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Self-Avoiding Walks on Hyperbolic Graphs

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We study self-avoiding walks (SAWs) on non-Euclidean lattices that correspond to regular tilings of the hyperbolic plane ('hyperbolic graphs'). We prove that on all but at most eight such graphs, (i) there are exponentially fewer N-step self-avoiding polygons than there are N-step SAWs, (ii) the number of N-step SAWs grows as μ_w^N within a constant factor, and (iii) the average end-to-end distance of an N-step SAW is approximately proportional to N. In terms of critical exponents from statistical physics, (ii) says that $\gamma = 1$ and (iii) says that $\nu = 1$. We also prove that γ is finite on all hyperbolic graphs, and we prove a general identity about non-reversing walks that had previously been discovered for certain special cases.

1. Introduction

The study of models in statistical mechanics such as Ising systems, self-avoiding walks (SAWs) and percolation has been mainly focused on the hypercubic lattice $\mathscr{L}^d$ during the past half century. However, in recent years increasing attention has been dedicated to the study of these models on other graphs. A sample of these efforts can be found in the papers [1, 2, 4, 5, 14, 17, 23, 24, 26] and the references therein. The graphs considered in these papers have the property that they are infinite-dimensional, in the sense that the graphs grow exponentially fast, but they are not trees, *i.e.*, they have cycles. The motivation for this trend of studying the models on general graphs is at least twofold. On one hand, it is clear from the results in the papers quoted above that on graphs different

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from the commonly studied hypercubic lattice there is new mathematical structure to be found. On the other hand, the techniques and results obtained from the study of the models on more general graphs have in some instances led to results which were not yet available for the most studied particular graphs.

In this paper we study SAWs and self-avoiding polygons on hyperbolic graphs. A self-avoiding walk is a path in a graph which visits no site of the graph more than once; a self-avoiding polygon starts and ends at the same site but is otherwise self-avoiding. We will give more formal definitions later in this introduction. Hyperbolic graphs are planar and transitive (*i.e.*, every site is like every other site), and can be briefly described as follows. (For their detailed constructions see Section 2 of [23], Chapter 3 of [27], or [21].) Consider the regular tilings of the three two-dimensional spaces: the sphere $\mathbf{S}^2$, the Euclidean plane $\mathbf{R}^2$ and the hyperbolic plane $\mathbf{H}^2$. Each tiling corresponds to a graph whose edges and vertices (or sites) are edges and vertices of the faces of the tiling. These tilings and graphs can be characterized by two integers, both greater than or equal to 3: v, the number of neighbours of each vertex; and f, the number of sides of each face. We will write (v, f) to denote the tiling. Then (v, f) is a tiling of $\mathbf{S}^2$, $\mathbf{R}^2$ or $\mathbf{H}^2$ respectively, according to whether the quantity $(v-2)(f-2)$ is smaller than, equal to or larger than four respectively. The (five) tilings of $\mathbf{S}^2$ have only a finite number of vertices; *e.g.*, the $(3, 3)$ tiling is a tetrahedron. On $\mathbf{R}^2$ there are three (infinite) tilings: triangular (6, 3), square (4, 4) and hexagonal (3, 6). Self-avoiding walks on these three lattices and other hypercubic lattices $\mathscr{Z}^d$ have received intensive study: see [18] and Chapter 7 of [12]. When tiling the hyperbolic plane, there are infinitely many choices of (v, f). We denote these tilings and the corresponding graphs by $\mathscr{H}(v, f)$ when $(v-2)(f-2) > 4$. The hyperbolic graph $\mathscr{H}(5, 5)$, which is self-dual, is shown in Figure 1 as an example.

SAWs on hyperbolic graphs were first studied in [26], where numerical results were obtained on the two graphs $\mathscr{H}(7, 3)$ and $\mathscr{H}(5, 5)$ (as well as two other non-Euclidean lattices). They showed numerically that the number of N-step self-avoiding polygons was exponentially smaller than the number of N-step self-avoiding walks in these lattices (in notation introduced below, this says that $\mu_p < \mu_w$), and that the critical exponent γ (defined in (1.2) below) equals 1. The result that $\gamma = 1$ has been rigorously proved in high-dimensional Euclidean space, specifically in $\mathscr{Z}^d$ for $d \geqslant 5$ [10]. The proof uses the lace expansion, and it reflects the fact that ordinary random walks in high dimensions are 'almost' self-avoiding in some asymptotic sense. In contrast, it has been proved that $\mu_p = \mu_w$ in $\mathscr{Z}^d$ for every d. However, the inequality $\mu_p < \mu_w$ does have a natural correspondence in the theory of ordinary random walks on hyperbolic graphs. On these graphs (indeed on any graph with nonzero isoperimetric constant – see Section 2), the probability that a random walk is at its starting point at the Nth step decays exponentially in N (see, for example, Chapter II of [28]). We can think of this as happening because the number of sites within N steps of a given site grows exponentially in N on hyperbolic graphs, instead of polynomially as in $\mathscr{Z}^d$. In a hyperbolic graph, the distance of a random walk from the origin behaves rather like a random walk on the nonnegative integers with a positive drift. Indeed, the distance of a walker from its starting point after N steps is almost surely asymptotically proportional to N (Theorem 8.14 of [28]). It is reasonable to expect the same behaviour for self-avoiding walks on hyperbolic graphs. In terms of the

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critical exponent ν defined below (see (1.6)), this says that we expect $\nu = 1$. This contrasts with the Euclidean case of $\mathscr{Z}^d$, where it is rigorously known that $\nu = 1/2$ for $d \geqslant 5$ [10].

In this paper, we prove that on $\mathscr{H}(v, f)$ with sufficiently large v or f (*e.g.*, either $v \geqslant 10$ or $f \geqslant 10$ suffices: for details see Condition LARGE, just before Theorem 1.2), self-avoiding walks do indeed behave as predicted in the previous paragraph. To make the statements more precise, we introduce some terminology and notation for self-avoiding walks on a given infinite graph $\mathscr{G}$ that has a vertex labelled '0' that serves as an 'origin'. Let N be a nonnegative integer. An N-step (ordinary) walk is a sequence $\omega = (\omega(0), \omega(1), \dots, \omega(N))$ of $N+1$ sites of $\mathscr{G}$ such that $\omega(i)$ and $\omega(i+1)$ are neighbours for each $i = 0, \dots, N-1$. In this case, we say that N is the *length* of ω. An N-step SAW is an N-step ordinary walk in which the $N+1$ sites are all different. Let c_N be the number of N-step self-avoiding walks ω such that $\omega(0) = 0$. For vertices x and y of $\mathscr{G}$, we define $c_N(x, y)$ to be the number of N-step self-avoiding walks from x to y (*i.e.*, SAWs ω such that $\omega(0) = x$ and $\omega(N) = y$). We define the graph distance between two vertices x and y to be the length of the shortest walk from x to y, and we denote it $d(x, y)$.

An N-step (rooted) self-avoiding polygon is an N-step ordinary walk ω in which $\omega(0) = \omega(N)$ and in which $\omega(0), \omega(1), \dots, \omega(N-1)$ are all different. Let p_N be the number of rooted N-step self-avoiding polygons with $\omega(0) = 0$. Evidently

$$p_N = \sum_{x\,:\,d(0,x)=1} c_{N-1}(0, x)$$

for $N \geqslant 3$.

Finally, we define the *connective constants* for self-avoiding walks and polygons:

$$\mu_w = \limsup_{n\to\infty} (c_N)^{1/N},$$
$$\mu_p = \limsup_{n\to\infty} (p_N)^{1/N}.$$

We note that when the graph is $\mathscr{Z}^d$ or $\mathscr{H}(v, f)$, the limit defining μ_w exists, and we have [8]

$$\mu_w = \lim_{N\to\infty} (c_N)^{1/N} = \inf_{N\geqslant 1} (c_N)^{1/N}, \tag{1.1}$$

which implies $c_N \geqslant \mu_w^N$. Equation (1.1) follows from the inequality $c_{N+M} \leqslant c_N c_M$ (which is true because any $(N+M)$-step SAW can be split into an N-step SAW followed by an M-step SAW), and the well-known *subadditivity lemma.*

Lemma 1.1. *Let* $\{a_n : n \geqslant 1\}$ *be a sequence that satisfies* $a_{n+m} \leqslant a_n + a_m$ *for all* $n, m \geqslant 1$. *Then the limit* $\lim_{n\to\infty} a_n/n$ *exists in* $[-\infty, \infty)$ *and equals* $\inf_{n\geqslant 1} a_n/n$.

It is believed that in $\mathscr{Z}^d$ there is a *critical exponent* γ (depending on d) such that

$$c_N \sim A \mu_w^N N^{\gamma-1} \tag{1.2}$$

(where A is a constant depending on d). It has been proved rigorously that this holds with $\gamma = 1$ when $d \geqslant 5$ [10]. It is widely expected that the value of γ is different from 1 in $\mathscr{Z}^2$ and $\mathscr{Z}^3$, and that $N^{\gamma-1}$ is replaced by a power of $\log N$ in $\mathscr{Z}^4$, but little has been proved rigorously (see [18] for a review). In fact, for $\mathscr{Z}^2$, $\mathscr{Z}^3$, and $\mathscr{Z}^4$, it is a major open question

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even to prove that 'γ is finite' in the (weak) sense that there exist finite constants A and p (depending on d) such that

$$\mu_w^N \leqslant c_N \leqslant A\mu_w^N N^p \quad \text{for all } N \geqslant 1. \tag{1.3}$$

The lower bound in (1.3) always holds, as explained above, but the best upper bound known for c_N in $\mathscr{Z}^2$ is of the form $\mu_w^N \exp(A\sqrt{N})$, due to Hammersley and Welsh in 1962 [9]. Slightly better upper bounds are known in $\mathscr{Z}^3$ and $\mathscr{Z}^4$ (see [16] and Section 3.3 of [18]), but nothing as good as equation (1.3).

