1, x_2, \ldots, x_n \mid f_1, f_2, \ldots, f_r \rangle.$$

So n is number of generators and r is the number of relations in the presentation. If H is an abstract group with a finite presentation, then the presentation of H also defines a pro-p group isomorphic to the pro-p completion $\hat{H}$, so that $\hat{H}$ is finitely presented.

A pro-p group G is *soluble* if and only if there is a finite series

$$G = G_0 \geqslant G_1 \geqslant \ldots \geqslant G_n = 0$$

such that each G_{i+1} is a closed normal subgroup of G_i and the factors G_i/G_{i+1} are abelian (possibly trivial); G is *metabelian* if and only if there is such a series with $n = 2$. If H is a soluble abstract group then its pro-p completion $\hat{H}$ is soluble. The converse holds if H is residually finite-p, but not in general.

2. *Examples of finitely presented metabelian pro-p groups*

We begin by proving Theorem A.

Proof of Theorem A. In Theorem C of [**Kg2**] we showed that such an extension of A by Q is finitely presented if and only if A and $A \wedge_{\mathbb{Z}_p} A$ are finitely generated as Q-modules. Here $A \wedge_{\mathbb{Z}_p} A$ is the completed exterior square of A and Q acts diagonally. It will certainly suffice to show that the completed tensor square $A \otimes_{\mathbb{Z}_p} A$ is finitely generated. If A has rank r then $A \otimes_{\mathbb{Z}_p} A$ is isomorphic as a diagonal Q-module to a direct sum of r^2 copies of $B \otimes_{\mathbb{Z}_p} B$, where B has rank 1; so we may reduce to the case where A has rank 1. We may reduce to the case $k = \mathbb{F}_p$ since $A \otimes_{\mathbb{Z}_p} A$ is finitely generated as a $\mathbb{Z}_p Q$-module if and only if $B \otimes_{\mathbb{Z}_p} B$ is finitely generated as a $\mathbb{F}_p Q$-module, where $B = A \otimes_{\mathbb{Z}_p} \mathbb{F}_p$.

We may identify $A \otimes_{\mathbb{Z}_p} A$ with the formal power series over $\mathbb{F}_p$ on the set $\{X_j, W_j \mid 1 \leqslant j \leqslant d\}$ and the diagonal action of Q is given by

$$f.x_j = f(1 + X_j)(1 + W_j), \qquad f.y_j = f(1 + 2X_j)(1 + 2W_j).$$

Now $A \otimes_{\mathbb{Z}_p} A$ is finitely generated as a Q-module if and only if $A \otimes_{\mathbb{Z}_p} A/[A \otimes_{\mathbb{Z}_p} A, Q]$ is finite-dimensional over $\mathbb{F}_p$. But in this trivialization we have

$$0 \equiv 1.x_j - 1 \equiv X_j + W_j + X_j W_j, \qquad 0 \equiv 1.y_j - 1 \equiv 2(X_j + W_j) + 4X_j W_j.$$

We conclude that $X_j \equiv -W_j$ and $X_j^2 \equiv 0 \equiv W_j^2$ for all j, which ensures that $A \otimes_{\mathbb{Z}_p} A/[A \otimes_{\mathbb{Z}_p} A, Q]$ is finite-dimensional.

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For the explicit presentation in the case $d = 1$, first we claim that $Q \ltimes A$ has infinite presentation

$$\langle a, x, y \mid a^{2x} = aa^y, [x, y] = 1 = [a, a^{y^j}] \quad \text{for } j > 0 \rangle.$$

Let G be the pro-p group with this presentation. Then the closed subgroup A_0 of G generated by $\{a^{y^j} \mid j \in \mathbb{Z}\}$ is an abelian pro-p group invariant under the action of y. Since 2 is a unit in $\mathbb{Z}_p$, we have $a^x = a^{\frac{1}{2}}(a^y)^{\frac{1}{2}}$, so that A_0 is invariant under the action of x. Thus A_0 is a normal subgroup and G/A_0 is isomorphic to Q. Moreover, A_0 is isomorphic to A as a Q-module. Finally we note that the relations $[a, a^{y^j}] = 1$ are redundant for $j > 1$. For suppose that a commutes with $a^y, a^{y^2}, \ldots, a^{y^n}$ for some $n \geqslant 1$. Then

$$1 = [a^2, a^{2y^n}]^x = [a^{2x}, a^{2xy^n}] = [aa^y, a^{y^n} a^{y^{n+1}}] = [a, a^{y^{n+1}}]$$

which establishes an induction. □

The abstract group with the presentation from Theorem A cannot be metabelian, as the next result shows.

