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Invariant circles and depinning transition

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Abstract. We associate the existence or non-existence of rotational invariant circles of an area-preserving twist map on the cylinder with a physically motivated quantity, the depinning force, which is a critical value in the depinning transition. Assume that $H : \mathbb{R}^2 \mapsto \mathbb{R}$ is a C^2 generating function of an exact area-preserving twist map $\bar{\varphi}$ and consider the tilted Frenkel–Kontorova (FK) model:

$$\dot{x}_n = -D_1 H(x_n, x_{n+1}) - D_2 H(x_{n-1}, x_n) + F, \quad n \in \mathbb{Z},$$

where $F \geq 0$ is the driving force. The depinning force is the critical value $F_d(\omega)$ depending on the mean spacing ω of particles, above which the tilted FK model is sliding, and below which the particles are pinned. We prove that there exists an invariant circle with irrational rotation number ω for $\bar{\varphi}$ if and only if $F_d(\omega) = 0$. For rational ω, $F_d(\omega) = 0$ is equivalent to the existence of an invariant circle on which $\bar{\varphi}$ is topologically conjugate to the rational rotation with rotation number ω. Such conclusions were claimed much earlier by Aubry *et al*. We also show that the depinning force $F_d(\omega)$ is continuous at irrational ω.

1. *Introduction*

1.1. *Motivation.* Existence or non-existence of homotopically non-trivial invariant circles for area-preserving monotone twist maps on the annulus or cylinder has been extensively studied, since it is related to stability or breakdown of stability of systems [**34**]. The celebrated KAM theory [**33**] implies that invariant circles with sufficiently irrational frequency persist under small exact symplectic perturbations. But, as the perturbation gets large, invariant circles will disintegrate and system stability will deteriorate. The converse KAM theory [**26**] gives conditions ensuring that no invariant circle with given frequency can exist or no invariant circle can pass through a point or region of the phase space of an area-preserving twist map. The Aubry–Mather theory [**3**, **22**, **29**] succeeds in describing what happens if the invariant circles with irrational rotation numbers are destroyed: motion on each invariant circle, if destroyed, remains as quasi-periodic, but the circle itself becomes a Cantor set.

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The goal of this paper is to associate the existence or non-existence of invariant circles for area-preserving twist maps with a physically motivated quantity: the depinning force, which is a critical value separating the pinned and sliding regions, and related to a critical phenomenon, the depinning transition, in a variety of physical systems [**9**, **17**].

Among various criteria (see [**10**, **26**, **32**] for a summary) for existence or non-existence of invariant circles, the most relevant to our results is Mather's criterion [**30**] involving a quantity ΔW_ω, which is defined as the difference in action between minimax and minimal Birkhoff periodic orbits of type (p, q) if $\omega = p/q$ is rational, or the limit of $\Delta W_{p/q}$ as $p/q \to \omega$ if ω is irrational; see also the last paragraph in [**5**, §4] and [**19**, §5]. It was proved [**30**] that for irrational ω, there is an invariant circle with rotation number ω if and only if $\Delta W_\omega = 0$. Moreover, ΔW_ω is continuous with respect to ω at irrational points. Mather's criterion has been generalized to models in statistical mechanics by de la Llave and Valdinoci [**14**, **15**].

The fascinating developments of the Aubry–Mather theory were begun independently by Aubry [**3**] and Mather [**29**] with different motivations and different methods but led to similar results. While Mather studied area-preserving annulus maps, Aubry investigated the Frenkel–Kontorova (FK) model in solid-state physics.

The standard FK model describes a chain of particles or oscillators in a sinusoidal on-site potential and harmonic interaction potential. In order to describe the commensurate and incommensurate structures of ground states of this mechanical model, physicists [**9**, **11**, **12**, **17**, **25**, **37**] introduced some physically motivated quantities, such as the gap in the phonon spectrum, the coherence length of the ground state, the Peierls energy barrier, and the depinning force.

According to the Aubry–Mather theory, for each rational or irrational ω, there exist minimal energy configurations, i.e., rotationally ordered equilibrium states, with rotation number ω. If we apply a uniform force F to each particle of the chain (for example, tip the corrugated table on which the particles rest, if the potential is gravitational), then the equilibrium states may disappear and the chain begins to slide if the driving force increases beyond a critical value, called the depinning force F_d, depending on the mean spacing of particles.

Such mechanical intuition gives rise to the following claim [**9**, **11**, **12**, **17**, **25**, **37**]: if the ground states (recurrent minimal energy configurations) with rotation number ω form a continuous one-dimensional family, which means that there are no gaps between the ground states, then even an infinitesimal force can make the chain move. As a result, the depinning force F_d vanishes. On the contrary, if there exist gaps in the family of ground states, then small forces do not lead to the motion of the particle chain unless the driving force is greater than some critical value.

From the classical Aubry–Mather theory [**5**], we know that if there are no gaps in the family of ground states with irrational rotation number ω, then the corresponding cylinder map has an invariant circle on which the twist map is topologically transitive. So the above claim amounts to saying that the depinning force vanishes if there is a transitive invariant circle with irrational rotation number for the twist map.

We should emphasize that there are two cases for an area-preserving twist map to have an invariant circle with irrational rotation number. One is that the twist map is topologically

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transitive restricted on this circle, which implies that there are no gaps for the ground states. The other is that the twist map is not topologically transitive on this circle, which means that there are gaps for the ground states. However, for the latter case, the equilibrium states still form a continuous one-dimensional family since each point in the gaps corresponds to an equilibrium state. Then one can conjecture from mechanical intuition that in this case the particle chain still can move freely, and hence the depinning force vanishes.

We remark that another difference between these two cases is that the hull function for the ground states is continuous in the former case, and it is discontinuous on a countable set for the latter case. If the hull function for the ground states is continuous, it has been proved [**39**] that the corresponding depinning force vanishes.

The aim of this paper is to give a rigorous proof for the above conjecture for the second case, that is, we do not know whether the twist map is topologically transitive or not on its invariant circle with irrational rotation number. We show that the converse is also true, that is, if the depinning force corresponding to some irrational rotation number ω vanishes, then the twist map has an invariant circle with rotation number ω. We also study the rational rotation number case.

We remark that for a C^r $(r \geq 2)$ symplectic twist map, an invariant circle with irrational rotation number is not necessarily topologically transitive. In fact, Herman [**21**] gave an example of class $C^{3-\varepsilon}$. Mather asked whether there is a C^3 counterexample of this kind; see [**27**, Problem 3.1.3]. Recently, Arnaud [**2**] constructed a C^1 symplectic twist map of the annulus that has an invariant circle such that the dynamics on it is conjugate to a Denjoy counterexample.

1.2. *Setting and results.* An exact area-preserving twist map of the cylinder is a C^1 diffeomorphism $\bar{\varphi} : \mathbb{R}/\mathbb{Z} \times \mathbb{R}$ to itself which preserves area, orientation, and topological ends of the cylinder and which satisfies the zero-flux condition, i.e., the algebraic area enclosed between a loop and its image of $\bar{\varphi}$ is zero, and the twist condition $\partial x'/\partial y \geq c > 0$, where $\bar{\varphi}(x, y) = (x', y')$. There exists for $\bar{\varphi}$ a C^2 generating function $H : \mathbb{R}^2 \to \mathbb{R}$ satisfying (1.2) and (1.3).

Consider the tilted FK model

$$\dot{x}_n = -D_1 H(x_n, x_{n+1}) - D_2 H(x_{n-1}, x_n) + F, \quad n \in \mathbb{Z}, \tag{1.1}$$

where $D_1 H$ and $D_2 H$ denote the first partial derivatives of H with respect to the first and second arguments, respectively, $x_n(t)$ indicates the displacement of the nth particle at time t, along the chain, $F \geq 0$ measures the external driving force, H is the C^2 generating function of $\bar{\varphi}$ satisfying

$$H(x + 1, y + 1) = H(x, y) \quad \text{for all } x, y \in \mathbb{R} \tag{1.2}$$

and

$$-D_1 D_2 H(x, y) \geq \delta > 0 \quad \text{for all } x, y \in \mathbb{R}, \tag{1.3}$$

in which $D_1 D_2 H$ denotes the mixed second partial derivative of H.

Note that condition (1.3) corresponds to the twist condition of $\bar{\varphi}$. On the other hand, it ensures that the gradient system (1.1) is strictly monotone. The monotonicity techniques of system (1.1) were used to discuss the periodic points of an area-preserving twist map by

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Angenent [**1**], to present an alternative proof of the Aubry–Mather theorem by Golé [**18**], and to study sliding dynamics of the tilted FK model by Baesens and MacKay [**4**].

A particle chain is said to have mean spacing ω if the limit

$$\omega = \lim_{m-n\to\infty} \frac{x_m - x_n}{m-n}$$

exists, where $\mathbf{x} = (x_n) \in \mathbb{R}^{\mathbb{Z}}$ denotes the position coordinates of particles. The reciprocal of the mean spacing $1/\omega$ measures the average particle density. In this paper, we call ω the rotation number of the configuration $\mathbf{x}$ in order to be consistent with the term in dynamical systems literature and we only consider $\omega > 0$.

An equilibrium state $\mathbf{x} = (x_n) \in \mathbb{R}^{\mathbb{Z}}$ of (1.1) satisfies the difference equation

$$-D_1 H(x_n, x_{n+1}) - D_2 H(x_{n-1}, x_n) + F = 0. \tag{1.4}$$

If $F = 0$, it is well known that there is one-to-one correspondence between the equilibrium states of the particle chain and the orbits of an area-preserving twist map on the cylinder.

In order to state our main results, we need to introduce some notation. One of the most important quantities characterizing the tilted FK model is the average velocity for each solution $\mathbf{x}(t) = (x_n(t))$ of (1.1) defined as the limit

$$\bar{v} = \lim_{\substack{m-k\to\infty \\ t\to+\infty}} \frac{1}{t(m-k)} \int_0^t \sum_{n=k}^{m-1} \dot{x}_n(t)\, dt \tag{1.5}$$

if it exists. It is an important quantity separating sliding states and pinning states in particle chains [**17**]. The existence of the average velocity for system (1.1) will be demonstrated in §4.

If the initial values of (1.1) are restricted to be rotationally ordered (Birkhoff sequences) and have the same rotation number $\omega > 0$, then the average velocity exists and is unique for all the trajectories with the above restricted initial values. Furthermore, the average velocity $\bar{v}$ is continuous and non-decreasing with respect to the driving force F, that is, $\bar{v}(F_1) \le \bar{v}(F_2)$ for $F_1 \le F_2$. Meanwhile, $\bar{v}$ vanishes if and only if there exist for (1.1) rotationally ordered equilibrium states and $\bar{v} > 0$ is equivalent to the existence of uniform sliding states of form

$$x_n(t) = u(n\omega + v\, t + \alpha), \tag{1.6}$$

where $v > 0$, $\alpha \in \mathbb{R}$, and u is a strictly increasing and continuous function satisfying $u(x+1) = u(x) + 1$ for all $x \in \mathbb{R}$. The function u is called the dynamical hull function of the uniform sliding states.