In this paper we prove that $\gamma = 1$ for most hyperbolic graphs, in the (weak) sense that

$$\mu_w^N \leqslant c_N \leqslant A\mu_w^N \tag{1.4}$$

for all $N \geqslant 1$. We also prove that on *every* hyperbolic graph $\mathscr{H}(v, f)$, γ is finite in the sense of generating functions (which is a bit weaker than equation (1.3), but still of substantial interest); see the discussion following Theorem 1.2 below.

It has been proved that $\mu_w = \mu_p$ in $\mathscr{Z}^d$ ($d \geqslant 2$) [7, 15]. In this paper we show that this is *false* on most hyperbolic graphs, *i.e.*, that $\mu_w > \mu_p$.

We also introduce another critical exponent, ν, which measures the spatial displacement of typical SAWs. For each $t > 0$, let

$$D_N^{(t)} = \frac{1}{c_N} \sum_y [d(0, y)]^t c_N(0, y), \tag{1.5}$$

which is the tth moment of the end-to-end distance of a random N-step self-avoiding walk (using graph distance). The exponent ν is usually defined by the asymptotic relation

$$D_N^{(2)} \sim A_D^{(2)} N^{2\nu} \tag{1.6}$$

(where $A_D^{(2)}$ is a constant). It is known that $\nu = 1/2$ on $\mathscr{Z}^d$ for $d \geqslant 5$ [10]. Other values are expected in lower-dimensional Euclidean lattices (see [18] for a review). In this paper we show that $\nu = 1$ on most hyperbolic graphs, in the (weak) sense that, for every t,

$$A^{(t)} N^t \leqslant D_N^{(t)} \leqslant N^t \tag{1.7}$$

for every $N \geqslant 1$ (for some constant $A^{(t)}$). Actually we prove a stronger fact: that for some $\epsilon > 0$, the fraction of N-step SAWs whose end-to-end distance is *less than* ϵN decays exponentially in N.

The class of hyperbolic graphs $\mathscr{H}(v, f)$ for which the above results are proved to hold consists of those corresponding to (v, f) that satisfy the following condition.

Condition LARGE. The pair of positive integers (v, f) satisfies $(v-2)(f-2) > 4$ and one of the following:

(i) $f = 3$, $v \geqslant 10$,
(ii) $f = 4$, $v \geqslant 6$,
(iii) $f = 5$, $v \geqslant 5$,
(iv) $f \in \{6, 7, 8, 9\}$, $v \geqslant 4$,
(v) $f \geqslant 10$, $v \geqslant 3$.

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Observe that there are only eight hyperbolic graphs $\mathscr{H}(v,f)$ that do not satisfy Condition LARGE, namely: $\mathscr{H}(v,3)$ with $v \in \{7,8,9\}$; $\mathscr{H}(4,5)$; $\mathscr{H}(5,4)$; and $\mathscr{H}(3,f)$ with $f \in \{7,8,9\}$.

Our main theorem is the following.

Theorem 1.2. *Assume that $\mathscr{H}(v,f)$ is a hyperbolic graph for which Condition LARGE holds. Then*

(a) $\mu_p < \mu_w$,

(b) *there exists a constant A_γ such that*

$$\mu_w^N \leqslant c_N \leqslant A_\gamma \mu_w^N \quad \textit{for every } N \geqslant 1,$$

(c) *for every $t \geqslant 1$, there is a constant $A^{(t)}$ such that*

$$A^{(t)} N^t \leqslant \frac{1}{c_N} \sum_y [d(0,y)]^t c_N(0,y) \leqslant N^t \quad \textit{for every } N \geqslant 1.$$

Parts (a) and (b) confirm the numerical conclusions of Swierczak and Guttmann [26], albeit for a somewhat restricted set of graphs. We expect that properties (a), (b) and (c) hold for all hyperbolic graphs, not just those satisfying Condition LARGE. More generally, they should hold for a large class of nonamenable graphs [28]. Also we expect that equations (1.2) and (1.6) hold (necessarily with $\gamma = 1$ and $\nu = 1$). But we are not yet able to prove these things.

There are other results that we can prove on every hyperbolic graph (see Section 7), including the fact that γ is always finite in the sense of generating functions. To explain this, define the generating function

$$\chi(z) = \sum_{N=0}^{\infty} c_N z^N \quad \text{for } z \geqslant 0.$$

If equation (1.2) holds, then $\chi(z) \sim K(1-\mu_w z)^{-\gamma}$ for some constant K as $z \nearrow \mu_w^{-1}$. In Theorem 7.4, we prove that for any hyperbolic graph $\mathscr{H}(v,f)$ there are constants A and p (depending on v and f) such that

$$\chi(z) \leqslant A(1-\mu_w z)^{-p} \quad \text{for } 0 < z < \mu_w^{-1}.$$

Thus $\gamma \leqslant p < \infty$ if equation (1.2) holds.

We remark that our arguments rely in places on the geometry of hyperbolic space, but they do not seem to rely on the planarity of the graph. In particular, it is probably not too hard to extend our arguments to SAWs on tilings of hyperbolic three-space.

The rest of the paper is organized as follows. In Section 2, we obtain bounds on μ_p and μ_w, which are key ingredients for proving that $\mu_p < \mu_w$ in Theorem 1.2. Indeed, these bounds enable us to prove that $\mu_p < \mu_w$ when both v and f are sufficiently large (the weaker restriction of Condition LARGE is achieved by using non-reversing walks to bound the number of self-avoiding polygons, with the help of the technical results of Section 5). Section 3 reviews some geometry of hyperbolic space, and proves some basic results about the geometry of our tilings. This section also introduces *bridges*, an

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important class of self-avoiding walks. Briefly, a bridge is a SAW that stays between two non-intersecting lines that contain its endpoints. Section 3 proves some basic results about the number of N-step bridges. Properties of area in hyperbolic geometry play important roles in our proofs. Section 4 presents the main argument for proving (b) and (c) in Theorem 1.2 for ν and γ. These arguments need to assume something a bit stronger than $\mu_p < \mu_w$, namely that $\sup_y c_N(0, y)$ is exponentially smaller than c_N. It turns out that this holds under Condition LARGE. Section 5 studies non-reversing walks on rather general graphs. The 'two-point function' of ordinary walks is related to that of non-reversing walks and, as a consequence, the connective constant for non-reversing walks is related to the spectral radius of the graph. Section 6 ties together the pieces and completes the proof of Theorem 1.2. Section 7 proves two results for all hyperbolic graphs, without assuming Condition LARGE: firstly, that the connective constant μ_B of bridges equals μ_w (*i.e.*, bridges are not an exponentially small subset of SAWs), and secondly, that γ is finite for every hyperbolic graph. Section 8 is a brief conclusion and summary.

2. Bounds for μ_w and μ_p

Consider an infinite, locally finite, connected graph $\mathcal{G} = (V, E)$. Write A for its adjacency matrix, *i.e.*, the (x, y)-entry of A takes the value 1 or 0 according to whether or not the vertices x and y are neighbours in $\mathcal{G}$. Write A^n for the nth power of A and $A^n(x, y)$ for the (x, y)-entry of A^n. It is not hard to see that $A^n(x, y)$ also equals the number of (ordinary) walks of length n from x to y. It is easy to see that

$$R = R(\mathcal{G}) = \limsup_{n\to\infty} (A^n(x, y))^{1/n}$$

does not depend on x and y. The quantity R is called the spectral radius associated to the adjacency matrix of $\mathcal{G}$. It is proved in [20] that

$$A^n(x, y) \leqslant (R(\mathcal{G}))^n \quad \text{for any } x, y \in V. \tag{2.1}$$

Let $K \subset V$ be a subset of sites and

$$\partial K = \{b \in E : \text{ one and only one end-point of } b \text{ is in } K\}$$

be the boundary edge set of K. Denote by $|B|$ the cardinality number of B. The (edge) isoperimetric constant of $\mathcal{G}$ is defined by

$$I = I(\mathcal{G}) = \inf\left\{ \frac{|\partial K|}{|K|} : K \subset V \text{ is finite } \right\}.$$

It is calculated in [5] that the isoperimetric constant of $\mathcal{H}(v, f)$ is given by

$$I(\mathcal{H}(v, f)) = (v - 2)\sqrt{1 - \frac{4}{(v - 2)(f - 2)}}. \tag{2.2}$$

Write d_x for the degree of a vertex x in $\mathcal{G}$, that is, the number of edges incident to x, and write

$$D = D(\mathcal{G}) = \sup\{d_x : x \in V\}$$

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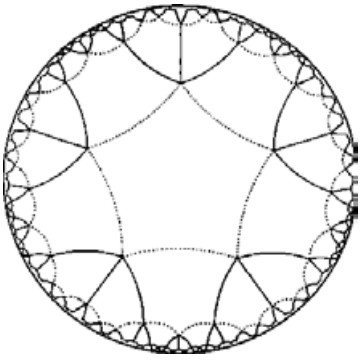


Figure 1. $\mathscr{H}(5,5)$

for the (maximum) degree of $\mathscr{G}$. The following inequality, proved in [19], Theorem 2.1(a), relates the spectral radius, the isoperimetric constant and the degree of the graph $\mathscr{G}$:

$$I^2 + R^2 \leqslant D^2. \tag{2.3}$$

This implies that $R < D$ if $I > 0$. We remark that the converse is also true in the case that $\mathscr{G}$ is D-regular (*i.e.*, $d_x = D$ for every x), as shown by Theorem 10.3(a),(c) of [28]. (Note that the spectral radius $\rho(P)$ of [28] is that of a Markov chain transition matrix P with entries $P(x,y) = A(x,y)/D$ in the case of random walk on a D-regular graph $\mathscr{G}$. Therefore $\rho(P)$ of [28] equals our $R(\mathscr{G})/D$.)