PROPOSITION 1. *The abstract group G given by the presentation*

$$\langle a, x, y \mid a^{2x} = aa^y, [x, y] = 1 = [a, a^y] \rangle$$

contains a non-abelian free subgroup and is not residually finite-p.

Proof. Let Q be a free abelian abstract group on the set $\{x, y\}$ and let $A = \mathbb{F}_2\langle x, y\rangle/(2x - (1 + y))$. Then $Q \ltimes A$ is a metabelian image of G. In [**BiS**], Bieri and Strebel showed that any metabelian image of a finitely presented group that contains no non-abelian free subgroups is finitely presented. So it will suffice to show that $Q \ltimes A$ cannot be finitely presented.

Note that A is isomorphic to $\mathbb{F}_2\langle x, y\rangle/(1 + y)$, so that $Q \ltimes A$ decomposes as a direct product of $\langle y \rangle$ and $\langle x \rangle \ltimes \mathbb{F}_2\langle x \rangle$. If $Q \ltimes A$ were finitely presented then its image $\langle x \rangle \ltimes \mathbb{F}_2\langle x \rangle$ would be finitely presented, but it is easy to show that this is not so. The last claim follows since the pro-p completion of G is metabelian by Theorem A. □

It is unsatisfactory that Theorem A gives us no examples of finitely presented pro-2 groups. We now rectify this.

Proof of Theorem B. This follows the proof of Theorem A closely. We may reduce to the case $k = \mathbb{F}_p$ and A has rank 1 as before. We may identify $A \otimes_{\mathbb{Z}_p} A$ with the formal power series over $\mathbb{F}_p$ on the set $\{X_j, W_j \mid 1 \leqslant j \leqslant d\}$ and the diagonal action of Q is given by

$$f \,.\, x_j = f(1 + X_j)(1 + W_j), \qquad f \,.\, y_j = f(1 + X_j + X_j^2)(1 + W_j + W_j^2).$$

So in the trivialization we have

$$0 \equiv 1 \,.\, x_j - 1 \equiv X_j + W_j + X_j W_j$$

and

$$0 \equiv 1 \,.\, y_j - 1 \equiv X_j + W_j + X_j^2 + W_j^2 + X_j W_j + X_j W_j (X_j + W_j) + X_j^2 W_j^2.$$

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Hence $X_jW_j \equiv X_j + W_j$ and $X_j^2W_j^2 \equiv X_j^2 + W_j^2$; thus $X_jW_j(X_j + W_j) \equiv 0$. Hence $X_j^2W_j^2 \equiv 0$ and $X_j^2 \equiv W_j^2$, which ensures that $A \otimes_{\mathbb{Z}_p} A/[A \otimes_{\mathbb{Z}_p} A, Q]$ is finite-dimensional.

The explicit presentation in the case $d = 1$ can be established much as in Theorem A, so we omit the proof.

3. *Embedding metabelian pro-*p *groups*

In this section we use the examples from Section 2 to prove an embedding theorem for finitely generated metabelian pro-p groups. We shall require the next result.

