PARAMETRIC SURFACES

I. AREA

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1. In this paper I consider various properties of the area of a surface including the relationship between Hausdorff measure and the Lebesgue area. The results here obtained will enable me, in a subsequent paper, to investigate the tangential properties of parametric surfaces. For simplicity of exposition I confine my attention to surfaces homoeomorphic to a disk, but the general case follows easily from cyclic element theory.

2. Let $P = \phi(M)$ be a continuous function, where M varies on a disk H, and P is a point of three-dimensional space. *Any saturated continuum Q in H on which ϕ is constant is called a ϕ-element.* Thus $\phi(M)$ establishes a representation of H as a sum of different ϕ-elements, some of which may of course be single points.

We form the set of pairs

$$S = \Sigma(P, Q), \tag{1}$$

where Q runs through all ϕ-elements of H and $P = \phi(Q)$. Then S is called a parametric surface, (P, Q) a point of the surface.

Two points (P, Q), (P', Q') are identical only if $Q = Q'$. If exactly k different points of S have the same P we say P *has multiplicity k on S.* Let now G be a subset of M. By $\phi(G)$ I mean the set of points $\phi(M)$ of the surface for all $M \in G$. Thus $\phi(G)$ is the set of points (P, Q) for which Q meets G; if there are exactly k' such points with a given P I say P *has multiplicity k'* on G. Further, I denote by $|\phi(G)|$ the set of points of three-dimensional space, without counting multiplicity, where $P = \phi(M)$ and $M \in G$.

Let $X_1 = (P_1, Q_1)$ and $X_2 = (P_2, Q_2)$ be a pair of points of the parametric surface S, and C any continuum in H joining Q_1, Q_2.

We define the S-distance between X_1 and X_2 as the lower bound of the diameter of the ϕ image of C. That is,

$$D_S(X_1, X_2) = \inf d\{\phi(C)\}. \tag{2}$$

Also we define $\delta_S(X_1, X_2)$ by

$$\delta_S(X_1, X_2) = \inf \Lambda\phi(C). \tag{3}$$

Clearly

$$\delta_S(X_1, X_2) \geqslant D_S(X_1, X_2) \geqslant D_{\text{Euc}}(X_1, X_2). \tag{4}$$

3. We say two surfaces $S_1 = \phi_1(H)$, $S_2 = \phi_2(H)$ are identical if the Fréchet distance between them is zero. That is, if for any $\epsilon > 0$ there exists an automorphism of H, $M = \psi(M')$ such that for any M' of H

$$D_{\text{Euc}}\{\phi_1\psi(M'), \phi_2(M')\} < \epsilon. \tag{5}$$

Suppose that Q, Q_n are ϕ-elements; then if the Euclidian distance between Q, Q_n tends to zero as n tends to infinity we say

$$Q_n \to Q. \tag{6}$$

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A (1-1) correspondence $Q'' = f(Q')$ between ϕ_1-elements and ϕ_2-elements is termed a homoeomorphism if

$$Q'_n \to Q' \rightleftharpoons f(Q'_n) \to f(Q'). \tag{7}$$

If there exists such a homoeomorphism where, in addition,

$$\phi_2 f(Q') = \phi_1(Q'), \tag{8}$$

then we say the surfaces are K-equivalent. By (9) identical surfaces are K-equivalent.

Suppose now that all ϕ-elements are simply connected; then, by (4), there exists a representation of the surface in which all ϕ-elements are single points and the surface is homoeomorphic to a disk. *For simplicity we suppose throughout this paper that $S = \phi(H)$ is homoeomorphic to a disk and that ϕ is a representation for which all ϕ-elements are single points.*

We define the Λ^2 *measure of* $\phi(G)$ *by*

$$\Lambda^2\phi(G) = \sum_1^\infty k\Lambda^2 E_k, \tag{9}$$

where E_k is the set of points P of multiplicity k on G and $\Lambda^2 E_k$ is the Hausdorff two-dimensional measure†. The term corresponding to $k = \infty$ is taken zero or infinite according as $\Lambda^2 E_\infty$ is zero or not. We define $\Lambda\phi(G)$ similarly using Hausdorff linear measure.

Let $L(S)$ *be the Lebesgue area.* $L\phi(R)$ is defined for any finitely connected Jordan domain R in H; define $L\phi(G)$ on any open G as

$$\sup \Sigma L\phi(R), \tag{10}$$

the sum being taken over any set of disjoint finitely connected Jordan domains R in G.

I now define $L^*(X)$ for any X in H°, the interior of H, by

$$L^*\phi(X) = L^*(X) = \inf_{G \supset X} L\phi(G). \tag{11}$$

This is a Caratheodory outer measure on H°. If D is a finitely connected Jordan domain $L^*(D) = L\phi(D)$. I use the notation $L^*(X)$ whenever it is clear what ϕ is involved.

4. Suppose that D is a domain in H and ϕ^* a continuous function on H. Then I denote by

$$\{\phi^*(D)\}^\circ \tag{12}$$

the interior points of $\phi^*(D)$. That is, the set of points (P^*, Q^*) of $S^* = \phi^*(H)$ such that the ϕ^*-element Q^* is contained in D.

Consider now the class Δ of functions ϕ^* continuous in H and equal to ϕ on $H - D$. Then I define $\mu\{\phi, D\}$ by

$$\mu\{\phi, D\} = \inf_\Delta \Lambda^2\{\phi^*(D)\}^\circ. \tag{13}$$

By (7), $\mu\{\phi, D\}$ is attained whenever D is a finitely connected Jordan domain.