We now state a proposition which gives a lower bound for μ_w.

Proposition 2.1. *For self-avoiding walks on $\mathscr{H}(v,f)$, we have*

(a) *for* $f = 3$, $\mu_w \geqslant \sqrt{(v-2)(v-3)}$,
(b) *for* $f = 4$, $\mu_w \geqslant ((v-1)(v-2)^2)^{1/3}$,
(c) *for* $f > 4$, $\mu_w \geqslant \sqrt{(v-1)(v-2)}$. *More generally* $\mu_w \geqslant ((v-1)^{f-4}(v-2))^{1/(f-3)}$. *Therefore,* μ_w *approaches* $v-1$ *as* f *approaches* ∞.

We remark that $\mu_w \leqslant v-1$ for every v and f. Also, the $f \to \infty$ result of (c) makes sense since $\mathscr{H}(v,f)$ becomes more and more like an infinite v-regular tree as f gets large, and μ_w for this tree is obviously $v-1$.

Proof of Proposition 2.1. We will give a proof for (c). The proof for (a) and (b) can be constructed in a similar manner. We first partition the sites of $\mathscr{H}(v,p)$ into different layers: see Figure 1, where the vertices in each layer are connected by dotted edges. The first layer consists of vertices on the central polygon. The second layer consists of those vertices which are not on the first layer but on a polygon which has a vertex in common with the first layer. The third, fourth, *etc.*, layers are formed in a similar way.

For $f > 4$, each site on any layer other than the first one has either zero or one neighbour in the previous layer. Starting at a site in the first layer, define a walk according to the following rules. At the first step, the walk is allowed to move to a neighbour site in the

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next layer; so it has $v-2$ choices to do so. Each time the walk reaches a site x on a new layer, it is allowed to have $v-1$ choices to move at its next step: $v-3$ choices to move to the next layer and two choices to move within the same layer, to the anticlockwise neighbour (denoted by $x+1$) or the clockwise neighbour (denoted by $x-1$). Notice that $x+1$ and $x-1$ each have no neighbour in the previous layer and $v-2$ neighbours in the next layer. If the walk reaches $x+1$ (respectively $x-1$), then from there it is allowed to have $v-2$ choices to move to the next layer. The walk defined above is self-avoiding and has the property that at each step the walk has either $v-1$ or $v-2$ choices to move and there are no two consecutive steps at which the walk has $v-2$ choices to move; that is, if the walk has $v-2$ choices to move at a step, then it must have $v-1$ choices to move at the previous (and the next) step. It follows from this property that $c_{2n} \geqslant (v-2)^n(v-1)^n$, which implies that $\mu_w \geqslant \sqrt{(v-2)(v-1)}$.

To prove the second statement of part (c), let $x+i$ (respectively $x-i$) denote the ith site from x along the same layer of x in the anticlockwise (respectively clockwise) direction. Notice that each time the walk reaches a site x on a new layer, then $x+i$ and $x-i$ each have no neighbour in the previous layer and $v-2$ neighbours in the next layer for $i = 1, 2, \ldots, f-4$. Modify the rules of the walk defined above as follows. As before, each time the walk reaches a site x on a new layer, it is allowed to have $v-1$ choices to move at its next step: $v-3$ choices to move to the next layer and two choices to move within the same layer: to $x+1$ or $x-1$. If the walk reaches $x+1$ (respectively $x-1$), then from there it is allowed to have $v-1$ choices to move: $v-2$ choices to the next layer and one choice to site $x+2$ (respectively $x-2$). In general, if the walk reaches site $x+(i-1)$ (respectively $x-(i-1)$) for $i = 2, 3, \ldots, f-4$, then it is allowed to have $v-1$ choices to move at the next step: $v-2$ choices to move to the next layer and one choice to move to site $x+i$ (respectively $x-i$). When the walk reaches site $x+(f-4)$ (respectively $x-(f-4)$), then at the next step it is only allowed to move to the next layer: $v-2$ choices. The defined walk is self-avoiding and has the property that if at a step it has $v-2$ choices to move, then it must have $v-1$ choices to move at the previous $f-4$ steps. It follows that $c_{(f-3)n} \geqslant ((v-2)(v-1)^{f-4})^n$ which implies the last inequality in part (c). □

Next we show how Proposition 2.1 implies that $\mu_p < \mu_w$ for some hyperbolic graphs. The stronger result of Theorem 1.2, that this holds whenever Condition LARGE is true, requires a better upper bound on μ_p that will be derived in Section 5; see Section 6 for the final proof of Theorem 1.2.

Proposition 2.2. *Consider a hyperbolic graph $\mathscr{H}(v, f)$ for which $v \geqslant 6$ and f is sufficiently large. Then $\mu_p \leqslant R < \mu_w$.*

Proof. The bound $\mu_p \leqslant R$ follows immediately from equation (2.1) and the definition of μ_p. In the limit as $f \to \infty$, Proposition 2.1 implies that the lower bound of μ_w approaches $v-1$, while inequalities (2.2) and (2.3) imply that the upper bound of R approaches $2\sqrt{v-1}$. Since $2\sqrt{v-1} < v-1$ whenever $v \geqslant 6$, we must have $R < \mu_w$ for sufficiently large f. □

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3. Lines and bridges

In the study of self-avoiding walks in Euclidean lattices, an important role is played by a subset of SAWs known as 'bridges'. For example, in the hypercubic lattice $\mathscr{L}^d$, a bridge is a SAW for which there exist two parallel hyperplanes P_1 and P_2 having the following properties: P_1 contains the first site of the SAW and no other site of the SAW; P_2 contains the final site of the SAW, and possibly some other sites; and all sites of the SAW lie on or between P_1 and P_2. The extension to hyperbolic geometry is slightly problematic because of the multiplicity of parallel (and 'ultraparallel') lines. We shall see that a natural generalization of the Euclidean definition will be very useful for our purposes.

The best way to understand the geometry of the hyperbolic graph $\mathscr{H}(v, f)$ is to identify it with the geometry of the hyperbolic plane. We consider a tiling of the hyperbolic plane by regular f-sided polygons, in which every vertex of every polygon is incident on exactly v of these polygons. Such a tiling can be constructed in the hyperbolic plane whenever $(f-2)(v-2) > 4$. We call the f-sided polygons the *faces* of the tiling. (When referring to a face as a subset of the hyperbolic plane, we include its vertices and edges as well as its interior. In particular, faces are closed sets.)

One useful model for the hyperbolic plane is the Poincaré disk, or conformal disk, model. The points of the hyperbolic planes correspond to points in the interior of a fixed disk in $\mathbf{R}^2$. Lines in the hyperbolic plane correspond to lines in the disk that pass through the centre of the disk, as well as to circular arcs that have endpoints on the boundary of the disk, and meet the boundary at right angles. If two lines do not intersect in the hyperbolic plane, then they could still intersect 'at infinity', in which case they are said to be *parallel*. In the disk model, two circular arcs that meet at the same boundary point of the disk are parallel. If two lines do not intersect in the plane and do not intersect 'at infinity', then they are said to be *ultraparallel*. Some introductory references to hyperbolic geometry include [13] and Chapter 4 of [25].

It is a fundamental result of hyperbolic geometry that there do not exist triangles of arbitrarily large area. Indeed, under the standard measure of area, the area of a triangle whose internal angles are θ_1, θ_2, and θ_3 is exactly $\pi - (\theta_1 + \theta_2 + \theta_3)$ (Section 4.7 of [25]). This also applies to 'asymptotic' triangles – triangles in which one or more pairs of sides intersect 'at infinity', in which case the corresponding internal angle is 0. The general result for polygons is part (a) of the following proposition.

Proposition 3.1.

(a) *Consider a k-sided polygon in the hyperbolic plane whose internal angles are $\theta_1, \dots, \theta_k$. The area of this polygon equals*

$$(k-2)\pi - (\theta_1 + \cdots + \theta_k).$$

(b) *In a (v, f) tiling of the hyperbolic plane, every interior angle of every face equals $2\pi/v$, and the area of every face equals*

$$(f-2)\pi - \frac{2\pi f}{v}.$$

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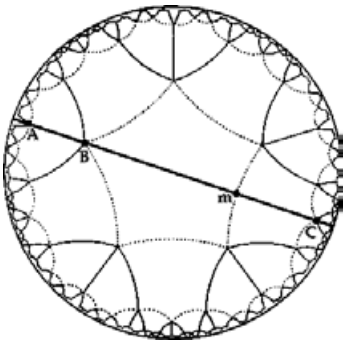


Figure 2. $\mathscr{H}(5,5)$. The bold line is a lattice-line in which segment AB is an edge of a face and segment BC passes through m, the middle of the edge common to the two adjacent faces

Proof. Part (a) is proved by dividing the polygon into $k-2$ triangles and using the area formula for triangles (Exercise 4.7.2 of [25]). Part (b) follows directly. □

We need to deal with lines in the hyperbolic plane in order to define and work with bridges. For a given tiling, an important role is played by those lines that are consistent with the tiling, in the following way.

Definition. Consider a (v,f) tiling of the hyperbolic plane. A *lattice-line* in the hyperbolic plane is a line that contains some edge of a face of the tiling. For clarity, we shall sometimes use the term *plane-line* to denote any line in the hyperbolic plane, not necessarily a lattice-line.