Therefore, we define the depinning force $F_d(\omega)$ to be the critical value below which the average velocity vanishes and above which the system admits uniform sliding states of the form (1.6).

Our main results are summarized as follows.

THEOREM A. *For irrational ω, the area-preserving twist map induced by the generating function H satisfying (1.2) and (1.3) has an invariant circle with rotation number ω if and only if $F_d(\omega) = 0$. For rational $\omega = p/q$ in lowest terms, $F_d(\omega) = 0$ if and only if there exists an invariant circle on which the twist map is topologically conjugate to the rational rotation with rotation number ω.*

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We also study the continuity property of the depinning force with respect to the rotation number ω.

THEOREM B. *$F_d(\omega)$ is continuous at irrational points.*

In fact we have shown in [**39**] that $F_d(\omega)$ is upper semi-continuous with respect to ω and hence we focus on in this paper the lower semi-continuity at irrational points. We remark that $F_d(\omega)$ is not necessarily continuous at rational points. For example, in the billiard ball problem in the ellipse, $F_d(\omega) > 0$ when $\omega = 1/2$, $F_d(\omega) = 0$ for $\omega \neq 1/2$ (see [**23**, §9.2]).

As remarked by Mather [**30**], the quantity ΔW_ω is closely related to a physical quantity, the Peierls energy barrier $P_\omega(\xi)$. The former is an upper bound for $P_\omega(\xi)$ [**31**]. So our conclusions are remarkably similar to Mather's criterion and we establish a relationship between the two physical quantities, the depinning force and the Peierls energy barrier, in the sense that $F_d(\omega) = 0$ if and only if $\Delta W_\omega = 0$ for irrational ω.

We should mention that Mather also obtained a modulus of continuity for the Peierls energy barrier [**31**], and Mramor and Rink [**36**] extended these estimates to monotone recurrence relations. In this paper we do not have such stronger results for the depinning force.

1.3. *Outline.* This paper is organized as follows. In §2 we introduce some preliminaries, including the Aubry–Mather theory, notation for area-preserving twist maps, the existence, uniqueness, continuous dependence, and in particular the monotonicity of the solutions for the tilted FK model, and the Hausdorff metric which will be used to construct invariant strictly ordered circles for irrational rotation numbers.

In §3 we construct for (1.1) an invariant strictly ordered circle, in which each element has the same rotation number. We first deal with rational rotation numbers and then by passing to the limit in the Hausdorff metric we obtain the results for irrational rotation numbers. We remark that these invariant strictly ordered circles are much like 'ghost circles' coined and constructed by Golé in [**19**] and our approach in some respects is inspired by Golé's work [**20**]. We should also mention that for periodic boundary conditions corresponding to rational ω, Le Calvez has already constructed these ghost circles [**24**].

Section 4 is devoted to the properties of the depinning transition of the tilted FK model. First we prove the existence of the average velocity for solutions with initial values being rotationally ordered. Since the average velocity is non-decreasing with respect to the driving force F, we define the depinning force $F_d(\omega)$ to be the critical value below which the average velocity vanishes. We also discuss the continuity property of the average velocity with respect to the rotation number and driving force.

We remark that some of the results in §4 were reported earlier in [**38**, **39**] for a special generating function H and similar conclusions in §3 have been reported in [**40**], but we need these conclusions and the corresponding notation in the proof of our main theorems.

Based on these preparations, we present the proofs of our main results in §§5 and 6. Section 7 consists of a discussion and some remarks.

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2. *Preliminaries*

In this section, we recall the classical Aubry–Mather theory, introduce some notation for area-preserving twist maps, discuss the existence, uniqueness, continuous dependence, and in particular monotonicity and its consequences of the tilted FK model (1.1), and state a result on the Hausdorff metric.

2.1. *Aubry–Mather theory.*

The classical Aubry–Mather theory can be found in [**3**, **5**, **18**, **22**, **29**] and many other references. The following material is taken from Bangert's excellent review article [**5**].

Assume that $H : \mathbb{R}^2 \to \mathbb{R}$ is C^2 smooth and satisfies (1.2) and (1.3). We extend H to arbitrary finite segments $(x_j, \ldots, x_k)$, $j < k$, of a configuration $\mathbf{x} \in \mathbb{R}^{\mathbb{Z}}$ by

$$H(x_j, \ldots, x_k) = \sum_{n=j}^{k-1} H(x_n, x_{n+1}).$$

The segment $(x_j, \ldots, x_k)$ is said to be minimal with respect to H if

$$H(x_j, \ldots, x_k) \le H(x_j^*, \ldots, x_k^*)$$

for all $(x_j^*, \ldots, x_k^*)$ with $x_j = x_j^*$ and $x_k = x_k^*$. A configuration $\mathbf{x} = (x_n) \in \mathbb{R}^{\mathbb{Z}}$ is said to be a minimal energy configuration (m.e.c.) if every finite segment of $\mathbf{x}$ is minimal.

We denote by $\mathscr{M} = \mathscr{M}(H)$ the set of minimal energy configurations. It is obvious that each $\mathbf{x} \in \mathscr{M}$ is an equilibrium of (1.1) with $F = 0$.

A partial order '$\le$' in $\mathbb{R}^{\mathbb{Z}}$ is defined as follows: for $\mathbf{x} = (x_n)$ and $\mathbf{y} = (y_n) \in \mathbb{R}^{\mathbb{Z}}$, we say that $\mathbf{x} \le \mathbf{y}$ if $x_n \le y_n$ for all $n \in \mathbb{Z}$; $\mathbf{x} < \mathbf{y}$ if $\mathbf{x} \le \mathbf{y}$ and $\mathbf{x} \ne \mathbf{y}$; $\mathbf{x} \prec \mathbf{y}$ if $x_n < y_n$ for all $n \in \mathbb{Z}$. We also denote $\mathbf{y} \ge \mathbf{x}$, $\mathbf{y} > \mathbf{x}$, and $\mathbf{y} \succ \mathbf{x}$ if $\mathbf{x} \le \mathbf{y}$, $\mathbf{x} < \mathbf{y}$, and $\mathbf{x} \prec \mathbf{y}$, respectively.

Let $\{\sigma_{m,k} \mid (m, k) \in \mathbb{Z}^2\}$ denote the group of transformations

$$(\sigma_{m,k}\, \mathbf{x})_n = x_{n+m} - k, \quad n \in \mathbb{Z}.$$

We denote the space shift map $\sigma_{1,0}$ by σ for simplicity.

A configuration $\mathbf{x} \in \mathbb{R}^{\mathbb{Z}}$ is said to be weakly rotationally ordered if for all $m, k \in \mathbb{Z}$, either $\sigma_{m,k}\mathbf{x} \le \mathbf{x}$ or $\mathbf{x} \le \sigma_{m,k}\mathbf{x}$. It is said to be rotationally ordered if for all $m, k \in \mathbb{Z}$, $\sigma_{m,k}\mathbf{x} \prec \mathbf{x}$, $\sigma_{m,k}\mathbf{x} = \mathbf{x}$, or $\mathbf{x} \prec \sigma_{m,k}\mathbf{x}$.

Let $p_n : \mathbb{R}^{\mathbb{Z}} \to \mathbb{R}$ denote the projection $\mathbf{x} = (x_n) \to x_n$. Then p_n is continuous and open for the product topology in $\mathbb{R}^{\mathbb{Z}}$ and order preserving for the partial order '$\le$'.

We say that $\mathbf{x} = (x_n)$ and $\mathbf{x}^* = (x_n^*) \in \mathbb{R}^{\mathbb{Z}}$ cross at $n \in \mathbb{Z}$ if $x_n = x_n^*$ and $(x_{n-1} - x_{n-1}^*)(x_{n+1} - x_{n+1}^*) < 0$, or they cross between n and $n+1$ if $(x_n - x_n^*)(x_{n+1} - x_{n+1}^*) < 0$.

Let $\tilde{f}$ be an orientation-preserving circle homeomorphism and f its lift, i.e., $f : \mathbb{R} \to \mathbb{R}$ is continuous and strictly increasing, and satisfies $f(x+1) = f(x) + 1$ for all $x \in \mathbb{R}$. Let $\mathrm{Rec}(\tilde{f})$ denote the recurrent set of $\tilde{f}$. If the rotation number of $\tilde{f}$ is irrational, define

$$\mathrm{Rec}(f) = \text{ set of accumulation points of } \{f^n(x) + k \mid (n, k) \in \mathbb{Z}^2\}.$$

Then $\mathrm{Rec}(f)$ does not depend on the choice of $x \in \mathbb{R}$. $\mathrm{Rec}(f)$ is either a periodic Cantor set or $\mathrm{Rec}(f) = \mathbb{R}$. Moreover, $\mathrm{Rec}(f)$ projects to $\mathrm{Rec}(\tilde{f})$.

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Since H satisfies (1.2) and (1.3), we have the following conclusions on $\mathscr{M}$ (see [**5**]).

(i) Each $\mathbf{x} \in \mathscr{M}$ is rotationally ordered and hence has a rotation number $\omega = \omega(\mathbf{x}) = \lim_{n\to\infty} x_n/n$. Moreover, it follows that $|x_m - x_n - (m-n)\omega| < 1$ for all $m, n \in \mathbb{Z}$.

(ii) For all $\omega \in \mathbb{R}$, the set $\mathscr{M}_\omega = \{\mathbf{x} \in \mathscr{M} \mid \omega(\mathbf{x}) = \omega\}$ is not empty.

(iii) For a rational rotation number $\omega = p/q$, $\mathscr{M}_\omega = \mathscr{M}_\omega^{\mathrm{per}} \cup \mathscr{M}_\omega^+ \cup \mathscr{M}_\omega^-$, where $\mathscr{M}_\omega^{\mathrm{per}}$ denotes the (p, q) periodic configurations in $\mathscr{M}_\omega$, and $\mathbf{x} \in \mathscr{M}_\omega^+$ (or $\mathscr{M}_\omega^-$) is α-asymptotic and ω-asymptotic to neighboring periodic minimals in $\mathscr{M}_\omega^{\mathrm{per}}$. $p_0(\mathscr{M}_\omega^{\mathrm{per}}) = A_\omega^{\mathrm{per}}$ is a closed subset and there is the exceptional case $A_\omega^{\mathrm{per}} = \mathbb{R}$ and then $\mathscr{M}_\omega = \mathscr{M}_\omega^{\mathrm{per}}$ is totally ordered. Otherwise there will exist neighboring elements in $\mathscr{M}_\omega^{\mathrm{per}}$ joined by the heteroclinic orbits in $\mathscr{M}_\omega^+$ and in $\mathscr{M}_\omega^-$.