5. DEFINITION. *Consider the class R of sets $\{r_i\}$ of disjoint simply-connected Jordan domains in H. Then I define $A(S)$ by*

$$A(S) = \sup_R \sum_i \mu\{\phi, r_i\}. \tag{14}$$

THEOREM 1. *$A(S)$ is independent of the representation ϕ and is lower semicontinuous.*

† Throughout the proofs in this paper I shall use Hausdorff *spherical* measure; it will be evident that the results are equally true with the ordinary Hausdorff measure.

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For the present denote $A(S)$, where $S = \phi(H)$, by $A_\phi(S)$. Let $S_n = \phi_n(H)$ be a sequence of parametric surfaces and suppose $\phi_n(M)$ tends uniformly to $\phi(M)$ for $M \subset H$. Then if r is any simply-connected Jordan domain in H

$$\mu\{\phi, r\} \leqslant \underline{\lim}\, \mu\{\phi_n, r\}. \tag{15}$$

For to prove this we may assume that $\mu\{\phi_n, r\}$ is bounded so that it is attained by (7). Let S_n^* be a surface subtending $\phi_n(\gamma)$, where γ is the boundary of r, and such that

$$\Lambda^2(S_n^*)^\circ = \mu\{\phi_n, r\}. \tag{16}$$

We may suppose S_n^* is smooth and hence (compare (7), p. 244) the S_n^* are compact and converge lower-semi-continuously in a subsequence to a surface S^*. S^* will subtend $\phi(\gamma)$ and hence

$$\mu\{\phi, r\} \leqslant \Lambda^2(S^*)^\circ \leqslant \underline{\lim}\, \Lambda^2(S_n^*)^\circ = \underline{\lim}\, \mu\{\phi_n, r\},$$

which is (15).

Considering now the sum $\sum_i \mu\{\phi, r_i\}$ taken over any set of disjoint r_i we obtain

$$A_\phi(S) \leqslant \underline{\lim}\, A_{\phi_n}(S_n). \tag{17}$$

Suppose now that the two surfaces $S_1 = \phi_1(H)$ and $S_2 = \phi_2(H)$ are identical. Then there exists a sequence ψ_n of automorphisms of H such that $\phi_1\psi_n(M)$ tends uniformly to $\phi_2(M)$ for $M \subset H$.

Now as ψ_n is an automorphism we have $A_{\phi_1\psi_n}(S_1) = A_{\phi_1}(S_1)$ and so

$$A_{\phi_2}(S_2) \leqslant \underline{\lim_n}\, A_{\phi_1\psi_n}(S_1) \leqslant \underline{\lim_n}\, A_{\phi_1}(S_1) = A_{\phi_1}(S_1). \tag{18}$$

Similarly, we have that $A_{\phi_1}(S_1) \leqslant A_{\phi_2}(S_2)$, and so $A(S)$ is independent of the representation ϕ.

COROLLARY. $$A(S) \leqslant L(S). \tag{19}$$

The corollary follows at once as for a polyhedron $A(S) = L(S)$.

6. THEOREM 2. *Given $\epsilon > 0$ there exists a $\delta > 0$ such that for any set of disjoint simply-connected Jordan domains $\{r_i\}$ in H for which (a) the diameter of r_i is at most δ, (b) $\Lambda^2\phi(H^\circ - \sum_i (r_i)) = 0$, we have*†

$$\sum_i \mu\{\phi, r_i\} > A(S) - \epsilon. \tag{20}$$

There exists a finite set of disjoint simply-connected Jordan domains R_i with boundaries C_i such that

$$\sum_i \mu\{\phi, R_i\} > A(S) - \tfrac{1}{2}\epsilon. \tag{21}$$

Suppose that C'_{in} is a sequence of simple closed Jordan curves in R_i such that $\phi C'_{in}$ tends to ϕC_i in the sense of Fréchet, as $n \to \infty$; then

$$\mu\{\phi, R_i\} \leqslant \underline{\lim_n}\, \mu\{\phi, R'_{in}\}, \tag{22}$$

where R'_{in} is the interior domain of C'_{in}. For by (7) there exists a smooth surface S_{in} subtending $\phi C'_{in}$ and having

$$\Lambda^2 S_{in}^\circ = \mu\{\phi, R'_{in}\}. \tag{23}$$

† When $(AS) = \infty$ read $1/\epsilon$ for $(AS) - \epsilon$.

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To prove (22) we may assume that $\Lambda^2 S_{in}^\circ$ is bounded, whence by p. 244 of (7) the surfaces S_{in} form a compact set and converge lower-semi-continuously in a subsequence to some surface S_i. S_i will then subtend the contour ϕC_i and so

$$\mu\{\phi, R_i\} \leqslant \Lambda^2 S_i^\circ \leqslant \underline{\lim_n} \Lambda^2 S_{in}^\circ = \underline{\lim_n} \mu\{\phi, R'_{in}\}. \tag{24}$$

which gives (22).