Lattice-lines are of three possible types, depending upon the parity of v and of f.

- If v is even, then every lattice-line is a union of edges of faces. In this case, every lattice-line corresponds to a doubly infinite path of edges and sites in the graph $\mathscr{H}(v,f)$ such that, for every site x on the path, the two edges of the path incident on x make an angle of π.
- If v is odd and f is even, then every lattice-line contains infinitely many vertices, and the segment of the lattice-line between two consecutive vertices on the line is either (i) an edge of a face or (ii) a line segment that joins opposite vertices of a single face. These two kinds of segment alternate as one moves along a given lattice-line.
- If v and f are both odd, then every lattice-line contains infinitely many vertices, and the segment of the lattice-line between two consecutive vertices on the line is either (i) an edge of a face or (ii) a line segment that joins a vertex of one face to a vertex on an adjacent face, passing through the middle of the edge common to the two faces (see Figure 2). Moreover, in case (ii), the line segment cuts each of the two faces through which it passes into two regions that are reflections of one another. These two kinds of segment alternate as one moves along a given lattice-line.

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Observe that every site of $\mathscr{H}(v,f)$ lies in exactly $v/2$ lattice-lines if v is even, and in exactly v lattice-lines if v is odd. Also, two distinct lines cannot intersect in more than one point (Exercise 4.7.2 of [25]).

Definition. A *half-plane* is a subset of the hyperbolic plane consisting of a plane-line L and all of the points on one side of L. The boundary of this half-plane is the line L. A *lattice-half-plane* is a half-plane whose boundary is a lattice-line.

Given a site x_1 of $\mathscr{H}(v,f)$ on the boundary of a lattice-half-plane H_1, and another site x_2 on the boundary of a lattice-half-plane H_2, there exists an automorphism of $\mathscr{H}(v,f)$ (indeed, an isometry of $\mathbf{H}^2$) that maps x_1 to x_2 and H_1 to H_2. This automorphism/isometry could be orientation-preserving or orientation-reversing (see Sections 4.4–4.5 of [25]). The automorphism is unique if v is odd, and there are two automorphisms if v is even.

It will often be necessary to know that we can find a lattice-line near, and ultra-parallel to, a given plane-line. The following definition and lemma will be useful for this.

Definition. Fix a tiling $\mathscr{H}(v,f)$ of the hyperbolic plane, and let S be any subset of the hyperbolic plane. The *neighbourhood of* S, denoted $\mathscr{N}(S)$, is the union of all faces that have nonempty intersection with S.

For example, the neighbourhood of a vertex of a face is the union of v faces.

We also have the following result, which will be useful for us.

Lemma 3.2. *Fix a tiling $\mathscr{H}(v,f)$. Let L be a plane-line, and let z be a site that is not in its neighbourhood $\mathscr{N}(L)$. Then there is a lattice-line L_1 containing z that does not intersect L. (That is, L_1 is parallel or ultraparallel to L.)*

Proof. Hyperbolic geometry tells us that there are two plane-lines L_A and L_B through the point z that meet L 'at infinity', one at each end of L (Section 4.2 of [25]). These three lines form a triangle Δ (a 'doubly asymptotic' triangle). Let θ be the interior angle of this triangle at z. Then the area of Δ equals $\pi - \theta$ (note that the two angles at infinity are both 0). To prove the lemma, we shall show that there is a lattice-line through z that does not intersect the interior of Δ. We consider two cases, according to the parity of v.

First assume v is even. There are $v/2$ lattice-lines through z, with an angle of $2\pi/v$ between any 'nearest pair' of lines. If we can show that

$$\theta < \left(\frac{v}{2} - 1\right)\frac{2\pi}{v},$$

then this would show that fewer than $v/2$ of the lattice-lines through z can pass through the interior of Δ, which would give the desired result. So let us assume that $\theta \geqslant (\frac{v}{2} - 1)2\pi/v$ and try to get a contradiction. Let $\mathscr{F}$ denote the union of the v faces that have z as a vertex. Then the area of $\mathscr{F}$ equals vA_F, where

$$A_F = (f-2)\pi - \frac{2\pi f}{v} = \frac{\pi}{v}[(v-2)(f-2) - 4]$$

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is the area of one face. Observe that $\mathscr{F} \cap L = \emptyset$ (since $z \notin \mathscr{N}(L)$). Let C be a cone of angle $2\pi/v$ with vertex z (*i.e.*, C is the region bounded by two half-lines emanating from z making an angle of $2\pi/v$). Then the area of $C \cap \mathscr{F}$ is exactly A_F. (To see this, note that if C_i is the rotation of C about z by the angle $2\pi i/v$ [for $i = 0, 1, \ldots, v-1$], then the $C_i \cap \mathscr{F}$s are congruent to each other, have disjoint interiors, and their union is $\mathscr{F}$; so each one's area is equal to $1/v$ times the area of $\mathscr{F}$.) Since $\theta \geqslant (\frac{v}{2} - 1)2\pi/v$, and since $\mathscr{F} \cap L = \emptyset$, we conclude that the area of $\Delta \cap \mathscr{F}$ is at least $(\frac{v}{2} - 1)A_F$. The area of Δ is strictly greater than this (since Δ is unbounded, unlike $\Delta \cap \mathscr{F}$), which implies

$$\pi - \theta > \left(\frac{v}{2} - 1\right)\frac{\pi}{v}[(v-2)(f-2) - 4].$$

Combining this with $\theta \geqslant (\frac{v}{2} - 1)2\pi/v$ and some algebra, we obtain

$$4 > (v-2)[(v-2)(f-2) - 4].$$

By inspection, this last inequality has no solutions for even integers $v \geqslant 4$ and integers f such that $(v-2)(f-2) > 4$. Thus we have a contradiction, which proves our result in the case that v is even.

Now assume that v is odd. Then there are v lattice-lines through z, with an angle of π/v between any 'nearest pair'. As in the even case, it suffices to prove $\theta < (v-1)\pi/v$. So we shall assume $\theta \geqslant (v-1)\pi/v$ and derive a contradiction. Writing this as $\theta \geqslant (\frac{v-1}{2})(2\pi/v)$, and noting that $\frac{v-1}{2}$ is an integer, we use the argument of the previous paragraph to conclude that the area of Δ is strictly greater than $(\frac{v-1}{2})A_F$. Since the area of Δ is $\pi - \theta$, we obtain

$$\left(\frac{v-1}{2}\right)\frac{\pi}{v}[(v-2)(f-2) - 4] < \pi - \theta \leqslant \pi - \left(\frac{(v-1)\pi}{v}\right),$$

which reduces to

$$(v-1)[(v-2)(f-2) - 4] < 2.$$

Since $v \geqslant 3$ and $[(v-2)(f-2) - 4]$ is a positive integer, this last equation is a contradiction. This completes the proof. □

Now we define two kinds of bridges on the graph $\mathscr{H}(v, f)$.

Definition. An *N-step lattice bridge* (respectively *plane-bridge*) in $\mathscr{H}(v, f)$ is an N-step self-avoiding walk $\omega = (\omega(0), \omega(1), \ldots, \omega(N))$ for which there exist two disjoint lattice-lines (respectively plane-lines) L_0 and L_N with the following properties: $\omega(0) \in L_0$; $\omega(N) \in L_N$; $\omega(i) \notin L_0$ for every $i \geqslant 1$; and every site of ω lies in the set formed by L_0, L_N, and all points between these two lines. Let b_N (respectively b_N^*) denote the number of N-step lattice-bridges (respectively plane-bridges) ω such that $\omega(0) = 0$.

Proposition 3.3. *Fix a hyperbolic tiling* $\mathscr{H}(v, f)$.

(a) *There exists a constant* μ_B *such that*

$$\lim_{N \to \infty} (b_N)^{1/N} = \mu_B$$

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and

$$b_N \leqslant 2v(\mu_B)^N \quad \textit{for every } N \geqslant 1.$$

(b) *There exists a constant* K_B *such that*

$$b_N \leqslant b_N^* \leqslant K_B(\mu_B)^N \quad \textit{for every } N \geqslant 1.$$

Proof. **(a)** The proof is similar to the Euclidean case ([18], p. 11). Consider an N-step lattice-bridge ω^A (with lattice-lines L_0^A and L_N^A from the definition of 'bridge') and an M-step lattice-bridge ω^B (with lattice-lines L_0^B and L_M^B), both starting at the origin. Then there exists an automorphism of $\mathscr{H}(v,f)$ that maps the initial site $\omega^B(0)$ to the end site $\omega^A(N)$, maps the line L_0^B to the line L_N^A, and takes ω^B into the half-space determined by L_N^A which does not contain ω^A. The result is an $(N+M)$-step lattice-bridge (since the line L_M^B is mapped to a new lattice-line that does not intersect L_0^A). Notice that for a given $(N+M)$-step lattice-bridge that has been produced in this way, the walk ω^A is uniquely determined, but there are at most $2v$ possibilities for ω^B, corresponding to the v different possibilities for its first step and to whether or not the automorphism included a reflection. Therefore $b_{N+M} \geqslant b_N\, b_M/2v$, which is equivalent to $(b_{N+M}/2v) \geqslant (b_N/2v)(b_M/2v)$. The rest of the proof follows from subadditivity arguments (Lemma 1.1), exactly as in [18].

(b) Since every lattice-bridge is a plane-bridge, it is obvious that $b_N \leqslant b_N^*$. Now consider an arbitrary N-step plane-bridge ω, with corresponding plane-lines L_0 and L_N. We shall argue that ω can be extended by at most $2f$ steps at each end to make a lattice-bridge.