(iv) For each irrational ω, $\mathscr{M}_\omega$ is totally ordered and hence there exists a lift f of an orientation-preserving circle homeomorphism such that $\mathscr{M}_\omega$ consists of orbits of f contained in $A_\omega = p_0(\mathscr{M}_\omega)$. If we denote the recurrent configurations in $\mathscr{M}_\omega$ by

$$\mathscr{M}_\omega^{\mathrm{rec}} = \{\mathbf{x} \in \mathscr{M}_\omega \mid \text{ there exist } (m_i, k_i) \in \mathbb{Z}^2 \backslash \{(0,0)\} \text{ such that } \mathbf{x} = \lim_{i\to\infty} \sigma_{m_i,k_i}\mathbf{x}\},$$

then $\mathbf{x} = (x_n) \in \mathscr{M}_\omega^{\mathrm{rec}}$ if and only if x_0 is a recurrent point of the corresponding circle map f, i.e., $p_0(\mathscr{M}_\omega^{\mathrm{rec}}) = \mathrm{Rec}(f)$. A recurrent m.e.c. is called a ground state. It follows that either $\mathrm{Rec}(f) = \mathbb{R}$ or $\mathrm{Rec}(f)$ is a Cantor set. Moreover, there exist two strictly increasing functions g^+ and g^- with $g^\pm(x+1) = g^\pm(x) + 1$ and g^+ being continuous from the right and g^- from the left such that for each $\mathbf{x} = (x_n) \in \mathscr{M}_\omega^{\mathrm{rec}}$, there exist $\alpha \in \mathbb{R}$ and $\kappa \in \{+, -\}$ such that $x_n = g^\kappa(n\omega + \alpha)$. Conversely, for any α, $x_n = g^\kappa(n\omega + \alpha)$ $(\kappa \in \{+, -\})$ is a ground state. The functions $g^\pm$ are called the hull functions for the ground states and in the case of $\mathrm{Rec}(f) = \mathbb{R}$, $g^+ = g^- = g$ is continuous.

We remark that the existence of an invariant circle with rotation number ω corresponds to the case $p_0(\mathscr{M}_\omega) = \mathbb{R}$.

The following lemma can be found in [**5**, §2] and is crucial to the proof of our main conclusions.

LEMMA 2.1. *Suppose that f and g are lifts of two orientation-preserving circle homeomorphisms and their rotation numbers are identical and irrational. Then either* $\mathrm{Rec}(f) = \mathrm{Rec}(g)$ *and* $f|_{\mathrm{Rec}(f)} = g|_{\mathrm{Rec}(g)}$ *or there exist* $x_0 \in \mathrm{Rec}(f)$ *and* $x_0^* \in \mathrm{Rec}(g)$ *such that the orbits* $\mathbf{x} = (x_n) = (f^n(x_0))$ *and* $\mathbf{x}^* = (x_n^*) = (g^n(x_0^*))$ *cross infinitely often.*

2.2. *Area-preserving twist maps.* In this section, we shall introduce some well-known results about the exact area-preserving twist maps [**5**, **20**, **23**]. Then we present a result of the non-exact twist maps which will be needed in the proof of Theorems A and B.

Let $\bar{\varphi}$ be an exact area-preserving twist map of the cylinder $(\mathbb{R}/\mathbb{Z}) \times \mathbb{R}$ into itself with generating function H satisfying (1.2) and (1.3). Let φ be a lift of $\bar{\varphi}$. Then $\varphi(x_0, y_0) = (x_1, y_1)$ if and only if

$$y_0 = -D_1 H(x_0, x_1) \quad \text{and} \quad y_1 = D_2 H(x_0, x_1). \tag{2.1}$$

There is a one-to-one correspondence between orbits $\{(x_n, y_n)\} = \{\varphi^n(x_0, y_0)\}$ of φ and sequences $\mathbf{x} = (x_n) \in \mathbb{R}^{\mathbb{Z}}$ satisfying

$$D_1 H(x_n, x_{n+1}) + D_2 H(x_{n-1}, x_n) = 0 \quad \text{for all } n \in \mathbb{Z}. \tag{2.2}$$

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The correspondence is given by

$$y_n = -D_1 H(x_n, x_{n+1}). \tag{2.3}$$

The twist map $\bar{\varphi}$ has the following properties due to (1.2) and (1.3).

(i) $\bar{\varphi}$ preserves each end of the cylinder, i.e., $\bar{\varphi}$ maps points near the top end of the cylinder to points near the top end, and likewise for points near the bottom end.

(ii) The form $y'dx' - y\,dx$ on $(\mathbb{R}/\mathbb{Z}) \times \mathbb{R}$ is exact if we set $\varphi(x, y) = (x', y')$, and hence $\bar{\varphi}$ is area preserving and orientation preserving, and the flux $\int y'\,dx' - y\,dx$ vanishes, where the integral is taken over any curve going around the cylinder.

For each $\mathbf{x} = (x_n) \in \mathbb{R}^{\mathbb{Z}}$ satisfying (2.2), we define a map $\pi : \mathbb{R}^{\mathbb{Z}} \to \mathbb{R}^2$ by

$$\pi(\mathbf{x}) = (x_0, -D_1 H(x_0, x_1)). \tag{2.4}$$

Then $\varphi \circ \pi(\mathbf{x}) = \pi \circ \sigma(\mathbf{x})$, in which σ is the space shift map.

Since each m.e.c. $\mathbf{x} = (x_n) \in \mathscr{M} = \mathscr{M}(H)$ is stationary with respect to H, i.e., satisfies (2.2), $p_0(\mathscr{M}_\omega) = \mathbb{R}$ implies the existence of an invariant circle $\Gamma = \pi(\mathscr{M}_\omega)$ for $\bar{\varphi}$ with rotation number ω. On the other hand, if Γ is a $\bar{\varphi}$-invariant circle on the cylinder and $\bar{\varphi}|_\Gamma$ has rotation number ω, then $\Gamma \subset \pi(\mathscr{M}_\omega)$ and $\Gamma = \pi(\mathscr{M}_\omega)$ if ω is irrational. Moreover, if $p_0(\mathscr{M}_\omega^{\mathrm{rec}}) = \mathbb{R}$ for irrational ω, then $\bar{\varphi}|_\Gamma$ is topologically transitive [**5**].

If there is an invariant circle Γ for $\bar{\varphi}$, according to Birkhoff's theorem [**8**], Γ is the graph of a function and it separates the cylinder into two parts. The lower (upper) part is invariant for $\bar{\varphi}$.

Note that each equilibrium state $\mathbf{x} = (x_n)$ of (1.1) satisfies (1.4), from which we define another area-preserving twist map $\bar{\varphi}_F$ with its lift $\varphi_F : (x_0, y_0) \mapsto (x_1, y_1)$ by

$$y_0 = -D_1 H(x_0, x_1), \quad y_1 = D_2 H(x_0, x_1) - F. \tag{2.5}$$

Similarly, there is a one-to-one correspondence between orbits $\{(x_n, y_n)\} = \{\varphi_F^n(x_0, y_0)\}$ of φ_F and configurations $\mathbf{x} = (x_n) \in \mathbb{R}^{\mathbb{Z}}$ satisfying (1.4). The correspondence is also given by (2.3).

We remark that φ is identical to φ_0 defined above. φ and φ_F $(F > 0)$ are lifts of area-preserving twist maps $\bar{\varphi}$ and $\bar{\varphi}_F$, respectively. The difference between them is that the flux for $\bar{\varphi}$ is zero, and hence it is exact, while the flux for $\bar{\varphi}_F$ is $-F \neq 0$, and hence $\bar{\varphi}_F$ has no invariant circles on the cylinder.

In this paper we always denote by φ and φ_F the lifts of the area-preserving twist maps $\bar{\varphi}$ induced by H and $\bar{\varphi}_F$ $(F > 0)$ defined by (2.5), respectively.

We observe that the image of (x_0, y_0) under φ_{F_1} lies above that under φ_{F_2} if $F_2 > F_1$.

LEMMA 2.2. *Assume that* $\varphi_{F_1} : (x_0, y_0) \mapsto (x_1, y_1)$ *and* $\varphi_{F_2} : (x_0, y_0) \mapsto (x_2, y_2)$. *Then* $x_1 = x_2$ *and* $y_1 > y_2$ *if* $F_2 > F_1$.

Proof. We deduce from (2.5) that

$$\begin{aligned} y_0 &= -D_1 H(x_0, x_1), \quad y_1 = D_2 H(x_0, x_1) - F_1, \\ y_0 &= -D_1 H(x_0, x_2), \quad y_2 = D_2 H(x_0, x_2) - F_2 \end{aligned}$$

and hence $x_1 = x_2$ and $y_1 - y_2 = F_2 - F_1 > 0$. □

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2.3. *Tilted FK model.* In this subsection, we list some properties of the tilted FK model with phase space

$$\mathscr{X} = \{\mathbf{x} \in \mathbb{R}^{\mathbb{Z}} \mid \|\mathbf{x}\| < \infty\},$$

where $\mathbf{x} = (x_n) \in \mathbb{R}^{\mathbb{Z}}$ and $\|\mathbf{x}\| = \sum_{n\in\mathbb{Z}} |x_n|/2^{|n|}$. System (1.1) can be rewritten compactly as

$$\dot{\mathbf{x}} = \mathcal{F}(\mathbf{x}, F), \quad \mathbf{x} \in \mathscr{X}, \quad F \in [0, +\infty), \tag{2.6}$$

where $\mathcal{F} : \mathscr{X} \times [0, +\infty) \mapsto \mathscr{X}$ is locally Lipschitz continuous in $\mathbf{x} \in \mathscr{X}$. Hence, we have the following results on solutions to (2.6) (or (1.1)).

LEMMA 2.3. *The solution* $\mathbf{x}(t)$ *to (2.6) with initial value* $\mathbf{x}(0) \in \mathscr{X}$ *exists locally and is unique, and it depends continuously on initial values and parameters.*

Let $\phi_F^t\mathbf{x}$ denote a solution to (2.6) with initial value $\mathbf{x}$ at $t = 0$. We remark that ϕ_F^t commutes with $\sigma_{m,k}$:

$$\sigma_{m,k}\phi_F^t\mathbf{x} = \phi_F^t\sigma_{m,k}\mathbf{x} \quad \text{for all } m, k \in \mathbb{Z} \text{ and } \mathbf{x} \in \mathscr{X} \tag{2.7}$$

for $t \in \mathbb{R}$ for which both sides are defined. In particular, we have

$$\phi_F^t(\mathbf{x} + \mathbf{1}) = \phi_F^t\mathbf{x} + \mathbf{1}, \tag{2.8}$$

where $\mathbf{1}$ denotes the element in $\mathscr{X}$ with each component being 1.

Definition 2.4. System (2.6) (or (1.1)) is said to be monotone if $\mathbf{x} \le \mathbf{y} \Rightarrow \phi_F^t\mathbf{x} \le \phi_F^t\mathbf{y}$ for $t > 0$ for which both are defined. It is said to be strictly monotone if $\mathbf{x} < \mathbf{y} \Rightarrow \phi_F^t\mathbf{x} \prec \phi_F^t\mathbf{y}$ for $t > 0$ for which both are defined.