Hence there exist δ_i so small that if the Fréchet distance between $\phi C'_i$ and ϕC_i is less than δ_i, where C'_i is a simple closed Jordan curve in R_i enclosing a domain R'_i, then

$$\sum_i \mu\{\phi, R'_i\} \geqslant \sum_i \mu\{\phi, R_i\} - \tfrac{1}{2}\epsilon. \tag{25}$$

As ϕ is continuous there exists $\delta_1 > 0$ such that if X is any set in H of diameter less than δ_1 then $\phi(X)$ will have diameter less than $\frac{1}{2} \min_i (\delta_i)$.

Each C_i is a Jordan curve and so uniformly locally connected. Thus we may choose $\delta < \frac{1}{2}\delta_1$ so that for any $x \in C_i$ there is a connected set in $c(x, \frac{1}{4}\delta_1) C_i$ containing all of $c(x, 2\delta) C_i$, where $c(x, r)$ denotes the circle of radius r and centre x.

Suppose now that $\{r_j\}$ is any set of disjoint simply-connected Jordan domains satisfying conditions (a) and (b) of this theorem; we may assume without loss of generality that it is a finite set. Suppose that r_j meets C_i; then by the above there will be an arc C_{ij} of C_i containing all of $C_i r_j$ and of diameter less than $\frac{1}{2}\delta_1$. There will be only one component of $R_i - \bar{r}_j$ having boundary points on $C_i - C_{ij}$; denote it by R_{ij}. Then

$$d\{R_i - R_{ij}\} < \delta_1 \quad \text{and} \quad R_i - R_{ij} - R_{ij'} = 0,$$

provided that $j \neq j'$. Hence $R''_i = \prod_j R_{ij}$ will contain a component R'_i such that if C'_i is the boundary of R'_i then C'_i will be a simple closed Jordan curve within Fréchet distance δ_1 of C_i. Thus by the choice of δ_1 and by (25) and (21) we have

$$\sum_i \mu\{\phi, R'_i\} \geqslant \sum_i \mu\{\phi, R_i\} - \tfrac{1}{2}\epsilon \geqslant A(S) - \epsilon. \tag{26}$$

But any r_j meeting R'_i is contained in it and $\Lambda^2\phi(R'_i - \Sigma r_j) = 0$ by (b), and so

$$\sum_j \mu\{\phi, r_j\} \geqslant \sum_i \mu\{\phi, R'_i\} \geqslant A(S) - \epsilon,$$

which establishes the theorem.

7. We will need two lemmas proved elsewhere.

LEMMA 1. *If E is any set (whether containing multiple points or not) and $E(r)$ is its intersection with the surface of the sphere of centre O and radius r, then*

$$\Lambda^2 E \geqslant \int_0^\infty \Lambda E(r)\, dr. \tag{27}$$

This is Lemma 3·1 of (6).

LEMMA 2. *From any set of equi-continuous uniformly bounded functions on a bounded set a uniformly convergent subsequence can be selected.*

This is shown in (10).

LEMMA 3. *If all ϕ-elements are single points and C is a continuum in H such that $\Lambda\phi(C) < \infty$ then C is a Peano space.*

$\phi(C)$ contains no continuum of convergence, counting multiplicity, and hence neither does C itself; that is, it is locally connected and hence a Peano space.

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8. *Suppose that D_1 is a domain; then I denote by D' the interior points of the closure $\overline{D_1}$ of D_1. Further, let D be the simply-connected domain consisting of the interior points of the simply-connected closed set obtained from $\overline{D}'$ by adding all its interior complementary domains. Denote the boundary of D by l.*

If $\Lambda\phi l < \infty$ then l is a simple closed Jordan curve. For by Lemma 3 it is a Peano space and it separates the plane into two domains, of both of which it is the boundary and one of which, D, is simply connected.

If $D_1 = D_1(i,j,\ldots)$ depends on a set of variables $i,j,\ldots$, then I write

$$D' = D'(i,j,\ldots), \quad D = D(i,j,\ldots) \quad \text{and} \quad l = l(i,j,\ldots).$$

I use this convention throughout this paper.

9. Denote the closed sphere of centre P and radius r by $S(P,r)$, and let $S^{-1}(P,r)$ denote the points of H° whose ϕ-image is *interior* to $S(P,r)$. Suppose that $D_1(r)$ is a component of $S^{-1}(P,r)$.

Then we have the following lemma:

LEMMA 4. *Suppose that $S = \phi(H)$ is homoeomorphic to a disk, and that $r_1 > r_2$ while*

$$D(r_1) \supset D(r_2).$$

Then if $D_1(h)$ is the component of $S^{-1}(P,h)$ which contains $D_1(r_2)$, we have

$$L^*\phi\{D'(r_1) - \overline{D'(r_2)}\} \geqslant \int_{r_2}^{r_1} \Lambda\phi l(h)\, dh. \tag{28}$$

If $S'_n = \phi'_n(R)$ and $S' = \phi'(R)$ are surfaces homoeomorphic to a finitely connected Jordan domain R, and $S'_n \to S'$, then by the definition of Fréchet distance there exist representations $S'_n = \phi_n(R)$ of the surfaces S'_n such that all ϕ_n-elements are single points and $\phi_n(x) \to \phi'(x)$ for all x of R. Moreover, if $S''_n = \phi''_n(R)$ is a polyhedral surface there will exist a polyhedral surface homoeomorphic to R approximating to S''_n as closely as we please both in area and in Fréchet distance. For a polyhedral surface has at most a finite number of cyclic elements, and so we may deform the surface slightly near each point common to two different cyclic elements in such a manner that the two cyclic elements become one.