Let H be the half-plane with boundary L_N that does not contain $\omega(0)$. Then the site $\omega(N)$ is the vertex of some face F that is entirely contained in H. And F must have at least one vertex in common with another face F' that does not intersect L_N. Let z be a vertex on F' that is not in $\mathscr{N}(L_N)$. By Lemma 3.2, there exists a lattice-line L' through z that does not intersect L_N. It is not hard to see that there is a self-avoiding walk of length $2f$ or less, from $\omega(N)$ to z, that stays strictly between L_N and L'. (Notice that f steps is enough to go completely around a face, and that we can go around in either direction in case one way is 'blocked' by L_N or L'.) There is a similar extension of the beginning of ω (on the side of L_0 that does not contain $\omega(N)$). These two extensions of ω produce a lattice-bridge ω' whose length at most $N+4f$, and which starts at a vertex within $2f$ steps of the origin. The upper bound of (b) follows from this and from part (a). □

4. Results for ν and γ

The next result gives a sufficient condition for asserting that the critical exponents ν and γ both equal 1 in a weak sense. Recall the definition of $D_N^{(t)}$ from equation (1.5).

Theorem 4.1. *Assume that there exist constants* M *and* ρ *such that* $\rho < \mu_w$ *and*

$$c_N(x,y) \leqslant M\rho^N \quad \textit{for every } N \geqslant 0 \textit{ and all vertices } x \textit{ and } y. \tag{4.1}$$

Then the following bounds hold.

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(a) *For any* $\tilde{\rho}$ *such that* $\rho < \tilde{\rho} < \mu_w$, *there exist* $\epsilon > 0$ *and* $\tilde{M} > 0$ *such that*

$$\sum_{y:d(0,y)<\epsilon N} c_N(0,y) \leqslant \tilde{M}\tilde{\rho}^N \tag{4.2}$$

for every $N \geqslant 1$. *This implies that for every* $t > 0$ *there exists a constant* $A^{(t)} > 0$ *such that*

$$A^{(t)}N^t \leqslant D_N^{(t)} \leqslant N^t \quad \text{for every } N \geqslant 1. \tag{4.3}$$

(b) *There exists a constant* M' *such that*

$$(\mu_w)^N \leqslant c_N \leqslant M'(\mu_w)^N$$

for every $N \geqslant 0$.

Remark. In Section 6 we shall show that (4.1) holds whenever Condition LARGE is true.

Proof of Theorem 4.1. **(a)** For any hyperbolic graph, there exist constants A and Λ such that the number of sites y within a graph-distance N of the origin is at most $A\Lambda^N$ for every $N \geqslant 0$. Therefore

$$\sum_{y:d(0,y)<\epsilon N} c_N(0,y) \leqslant A\Lambda^{\epsilon N} M\rho^N.$$

The bound (4.2) follows upon taking ϵ small enough to satisfy $\Lambda^\epsilon \rho \leqslant \tilde{\rho}$, and letting $\tilde{M} = AM$. The upper bound in (4.3) is trivial, and the lower bound follows from (4.2), $c_N \geqslant \mu_w^N$, and

$$\sum_y [d(0,y)]^t c_N(0,y) \geqslant (\epsilon N)^t \left(c_N - \sum_{y:d(0,y)<\epsilon N} c_N(0,y) \right).$$

(b) We define a *line-walk* to be a self-avoiding walk whose final site lies in the neighbourhood of a lattice-line that contains the initial site. Let l_N be the number of N-step line-walks that start at the origin. Observe that the neighbourhood of each lattice-line through the origin contains at most $K_L N$ sites within N steps of the origin (besides the origin itself) for some constant K_L. Since there are at most v lattice-lines passing through the origin, it follows from our hypothesized bound (4.1) that

$$l_N \leqslant (vK_L N + 1)M\rho^N \quad \text{for every } N \geqslant 0. \tag{4.4}$$

Fix a lattice-line L_0 that contains the origin. For every site y that is not in $\mathcal{N}(L_0)$, Lemma 3.2 allows us to choose a lattice-line $L(y)$ containing y that does not intersect L_0. Consider an N-step self-avoiding walk ω that starts at 0 and ends at a point y not in $\mathcal{N}(L_0)$. Let H_0 be the half-plane bounded by L_0 that does not contain $L(y)$, and let H_y be the half-plane bounded by $L(y)$ that does not contain L_0. Recall that half-planes are closed sets. Define

$$t_1 = \max\{\, i \,:\, 0 \leqslant i \leqslant N,\ \omega(i) \in H_0 \,\},$$
$$t_2 = \min\{\, i \,:\, t_1 \leqslant i \leqslant N,\ \omega(i) \in H_y \,\}.$$

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Observe that $t_1 < t_2 \leqslant N$ since H_0 and H_y are disjoint. Let

$$\omega^A = (\omega(0), \dots, \omega(t_1)),$$
$$\omega^B = (\omega(t_1), \dots, \omega(t_2)),$$
$$\omega^C = (\omega(t_2), \dots, \omega(N)).$$

Then ω^A, ω^B, and ω^C are self-avoiding walks (ω^A or ω^C could have 0 steps). Observe that ω^A and ω^C are line-walks, and that ω^B is a plane-bridge. (Notice that ω^B must be a lattice-bridge if v and f are not both odd. If v and f are both odd, then the first site of ω^B could be in the interior of H_0, in which case there exists a plane-line through this site that is ultra-parallel to L_0; similarly for the last site.) Adding up all the possibilities for ω and y, including $y \in \mathcal{N}(\omega)$ as well as $y \notin \mathcal{N}(\omega)$, gives

$$\begin{aligned} c_N &\leqslant l_N + \sum_{j_1, j_2 \,:\, 0 \leqslant j_1 + j_2 < N} \sum l_{j_1} b^*_{N-j_1-j_2} l_{j_2} \\ &\leqslant (vK_L N + 1) M \rho^N + \sum_{j_1, j_2 \,:\, 0 \leqslant j_1 + j_2 < N} \sum (vK_L j_1 + 1) M \rho^{j_1} \left(K_B \mu_B^{N-j_1-j_2}\right) (vK_L j_2 + 1) M \rho^{j_2} \\ &\qquad \text{(by equation (4.4) and Proposition 3.3)} \\ &\leqslant M' \mu_w^N \qquad \text{(since } \mu_B \leqslant \mu_w\text{)}, \end{aligned}$$

where M' is defined by

$$M' = M^2 K_B \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} (vK_L j_1 + 1)(vK_L j_2 + 1) \left(\frac{\rho}{\mu_w}\right)^{j_1 + j_2}.$$

We know that M' is finite because $\rho/\mu_w < 1$. This completes the proof of part (b). □

Remark. The proof of part (b) above could be easily modified to prove that $\mu_B = \mu_w$. It turns out that we can prove this without the assumption (4.1); see Theorem 7.3.

5. Non-reversing walks

In this section we introduce walks that are self-avoiding up to some finite memory τ, as follows. Let τ be a nonnegative integer. Then $c_N(x, y; \tau)$ is the number of N-step ordinary walks ω from x to y such that $\omega(i) \neq \omega(j)$ for every i and j such that $0 \leqslant i < j \leqslant N$ and $|j - i| \leqslant \tau$. In particular, $c_N(x, y; 0)$ is the number of ordinary walks from x to y, which is the same as $A^N(x, y)$ defined in Section 2. Also, $c_N(x, y; 2)$ is the number of ordinary walks from x to y that have no steps that are reversals of the immediately preceding step. We call such walks 'non-reversing'.

We also define the associated generating functions for nonnegative values of z:

$$G_z(x, y; \tau) = \sum_{N=0}^{\infty} c_N(x, y; \tau) z^N.$$

These are often called 'two-point functions' in statistical mechanics.

The next result is an identity that relates the two-point functions of ordinary walks and of non-reversing walks. It was proved for the hypercubic graph $\mathbf{Z}^d$ by Domb and

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Fisher [3] (as cited by [11]), but their proof relied on Fourier analysis and does not seem to carry over to hyperbolic or more general graphs. A statement equivalent to this was proved by Wu and Kunz [29] for *finite* graphs using, among other things, an explicit matrix inversion, but it is not clear how to carry over their proof to the case of an infinite graph. Our proof is more direct.

Proposition 5.1. *Consider any (finite or infinite) graph $\mathscr{G}$ in which every site has exactly v neighbours. Define the functions*

$$S(z) = \frac{2(v-1)}{v-2+v\sqrt{1-4(v-1)z^2}},$$
$$T(z) = \frac{2z}{1+\sqrt{1-4(v-1)z^2}}.$$

Then for every pair of sites x and y in the graph, we have

$$G_z(x,y;0) = S(z)G_{T(z)}(x,y;2). \tag{5.1}$$

Alternatively, define the function $Q(t) = t/(1+(v-1)t^2)$. Then

$$G_t(x,y;2) = \frac{1-t^2}{1+(v-1)t^2} G_{Q(t)}(x,y;0). \tag{5.2}$$

Before we prove this result, we shall show how to use it to get a bound on μ_p. Define

$$\mu_{p,2} = \limsup_{N\to\infty} c_N(0,0;2)^{1/N}.$$

Since $p_N \leqslant c_N(0,0;2)$ for every N, it follows that $\mu_p \leqslant \mu_{p,2}$. The reciprocal $1/\mu_{p,2}$ is the radius of convergence of $G_z(0,0;2)$, and the following result shows that we can bound this using the identities of the preceding proposition.