Proposition 1 (Remeslennikov [**R2**] Theorem 2). *Let F be a free pro-p group on the set $\{x_i \mid i \in I\}$ and suppose that R is a normal subgroup of F. Let A be the free $\mathbb{Z}_p(F/R)$-module on $\{a_i \mid i \in I\}$. Then the homomorphism $\beta\colon F \to (F/R) \ltimes A$ induced by $x_i \mapsto (x_iR)a_i$ has kernel $[R, R]$.*

Corollary 2. *Let G be a finitely generated metabelian pro-p group. Then G embeds in $Q \ltimes A/B$, where $Q = G/[G, G]$, A is a free $\mathbb{Z}_pQ$-module of finite rank and B is a Q-submodule of A.*

Proof. Let F be a free pro-p group on the set $\{x_i \mid i \in I\}$ and $\pi\colon F \to G$ be an epimorphism. Let R be the inverse image of $[G, G]$, so that $[R, R] \leqslant \ker\pi \leqslant R$. Let $Q = F/R$ and let A be the free $\mathbb{Z}_pQ$-module on $\{a_i \mid i \in I\}$. By Proposition 1, $F/[R, R]$ embeds in $Q \ltimes A$. The image of $\ker\pi$ lies in A; since it is normalized by $(x_iR)a_i$ and centralized by a_i, it is a Q-submodule of A. □

Thus we may concentrate on embedding split extensions.

Lemma 3. *Let Q be a finitely generated abelian pro-p group and A be a free $\mathbb{Z}_pQ$-module of finite rank. Then $Q \ltimes A$ embeds in a finitely presented metabelian group $(Q \oplus Q_1) \ltimes A$ such that Q_1 is abelian and every Q-submodule of A is invariant under the action of Q_1.*

Proof. Note that Q has a free abelian subgroup Q_0 of finite index and A is a free $\mathbb{Z}_pQ_0$-module of finite rank. Let Q_0 have basis $\{x_1, x_2, \ldots, x_d\}$. Let Q_1 be a free abelian pro-p group on the basis $\{y_1, y_2, \ldots, y_d\}$ and let Q_1 act on A by $a.y_j = a.(2x_j - 1)$ (or $a.(1 - x_j + x_j^2)$ in the case $p = 2$). Then every Q-submodule of A is invariant under the action of Q_1. Now $(Q_0 \oplus Q_1) \ltimes A$ is finitely presented by Theorem A (by Theorem B in the case $p = 2$). But $Q_0 \oplus Q_1$ has finite index in $Q \oplus Q_1$, so that $(Q \oplus Q_1) \ltimes A$ must be finitely presented. □

We may now prove Theorem C.

Proof of Theorem C. The group $Q \ltimes A/B$ of Corollary 2 embeds in the image $(Q \oplus Q_1) \ltimes A/B$ of the finitely presented group of Lemma 3. But finitely generated metabelian pro-p groups satisfy the maximal condition on (closed) normal subgroups (see [**R2**]). Hence any image of $(Q \oplus Q_1) \ltimes A$ is finitely presented. □

Remeslennikov attempted to prove the above theorem in [**R2**]. Unfortunately the groups he claims to be finitely presented pro-p groups in his Lemma 13 are not pro-p groups because the action he gives is not continuous.

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In Section 4 we shall adapt Theorem C for a class of linear pro-p groups which are nilpotent-by-abelian. It would be good to generalize it to abelian-by-nilpotent pro-p groups, but there are no such results known for abstract groups. There is no embedding theorem for general finitely generated abelian-by-polycyclic abstract groups, since Brookes, Roseblade and Wilson showed in Theorem A of [**BRW**] that every finitely presented abelian-by-polycyclic abstract group is nilpotent-by-nilpotent-by-finite.

4. *Linear pro-*p *groups*

First we shall define linear pro-p groups and establish some of their elementary properties.

All rings we consider have a 1. A *profinite ring* is a topological ring given by the inverse limit of a system of finite rings. A profinite ring is a *pro-p ring* if each ring in the inverse limit is a finite-p ring.

Let A be a ring. We write $A^\times$ for the multiplicative group of units of A, and $R(A)$, or merely R, for the (*Jacobson*) *radical* of A, which is the intersection of all maximal right ideals of A. It is well known that for any ring

$$R = \{a \in A \mid 1 + ax \in A^\times \quad \text{for all } x \in A\}.$$

Consequently R is a two-sided ideal of A, $1 + R$ is a normal subgroup of $A^\times$ and the quotient $A^\times/(1 + R)$ is isomorphic to $(A/R)^\times$.