Thus if D_m is a finitely connected Jordan domain in H and $S = \phi(H)$ is a given surface, there exists a sequence $S_{km} = \phi_{km}(H)$ of polyhedral surfaces such that each ϕ_{km} is bicontinuous and

$$\phi_{km}(x) \to \phi(x) \quad \text{for each} \quad x \subset H, \tag{29}$$

while

$$\Lambda^2\phi_{km}(D_m) \to L^*\phi(D_m). \tag{30}$$

Consider now $D'(r_1) - \overline{D'(r_2)}$. Take $r_2 < r''_2 < r''_1 < r_1$. Denote the complementary domains of $\overline{D'(r_1) - \overline{D'(r_2)}}$ by G_i $(i = 1,2,3,\ldots)$. Then $D'(r_1) - \overline{D'(r_2)} + \sum_m^\infty G_i$ will be a finitely connected domain containing $\overline{D'(r''_1) - \overline{D'(r''_2)}}$.

Thus we can define a decreasing sequence D_m of finitely connected Jordan domains such that

$$D'(r''_1) - \overline{D'(r''_2)} \subset D_m \subset D'(r_1) - \overline{D'(r_2)} + \sum_m^\infty G_i. \tag{31}$$

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Then† $$\lim L^*\phi(D_m) \leqslant L^*\phi\{D'(r_1) - \overline{D'(r_2)}\}. \tag{32}$$

By equations (29)–(32) we can choose a subsequence ϕ_n of the double sequence ϕ_{km} such that $$\overline{\lim}\, \Lambda^2\phi_n\{D'(r_1'') - \overline{D'(r_2'')}\} \leqslant L^*\phi\{D'(r_1) - \overline{D'(r_2)}\}. \tag{33}$$

Now take $$r_2 < r_2'' < r_2' < r_1' < r_1'' < r_1.$$

There exists n'' such that for $n > n''$, $\phi_n \overline{D_1(r_2'')} \subset S^\circ(P, r_2')$. For $h \geqslant r_2'$ let $D_1(h, n)$ be the component of $\phi_n^{-1}S^\circ(P, h)$ which contains $\overline{D_1(r_2'')}$. For $n > n''$ and $h \geqslant r_2'$ $D'(h, n)$ and $l(h, n)$ are then defined, while $D'(h, n) \supset \overline{D'(r_2'')}$. There exists $n' > n''$ such that for $n > n'$ the boundary of $\phi_n D'(r_1'')$ lies outside $S(P, r_1')$, and hence

$$D'(r_1', n) \subset D'(r_1''). \tag{34}$$

Let $S(P, h)$ be the surface of $S(P, h)$. Then

$$\phi_n l(h, n) \subset s(P, h)\, \phi_n\{D'(r_1', n) - \overline{D'(r_2', n)}\} \subset s(P, h)\, \phi_n\{D'(r_1'') - \overline{D'(r_2'')}\} \tag{35}$$

for $n > n'$ and $r_2' < h < r_1'$. Hence by Lemma 1

$$\int_{r_2'}^{r_1'} \Lambda\phi_n l(h, n)\, dh \leqslant \Lambda^2\phi_n\{D'(r_1'') - \overline{D'(r_2'')}\}. \tag{36}$$

Let $n \to \infty$; then in virtue of (33)

$$\overline{\lim} \int_{r_2'}^{r_1'} \Lambda\phi_n l(h, n)\, dh \leqslant L^*\phi\{D'(r_1) - \overline{D'(r_2)}\}. \tag{37}$$

By Fatou's theorem, for almost all h

$$\underline{\lim}\, \Lambda\phi_n l(h, n) < \infty, \tag{38}$$

and so, for almost any fixed h, there exists a subsequence of the n where $\Lambda\phi_n l(h, n)$ is bounded in n.

Consider this subsequence and $r_1' < h < r_2'$. By Lemma 3 $l(h, n)$ is a Jordan curve. Take a fixed point O on $l(h, n)$ and for any point x on $l(h, n)$ define $S(x)$ to be the length of the arc $\phi_n(\widehat{Ox})$ taken in a fixed direction. Put

$$t = \frac{2\pi S(x)}{\Lambda\phi_n l(h, n)}, \tag{39}$$

and write $x = f_n(t)$ for the inverse function.

Then as t varies round the unit circle C from 0 to 2π, x runs round $l(h, n)$.

The functions $f_n(t)$ are equicontinuous.

For $$D_{S_n}[\phi_n f_n(t),\, \phi_n f_n(t')] \leqslant \frac{|t - t'|}{2\pi} \sup_m \Lambda\phi_m l(h, m), \tag{40}$$

and so unless the $f_n(t)$ are equicontinuous there exists $\epsilon_0 > 0$ such that for any $\delta > 0$ there exists $n = n(\delta)$ and (P_n, Q_n), (P_n', Q_n') on S_n such that

$$D_{S_n}[(P_n, Q_n);\, (P_n', Q_n')] < \delta, \quad \text{but} \quad D_{\text{Euc}}(Q_n, Q_n') > \epsilon_0. \tag{41}$$

Now there exists a subsequence of the n such that $Q_n \to Q$, $Q_n' \to Q_n'$, where $Q, Q' \subset H$.