Proposition 5.2. *Let R be the spectral radius of the hyperbolic graph $\mathscr{H}(v,f)$. Then*

$$\mu_p \leqslant \mu_{p,2} = \frac{R+\sqrt{R^2-4(v-1)}}{2} < R-1. \tag{5.3}$$

Proof of Proposition 5.2. Since the left-hand side of equation (5.1) converges for $z < 1/R$ and diverges for $z > 1/R$, so must the right-hand side. The radius of convergence of $S(z)$ is $1/\sqrt{4(v-1)}$, which is strictly greater than $1/R$ (by Corollary 11.7 of [28]). Therefore the radius of convergence of $G_z(x,y;2)$ equals

$$T\left(\frac{1}{R}\right) = \frac{2}{R\left(1+\sqrt{1-4(v-1)/R^2}\right)} = \frac{2}{R+\sqrt{R^2-4(v-1)}}. \tag{5.4}$$

Now the first two relations in (5.3) follow from the discussion preceding the proposition, and the final inequality follows from the fact that $v > R$. □

Proof of Proposition 5.1. Consider an infinite tree $\mathscr{T}_v$ in which every site has exactly v neighbours. Let e be a specified site of the tree, called the *root*. There is a map Φ from

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the vertices of $\mathscr{T}_v$ to the vertices of $\mathscr{G}$ that has the following properties:

(a) $\Phi(e) = 0$, and

(b) if x and y are neighbours in $\mathscr{T}_v$, then $\Phi(x)$ and $\Phi(y)$ are neighbours in $\mathscr{G}$, and

(c) if y_1 and y_2 are distinct neighbours of x in $\mathscr{T}_v$, then $\Phi(y_1) \neq \Phi(y_2)$.

Notice that Φ is necessarily a surjection. (This is an example of a 'covering map', as described on p. 120 of [28].)

Observe that for every nonnegative integer N, Φ induces a one-to-one correspondence between N-step ordinary walks in $\mathscr{T}_v$ starting at e and N-step ordinary walks in $\mathscr{G}$ starting at 0. In addition, this correspondence matches non-reversing walks in $\mathscr{T}_v$ with non-reversing walks in $\mathscr{G}$. These observations lead to the following relations between the two-point functions on $\mathscr{G}$ (denoted $G_z^{\mathscr{G}}$) and on $\mathscr{T}_v$ (denoted $G_z^{\mathscr{T}_v}$):

$$G_z^{\mathscr{G}}(0, y; \tau) \;=\; \sum_{u|y} G_z^{\mathscr{T}_v}(e, u; \tau) \quad \text{for } y \in V \text{ and } \tau = 0, 2, \tag{5.5}$$

where $\sum_{u|y}$ denotes the sum over all vertices u of $\mathscr{T}_v$ such that $\Phi(u) = y$.

Now, in a tree, the only non-reversing walk from one site to another is the unique self-avoiding walk joining those two sites. Therefore,

$$G_z^{\mathscr{T}_v}(e, u; 2) \;=\; z^{d'(e,u)} \tag{5.6}$$

where d' is graph distance in $\mathscr{T}_v$. Next, we use the following identity for trees:

$$G_z^{\mathscr{T}_v}(e, u; 0) \;=\; S(z)T(z)^{d'(e,u)} \tag{5.7}$$

([28], Lemma 1.24). Finally, we have the following:

$$\begin{aligned} G_z^{\mathscr{G}}(0, y; 0) &= \sum_{u|y} G_z^{\mathscr{T}_v}(e, u; 0) && \text{(by (5.5))} \\ &= \sum_{u|y} S(z)T(z)^{d'(e,u)} && \text{(by (5.7))} \\ &= S(z) \sum_{u|y} G_{T(z)}^{\mathscr{T}_v}(e, u; 2) && \text{(by (5.6))} \\ &= S(z) G_{T(z)}^{\mathscr{G}}(0, y; 2) && \text{(by (5.5)).} \end{aligned}$$

This proves equation (5.1). Equation (5.2) follows from this by inverting $T(z) = t$ to obtain $z = Q(t)$. □

6. Proof of the main theorem

In this section, we tie the results together that allow us to prove Theorem 1.2. First we complete the proof that $\mu_p < \mu_w$ whenever Condition LARGE holds.

Proof of Theorem 1.2(a). From Proposition 5.2 and inequality (2.3), we only need to show

$$\frac{a + \sqrt{a^2 - 4(v-1)}}{2} < \mu_w \tag{6.1}$$

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where $a = \sqrt{D^2 - I^2}$. Substitute v for D, the right-hand side of (2.2) for I, and the lower bound of μ_w given in Proposition 2.1 for μ_w in inequality (6.1). Then the solution of the resulting inequality is the set of v and f given in Condition LARGE. □

Next we need a short lemma that is important for verifying the hypothesis (4.1) in Theorem 4.1.

Lemma 6.1. *Consider a hyperbolic graph $\mathscr{H}(v, f)$. Then, for every $N \geqslant 0$ and every pair of sites x and y,*

$$c_{2N}(x, y; 2) \leqslant c_{2N+f}(0, 0; 2) \quad \textit{and} \quad c_{2N+1}(x, y; 2) \leqslant v\, c_{2N+f}(0, 0; 2), \tag{6.2}$$

and

$$c_N(x, y; 2) \leqslant \frac{v^2}{v-1}\, \mu_{p,2}^{N+f-1}. \tag{6.3}$$

Proof of Lemma 6.1. First observe that any $2N$-step non-reversing walk is the concatenation of two N-step non-reversing walks. This and the Schwarz inequality gives

$$\begin{aligned} c_{2N}(x, y; 2) &\leqslant \sum_u c_N(x, u; 2) c_N(u, y; 2) \\ &\leqslant \left(\sum_u c_N(x, u; 2)^2 \right)^{1/2} \left(\sum_u c_N(u, y; 2)^2 \right)^{1/2} \end{aligned} \tag{6.4}$$

where the sum is over all vertices u in the graph. Consider two N-step non-reversing walks, ω from x to u and ω' from u to x. It is always possible to choose an f-step walk ρ from u to u that goes around one face of $\mathscr{H}(v, f)$ in such a way that the first step of ρ uses a different edge from the last step of ω, and the last step of ρ uses a different edge from the first step of ω'. Therefore the $(2N + f)$-step walk formed by ω, ρ, and then ω' is non-reversing. This shows that

$$\begin{aligned} c_{2N+f}(x, x; 2) &\geqslant \sum_u c_N(x, u; 2) c_N(u, x; 2) \\ &= \sum_u c_N(x, u; 2)^2 \quad (\text{since } c_N(x, u; 2) = c_N(u, x; 2)). \end{aligned}$$

Since $c_{2N+f}(x, x; 2)$ is independent of x, applying the above inequality to (6.4) proves the first inequality in (6.2). The second inequality in (6.2) follows immediately from the first because

$$c_{2N+1}(x, y; 2) \leqslant \sum_{u\,:\,d(u,y)=1} c_{2N}(x, u; 2).$$

Consider two non-reversing walks ω and ω' from 0 to 0. Joining them produces a non-reversing walk from 0 to 0 if the first step of ω' uses a different edge from the last step of ω. Therefore

$$c_{N+M}(0, 0; 2) \geqslant c_N(0, 0; 2) \left(\frac{v-1}{v} c_M(0, 0; 2) \right)$$

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for every $N, M \geqslant 1$. By subadditivity (Lemma 1.1), we see that

$$\mu_{p,2} = \sup_{N \geqslant 1} \left(\frac{v-1}{v} c_N(0,0;2) \right)^{1/N},$$

i.e.,

$$c_N(0,0;2) \leqslant \frac{v}{v-1} \mu_{p,2}^N \quad \text{for every } N \geqslant 1.$$

Equation (6.3) follows from this inequality together with (6.2) and the fact that $\mu_{p,2} \leqslant v$. □

Finally we show that '$\gamma = 1$' and '$\nu = 1$' when Condition LARGE holds.

Proof of Theorem 1.2(b),(c). By Theorem 4.1, it suffices to show that Condition LARGE implies the bound (4.1). Since $c_N(x,y) \leqslant c_N(x,y;2)$, Lemma 6.1 shows that it suffices to prove that Condition LARGE implies $\mu_{p,2} < \mu_w$. But this is exactly the calculation that was described in the proof of part (a) above, so we are done. □

7. The bridge generating function

In this section we prove that μ_B equals μ_w and that 'γ is finite'. These proofs hold for all hyperbolic graphs $\mathscr{H}(v,f)$, in contrast to the results $\gamma = 1$ and $\nu = 1$ of preceding sections, which were only proved when Condition LARGE holds.

We introduce a class of SAWs that start on the boundary of a half-plane and stay inside that half-plane. This includes the set of lattice-bridges as a subset.

Definition. A *lattice half-space walk* (respectively *plane half-space walk*) in $\mathscr{H}(v,f)$ is a self-avoiding walk ω for which there exists a lattice-line (respectively plane-line) L containing $\omega(0)$ such that all of the sites of ω lie on the same side of L as $\omega(1)$, and $\omega(0)$ is the only site of ω that is in L. Let h_N (respectively h_N^*) denote the number of N-step lattice (respectively plane) half-space walks that start at the origin. We define $h_0 = 1 = h_0^*$.

We now introduce generating functions of several classes of SAWs.

Definition. For nonnegative z, define the generating functions

$$\chi(z) = \sum_{N=0}^{\infty} c_N z^N,$$

$$\beta(z) = \sum_{N=0}^{\infty} b_N z^N, \qquad \beta^*(z) = \sum_{N=0}^{\infty} b_N^* z^N,$$

$$H(z) = \sum_{N=0}^{\infty} h_N z^N, \qquad H^*(z) = \sum_{N=0}^{\infty} h_N^* z^N.$$

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Define the *critical point*

$$z_c = \frac{1}{\mu_w}.$$

By equation (1.1), we see that z_c is the radius of convergence of χ, and that $c_N \geqslant \mu_w^N$ for every N. The latter inequality implies that

$$\chi(z) \geqslant \left(1 - \frac{z}{z_c}\right)^{-1}. \tag{7.1}$$

We will show that β and H have the same radius of convergence as χ. In particular, we will show that μ_B (defined in Proposition 3.3) equals μ_w.