Profinite rings behave in some ways like Artinian rings, though not in all respects (for example, they are not usually Noetherian). The facts in the next two propositions are essentially known but no reference is entirely satisfactory.

PROPOSITION 1. *Let A be a profinite ring with radical R and group of units $A^\times$. Then $A^\times$ and R are closed subsets of A. Moreover $A^\times$ and $1 + R$ are profinite groups under the subspace topology. If A is a pro-p ring then $1 + R$ is a pro-p group.*

Proof. We shall show that $A^\times$ is equal to $\varprojlim (A/I)^\times$, where the inverse limit is taken over all open ideals I of A; for this is a profinite group, hence compact and so closed in A. Now if $a \in A^\times$ then $a + I \in (A/I)^\times$ for all I. Conversely, if $a \notin A^\times$ then $1 \notin aA$, which is a compact subset of A. Thus there is some open ideal I of A such that $1 + I$ and aA are disjoint. Hence $a + I \notin (A/I)^\times$, as required. A similar argument establishes that R is equal to $\varprojlim R(A/I)$, which is compact. Since $1 + R$ is a closed subgroup of $A^\times$, it must be profinite.

Let A be a pro-p ring. If I is an ideal of A such that A/I is an additive finite-p group, then $R + I/I$ is an additive finite-p group, so that the multiplicative group $(1 + R + I)/I$ is finite-p. Thus the inverse limit $\varprojlim (1 + R + I)/I$, taken over all such I, is a pro-p group to which $1 + R$ maps surjectively (since $1 + R$ is compact and the image is dense), and injectively (since the intersection of all the I is zero). □

Recall that a *Sylow-p subgroup* of a profinite group is a maximal pro-p subgroup. As with finite groups, every profinite group has at least one Sylow-p subgroup, for each p. For fixed p, every pro-p group lies in a Sylow-p subgroup and all the Sylow-p subgroups are conjugate.

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PROPOSITION 2. *Let A be a commutative pro-p ring. Then $(A/R)^\times$ has order prime to p, so that $1+R$ is the unique Sylow-p subgroup of $A^\times$.*

Proof. Since A/R is a commutative semisimple profinite ring, it is a Cartesian product of finite fields by Corollary 32.8 of [**Wr**] (compare the structure theorem for commutative semisimple abstract Artinian rings). Clearly the fields all have characteristic p. So $(A/R)^\times$ is the Cartesian product of the multiplicative groups, which all have order prime to p. We complete by Proposition 1. □

If A is a topological ring then we can put a topology on the ring $\mathrm{M}_n(A)$ of $n \times n$ matrices by considering it as the Cartesian product of n^2 copies of A. The multiplication will be continuous with respect to this topology. If A is a profinite ring then so is $\mathrm{M}_n(A)$. So by Proposition 1, the general linear group $\mathrm{GL}_n(A) = \mathrm{M}_n(A)^\times$ is profinite. We shall call a profinite group (A-)*linear* if it is a subgroup of $\mathrm{GL}_n(A)$ for some profinite ring A and some n.

From now on we will assume that A is a commutative pro-p ring and n is a fixed positive integer. Let P and $M_1, M_2, \ldots, M_{n-1}$ be subsets of A. We write $\mathrm{T}_n(P, M_1, \ldots, M_{n-1})$ for the set of all upper triangular matrices of form

$$\begin{pmatrix} p_1 & s_{12} & s_{13} & \cdots & s_{1n} \\ & p_2 & s_{23} & \cdots & s_{2n} \\ & & p_3 & \cdots & s_{3n} \\ & & & \ddots & \vdots \\ & & & & p_n \end{pmatrix}$$

such that $p_i \in P$ and $s_{ij} \in M_{j-i}$. In the case that all M_i are equal to some M, we shall write $\mathrm{T}_n(P, M)$ for brevity; we shall sometimes write $\mathrm{T}_n(P, M_*)$ in the general case.