LEMMA 2.5. *System (2.6) is strictly monotone.*

We remark that system (2.6) is cooperative [**41**] and the strict monotonicity of (2.6) (or (1.1)) was proved in [**4**, **18**, **20**]. For $\mathbf{x} = (x_n) \in \mathscr{X}$ and $m \in \mathbb{Z}$, let

$$k^+(m) = \inf\{k \in \mathbb{Z} \mid \sigma_{m,k}\mathbf{x} \le \mathbf{x}\}, \quad k^-(m) = \sup\{k \in \mathbb{Z} \mid \mathbf{x} \le \sigma_{m,k}\mathbf{x}\}.$$

Then it follows that $k^-(m) \le x_{n+m} - x_n \le k^+(m)$, for all $n \in \mathbb{Z}$. $w(\mathbf{x}) = \sup\{k^+(m) - k^-(m) \mid m \in \mathbb{Z}\}$ is said to be the width of $\mathbf{x}$. If $w(\mathbf{x}) < +\infty$, then the width of $\mathbf{x}$ is said to be bounded. It follows immediately from (2.7) and Lemma 2.5 that if the width of $\mathbf{x}$ is bounded, then $w(\phi_F^t\mathbf{x}) \le w(\mathbf{x})$ for $t > 0$ for which the solution is defined.

Let

$$\mathscr{X}_\omega = \left\{\mathbf{x} = (x_n) \in \mathscr{X} \mid \sup_{n\in\mathbb{Z}} |x_n - n\omega| < +\infty\right\}.$$

The following conclusions are easy to check; see [**4**, **38**].

LEMMA 2.6. *For each* $\mathbf{x} = (x_n) \in \mathscr{X}_\omega$, *we have*

$$|x_{n+m} - x_n - m\omega| \le w(\mathbf{x}), \quad n, m \in \mathbb{Z}, \tag{2.9}$$

$\mathbf{x} \in \mathscr{X}_\omega$ *is weakly rotationally ordered if and only if* $w(\mathbf{x}) \le 1$, *and* $\phi_F^t\mathbf{x} \in \mathscr{X}_\omega$ *for* $t > 0$ *for which the solution is defined.*

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Let

$$\mathscr{X}^1_\omega = \{\mathbf{x} \in \mathscr{X}_\omega \mid w(\mathbf{x}) \le 1\}, \quad \mathscr{X}^1_{[a,b]} = \bigcup_{\omega\in[a,b]} \mathscr{X}^1_\omega, \quad 0 \le a < b < +\infty.$$

Assume that $\mathbf{x} \in \mathscr{X}^1_\omega$ and $\phi^t_F\mathbf{x}$ is a solution to (2.6). Then from the above discussion we deduce that $\phi^t_F\mathbf{x} \in \mathscr{X}^1_\omega$ for $t > 0$ for which the solution is defined. Note that $\mathcal{F}$ in (2.6) is Lipschitz continuous in $\mathscr{X}^1_\omega$ since $|x_{n+1} - x_n| \le 1 + \omega$, $n \in \mathbb{Z}$, for all $\mathbf{x} = (x_n) \in \mathscr{X}^1_\omega$. Consequently, $\phi^t_F\mathbf{x}$ is defined for all $t \in \mathbb{R}$ if $\mathbf{x} \in \mathscr{X}^1_\omega$. Therefore, we have the following conclusion.

LEMMA 2.7. *$\mathscr{X}^1_\omega$ is forward invariant for (2.6).*

LEMMA 2.8. *Each closed bounded set in $\mathscr{X}^1_{[a,b]}$ is compact.*

LEMMA 2.9. *The topology in $\mathscr{X}^1_{[a,b]}$ induced by $\|\cdot\|$ is equivalent to the product topology and hence $\mathscr{X}^1_{[a,b]}/\langle\mathbf{1}\rangle$ is compact.*

We remark that the proof of the above conclusions is similar to that in [**40**, §2.3] and hence omitted here.

2.4. *Hausdorff metric.* The Hausdorff metric is defined by setting [**23**]

$$d(A, B) = \sup\{d(x, B) \mid x \in A\} + \sup\{d(A, y) \mid y \in B\}$$

for any two closed sets A, B in a metric space X.

LEMMA 2.10. *The Hausdorff metric on the closed subsets of a compact metric space defines a compact topology.*

3. *Invariant ordered circles*

In this section, we construct invariant ordered circles which are invariant both for (1.1) and for the group of transformations $\{\sigma_{m,k}\}$. We should mention that the process is pretty much analogous to that in [**40**, §3] except that the local potential is more general in this paper.

For the tilted FK model (1.1) with periodic boundary condition

$$x_{n+q} = x_n + p, \quad n \in \mathbb{Z}, \tag{3.1}$$

where p and q are relatively prime integers, Baesens and MacKay proved [**4**] by monotonicity techniques the existence of a depinning force $F_d(p/q)$. Furthermore, they showed by Birkhoff's recurrence theorem the existence of uniform sliding states of form (1.6) for $F > F_d(p/q)$. For the irrational rotation number case, we obtained similar conclusions in [**38**] by applying a topological method for a linear interaction potential and in [**39**] by taking a limit procedure for a nonlinear interaction potential.

For both rational and irrational rotation number ω, we can construct from uniform sliding states a continuous curve in phase space $\mathscr{X}$, which is the image of a continuous function $h : \mathbb{R} \to \mathscr{X}$ with $p_n(h(s)) = u(n\omega + s)$ for $s \in \mathbb{R}$, where u is the dynamical hull function of uniform sliding states in (1.6). Then h is strictly increasing since u is [**4**, **38**, **39**], i.e., $h(s_1) \prec h(s_2)$ if $s_1 < s_2$, and $h(s + 1) = h(s) + \mathbf{1}$. Therefore, the image of h is invariant for $\{\sigma_{m,k}\}$ and (1.1), and is a circle in $\mathscr{X}/\langle\mathbf{1}\rangle$.

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In this section, we shall show that even for $0 \le F \le F_d(\omega)$, there exists such a curve in $\mathscr{X}^1_\omega$, called a strictly ordered circle, which is invariant for $\{\sigma_{m,k}\}$ and (1.1), and consists of more than one orbit, unlike the case $F > F_d(\omega)$. In fact, it contains at least one rotationally ordered equilibrium state of (1.1) with rotation number ω.

We should mention that for $F = 0$, the invariant strictly ordered circle we construct is much like the ghost circle coined and constructed by Golé [**19**] and our approach in some respects is inspired by Golé's work [**20**].

Definition 3.1. The image ℓ of a continuous function $h : \mathbb{R} \mapsto \mathscr{X}$ is called an ordered circle if $h(s_1) < h(s_2)$ for $s_1 < s_2$ and $h(s+1) = h(s) + \mathbf{1}$. It is said to be a strictly ordered circle if furthermore $h(s_1) \prec h(s_2)$ for $s_1 < s_2$. The function h is called a parametrization of ℓ.

An ordered circle ℓ is said to be invariant for the group of transformations $\{\sigma_{m,k}\}$ if for all $m, k \in \mathbb{Z}$, $\sigma_{m,k}\ell = \ell$. It is easy to see that the following conclusions hold.

LEMMA 3.2. *Each element in an ordered circle ℓ is weakly rotationally ordered if ℓ is invariant for $\{\sigma_{m,k}\}$. Each element in a strictly ordered circle ℓ is rotationally ordered if ℓ is invariant for $\{\sigma_{m,k}\}$.*

We first show for each rational number p/q, in $\mathscr{X}^1_{p,q}$, in which each element satisfies (3.1) and is weakly rotationally ordered, there exists an invariant curve, which is actually a circle in $\mathscr{X}^1_{p,q}/\langle\mathbf{1}\rangle$. By taking the limit in the Hausdorff metric, we then obtain the existence of an invariant strictly ordered circle in $\mathscr{X}^1_\omega$ with irrational ω.

In this section, we drop the subscript in ϕ^t_F since our conclusions hold for all $F \ge 0$.

3.1. *Periodic boundary conditions.* In this section, we shall study the existence of invariant strictly ordered circles for (1.1) with spatially periodic boundary condition (3.1). Consider the subspace $\mathscr{X}_{p,q}$ of $\mathscr{X}$:

$$\mathscr{X}_{p,q} = \{\mathbf{x} = (x_n) \in \mathscr{X} \mid x_{n+q} = x_n + p\},$$

where q and p are relatively prime positive integers. Let

$$\mathscr{X}^1_{p,q} = \mathscr{X}^1_{p/q} \cap \mathscr{X}_{p,q}.$$

Then each element in $\mathscr{X}^1_{p,q}$ is weakly rotationally ordered and satisfies (3.1), and it follows from (2.8) and Lemma 2.7 that $\phi^t(\mathscr{X}^1_{p,q}) \subset \mathscr{X}^1_{p,q}$ for all $t > 0$.

Definition 3.3. If the parametrization h of an ordered circle $\ell \subset \mathscr{X}^1_{p,q}$ satisfies

$$\sum_{n=1}^{q} h_n(s) = q\,s \quad \text{for all } s \in \mathbb{R},$$

in which $h_n(s) = p_n(h(s))$, $n \in \mathbb{Z}$, $s \in \mathbb{R}$, then it is called the standard parametrization.

For example, take

$$h_n(s) = s + n \cdot \frac{p}{q} - \frac{p(q+1)}{2q}, \quad s \in \mathbb{R}. \tag{3.2}$$

Then $\ell = \{\mathbf{x} = (h_n(s)) \mid s \in \mathbb{R}\}$ is an ordered circle invariant for $\{\sigma_{m,k}\}$ and $\ell \subset \mathscr{X}^1_{p,q}$ by Lemma 3.2. Moreover, the parametrization in (3.2) is standard.

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LEMMA 3.4. *Each ordered circle ℓ in $\mathscr{X}^1_{p,q}$ has a unique standard parametrization.*

Proof. Assume that $\ell \subset \mathscr{X}^1_{p,q}$ is an ordered circle and h the corresponding parametrization. Let

$$\bar{h}(s) = \frac{1}{q}\sum_{n=1}^{q} h_n(s), \quad s \in \mathbb{R}.$$

Then $\bar{h} : \mathbb{R} \to \mathbb{R}$ is strictly increasing and continuous, and satisfies $\bar{h}(s+1) = \bar{h}(s) + 1$. Let $s = \bar{h}^{-1}(\bar{s})$ denote the inverse function of $\bar{s} = \bar{h}(s)$. Setting $\eta_n(\bar{s}) = h_n(\bar{h}^{-1}(\bar{s}))$ yields the standard parametrization of ℓ. The uniqueness is easy to check. □

Let

$$\Omega = \{\ell \mid \ell \text{ is an ordered circle in } \mathscr{X}^1_{p,q} \text{ and invariant for } \{\sigma_{m,k}\}\}, \tag{3.3}$$

$$\mathscr{H} = \{h \mid h \text{ is the standard parametrization of } \ell \in \Omega\}, \tag{3.4}$$

and

$$\|h\|_{C^0} = \sup_{s\in[0,1]} \|h(s)\| = \sup_{s\in[0,1]} \sum_{n=-\infty}^{+\infty} \frac{|h_n(s)|}{2^{|n|}}$$

for each continuous h from [0, 1] to $\mathscr{X}$. Then it is easy to check that $\mathscr{H}$ is a non-empty subset of a Banach space with the above C^0 norm since the example in (3.2) belongs to $\mathscr{H}$.