The proof of Proposition 3.3(b) shows that any plane-bridge in $\mathscr{H}(v,f)$ can be extended to a lattice-bridge by adding at most $2f$ steps to each end of the walk. Hence we have

$$\beta(z) \leqslant \beta^*(z) \leqslant (1 + z^{-1} + \cdots + z^{-4f})\beta(z) \qquad \text{for all } z \geqslant 0.$$

Therefore there is a constant K_β^* such that

$$\beta(z) \leqslant \beta^*(z) \leqslant K_\beta^* \beta(z) \qquad \text{for all } z \in [z_c/2, z_c]. \tag{7.2}$$

Similarly, any plane half-space walk in $\mathscr{H}(v,f)$ can be extended to a lattice half-space walk by adding at most $2f$ steps to the beginning of the walk, so

$$H(z) \leqslant H^*(z) \leqslant K_H^* H(z) \qquad \text{for all } z \in [z_c/2, z_c]. \tag{7.3}$$

for some constant K_H^*. In particular, $H(z)$ and $H^*(z)$ converge for the same positive z values (and similarly for $\beta(z)$ and $\beta^*(z)$).

We now require some geometry. Let S be a subset of the hyperbolic plane. The *convex hull* of S, denoted conv(S), is defined to be the intersection of all half-planes containing S. If S is a finite set, then conv(S) is a convex polygon. (See Sections 6.2–6.4 of [22] for a discussion of convex sets and polytopes in hyperbolic space.) We shall use the term *extreme point* to refer to a vertex of a convex polygon. The set conv(S) is also a polygon whenever S is the union of finitely many faces of a given tiling (since then conv(S) equals the convex hull of the set of vertices of the faces of S). Also, in this case, each extreme point of conv(S) is a vertex of one of the faces of S.

The next lemma says that the convex hull of the neighbourhood of a long self-avoiding walk must have many extreme points. Notice that this is false in Euclidean lattices. Also notice that the convex hull of a long straight self-avoiding walk is the walk itself, which has only two extreme points; this shows why we include the neighbourhood of the walk.

Lemma 7.1. *Fix a hyperbolic tiling $\mathscr{H}(v,f)$. There exist $\epsilon_A > 0$ and $\epsilon_F > 0$ with the property that for every $N \geqslant 1$ and every N-step SAW ω, the area of* conv($\mathscr{N}(\omega)$) *is at least $\epsilon_A N$, and* conv($\mathscr{N}(\omega)$) *has at least $\epsilon_F N$ extreme points.*

Proof. The area of conv($\mathscr{N}(\omega)$) is greater than or equal to the area of $\mathscr{N}(\omega)$, which is proportional to the number of faces in $\mathscr{N}(\omega)$. This in turn is bounded below by Nv/f since every site of ω is incident on v faces in $\mathscr{N}(\omega)$, and every face in $\mathscr{N}(\omega)$ is incident

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on at most f sites of ω. So we can take ϵ_A to be v/f times the area of one face. By Proposition 3.1(a), the number of extreme points of $\mathrm{conv}(\mathcal{N}(\omega))$ is at least the area of $\mathrm{conv}(\mathcal{N}(\omega))$ divided by π. Therefore we can take $\epsilon_A = \epsilon_F/\pi$. □

Proposition 7.2. *There is a constant A_H such that*

$$H(z) \geqslant \frac{A_H}{z_c - z} \qquad (0 < z < z_c).$$

Proof. The key geometric idea is that there are many ways to break any N-step SAW into two pieces that are both (almost) half-space walks. Let ω be an N-step SAW, and let y be an extreme point of $\mathrm{conv}(\mathcal{N}(\omega))$. By convexity, there is a half-plane H_y that contains $\mathrm{conv}(\mathcal{N}(\omega))$ and whose boundary contains y. As noted above, y is a vertex of some face F of $\mathcal{N}(\omega)$. We know that at least one vertex of F is a site of ω. Let ψ be the SAW that starts at y and traverses the boundary of F clockwise until it first reaches a site of ω; let i be the index such that $\omega(i)$ is the final site of ψ. Let N_ψ be the length of ψ. Then $1 \leqslant N_\psi < f$ and $\psi(N_\psi) = \omega(i)$.

Consider the two walks τ^A and τ^B that start with ψ and then follow ω from $\omega(i)$ either to the end or to the beginning, *i.e.*,

$$\tau^A = (\psi(0), \ldots, \psi(N_\psi - 1), \omega(i), \omega(i+1), \ldots, \omega(N)),$$
$$\tau^B = (\psi(0), \ldots, \psi(N_\psi - 1), \omega(i), \omega(i-1), \ldots, \omega(0)).$$

Then τ^A and τ^B are plane half-space walks of lengths $N - i + N_\psi$ and $i + N_\psi$ respectively.

Define $\tilde{h}_m = h^*_m + \cdots + h^*_{m+f}$ for $m \geqslant 0$. In the above argument, there are $\epsilon_F N$ ways to choose the extreme point y, so we conclude that

$$\epsilon_F N c_N \leqslant \sum_{m=0}^{N} \tilde{h}_m \tilde{h}_{N-m}.$$

Multiply this equation by z^N and sum over N, to get

$$\epsilon_F z \chi'(z) \leqslant \left(\sum_{k=0}^{\infty} \tilde{h}_k z^k \right)^2$$
$$\leqslant ([1 + z^{-1} + \cdots + z^{-f}] H^*(z))^2$$

for $z \geqslant 0$. The above inequality and (7.3) imply that

$$\chi'(z) \leqslant K_2 H(z)^2 \quad \text{for all } z \in [z_c/2, z_c] \tag{7.4}$$

for some constant K_2. In particular, since $\chi'(z_c) = +\infty$ (this is because $c_N \geqslant \mu_w^N = z_c^{-N}$), we deduce that $H(z_c) = +\infty$. Next, since $H'(z) \leqslant \chi'(z)$ (because half-space walks are a subset of all SAWs), the inequality (7.4) implies that

$$H'(z) \leqslant K_3 H(z)^2 \quad \text{for all } z \in [z_c/2, z_c]$$

for some constant K_3. That is,

$$K_3 \geqslant \frac{H'(z)}{H(z)^2} = -\frac{d}{dz}\left(\frac{1}{H(z)}\right) \quad \text{for all } z \in [z_c/2, z_c].$$

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Now integrate both sides of this inequality from z to z_c, where $z \in (z_c/2, z_c)$. This gives

$$K_3(z_c - z) \geqslant -\frac{1}{H(z_c)} + \frac{1}{H(z)} = \frac{1}{H(z)} \quad \text{for all } z \in (z_c/2, z_c),$$

which proves

$$H(z) \geqslant \frac{K_3^{-1}}{z_c - z} \quad \text{for all } z \in (z_c/2, z_c).$$

Since $H(z) \geqslant H(0) = 1$ for all $z \geqslant 0$, the proposition follows. □

Theorem 7.3. *The bridge generating function $\beta(z)$ diverges at $z = z_c$. Therefore $\mu_B = \mu_w$.*

Proof. We begin with a construction similar to the one in the proof of Proposition 7.2, showing that there are many (on the order of N^2) ways to break any N-step SAW into three pieces, consisting approximately of a bridge between two half-space walks. Let ω be an N-step SAW. For each extreme point q of $\text{conv}(\mathcal{N}(\omega))$, let L_q be a plane-line that passes through q but does not contain any other point of $\text{conv}(\mathcal{N}(\omega))$. Let H_q be the half-plane whose boundary is L_q that contains $\text{conv}(\mathcal{N}(\omega))$. We claim that for each extreme point q of $\text{conv}(\mathcal{N}(\omega))$, there are at least $(\epsilon_A/\pi)N - 3$ other extreme points y such that $L_y \cap L_q = \emptyset$ (where ϵ_A was defined in Lemma 7.1). We shall verify this claim at the end of the proof.

Let y and u be two extreme points of $\text{conv}(\mathcal{N}(\omega))$ such that $L_y \cap L_u = \emptyset$. Let F_y (respectively F_u) be a face of $\mathcal{N}(\omega)$ containing y (respectively u). Let ψ (respectively ζ) be the SAW that starts at y (respectively u) and traverses the boundary of F_y (respectively F_u) clockwise until it first reaches a site of ω. Let i (respectively j) be the index such that $\omega(i)$ (respectively $\omega(j)$) is the final point of ψ (respectively ζ). Let N_ψ and N_ζ be the lengths of ψ and ζ. Without loss of generality, assume $i \leqslant j$.