LEMMA 3. *Let P be a (closed) subgroup of $A^\times$ and $M_1, \ldots, M_{n-1}$ be (closed) P-submodules of A satisfying $M_i M_j \subseteq M_{i+j}$ for $i + j \leqslant n - 1$. Then the set $\mathrm{T}_n(P, M_1, M_2, \ldots, M_{n-1})$ is a profinite group. Moreover, it is a pro-p group if and only if P is contained in $1 + R$.*

Proof. It is clear that the set $\mathrm{T}_n(P, M_*)$ is a closed subset of $\mathrm{M}_n(A)$ and closed under multiplication. Also, it is the product of the sets $\mathrm{T}_n(P, 0)$ and $\mathrm{T}_n(1, M_*)$. The former is certainly a subgroup. The latter is a subgroup since $\mathrm{T}_n(0, M_*)$ is an ideal of $\mathrm{T}_n(A^\times, A)$ contained in the radical. Moreover, $\mathrm{T}_n(P, M_*)$ is a pro-p group if and only if $\mathrm{T}_n(P, 0)$ is a pro-p group. But the unique Sylow-p subgroup of $\mathrm{T}_n(A^\times, 0)$ is $\mathrm{T}_n(1 + R, 0)$ by Proposition 2. □

The most familiar case of Lemma 3 is when S is a subring of A, $M_i = S$ for all i and P is a subgroup of $S^\times$.

LEMMA 4. *Let P and $M_1, M_2, \ldots, M_{n-1}$ be as in Lemma 3. Then the group $\mathrm{T}_n(P, M_1, M_2, \ldots, M_{n-1})$ has a subnormal series with metabelian factors isomorphic to*

$$P \ltimes (M_1 \oplus \ldots \oplus M_{n-1}), \ldots, P \ltimes (M_1 \oplus M_2), P \ltimes M_1, P.$$

In particular, if P is contained in $1 + R$ and each of these factors is finitely presented then $\mathrm{T}_n(P, M_*)$ is a finitely presented pro-p group.

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Proof. Note that there is an epimorphism from $\mathrm{T}_n(P, M_1, \ldots, M_{n-1})$ onto $\mathrm{T}_{n-1}(P, M_1, \ldots, M_{n-2})$ given by removing the first row and column. The kernel is isomorphic to $P \ltimes (M_1 \oplus \ldots \oplus M_{n-1})$, which is metabelian. An easy induction gives the subnormal series. The last claim follows since any extension of a finitely presented pro-p group by another finitely presented pro-p group is finitely presented (this is an analogue of Hall's well-known theorem, and is proved similarly). □

PROPOSITION 5. *Suppose that $X_1, X_2, \ldots, X_r$ lie in R. Let Q be the subgroup of $1+R$ generated by $\{1 + X_i\}$ and P the subgroup generated by $\{1 + X_i, 1 + 2X_i\}$ (in the case $p = 2$ take P to be generated by $\{1 + X_i, 1 + X_i + X_i^2\}$). Let $M_1, M_2, \ldots, M_{n-1}$ be finitely generated Q-modules satisfying $M_i M_j \subseteq M_{i+j}$ for $i + j \leqslant n - 1$. Then $\mathrm{T}_n(P, M_1, M_2, \ldots, M_{n-1})$ is a finitely presented pro-p group.*

Proof. Now P is finitely presented. So by Lemma 4 it will suffice to show that $P \ltimes (M_1 \oplus \ldots \oplus M_m)$ is finitely presented for $m \geqslant 1$. But this group is an image of the finitely presented group $(Q \oplus Q_1) \ltimes A$ which appears in Lemma 3.3 and so it is finitely presented. □

We are now in a position to prove Theorem D.