† We are assuming that there is some set of finite L^* measure containing D_m for all large m. This involves no loss of generality, as we may assume that $L^*\{D'(r_1) - \overline{D'(r_2)}\} < \infty$. This enables us to prove the result of the lemma for any $r_1 > r_1^* > r_2^* > r_2$, and we may then let $r_1^* \to r_1$ and $r_2^* \to r_2$.

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Then $$D_{\text{Euc}}(Q, Q') \geqslant \epsilon_0. \tag{42}$$

But $$D_S[(\phi(Q), Q); (\phi(Q'), Q')] = \lim D_{S_n}[(\phi(Q), Q) ; (\phi(Q'), Q')] \leqslant \delta, \tag{43}$$

since the ϕ_n are equicontinuous by (29).

Equations (42) and (43) give a contradiction as ϕ is (1-1), and hence (40) is established.

By Lemma 2 the functions $f_n(t)$ converge in a subsequence to a continuous function $f(t)$. Consider $f(C)$. $\phi_n f_n(C)$ lies on $s(P, h)$, and so $\phi f(C)$ will lie on $s(P, h)$ also. Thus $f(C)$ does not meet $D'(r_2)$. Let D_1 be the component of $H^\circ - f(C)$ which contains $D'(r_2)$. D_1 is a domain whose boundary is part of $f(C)$. Let D be the domain derived from D_1 as in § 8; then the boundary of D is a simple closed Jordan curve, $\Gamma(h)$. As $\phi_n f_n(t) \to \phi f(t)$, we have for almost all h

$$\Lambda\phi\Gamma(h) \leqslant \varlimsup_n \Lambda\phi_n l(h, n). \tag{44}$$

Now $$\Gamma(h) = l(h). \tag{45}$$

For $D \supset D'(r_2)$ and $\phi\Gamma(h)$ lies on $s(P, h)$ so that $D \supset D'(h)$, and hence, as D is a simply-connected Jordan domain, $D \supset D(h)$. On the other hand, if $h_1 > h$ then for large n, $D(h, n) \subset D(h_1)$, and so $l(h, n) \subset D(h_1)$. Going to the limit we obtain

$$\Gamma(h) \subset \overline{D(h_1)}. \tag{46}$$

But except for a denumerable set of h

$$\prod_{h_1 > h} \{D(h_1) - \overline{D(h)}\}$$

can have no interior points lest we obtain a non-denumerable set of disjoint open sets.

Thus, for almost all h, equation (46) and the construction of $D(h)$ give

$$D \subset \{\overline{D(h)}\}^\circ, \tag{47}$$

which implies (45). Hence by equations (37) and (44),

$$\int_{r_1'}^{r_1'} \Lambda\phi l(h)\, dh \leqslant \int_{r_1'}^{r_1'} \varlimsup \Lambda\phi_n l(h, n)\, dh \leqslant \varlimsup \int_{r_1'}^{r_1'} \Lambda\phi_n l(h, n)\, dh \leqslant L^*\phi\{D'(r_1) - \overline{D'(r_2)}\}. \tag{48}$$

But r_1' and r_2' are as near as we please to r_1 and r_2, and so, on going to the limit, the Lemma follows.

10. Given $s = \phi(H)$, a surface homoeomorphic to a disk, let $D_1(i, h)$ be a component of $S^{-1}(P, h)$. Consider $D'(i, h)$ and $l(i, h)$ defined as in § 11. Denote by $\Delta(h)$ the class of i for which $D_1(i, h) \subset D'(r_1)$. Then we have

THEOREM 3. $$\int_{r_0}^{r_1} \sum_{\Delta(h)} \Lambda\phi l(i, h)\, dh \leqslant L^*\{D'(r_1) - \sum_{\Delta(r_0)} D'(i, r_0)\}. \tag{49}$$

In particular, $$\int_0^{r_1} \sum_{\Delta(h)} \Lambda\phi l(i, h)\, dh \leqslant L^* D'(r_1). \tag{50}$$

Take $r_0 < r_0^* < r_1^* < r_1$ and let $\Delta(r_0^*, n)$ be a finite subset, of n members, of $\Delta(r_0^*)$.

For $r > r_0^*$ denote by $\Delta(r, n)$ the class of those i of $\Delta(r)$ for which there is a $j \in \Delta(r_0^*, n)$ such that

$$D'(i, r) \supset D'(j, r_0^*). \tag{51}$$

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$\Delta(r, n)$ contains at most n members. Suppose that it contains exactly $n(r)$ members; then $n(r)$ is a decreasing function. Thus there will exist

$$r_0^* = r_0' < r_1' < r_2' \ldots < r_{n+1}' = r_1^*, \tag{52}$$

such that for $r_k' < r < r_{k+1}'$ $n(r)$ will be constant.