LEMMA 3.5. *$\mathscr{H}$ is convex and compact with the C^0 norm.*

Proof. To verify the convexity, it suffices to show that the ordered circle parametrized by $(1-\lambda)h + \lambda\hat{h}$ with $\lambda \in [0, 1]$ and $h, \hat{h} \in \mathscr{H}$ is invariant for $\{\sigma_{m,k}\}$. To this end, it is enough to show that there is a common value θ such that $\sigma h(s) = h(s+\theta)$ for all $h \in \mathscr{H}$ and for all $s \in \mathbb{R}$. Indeed, since $\ell = \{h(s) \mid s \in \mathbb{R}\}$ with $h \in \mathscr{H}$ is invariant for the shift map σ, then, for each $s \in \mathbb{R}$, there exists $s' \in \mathbb{R}$ such that $\sigma h(s) = h(s')$, i.e., $h_{n+1}(s) = h_n(s')$, implying by (3.1) that

$$qs + p = h_2(s) + \cdots + h_{q+1}(s) = h_1(s') + \cdots + h_q(s') = qs'$$

and hence $s' = s + p/q$, i.e., the common value $\theta = p/q$.

The proof of the compactness is similar to that of [**40**, Lemma 3.3] and hence omitted here. □

From Lemma 2.7, we know that (1.1) induces in $\mathscr{X}^1_{p,q}$ a semi-flow $\{\phi^t\}_{t\geq 0}$ and ϕ^t is Lipschitz continuous with Lipschitz constant L depending on t, i.e., $\|\phi^t\mathbf{x} - \phi^t\mathbf{y}\| \leq L\|\mathbf{x} - \mathbf{y}\|$ for $\mathbf{x}, \mathbf{y} \in \mathscr{X}^1_{p,q}$.

Meanwhile, fixing $t > 0$, we introduce from ϕ^t a map $\hat{\phi}^t$ from $\mathscr{H}$ to $\mathscr{H}$ as follows. For each $h \in \mathscr{H}$, let $\eta(s) = \phi^t h(s)$, $s \in \mathbb{R}$. Then the image of η is an ordered circle in $\mathscr{X}^1_{p,q}$ invariant for $\{\sigma_{m,k}\}$ due to (2.7), (2.8), and Lemma 2.5. Let ξ denote the standard parametrization of this ordered circle, i.e.,

$$\xi(\bar{s}) = \eta(s), \quad \text{in which } \bar{s} = \bar{\eta}(s) = \frac{1}{q}\sum_{n=1}^{q} \eta_n(s).$$

Defining $\hat{\phi}^t h = \xi$, we have a map from $\mathscr{H}$ to $\mathscr{H}$.

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LEMMA 3.6. *$\hat{\phi}^t$ is continuous for each $t > 0$.*

Proof. Assume that $h^1, h^2 \in \mathscr{H}$. Let $\eta^k(s) = \phi^t h^k(s)$, and $\bar{\eta}^k(s) = (1/q)\sum_{n=1}^{q} \eta_n^k(s)$, $k = 1, 2$, $s \in \mathbb{R}$. If we denote the standard parametrization of η^k by ξ^k, $k = 1, 2$, then

$$\xi^1(\bar{s}) = \eta^1(s) \quad \text{and} \quad \xi^2(\tilde{s}) = \eta^2(s),$$

in which $\bar{s} = \bar{\eta}^1(s)$, $\tilde{s} = \bar{\eta}^2(s)$.

For each $\bar{s} \in [0, 1]$, there is an $s \in \mathbb{R}$ such that $\bar{s} = \bar{\eta}^1(s)$. Let $\tilde{s} = \bar{\eta}^2(s)$. Then

$$\|\xi^1(\bar{s}) - \xi^2(\bar{s})\| \le \|\xi^1(\bar{s}) - \xi^2(\tilde{s})\| + \|\xi^2(\tilde{s}) - \xi^2(\bar{s})\|.$$

Since for $n = 1, 2, \ldots, q$,

$$\begin{aligned}\frac{|\eta_n^1(s) - \eta_n^2(s)|}{2^n} &\le \|\eta^1(s) - \eta^2(s)\| = \|\phi^t h^1(s) - \phi^t h^2(s)\| \\ &\le L\|h^1(s) - h^2(s)\| \le L\|h^1 - h^2\|_{C^0},\end{aligned}$$

in which L denotes the Lipschitz constant of ϕ^t depending on t,

$$\begin{aligned}|\tilde{s} - \bar{s}| &\le \frac{1}{q}\sum_{n=1}^{q} |\eta_n^2(s) - \eta_n^1(s)| \le \frac{L}{q}(2 + \cdots + 2^q)\|h^2 - h^1\|_{C^0} \\ &\le (2^{q+1}L/q)\|h^2 - h^1\|_{C^0}.\end{aligned}$$

Note that for each $h = (h_n) \in \mathscr{H}$, $h_n (n \in \mathbb{Z})$ is non-decreasing and

$$h_1(s_2) - h_1(s_1) + \cdots + h_q(s_2) - h_q(s_1) = q(s_2 - s_1)$$

for $s_1 < s_2$. Hence,

$$|h_n(s_2) - h_n(s_1)| \le q|s_2 - s_1| \quad \text{for all } s_1, s_2 \in \mathbb{R}, n \in \mathbb{Z}.$$

It follows that

$$\|h(s_2) - h(s_1)\| = \sum_{n=-\infty}^{+\infty} \frac{|h_n(s_2) - h_n(s_1)|}{2^{|n|}} \le 3q|s_2 - s_1| \tag{3.5}$$

and hence all the elements in $\mathscr{H}$ are Lipschitz continuous with Lipschitz constant $3q$. It then follows that

$$\|\xi^2(\tilde{s}) - \xi^2(\bar{s})\| \le 3q|\tilde{s} - \bar{s}| \le 3L \cdot 2^{q+1}\|h^1 - h^2\|_{C^0}.$$

Meanwhile,

$$\begin{aligned}\|\xi^1(\bar{s}) - \xi^2(\tilde{s})\| &= \|\eta^1(s) - \eta^2(s)\| = \|\phi^t h^1(s) - \phi^t h^2(s)\| \\ &\le L\|h^1(s) - h^2(s)\| \le L\|h^1 - h^2\|_{C^0}.\end{aligned}$$

Consequently, we have

$$\|\hat{\phi}^t h^1 - \hat{\phi}^t h^2\|_{C^0} = \sup_{\bar{s}\in[0,1]} \|\xi^1(\bar{s}) - \xi^2(\bar{s})\| \le (3 \cdot 2^{q+1} + 1)L\|h^1 - h^2\|_{C^0},$$

implying that $\hat{\phi}^t$ is Lipschitz continuous. □

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LEMMA 3.7. *For each $h \in \mathscr{H}$, $\hat{\phi}^t h$ is continuous with respect to $t \geq 0$. Meanwhile, $\hat{\phi}^t \circ \hat{\phi}^s = \hat{\phi}^{t+s}$ for $t \geq 0$ and $s \geq 0$.*

Proof. The proof of continuity of $\hat{\phi}^t h$ with respect to t follows from the uniform continuity of ϕ^t on the compact set of $\mathscr{X}^1_{p,q}$ and is similar to that of Lemma 3.6, and hence omitted here.

To verify that $\hat{\phi}^t \circ \hat{\phi}^s = \hat{\phi}^{t+s}$, it suffices to prove that $\hat{\phi}^t(\hat{\phi}^s h) = \hat{\phi}^{t+s} h$ for each $h \in \mathscr{H}$. From the uniqueness of the standard parametrization, it is enough to show that the ordered circles corresponding to $\hat{\phi}^t(\hat{\phi}^s h)$ and $\hat{\phi}^{t+s} h$ are identical, which is obvious since $\phi^{t+s} = \phi^t \circ \phi^s$. □

THEOREM 3.8. *There exists a strictly ordered circle $\ell \subset \mathscr{X}^1_{p,q}$ invariant for $\{\sigma_{m,k}\}$ and ϕ^t for all $t \geq 0$.*

Proof. Fix $T > 0$. From Lemmas 3.5 and 3.6 and the Schauder fixed point theorem, it follows that there exists for each positive integer k a fixed point $h^k \in \mathscr{H}$ for $\hat{\phi}^{T/2^k}$. Note that $\mathscr{H}$ is compact. Then there is a convergent subsequence of $\{h^k\}$, not relabeled, such that $h^k \to h \in \mathscr{H}$ as $k \to \infty$. We remark that for each positive integer k, h^k, h^{k+1}, ... are fixed points of $\hat{\phi}^{T/2^k}$ by Lemma 3.7, and hence h is in fact a fixed point of $\hat{\phi}^{T/2^k}$ for each positive integer k. Furthermore, $\hat{\phi}^t h = h$ for all $t \geq 0$.

Indeed, suppose that this is not true. Then there is a $t_0 > 0$ such that $\hat{\phi}^{t_0} h = \eta \neq h$. Let $\varepsilon_0 = \|h - \eta\|_{C^0} > 0$. Since $\hat{\phi}^t h$ is continuous at $t = t_0$ by Lemma 3.7, then, for $\varepsilon_0 > 0$, there exists $\delta > 0$, small enough, such that

$$\|\hat{\phi}^t h - \hat{\phi}^{t_0} h\|_{C^0} < \varepsilon_0 \quad \text{for } |t - t_0| < \delta.$$

For this $\delta > 0$, there exists a positive integer k such that $T/2^k < \delta$. We can find an appropriate positive integer m such that $|t - t_0| < \delta$ with $t = mT/2^k$. It then follows from Lemma 3.7 that

$$\|\hat{\phi}^t h - \hat{\phi}^{t_0} h\|_{C^0} = \|(\hat{\phi}^{T/2^k})^m\, h - \hat{\phi}^{t_0} h\|_{C^0} = \|h - \eta\|_{C^0} = \varepsilon_0,$$

which is a contradiction. Therefore, the ordered circle ℓ determined by h is invariant for $\{\sigma_{m,k}\}$ and ϕ^t for $t \geq 0$. The strict monotonicity of ϕ^t for $t > 0$ by Lemma 2.5 ensures that ℓ is a strictly ordered circle. □

Remark. We remark that the conclusion of Theorem 3.8, i.e., the existence of ghost circles for the periodic boundary condition (3.1), has already been demonstrated by Le Calvez in [**24**, Ch. 10]. We present the proof here only for completeness and introducing some notation which will be used in the remainder of this paper.