Consider the three walks τ^A, τ^B, and τ^C obtained by breaking ω at $\omega(i)$ and $\omega(j)$ and extending the ends to y and u, *i.e.*,

$$\begin{aligned}
\tau^A &= (\psi(0), \ldots, \psi(N_\psi - 1),\ \omega(i), \omega(i-1), \ldots, \omega(0)),\\
\tau^C &= (\zeta(0), \ldots, \zeta(N_\zeta - 1),\quad \omega(j), \omega(j+1), \ldots, \omega(N)),\\
\tau^B &= (\psi(0), \ldots, \psi(N_\psi - 1),\ \omega(i), \omega(i+1), \ldots, \omega(j),\quad \zeta(N_\zeta - 1), \ldots, \zeta(0)).
\end{aligned}$$

Then τ^A, τ^B, and τ^C are SAWs lying in $\mathcal{N}(\omega)$. We see that τ^A and τ^C are plane half-space walks of lengths $i + N_\psi$ and $N - j + N_\zeta$ respectively. Also, since $L_y \cap L_u = \emptyset$, we know that τ^B is a plane-bridge of length $j - i + N_\psi + N_\zeta$. By Lemma 7.1 and the claim at the end of the first paragraph of this proof, there are at least $(\epsilon_F N)([\epsilon_A/\pi]N - 3)/2$ (unordered) pairs of extreme points $\{y, u\}$ such that $L_y \cap L_u = \emptyset$. Therefore the above decomposition implies that, for sufficiently large N,

$$\hat{K} N^2 c_N \leqslant \sum_{i,j,k \geqslant 0 \,:\, N < i+j+k < N+4f} h_i^* b_j^* h_k^*,$$

where $\hat{K} = \epsilon_F \epsilon_A / 8$. By decreasing the constant $\hat{K}$ if necessary, we can make the above inequality hold for every nonnegative N. As we have argued before, this shows that there

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is a positive constant K^* such that

$$K^* z^2 \chi''(z) \leqslant H^*(z)^2 \beta^*(z) \quad \text{for all } z \in [z_c/2, z_c].$$

Therefore, equations (7.2) and (7.3) imply

$$\chi''(z) \leqslant K_4 H(z)^2 \beta(z) \quad \text{for all } z \in [z_c/2, z_c]. \tag{7.5}$$

Now consider a lattice-line L_0 passing through the origin. Consider two lattice half-space walks ρ^A and ρ^B of any lengths, starting at the origin and staying on opposite sides of L_0 from each other. Then the walk obtained by the union of ρ^A and ρ^B is a self-avoiding walk. This shows that

$$c_{i+j} \geqslant \frac{h_i}{v} \frac{h_j}{v} \quad \text{for all } i, j \geqslant 0, \tag{7.6}$$

which leads to

$$(N+1) c_N \geqslant \frac{1}{v^2} \sum_{i,j: i+j=N} h_i h_j.$$

This implies that there is a positive constant K_5 such that

$$\chi'(z) \geqslant K_5 H(z)^2 \quad \text{for all } z \in [z_c/2, z_c]. \tag{7.7}$$

Combining equations (7.5) and (7.7) implies

$$K_6 \beta(z) \geqslant \frac{\chi''(z)}{\chi'(z)} = \frac{d}{dz} \ln \chi'(z) \quad \text{for all } z \in [z_c/2, z_c]. \tag{7.8}$$

Since $z\chi'(z) \geqslant \chi(z) - 1$, we know from (7.1) that $\ln \chi'(z)$ diverges to $+\infty$ as $z \nearrow z_c$. Hence

$$\limsup_{z \nearrow z_c} \frac{d}{dz} \ln \chi'(z) = +\infty.$$

This and (7.8) imply that $\beta(z_c) = +\infty$. Hence the radius of convergence of $\beta(z)$, which is μ_B^{-1}, cannot be greater than z_c, which equals μ_w^{-1}. But $\mu_B \leqslant \mu_w$, so we must have $\mu_B = \mu_w$.

Finally, we must verify the geometric claim made at the end of the first paragraph of this proof. For brevity, write $\mathscr{C} = \text{conv}(\mathscr{N}(\omega))$. Fix an extreme point q of $\mathscr{C}$, and let M be the number of extreme points of $\mathscr{C}$. Let $q_1, q_2, \ldots, q_M$ be the extreme points of $\mathscr{C}$, starting with $q_1 = q$ and proceeding clockwise around the boundary of $\mathscr{C}$. Let θ_i be the interior angle of $\mathscr{C}$ at q_i $(i = 1, \ldots, M)$. By Lemma 7.1 and Proposition 3.1,

$$\epsilon_A N \leqslant (\text{area of } \mathscr{C}) = \sum_{i=1}^{M} (\pi - \theta_i) - 2\pi. \tag{7.9}$$

Let

$$J = \min\left\{ j \geqslant 2 \,:\, \sum_{i=2}^{j} (\pi - \theta_i) > \pi \right\} \quad \text{and} \quad K = \max\left\{ k \leqslant M \,:\, \sum_{i=k}^{M} (\pi - \theta_i) > \pi \right\}.$$

We shall show (i) $K - J - 1 > (\epsilon_A/\pi)N - 3$, and (ii) $L_q \cap L_{q_i} = \emptyset$ whenever $J < i < K$. Then these two assertions will immediately imply the geometric claim that we are trying to prove.

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First we shall verify assertion (i). Since $0 < \pi - \theta_i < \pi$ for every i, we have $J > 2$, $K < M$,

$$\sum_{i=2}^{J}(\pi - \theta_i) < 2\pi, \quad \text{and} \quad \sum_{i=K}^{M}(\pi - \theta_i) < 2\pi.$$

These inequalities together with equation (7.9) give

$$\begin{aligned}\epsilon_A N &\leqslant (\pi - \theta_1) + \sum_{i=2}^{J}(\pi - \theta_i) + \sum_{i=J+1}^{K-1}(\pi - \theta_i) + \sum_{i=K}^{M}(\pi - \theta_i) - 2\pi \\ &< \pi + 2\pi + (K - J - 1)\pi + 2\pi - 2\pi.\end{aligned}$$

Assertion (i) follows immediately.

Now for assertion (ii). Let i be an integer such that $J < i < K$. Assume that L_q and L_{q_i} intersect at a point y_0. Removing $\mathscr{C}$ from the wedge-shaped region $H_q \cap H_{q_i}$ produces two connected components, of which one is bounded and the other is unbounded. Let $\mathscr{R}$ be the bounded component of $(H_q \cap H_{q_i})\backslash\mathscr{C}$. Then $\mathscr{R}$ is a non-convex polygon, with vertices y_0, y_1, y_i, and either $y_2, y_3, \ldots, y_{i-1}$ or $y_{i+1}, y_{i+2}, \ldots, y_M$. Assume that it is the first case; the argument for the second case proceeds identically. Let ϕ_j be the internal angle of $\mathscr{R}$ at y_j, for each appropriate j. Then $\phi_j = 2\pi - \theta_j$ for $j = 2, \ldots, i-1$, so Proposition 3.1(a) tells us that

$$\begin{aligned}\text{Area of } \mathscr{R} &= (i-1)\pi - \phi_0 - \phi_1 - \phi_i - \sum_{j=2}^{i-1}(2\pi - \theta_j) \\ &< \pi - \sum_{j=2}^{i-1}(\pi - \theta_j) \\ &\leqslant \pi - \sum_{j=2}^{J}(\pi - \theta_j) \quad (\text{since } i - 1 \geqslant J) \\ &< 0 \quad (\text{by the definition of } J).\end{aligned}$$

Since areas are not negative, this gives a contradiction. Hence (ii) holds, and the proof of the Theorem is complete. □

The next theorem says essentially that the exponent γ is finite, at least in the sense of generating functions.

Theorem 7.4. *There are finite positive constants A_X and p such that*

$$\chi(z) \leqslant \frac{A_X}{(z_c - z)^p} \qquad (0 < z < z_c).$$

Proof. Proposition 3.3 and Theorem 7.3 imply that

$$\beta(z) \leqslant 2v \sum_{N=0}^{\infty} \mu_w^N z^N = \frac{2vz_c}{z_c - z}.$$

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Applying this bound to equation (7.8) gives

$$\frac{2vz_cK_6}{z_c - z} \geqslant \frac{d}{dz} \ln \chi'(z) \quad \text{for all } z \in [z_c/2, z_c].$$

Integrate both sides of this inequality from $z_c/2$ to t, where $t \in (z_c/2, z_c)$. This gives

$$2vz_cK_6(\ln(z_c/2) - \ln(z_c - t)) \geqslant \ln \chi'(t) - \ln \chi'(z_c/2) \quad \text{for all } t \in (z_c/2, z_c).$$

Let $p = 2vz_cK_6$ and $K_7 = p \ln(z_c/2) + \ln \chi'(z_c/2)$. Then we obtain

$$(z_c - t)^{-p} \exp(K_7) \geqslant \chi'(t) \geqslant \chi(t) - 1 \quad \text{for all } t \in (z_c/2, z_c).$$

The theorem now follows. □

8. Conclusions

We have studied asymptotic properties of self-avoiding walks on the hyperbolic graphs $\mathscr{H}(v, f)$. We have shown that all of these graphs, except for at most eight of them, have the properties that $\mu_p < \mu_w$, $\gamma = 1$, and $\nu = 1$. These results provide rigorous support for the numerical conclusions of Swierczak and Guttmann [26]. These properties reflect the behaviour of ordinary random walks on non-amenable graphs, as well as of self-avoiding walks on v-regular trees (in which $\mu_p = 0 < \mu_w$, $c_N = v(v-1)^N$, and $D_N^{(t)} = N^t$). We expect that these results should hold generically for 'non-amenable' graphs – *i.e.*, graphs in which the number of sites within N steps of the origin grows exponentially in N.

We have also proved on every hyperbolic graph that γ is finite (in the sense of generating functions) and that the connective constant for bridges equals μ_w. Lastly, we proved an identity relating the two-point function of non-reversed walks to the two-point function of ordinary walks. This identity, previously proved for $\mathscr{Z}^d$ and for finite regular graphs, is shown to hold on every (finite or infinite) regular graph.

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Note added in proof. For another approach to proving Proposition 5.1, which predates our own, see Bartholdi, L. (1999) Counting paths in graphs. *Enseign. Math.* **45** 83–131.

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