Then for $r_k' < \rho_k' \leqslant r \leqslant \rho_{k+1} < r_{k+1}'$ we have a finite number, $n(k)$ say, of increasing series of domains as in Lemma 4. Apply this lemma to each series and add:

$$\int_{\rho_k'}^{\rho_{k+1}} \sum_{\Delta(r,n)} \Lambda\phi l(i,r)\,dr \leqslant L^*\{\sum_{\Delta(\rho_{k+1},n)} D'(i,\rho_{k+1}) - \sum_{\Delta(\rho_k',n)} D'(i,\rho_k')\}$$

$$\leqslant L^*\{\sum_{\Delta(r_{k+1}',n)} D'(i,r_{k+1}') - \sum_{\Delta(r_k',n)} D'(i,r_k')\}. \tag{53}$$

Let $\rho_k' \to r_k' + 0$ and $\rho_{k+1} \to r_{k+1}' - 0$, and then sum over all the intervals (r_k', r_{k+1}'). We obtain

$$\int_{r_0^*}^{r_1^*} \sum_{\Delta(r,n)} \Lambda\phi l(i,r)\,dr \leqslant L^*\{\sum_{\Delta(r_1^*,n)} D'(i,r_1^*) - \sum_{\Delta(r_0^*,n)} D'(i,r_0^*)\}. \tag{54}$$

Let $\Delta(r, \infty)$ be the class of those i of $\Delta(r)$ such that there exists a $j \in \Delta(r_0^*)$ for which

$$D'(i,r) \supset D'(j,r_0^*). \tag{55}$$

Let $n \to \infty$ in such a manner that $\Delta(r_0^*, n) \to \Delta(r_0^*)$. Then

$$\lim_{n\to\infty} \sum_{\Delta(r,n)} \Lambda\phi l(i,r) = \sum_{\Delta(r,\infty)} \Lambda\phi l(i,r). \tag{56}$$

Thus we obtain

$$\int_{r_0^*}^{r_1^*} \sum_{\Delta(r,\infty)} \Lambda\phi l(i,r)\,dr \leqslant L^*\{\sum_{\Delta(r_1^*,\infty)} D'(i,r_1^*) - \sum_{\Delta(r_0^*)} D'(i,r_0^*)\}, \tag{57}$$

that is,

$$\int_{r_0^*}^{r_1^*} \sum_{\Delta(r,\infty)} \Lambda\phi l(i,r)\,dr \leqslant L^*\{\sum_{\Delta(r_1^*)} D'(i,r_1^*) - \sum_{\Delta(r_0^*)} D'(i,r_0^*)\}. \tag{58}$$

$\Delta(r, \infty)$ depends on r_0^*. Write it $\Delta(r, r_0^*)$ to express this dependence. Also write

$$L^*\{\sum_{\Delta(r)} D'(i,r)\} = X(r) \tag{59}$$

and

$$\sum_{\Delta(r,r')} \Lambda\phi l(i,r) = f(r,r') \quad (r' < r). \tag{60}$$

We then have by (58)

$$\int_{r_0^*}^{r_1^*} f(r,r_0^*)\,dr \leqslant X(r_1^*) - X(r_0^*). \tag{61}$$

Denote $\lim_{r'\to r-0} f(r,r')$ by $g(r)$; then by the definition of $S^{-1}(P,r)$ we obtain

$$g(r) = \sum_{\Delta(r)} \Lambda\phi l(i,r), \tag{62}$$

for given any $i \in \Delta(r)$ there exists $r' < r$ such that $D'(i,r) \supset D'(j,r')$ for some $j \in \Delta(r')$.

Divide (r_1, r_0) into intervals of length δ. For each interval we have

$$\int_{r_1''-\delta}^{r_1''} f(r, r-\delta)\,dr \leqslant \int_{r_1''-\delta}^{r_1''} f(r, r_1''-\delta)\,dr, \tag{63}$$

because $f(r,r')$ increases with r'. Hence

$$\int_{r_1''-\delta}^{r_1''} f(r, r-\delta)\,dr \leqslant X(r_1'') - X(r_1''-\delta). \tag{64}$$

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Summing this over all the intervals we obtain

$$\int_{r_0}^{r_1} f(r, r-\delta)\,dr \leqslant X(r_1) - X(r_0). \tag{65}$$

But by Fatou's theorem $\int_{r_0}^{r_1} g(r)\,dr \leqslant \varlimsup_{\delta \to 0} \int_{r_0}^{r_1} f(r, r-\delta)\,dr,$

which together with (62) and (65) establishes the theorem.

11. LEMMA 5. *For a smooth surface*

$$\Lambda^2 S^\circ = L(S). \tag{66}$$

This is equation (19) of (8).

THEOREM 4. *If $L(S) < \infty$ then $L(S) = A(S)$, and in any case*

$$\Lambda^2(S) \geqslant L(S).$$

It is clear from the definition of $A(S)$ that

$$\Lambda^2(S) \geqslant A(S). \tag{67}$$

By Lemmas 1 and 4 we see that if either of $L(S)$ or $\Lambda^2(S)$ is finite then we can divide H into domains as small as we please by Jordan curves whose ϕ-images are of finite length. We need only take a suitable aggregate of curves of the form $l(h)$, as defined in § 8.

Let $\{r_{in}\}$ be such a set of domains, and suppose that the diameter of $\phi(r_{in})$ is at most 2^{-n}.

Define a surface $S_n = \phi_n(H)$ as follows:

Let $\phi_n = \phi$ on $H - \sum_i r_{in}$, and on r_{in} let ϕ_n be a representation of a surface such that

$$\Lambda^2\{\phi_n(r_{in})\}^\circ = \mu\{\phi, r_{in}\}; \tag{68}$$

that is possible by (7), and the definition of $\mu\{\phi, r_{in}\}$.