3.2. *Irrational rotation number.* In what follows, by applying the property of the Hausdorff metric stated in Lemma 2.10, we show by passing to the limit the existence of a strictly ordered circle invariant for $\{\sigma_{m,k}\}$ and ϕ^t for all $t \geq 0$ in $\mathscr{X}^1_\omega$ with irrational ω. First we prove a lemma, a similar result of which has been obtained in [**40**, Theorem 3.5]; the proof given here is slightly simpler than that in [**40**]. Assume that $0 < a < b$.

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LEMMA 3.9. *Assume that $\ell_k \subset \mathscr{X}^1_{\omega_k} \subset \mathscr{X}^1_{[a,b]}$ is a sequence of strictly ordered circles invariant for $\{\sigma_{m,l}\}$ and $\phi^t_{F_k}$ with $\{F_k\}$ bounded. Then there exists a subsequence, not relabeled, such that $\ell_k \to \ell$ in the Hausdorff topology, $\omega_k \to \omega$, and $F_k \to F$ as $k \to \infty$. Moreover, $\ell \in \mathscr{X}^1_\omega$ is a strictly ordered circle invariant for $\{\sigma_{m,l}\}$ and ϕ^t_F for all $t > 0$.*

Proof. Let $\hat{\ell}_k = \ell_k/\langle \mathbf{1}\rangle$ denote the strictly ordered circles in $\mathscr{X}^1_{\omega_k}/\langle \mathbf{1}\rangle$. Note that $\mathscr{X}^1_{[a,b]}/\langle \mathbf{1}\rangle$ is compact by Lemma 2.9. Then the Hausdorff metric in $\mathscr{X}^1_{[a,b]}/\langle \mathbf{1}\rangle$ is compact by Lemma 2.10. Without loss of generality, we may assume that $\hat{\ell}_k \to \hat{\ell}$ as $k \to \infty$ in the Hausdorff metric.

Note that ℓ_k is periodic with period $\mathbf{1}$ and $\hat{\ell}_k$ is in fact a segment in ℓ_k. We extend $\hat{\ell}$ to ℓ by periodicity along $\mathbf{1}$. Then $\ell_k \to \ell$ as $k \to \infty$ in the Hausdorff metric and hence for each $\mathbf{x} \in \ell$, there exists a sequence of points $\mathbf{x}^k \in \ell_k$ such that $\lim_{k\to\infty} \|\mathbf{x}^k - \mathbf{x}\| = 0$.

Since $\sigma_{m,l}$ $(m, l \in \mathbb{Z})$ is Lipschitz continuous in $\mathscr{X}^1_{[a,b]}$, it follows that ℓ is invariant for $\{\sigma_{m,l}\}$, and ℓ is connected since ℓ_k is connected. Meanwhile, ℓ is invariant for ϕ^t_F for all $t \geq 0$.

Indeed, for each $t \geq 0$,

$$d(\phi^t_{F_k}\ell_k, \phi^t_F\ell) \leq d(\phi^t_{F_k}\ell_k, \phi^t_{F_k}\ell) + d(\phi^t_{F_k}\ell, \phi^t_F\ell).$$

Since ϕ^t_F is Lipschitz continuous with respect to initial values in $\mathscr{X}^1_\omega$ and the Lipschitz constant is independent of F if F is restricted in a bounded closed interval, the first term on the right-hand side tends to zero as $k \to \infty$. Meanwhile, due to the continuous dependence of (1.1) on parameters and the periodicity of ℓ, the second term on the right-hand side approaches zero as $k \to \infty$. Consequently, we have

$$\ell = \lim_{k\to\infty} \ell_k = \lim_{k\to\infty} \phi^t_{F_k}\ell_k = \phi^t_F\ell$$

and hence ℓ is invariant for ϕ^t_F for all $t \geq 0$.

It remains to show that ℓ is a strictly ordered circle in $\mathscr{X}^1_\omega$. First, we shall show that ℓ is strictly ordered. Assume that $\mathbf{x} = (x_n) \neq \mathbf{y} = (y_n)$, $\mathbf{x}, \mathbf{y} \in \ell$. We choose sequences $\{\mathbf{x}^k\}$ and $\{\mathbf{y}^k\} \subset \ell_k$ such that $\mathbf{x}^k \to \mathbf{x}$ and $\mathbf{y}^k \to \mathbf{y}$ as $k \to \infty$ in $\mathscr{X}$. Since $\mathbf{x} \neq \mathbf{y}$, there exists $n_0 \in \mathbb{Z}$ such that $x_{n_0} \neq y_{n_0}$, say $x_{n_0} < y_{n_0}$. Since $x^k_{n_0} \to x_{n_0}$ and $y^k_{n_0} \to y_{n_0}$ as $k \to \infty$, we have $x^k_{n_0} < y^k_{n_0}$ for sufficiently large k. It follows that $\mathbf{x}^k \prec \mathbf{y}^k$ since ℓ_k is strictly ordered. From the fact that ℓ_k is ϕ^{-T}-invariant for some $T > 0$, we have

$$\phi^{-T}\mathbf{x}^k \prec \phi^{-T}\mathbf{y}^k.$$

Taking $k \to \infty$, we derive $\phi^{-T}\mathbf{x} < \phi^{-T}\mathbf{y}$ and $\mathbf{x} \prec \mathbf{y}$ due to the strict monotonicity of ϕ^T. As a consequence, ℓ is strictly ordered, which implies that the continuous coordinate projection map p_0 is one-to-one.

Second, we shall show that $p_0 : \ell \to \mathbb{R}$ is a homeomorphism. Since ℓ is connected and p_0 is continuous, $p_0(\ell)$ is an interval in $\mathbb{R}$. Furthermore, we have that $p_0(\ell) = \mathbb{R}$ from the fact that $\sigma_{0,m}\ell = \ell$ for each $m \in \mathbb{Z}$. Now it suffices to show that p_0^{-1} is continuous.

Let $\mathbf{x}^k = p_0^{-1}(x^k_0)$ and $\lim_{k\to\infty} x^k_0 = x^*_0$. Then $\{x^k_0\}$ in bounded in $\mathbb{R}$ and hence $\{\mathbf{x}^k\}$ is bounded in $\mathscr{X}^1_\omega$. It then follows from Lemma 2.8 that there is a subsequence of $\{\mathbf{x}^k\}$,

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not relabeled, such that $\lim_{k\to\infty} \|\mathbf{x}^k - \mathbf{x}^*\| = 0$. We deduce from Lemma 2.9 that $x_0^k \to p_0(\mathbf{x}^*) = x_0^*$ as $k \to \infty$, implying that $p_0^{-1}(x_0^k) \to p_0^{-1}(x_0^*)$ as $k \to \infty$ and hence p_0^{-1} is continuous.

Consequently, $h = p_0^{-1}$ is a parametrization of ℓ since $h(s+1) = h(s) + \mathbf{1}$ and $h(s_1) \prec h(s_2)$ for $s_1 < s_2$. Thus, ℓ is a strictly ordered circle.

Finally, the rotation number of each element in ℓ is ω and hence $\ell \subset \mathscr{X}_\omega^1$. □

THEOREM 3.10. *For each irrational number $\omega > 0$, there exists a strictly ordered circle $\ell \subset \mathscr{X}_\omega^1$ invariant for $\{\sigma_{m,k}\}$ and ϕ_F^t for all $t \geq 0$.*

Proof. Let $\omega_k = p_k/q_k \in [a, b]$ be a sequence of rational numbers converging to ω. Let ℓ_k denote the strictly ordered circles obtained by Theorem 3.8 in $\mathscr{X}_{p_k,q_k}^1$. Then the conclusion follows from Lemma 3.9. □

4. *Depinning transition*

4.1. *Average velocity and depinning force.*

In the previous section, we demonstrated for rational or irrational rotation number ω the existence of a strictly ordered circle $\ell \subset \mathscr{X}_\omega^1$ invariant for $\{\sigma_{m,k}\}$ and ϕ^t for all $t \geq 0$. In fact, both the shift map $\sigma = \sigma_{1,0}$ and ϕ_F^T for some $T > 0$ induce on ℓ orientation-preserving circle homeomorphisms.

Note that the corresponding parametrization of the strictly ordered circle ℓ is $h = p_0^{-1}$ with the form

$$h(s) = (\ldots, h_{-1}(s), s, h_1(s), h_2(s), \ldots), \tag{4.1}$$

where $h_n(s)$ are strictly increasing and continuous, and satisfy $h_n(s+1) = h_n(s) + 1$ for $s \in \mathbb{R}$ and $n \in \mathbb{Z}$. Moreover, from the $\sigma_{n,0}$-invariance of ℓ, we deduce that

$$h_n(s) = h_1^n(s), \quad h_{-1}(s) = h_1^{-1}(s), \quad \text{and} \quad h_{-n}(s) = h_1^{-n}(s) \quad \text{for } n > 0. \tag{4.2}$$

Therefore, h_1 is a lift of an orientation-preserving circle homeomorphism with irrational rotation number ω and

$$\ell = \{\mathbf{x} = (x_n) \in \mathscr{X} \mid x_n = h_1^n(x_0),\ x_0 \in \mathbb{R},\ n \in \mathbb{Z}\}. \tag{4.3}$$

The action of ϕ_F^T ($T > 0$) on ℓ is also an orientation-preserving circle homeomorphism. Indeed, define a map G from $\mathbb{R}$ to $\mathbb{R}$ by

$$G(s) = h^{-1}\phi_F^T h(s), \quad s \in \mathbb{R}.$$

Then it is easy to see that G is strictly increasing and satisfies $G(s+1) = G(s) + 1$. Therefore, G is a lift of an orientation-preserving circle homeomorphism, and the rotation number

$$\rho = \rho(G) = \lim_{k\to\infty} \frac{G^k(s)}{k}$$

is well defined and independent of $s \in \mathbb{R}$.

In what follows, we study the existence and uniqueness of the average velocity defined by (1.5) of solutions of (1.1) in $\mathscr{X}_\omega^1$.