Proof of Theorem D. Note that $\mathrm{T}_n(1+R, A)$ is a pro-p group by Lemma 3. Suppose that G is generated by a finite set of matrices in $\mathrm{T}_n(1 + R, A)$ and let $\{1 + X_1, 1 + X_2, \ldots, 1 + X_r\}$ be the complete set of diagonal entries of the matrices. Let Q and P be as in Proposition 5. Let $\{a_1^{(i)}, a_2^{(i)}, \ldots, a_{m_i}^{(i)}\}$ be the complete set of entries on the ith superdiagonal of the matrices, for $1 \leqslant i \leqslant n - 1$. Let M_i be the Q-submodule of A generated by all products of form $a_{k_1}^{(j_1)} a_{k_2}^{(j_2)} \ldots a_{k_s}^{(j_s)}$ such that $j_1 + j_2 + \ldots + j_s$ equals i. By construction each M_i is finitely generated and $M_i M_j \subseteq M_{i+j}$ for $i + j \leqslant n - 1$. So G lies in the group $\mathrm{T}_n(P, M_*)$, which is finitely presented by Proposition 5. □

We could express the conclusion of this theorem by saying that $\mathrm{T}_n(1 + R, A)$ is a locally finitely presentable pro-p group. With Theorem D, compare the following result.

THEOREM 6 (Thomson [**T**] Theorem 1). *Let A be an abstract commutative ring and G be a finitely generated subgroup of $\mathrm{T}_n(A^\times, A)$. Then there is an extension ring B of A and a finitely presented subgroup H of $\mathrm{T}_n(B^\times, B)$ such that G lies in H.*

If the base ring A in Theorem 6 is not a field, it may need to be extended to contain sufficient units. The result for pro-p groups depends on the fact that pro-p rings already have plenty of units. In the abstract case, the finitely presented subgroups H can be taken to be of the form $\mathrm{T}_n(P, S)$, where S is a subring of A and P is a subgroup of $S^\times$. We shall show that this cannot be done over certain pro-p rings related to the examples of Proposition E.

Proof of Proposition E. One can see directly that the radical is zero. For each $i \in I$, choose a generator α_i for ${\mathbb{F}_{p^i}}^\times$, so that α_i also generates $\mathbb{F}_{p^i}$ as a ring. Let α be the element of C with ith component α_i. Suppose first that I is finite, and let j be the largest integer in I. Note that

$$\alpha^{p^{j-1}} - \alpha = (0, 0, \ldots, 0, \alpha_j^{p^{j-1}} - \alpha_j),$$
$$(\alpha^{p^{j-1}-1} - 1)^{p^j - 2} = (0, 0, \ldots, 0, (\alpha_j^{p^{j-1}-1} - 1)^{-1}).$$

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Hence the subring of C generated by α contains $(0, 0, \ldots, 0, \alpha_j)$ which generates the copy of $\mathbb{F}_{p^j}$ in C. An easy induction completes the finite case. For the general case, the above work shows that the abstract subring of C generated by α is dense in C, so that its closure is the whole of C. Finally note that each closed ideal is given by a Cartesian product of $\{\mathbb{F}_{p^i} \mid i \in J\}$ for some subset J of I. So each ideal is generated by a single element. The last claim is trivial. □

PROPOSITION 7. *Let A be a commutative pro-p ring with radical R such that $(A/R)^\times$ has infinite exponent. Then $\mathrm{T}_2(1 + R, A)$ has a 1-generator subgroup which lies in no finitely presented subgroup of form $\mathrm{T}_2(P, S)$ such that S is a subring of A and P is a subgroup of $S^\times$.*

Proof. As in Proposition 2, $(A/R)^\times$ is a Cartesian product of finite fields of characteristic p. By hypothesis, there must be fields involved of arbitrarily large size. In particular, A has an image C isomorphic to one of the infinite examples of Proposition E. Let $\alpha' \in A$ be a preimage of the element $\alpha \in C$ constructed in Proposition E. Suppose that S is a subring of A and P is a subgroup of $(1 + R) \cap S^\times$ such that $\mathrm{T}_2(P, S)$ contains the element

$$\begin{pmatrix} 1 & \alpha' \\ 0 & 1 \end{pmatrix}$$

so that $\alpha' \in S$. Then the image of S is the whole of C and so the image of $\mathrm{T}_2(P, S)$ is $\mathrm{T}_2(1, C)$. But $\mathrm{T}_2(1, C)$ is an abelian pro-p group isomorphic to C, which is not finitely generated. Hence $\mathrm{T}_2(P, S)$ cannot be finitely presented. □

The next result is a satisfying converse to Proposition 7.