The boundaries of the $\phi_n(r_{in})$ are rectifiable, and so

$$L(S_n) = \sum_i L\phi_n(r_{in}). \tag{69}$$

But $\phi_n(r_{in})$ is smooth, and so equations (66), (68) and (69) give

$$L(S_n) = \sum_i L\phi_n(r_{in}) = \sum_i \Lambda^2\{\phi_n(r_{in})\}^\circ = \sum_i \mu\{\phi, r_{in}\} \leqslant A(S). \tag{70}$$

Now the diameter of $\phi(r_{in})$ is at most 2^{-n}, and hence the diameter of $\phi_n(r_{in})$ is not greater than 2^{-n} either, so that

$$D_{\text{Euc}}(\phi(x), \phi_n(x)) < 2^{-n+1} \quad \text{for} \quad x \subset H. \tag{71}$$

That is, $\phi_n \to \phi$, whence $L(S) \leqslant \varliminf L(S_n) \leqslant A(S)$ by (70),

and so by the corollary to Theorem 1 the theorem is established.

12. LEMMA 6. *If $L(S) < \infty$ and given $\epsilon > 0$ then there exists a $\delta > 0$ such that if $\{D_i\}$ is any set of disjoint simply-connected Jordan domains in H such that the diameter of each D_i is at most δ and the boundary of $\phi(D_i)$ is rectifiable, then*

$$\sum_i L\phi(D_i) - \sum_i \mu\{\phi, D_i\} < \epsilon. \tag{72}$$

Let $\delta = \delta(\epsilon)$ be the δ of Theorem 2. Take any finite subset Δ of the D_i. Then we can divide $H - \sum_\Delta D_i$ into Jordan domains $\{r_j\}$ such that

$$\text{the diameter of } r_i \text{ is at most } \delta \tag{73}$$

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and
$$\Lambda\phi\{H^\circ - \sum_{\Delta} D_i - \sum_j r_j\} < \infty, \tag{74}$$

then by Theorem 2
$$\sum_{\Delta} \mu\{\phi, D_i\} + \sum_j \mu\{\phi, r_j\} > A(S) - \epsilon. \tag{75}$$

By Theorem 4 $A(S) = L(S)$ and
$$L\phi(r_j) \geqslant A\phi(r_j) \geqslant \mu\{\phi, r_j\}. \tag{76}$$

Hence by (75) and (76)
$$\sum_{\Delta} \mu\{\phi, D_i\} > L(S) - \sum_j L\phi(r_j) - \epsilon \geqslant \sum_{\Delta} L\phi(D_i) - \epsilon, \tag{77}$$

and the lemma follows.

13. THEOREM 5. *If $X \subset H$ there exists a Borel set B containing X such that*
$$\Lambda^2\phi(B) = \Lambda^2\phi(X). \tag{78}$$

We may suppose without loss of generality that $\Lambda^2\phi(X) < \infty$. Take an integer $k < \infty$. Divide H into a finite number of disjoint Borel sets D_{ik} each of diameter at most $1/k$. There exists a Borel set B'_{ik} in three-dimensional space such that
$$\Lambda^2 B'_{ik} = \Lambda^2 \,|\, \phi(D_{ik}X) \,| \tag{79}$$

and
$$B'_{ik} \supset |\, \phi(D_{ik}X) \,|.$$
$$D_{ik}\phi^{-1}(B'_{ik}S) = B_{ik}$$

is a Borel set and
$$D_{ik} \supset B_{ik} \supset D_{ik}X, \tag{80}$$

while
$$\Lambda^2 \,|\, \phi(B_{ik}) \,| = \Lambda^2 \,|\, \phi(D_{ik}X) \,|. \tag{81}$$

Let $B = \prod_{k=1}^{\infty} \sum_i B_{ik}$, then clearly $B \supset X$. Denote by E_∞ the set of those points of $|\, \phi(B) \,|$ which have infinite multiplicity on $\phi(B)$. Then
$$\Lambda^2 E_\infty = 0. \tag{82}$$

For let $E_\infty(k, n)$ be the set of points P of E_∞ such that there exist n ϕ-elements $Q_1 \dots Q_n$, no two of which are within $1/k$ of each other, and for which $\phi(Q_i) = P$. Then for any given n we can choose k so large that
$$\Lambda^2 E_\infty(k, n) \geqslant \tfrac{1}{2}\Lambda^2 E_\infty. \tag{83}$$

$Q_1 \dots Q_n$ must be in distinct sets B_{ik} as by (80) the B_{ik} all have diameter at most $1/k$.

That is, every point of $E_\infty(k, n)$ is in at least n of the sets $|\, \phi(B_{ik}) \,|$; but these are Borel and so
$$\tfrac{1}{2}n\Lambda^2 E_\infty \leqslant n\Lambda^2 E_\infty(k, n) \leqslant \sum_i \Lambda^2 \,|\, \phi(B_{ik}) \,|$$
$$= \sum_i \Lambda^2 \,|\, \phi(D_{ik}X) \,| \leqslant \sum_i \Lambda^2\phi(D_{ik}X) = \Lambda^2\phi(X). \tag{84}$$

But n is arbitrary and $\Lambda^2\phi(X) < \infty$, whence (82) follows.