THEOREM 4.1. *The average velocity $\bar{v}$ exists in $\mathscr{X}_\omega^1$ and it is independent of initial values.*

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Proof. Assume that $\mathbf{x} = h(s) \in \ell$ and $\mathbf{x}(t) = (x_n(t)) = \phi_F^t \mathbf{x}$ is a solution of (1.1). Note that

$$\frac{G^k(s)}{k} = \frac{h^{-1}\phi_F^{kT} h(s)}{k} = \frac{p_0 \mathbf{x}(kT)}{k} = \frac{x_0(kT)}{k}.$$

It follows that

$$\lim_{k\to+\infty} \frac{x_0(kT)}{kT} = \frac{\rho}{T}$$

and, furthermore,

$$\lim_{t\to+\infty} \frac{x_0(t)}{t} = \frac{\rho}{T}.$$

Besides, for each $n \in \mathbb{Z}$, there exists $s' \in \mathbb{R}$ such that $\sigma_{n,0} h(s) = h(s')$, and

$$x_n(kT) = p_0 \sigma_{n,0} \phi_F^{kT} h(s) = h^{-1} \phi_F^{kT} \sigma_{n,0} h(s) = h^{-1} \phi_F^{kT} h(s') = G^k(s').$$

Since the rotation number ρ is independent of $s \in \mathbb{R}$, we have

$$\lim_{k\to+\infty} \frac{x_n(kT)}{k} = \rho \quad \text{and} \quad \lim_{t\to+\infty} \frac{x_n(t)}{t} = \frac{\rho}{T}.$$

As a consequence, the average velocity for each solution $\phi_F^t \mathbf{x}$ with $\mathbf{x} \in \ell$ exists and is independent of $\mathbf{x} \in \ell$. If the initial value $\mathbf{x} \in \mathscr{X}_\omega^1$ but $\mathbf{x} \notin \ell$, then we choose $\bar{\mathbf{x}} \in \ell$ and large $k \in \mathbb{Z}$ such that

$$\bar{\mathbf{x}} \prec \mathbf{x} \prec \bar{\mathbf{x}} + k \cdot \mathbf{1}.$$

Then it follows that

$$\phi_F^t \bar{\mathbf{x}} \prec \phi_F^t \mathbf{x} \prec \phi_F^t \bar{\mathbf{x}} + k \cdot \mathbf{1}, \quad t > 0,$$

implying that the average velocity $\bar{\nu}$ of $\phi_F^t \mathbf{x}$ exists and $\bar{\nu} = \rho/T$. □

We remark that the proof is similar to that of [**40**, Theorem 4.1]. The average velocity $\bar{\nu}$ is in fact independent of T and depends on the driving force F and the rotation number ω. So we denote it by $\bar{\nu}(\omega, F)$.

Note that $\bar{\nu}(\omega, 0) = 0$ since there exist for (1.1) with $F = 0$ equilibrium states (minimal energy configurations) in $\mathscr{X}_\omega^1$ by the Aubry–Mather theory.

If $\bar{\nu}(\omega, F) = 0$, then there exists in ℓ a rotationally ordered equilibrium state of (1.1). Indeed, for some $T > 0$, there is a fixed point $\mathbf{x} \in \ell$ for ϕ_F^T since the rotation number $\rho = 0$ for ϕ_F^T on ℓ. Then we have $\phi_F^T \mathbf{x} = \mathbf{x}$ and furthermore $\mathbf{x}$ is an equilibrium state since (1.1) is a gradient system.

Let $(1.1)_F$ denote (1.1) with the external driving force being F.

LEMMA 4.2. *Assume that $F_1 < F_2$. Fix $T > 0$. Then there exists $b > 0$ such that*

$$\phi_{F_1}^T \mathbf{x} + b \cdot \mathbf{1} \le \phi_{F_2}^T \mathbf{x} \quad \textit{for all } \mathbf{x} \in \mathscr{X}_\omega^1.$$

Proof. The proof is similar to that of Lemma 2.5; see the appendix in [**4**]. Let $\mathbf{x}(\lambda, t)$ be a solution of $(1.1)_{(1-\lambda)F_1+\lambda F_2}$ with the initial value $\mathbf{x}(\lambda, 0) = \mathbf{x} \in \mathscr{X}_\omega^1$, where $\lambda \in [0, 1]$. Set

$$\Psi(\lambda, t) = \frac{\partial \mathbf{x}(\lambda, t)}{\partial \lambda} \triangleq (\ldots, \psi_{-1}(\lambda, t), \psi_0(\lambda, t), \psi_1(\lambda, t), \ldots).$$

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Then $\psi_n(\lambda, t)$ satisfy the following equations:

$$\begin{aligned}\dot{\psi}_n(\lambda, t) = &-D_1D_2H(x_n, x_{n+1})\psi_{n+1} - D_1D_2H(x_{n-1}, x_n)\psi_{n-1}\\ &-[D_1D_1H(x_n, x_{n+1}) + D_2D_2H(x_{n-1}, x_n)]\psi_n + F_2 - F_1.\end{aligned}$$

Note that $\psi_n(\lambda, 0) = 0$. Then $\psi_n(\lambda, t) \geq 0$ for $t \geq 0$ and $n \in \mathbb{Z}$ due to our assumptions $F_2 > F_1$ and (1.3). We remark that this is a fundamental result for monotone dynamical systems; see [**1**, **4**, **20**, **41**]. Consequently, we have

$$\dot{\psi}_n(\lambda, t) \geq -[D_1D_1H(x_n, x_{n+1}) + D_2D_2H(x_{n-1}, x_n)]\psi_n + F_2 - F_1.$$

Since $\mathbf{x}(\lambda, t) \in \mathscr{X}_\omega^1$ for $t \geq 0$, we have for $\lambda \in [0, 1]$, $t \geq 0$, and $n \in \mathbb{Z}$,

$$|x_{n+1}(\lambda, t) - x_n(\lambda, t)| \leq 1 + \omega \quad \text{and} \quad |x_n(\lambda, t) - x_{n-1}(\lambda, t)| \leq 1 + \omega.$$

Let

$$d = \sup_{|y-x|\leq 1+\omega} |D_1D_1H(x, y)| + \sup_{|y-x|\leq 1+\omega} |D_2D_2H(x, y)| > 0.$$

Then it follows that

$$\dot{\psi}_n(\lambda, t) \geq -d\ \psi_n(\lambda, t) + F_2 - F_1.$$

The Gronwall inequality implies that

$$\psi_n(\lambda, T) \geq \psi_n(\lambda, 0)\, e^{-dT} + \frac{F_2 - F_1}{d}(1 - e^{-dT}) = \frac{F_2 - F_1}{d}(1 - e^{-dT}) \triangleq b > 0.$$

Therefore, we deduce that

$$\phi_{F_2}^T\mathbf{x} - \phi_{F_1}^T\mathbf{x} = \mathbf{x}(1, T) - \mathbf{x}(0, T) = \int_0^1 \frac{\partial \mathbf{x}(\lambda, T)}{\partial \lambda}\, d\lambda = \int_0^1 \Psi(\lambda, T)\, d\lambda \geq b \cdot \mathbf{1}. \quad \square$$

We derive the following conclusion immediately from Lemma 4.2.

THEOREM 4.3. *$\bar{\nu}(\omega, F)$ is non-decreasing in F.*

Define

$$F_d(\omega) = \sup\{F \geq 0 \mid \bar{\nu}(\omega, F) = 0\}. \tag{4.4}$$

Then $\bar{\nu}(\omega, F) = 0$ for $0 \leq F < F_d(\omega)$ and $\bar{\nu}(\omega, F) > 0$ if $F > F_d(\omega)$. The critical value $F_d(\omega)$ is called the depinning force [**4**, **9**, **11**, **12**, **17**, **25**, **37**].

4.2. *Uniform sliding states.* A uniform sliding state is referred to as a solution of (1.1) which has the form (1.6). We remark that uniform sliding states exist in many mechanical models, such as the Fermi–Pasta–Ulam chain [**16**], other than the tilted FK model [**9**, **11**, **12**, **17**, **42**]. In this subsection, we investigate the dynamics in $\mathscr{X}_\omega^1$ of (1.1) with $F > F_d(\omega)$ and prove the following conclusion.

THEOREM 4.4. *There exist in $\mathscr{X}_\omega^1$ uniform sliding states of form (1.6) for (1.1) with $F > F_d(\omega)$.*

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Proof. From Theorems 3.8 and 3.10, we know that there is a strictly ordered circle $\ell \subset \mathscr{X}^1_\omega$ invariant for ϕ^t_F with $F > F_d(\omega)$. Take an $\mathbf{x} \in \ell$ and consider $\mathbf{x}(t) = (x_n(t)) = \phi^t_F \mathbf{x}$ for $t \geq 0$. It follows that $\mathbf{x}(t_1) \prec \mathbf{x}(t_2)$ if $0 \leq t_1 < t_2$. Otherwise we have $\bar{\nu}(\omega, F) \leq 0$. Meanwhile, there exists $\tau > 0$ such that $\mathbf{x}(\tau) = \mathbf{x} + \mathbf{1}$. Otherwise we have $\bar{\nu}(\omega, F) = 0$. Consequently, the solution $\mathbf{x}(t)$ is defined for all $t \in \mathbb{R}$ and furthermore $\ell = \{\mathbf{x}(t) \mid t \in \mathbb{R}\}$.

The σ-invariance of ℓ yields the existence of some $t_0 \in \mathbb{R}$ such that $\sigma \mathbf{x} = \mathbf{x}(t_0)$. From (2.7), we deduce that $\sigma \mathbf{x}(t) = \mathbf{x}(t + t_0)$ for all $t \in \mathbb{R}$ and hence $x_n(t) = x_0(t + nt_0)$ for all $n \in \mathbb{Z}$ and $t \in \mathbb{R}$. We conclude that $t_0/\tau = \omega$ due to the facts that $x_0(t + \tau) = x_0(t) + 1$ and $\mathbf{x}(0) = \mathbf{x} \in \mathscr{X}^1_\omega$, i.e.,

$$\lim_{n\to\infty} \frac{x_n(0) - x_0(0)}{n} = \lim_{n\to\infty} \frac{x_0(nt_0) - x_0(0)}{n} = \omega.$$

Let $u(s) = x_0(\tau s)$, $s \in \mathbb{R}$. Then u is strictly increasing and satisfies $u(s + 1) = u(s) + 1$. The solution $\mathbf{x}(t)$ now can be expressed as $x_n(t) = u(n\omega + \nu t)$, in which $\nu = 1/\tau$. □

Remark 1. The dynamical hull function u of the uniform sliding states satisfies an advance–delay differential equation

$$\nu u'(s) = -D_1 H(u(s), u(s + \omega)) - D_2 H(u(s - \omega), u(s)) + F. \tag{4.5}$$

Remark 2. $\nu = 1/\tau$ is actually the average velocity $\bar{\nu}(\omega, F)$ for $F > F_d(\omega)$.

Remark 3. Each solution $\mathbf{x}(t) = \phi^t_F \mathbf{x}$ for $\mathbf{x} \in \ell$ can be expressed as $x_n(t) = u(n\omega + \nu t + \alpha)$ for some $\alpha \in \mathbb{R}$.

Remark 4. If u is a dynamical hull function, then so is $u(s + \alpha)$ for $\alpha \in \mathbb{R}$. If we require that $\int_0^1 u(s)\, ds = 1$, then the dynamical hull function is unique; see [**39**].

We should mention that for rational ω the conclusion in Theorem 4.4 has already been obtained by Baesens and MacKay by Birkhoff's recurrence theorem [**4**]. For irrational rotation number ω, the existence of a dynamical hull function was studied by a topological method [**38**] for a special $H(x, y) = (K/2)(y - x)^2 + (1 - \cos 2\pi x)$, $K > 0$. For more general H, a similar conclusion has been obtained by discussing the advance–delay differential equation (4.5) and taking a limit procedure [**39**].