THEOREM 8. *Let A be a commutative pro-p ring with radical R such that $(A/R)^\times$ has finite exponent. Then any finitely generated subgroup of the group $\mathrm{T}_n(1 + R, A)$ lies in a finitely presented subgroup of the form $\mathrm{T}_n(P, S)$, where S is a subring of A and P is a subgroup of $S^\times$.*

Proof. As in Proposition 2, $(A/R)^\times$ is a Cartesian product of finite fields of characteristic p. By hypothesis, the fields involved must be of bounded size. So there is some e such that the exponent of $(A/R)^\times$ divides $p^e - 1$. Let G be a subgroup of $\mathrm{T}_n(1+R, A)$ generated by a finite set of matrices. Let S be the ring generated by all entries of the matrices. Let S_0 be the subring of S generated by the complete set of diagonal entries $1 + X_1, 1 + X_2, \ldots, 1 + X_m$ together with $X_i = a_i^{p^e} - a_i$ where $a_{m+1}, a_{m+2}, \ldots, a_r$ are the remaining entries. So S_0 is generated as a ring by $X_1, X_2, \ldots, X_r$ which lie in R, It is easy to show that S is finitely generated as an S_0-module (for example, apply Lemma 3.4 of [**Kg3**]). Let P be the subgroup of $1 + R$ generated by $\{1 + X_i, 1 + 2X_i\}$ (in the case $p = 2$ take P to be generated by $\{1 + X_i, 1 + X_i + X_i^2\}$). Then $\mathrm{T}_n(P, S)$ is finitely presented by Proposition 5, and it contains G by construction. □

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REFERENCES

[**Bm1**] G. BAUMSLAG. Subgroups of finitely presented metabelian groups. *J. Austral. Math. Soc.* **16** (1973), 98–110.

[**Bm2**] G. BAUMSLAG. Subalgebras of finitely presented solvable Lie algebras. *J. Algebra* **45** (1977), 295–305.

[**BiS**] R. BIERI and R. STREBEL. Valuations and finitely presented metabelian groups. *Proc. London Math. Soc. (3)* **41** (1980), 439–464.

[**BRW**] C. J. B. BROOKES, J. E. ROSEBLADE and J. S. WILSON. Exterior powers of modules for group rings of polycyclic groups. *J. London Math. Soc.*, to appear.

[**DDMS**] J. D. DIXON, M. P. F. DU SAUTOY, A. MANN and D. SEGAL. *Analytic pro-p groups*, *London Math. Soc. Lecture Notes Series* **157** (Cambridge University Press, 1991).

[**Kg1**] J. D. KING. *Finite presentability of Lie algebras and pro-p groups*, PhD dissertation (Cambridge University, 1995).

[**Kg2**] J. D. KING. Homological finiteness conditions for pro-p groups, preprint (1996).

[**Kg3**] J. D. KING. A geometric invariant for metabelian pro-p groups, preprint (1996).

[**R1**] V. N. REMESLENNIKOV. On finitely presented groups; *in* Proc. Fourth All-Union Symposium on the Theory of Groups (Novosibirsk, 1973), 164–169.

[**R2**] V. N. REMESLENNIKOV. Embedding theorems for profinite groups. *Math. USSR Izvestija* **14** (1980), 367–382.

[**St**] R. STREBEL. Finitely presented soluble groups; *in Group theory: essays for Philip Hall* (Academic Press, 1984), 257–314.

[**T**] M. W. THOMSON. Subgroups of finitely presented solvable linear groups. *Trans. Amer. Math. Soc.* **231** (1977), 133–142.

[**Wr**] S. WARNER. *Topological rings*, *North-Holland Math. Studies* **178** (Amsterdam, 1993).

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