Denote by E the set of points of $|\, \phi(B) \,|$ which have finite multiplicity on $\phi(B)$. Let E_k be those points P of E such that no two ϕ-elements Q', Q'', for which
$$\phi(Q') = \phi(Q'') = P,$$

are within $1/k$ of each other. Then $E_k \,|\, \phi(B_{ik}) \,|$ consists of points simple on $\phi(B_{ik})$. Thus
$$\Lambda^2\phi(B_{ik}\phi^{-1}E_k) = \Lambda^2 E_k \,|\, \phi(B_{ik}) \,| \leqslant \Lambda^2 \,|\, \phi(D_{ik}X) \,|,$$

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and therefore
$$\Lambda^2\phi(\sum_i B_{ik}\phi^{-1}E_k) \leqslant \Lambda^2\phi(X), \tag{85}$$

that is,
$$\Lambda^2\phi(B\phi^{-1}E_k) \leqslant \Lambda^2\phi(X). \tag{86}$$

Letting $k \to \infty$ and using (82) we obtain

$$\Lambda^2\phi(B) = \Lambda^2\phi(X),$$

which establishes the theorem.

14. THEOREM 6. *If* $L(S) < \infty$ *and* $X \subset H^\circ$, *then*

$$\Lambda^2\phi(X) \geqslant L^*(X).$$

We may assume $\Lambda^2\phi(X) < \infty$. By Theorem 5 it suffices to consider Borel measurable X. But both Λ^2 and L^* are additive functions on Borel subsets of X, and so any such set will contain a closed subset of nearly the same measure, from which it follows that we may assume, without loss of generality, that X is closed.

I say $S' = \phi'(H)$ is ϵ-smooth with respect to X if, whenever the image of the boundary of a simply-connected domain, D, containing no points of X, lies in some plane Π, then the whole of $\phi'(D)$ lies within ϵ of Π.

Now as in (7), p. 243, we can construct a surface $S_n = \phi_n(H)$ such that

(*a*) $\phi_n = \phi$ on X and on the boundary of H.
(*b*) $D_{\text{Euc}}(\phi_n(x), \phi_n(x')) \leqslant D_{\text{Euc}}(\phi(x), \phi(x'))$ for $x, x' \subset H$.
(*c*) S_n is $1/n$-smooth with respect to X.
(*d*) $L\phi_n(H - X) \leqslant L\phi(H - X)$.

By (*b*) the ϕ_n are equicontinuous, and so by Lemma 2 converge in a subsequence to a continuous function ϕ_0. Then

(*a'*) $\phi_0 = \phi$ on X and on the boundary of H.
(*b'*) $D_{\text{Euc}}(\phi_0(x), \phi_0(x')) \leqslant D_{\text{Euc}}(\phi(x), \phi(x'))$ for $x, x' \subset H$.
(*c'*) $S_0 = \phi_0(H)$ is smooth with respect to X.
(*d'*) $L\phi_0(H - X) \leqslant L\phi(H - X)$.

Now divide H into a finite set of disjoint simply-connected Jordan domains $\{r_{in}\}$ $(i = 1, 2, \ldots)$ by a set of Jordan curves whose ϕ-images are of finite length. This is possible in virtue of Lemma 4, and we may suppose that the diameter of $\phi(r_{in})$ is at most 2^{-n}.

I define $\phi'_n(x)$ on r_{in} as I defined ϕ_0 on H. If r_{in} contains no points of X, $\phi'_n(r_{in})$ will be simply smooth instead of smooth with respect to X. On

$$H - \sum_i (r_{in}), \quad \phi'_n(x) = \phi(x). \tag{87}$$

By (*b'*) the diameter of $\phi'_n(r_{in})$ is at most 2^{-n} and so

$$D_{\text{Euc}}(\phi'_n(x), \phi(x)) \leqslant 2^{-n+1} \quad \text{for} \quad x \subset H. \tag{88}$$

That is
$$\phi'_n(x) \to \phi(x) \quad \text{for} \quad x \subset H. \tag{89}$$

Also
$$\Lambda^2\{\phi'_n(H^\circ - X)\}^\circ \leqslant L\phi'_n(H - X). \tag{90}$$

For there exists a finitely connected Jordan domain $R_n \subset H^\circ - X$ such that $\Lambda^2\phi'_n R_n$ is as near as we please to $\Lambda^2\phi'_n(H^\circ - X)$, while in any case

$$L\phi'_n(R_n) \leqslant L\phi'_n(H - X),$$

so that it suffices to show that
$$\Lambda^2\phi'_n(R_n) \leqslant L\phi'_n(R_n). \tag{91}$$

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Now $\phi'_n(R_n)$ is smooth and R_n can be subdivided into simply-connected Jordan domains by arcs whose images are of finite length, so that we can apply Lemma 5 to establish the result.

By (90) and Theorem 4

$$L\{\phi'_n(H)\} \leqslant \Lambda^2\{\phi'_n(H)\}^\circ \leqslant L\phi'_n(H-X) + \Lambda^2\phi'_n(X). \tag{92}$$

Thus by (89) and (d') we obtain

$$L\phi(H) \leqslant L\phi(H-X) + \varlimsup_{n\to\infty} \Lambda^2\phi'_n(X). \tag{93}$$

Now $\phi'_n = \phi$ on X, and so $|\phi'_n(X)|$ is the same point set as $|\phi(X)|$; however, the multiplicity may be different. But as all ϕ-elements are single points no two distinct ϕ'_n-elements can meet the same ϕ-element. Thus

$$\Lambda^2\phi'_n(X) \leqslant \Lambda^2\phi(X), \tag{94}$$

and so (93) gives $\Lambda^2\phi(X) \geqslant L\phi(H) - L\phi(H-X) = L^*(X)$,

which is the required result.

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