THEOREM 4.5. *The average velocity $\bar{\nu}(\omega, F)$ is continuous with respect to $\omega > 0$ and $F \geq 0$.*

Proof. Assume that $\omega_k \to \omega$ and $F_k \to F$ as $k \to \infty$. According to Theorems 3.8 and 3.10, there exists a strictly ordered circle $\ell_k \subset \mathscr{X}^1_{\omega_k}$ invariant for σ and (1.1) with $F = F_k$, $k = 1, 2, \ldots$, and $\ell_k \to \ell \subset \mathscr{X}^1_\omega$, where ℓ is a strictly ordered circle invariant for σ and (1.1) by Lemma 3.9.

Suppose that $\bar{\nu}(\omega, F) = 0$. Then there exist an equilibrium state $\mathbf{x} \in \ell$, i.e., $\mathbf{x}(t) = \phi^t_F \mathbf{x} = \mathbf{x}$ for $t \in \mathbb{R}$, and a sequence $\{\mathbf{x}^k\}$ with $\mathbf{x}^k \in \ell_k$ such that $\lim_{k\to\infty} \|\mathbf{x}^k - \mathbf{x}\| = 0$.

For all $\varepsilon > 0$, there is a $T > 0$ such that $1/T < \varepsilon$, and there is a positive integer K such that for $k \geq K$ both $\|\mathbf{x}^k - \mathbf{x}\|$ and $\|\phi^T_{F_k} \mathbf{x}^k - \phi^T_F \mathbf{x}\|$ are small enough due to the continuous dependence of (1.1) on parameters and initial values. Consequently, we derive that $\|\phi^T_{F_k} \mathbf{x}^k - \mathbf{x}^k\|$ is small enough and hence

$$\mathbf{x}^k - \mathbf{1} \leq \phi^T_{F_k} \mathbf{x}^k \leq \mathbf{x}^k + \mathbf{1},$$

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implying by the uniqueness of the average velocity that $0 \le \bar{\nu}(\omega_k, F_k) \le 1/T < \varepsilon$ for $k \ge K$.

Suppose that $\bar{\nu}(\omega, F) > 0$. Then the strictly ordered circle ℓ is actually an orbit, i.e., $\ell = \{\mathbf{x}(t) \mid t \in \mathbb{R}\}$ and $\mathbf{x}(t+\tau) = \mathbf{x}(t) + \mathbf{1}$, in which $\tau = 1/\bar{\nu}(\omega, F) > 0$ and $\mathbf{x}(0) = \mathbf{x}$ is approximated by $\mathbf{x}^k \in \ell_k$, $k = 1, 2, \ldots$. Let $\mathbf{x}^k(t) = \phi^t_{F_k}\mathbf{x}^k$, $t \ge 0$.

For all $\varepsilon > 0$, there exists a positive integer $m \ge 2$ such that $T = m\tau > 1/\varepsilon$. Meanwhile, there exists a positive integer K such that for $k \ge K$ both $\|\mathbf{x}^k - \mathbf{x}\|$ and $\|\mathbf{x}^k(T) - \mathbf{x}(T)\|$ are small enough, implying that

$$\mathbf{x}^k + (m-1)\cdot\mathbf{1} \le \mathbf{x}^k(T) \le \mathbf{x}^k + (m+1)\cdot\mathbf{1},$$

from which we deduce that

$$\bar{\nu}(\omega, F) - \frac{1}{T} = \frac{m-1}{m\tau} = \frac{m-1}{T} \le \bar{\nu}(\omega_k, F_k) \le \frac{m+1}{T} = \frac{m+1}{m\tau} = \bar{\nu}(\omega, F) + \frac{1}{T}$$

and hence $|\bar{\nu}(\omega_k, F_k) - \bar{\nu}(\omega, F)| < \varepsilon$ for $k \ge K$. □

Remark. From Theorem 4.5, we know that $\bar{\nu}(\omega, F_d(\omega)) = 0$.

Let ℓ be a strictly ordered circle obtained by Theorem 3.8 or 3.10 for (1.1) with $F \in (0, F_d(\omega)]$, and

$$\ell^{\mathrm{rec}} = \Big\{\mathbf{x} \in \ell \mid \text{There exist } (m_i, k_i) \in \mathbb{Z}^2\backslash\{(0,0)\} \text{ such that } \mathbf{x} = \lim_{i\to\infty} \sigma_{m_i,k_i}\mathbf{x}\Big\}. \tag{4.6}$$

Then $\mathbf{x} = (x_n) \in \ell^{\mathrm{rec}}$ if and only if x_0 is a recurrent point of h_1, i.e., $x_0 \in \mathrm{Rec}(h_1)$, where h_1 is defined in (4.1) and (4.2) corresponding to ℓ.

Note that there exists an equilibrium state $\mathbf{x}^* \in \ell$ since the average velocity $\bar{\nu} = 0$ for $F \in (0, F_d(\omega)]$. Moreover, if ω is irrational, then

$$\ell^{\mathrm{rec}} \subset \mathrm{cl}\{\sigma_{m,k}\mathbf{x}^* \mid m, k \in \mathbb{Z}\},$$

in which 'cl' denotes the closure in the product topology. Therefore, each $\mathbf{x} \in \ell^{\mathrm{rec}}$ is a rotationally ordered equilibrium state of (1.1) with $F \in (0, F_d(\omega)]$.

5. *Proof of Theorem A*

First we need some lemmas.

LEMMA 5.1. *For each $\omega > 0$, there exists a strictly ordered circle $\ell \subset \mathscr{X}^1_\omega$ invariant for σ and (1.1) with $F = F_d(\omega)$ such that $\mathcal{F}(\mathbf{x}, F_d(\omega)) \ge \mathbf{0}$ for each $\mathbf{x} \in \ell$.*

Proof. For each $F_k = F_d(\omega) + 1/k$ $(k = 1, 2, \ldots)$, there exists by Theorem 3.8 or 3.10 a strictly ordered circle $\ell^k \subset \mathscr{X}^1_\omega$ invariant for both σ and $\phi^t_{F_k}$ $(t \ge 0)$, and $\ell^k \to \ell \subset \mathscr{X}^1_\omega$ by Lemma 3.9, in which ℓ is a strictly ordered circle and invariant for σ and $\phi^t_{F_d(\omega)}$ for all $t \ge 0$.

For each $\mathbf{x} \in \ell$, there exists a sequence $\mathbf{x}^k \in \ell^k$ such that $\lim_{k\to\infty}\mathbf{x}^k = \mathbf{x}$. Note that $\mathbf{x}^k(t) = \phi^t_{F_k}\mathbf{x}^k$ are in fact uniform sliding states and hence $\dot{\mathbf{x}}^k(t) \succ \mathbf{0}$ for $t \ge 0$ and $k = 1, 2, \ldots$, implying that $\lim_{k\to\infty}\dot{\mathbf{x}}^k(0) \ge \mathbf{0}$. It follows from the fact that $\lim_{k\to\infty}\mathbf{x}^k(0) = \mathbf{x}$ that $\mathcal{F}(\mathbf{x}, F_d(\omega)) \ge \mathbf{0}$. □

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LEMMA 5.2. *Assume that there exists in $\mathscr{X}^1_\omega$ a strictly ordered circle ℓ invariant for σ and (1.1) with $F \geq 0$, and $\mathcal{F}(\mathbf{x}, F) \geq \mathbf{0}$ for each $\mathbf{x} \in \ell$. Let $\Gamma = \pi(\ell)$. Then Γ is a circle on the cylinder satisfying the following property: for each $(x_0, y_0) \in \Gamma$, $\varphi_F(x_0, y_0)$ lies below or on Γ and $\varphi_F^{-1}(x_0, y_0)$ lies above or on Γ.*

Proof. For each $(x_0, y_0) \in \Gamma$, there exists $\mathbf{x} = (x_n) \in \ell$ such that $y_0 = -D_1 H(x_0, x_1)$. Let $(\bar{x}, \bar{y}) = \varphi_F(x_0, y_0)$. Then, by (2.5), we have

$$y_0 = -D_1 H(x_0, \bar{x}), \quad \bar{y} = D_2 H(x_0, \bar{x}) - F,$$

implying that $\bar{x} = x_1$ and hence $\bar{y} = D_2 H(x_0, x_1) - F$.

Note that ℓ is invariant for σ. Then $\sigma\mathbf{x} \in \ell$ and hence $\pi(\sigma\mathbf{x}) = (x_1, y_1) \in \Gamma$, in which $y_1 = -D_1 H(x_1, x_2)$.

From the assumption, we have that $\mathcal{F}(\mathbf{x}, F) \geq \mathbf{0}$, i.e.,

$$-D_1 H(x_n, x_{n+1}) - D_2 H(x_{n-1}, x_n) + F \geq 0 \quad \text{for all } n \in \mathbb{Z}. \tag{5.1}$$

Taking $n = 1$ yields

$$y_1 = -D_1 H(x_1, x_2) \geq D_2 H(x_0, x_1) - F = \bar{y},$$

implying that $(\bar{x}, \bar{y})$ lies below or on the circle Γ.

Let $(\bar{x}, \bar{y}) = \varphi_F^{-1}(x_0, y_0)$. Then $\varphi_F(\bar{x}, \bar{y}) = (x_0, y_0)$ and hence

$$\bar{y} = -D_1 H(\bar{x}, x_0), \quad y_0 = D_2 H(\bar{x}, x_0) - F.$$

Note that $y_0 = -D_1 H(x_0, x_1)$. Then we derive

$$-D_1 H(x_0, x_1) - D_2 H(\bar{x}, x_0) + F = 0.$$

Taking $n = 0$ in (5.1), we obtain

$$-D_1 H(x_0, x_1) - D_2 H(x_{-1}, x_0) + F \geq 0.$$

It then follows that $D_2 H(\bar{x}, x_0) \geq D_2 H(x_{-1}, x_0)$, implying that $\bar{x} \leq x_{-1}$ by (1.3).

Construct a point $\hat{\mathbf{x}} = (\hat{x}_n) \in \ell$ with $\hat{x}_0 = \bar{x} \leq x_{-1}$. Then $\hat{x}_1 \leq x_0$ since ℓ is a strictly ordered circle. Let $(\hat{x}_0, \hat{y}_0) = \pi(\hat{\mathbf{x}}) \in \Gamma$. Then $\hat{x}_0 = \bar{x}$ and

$$\hat{y}_0 = -D_1 H(\hat{x}_0, \hat{x}_1) = -D_1 H(\bar{x}, \hat{x}_1) \leq -D_1 H(\bar{x}, x_0) = \bar{y},$$

leading to the conclusion that $(\bar{x}, \bar{y})$ lies on or above Γ. □

LEMMA 5.3. *Assume that $\mathbf{x} = (x_n)$ and $\mathbf{x}^* = (x_n^*) \in \mathbb{R}^{\mathbb{Z}}$ and $x_k = x_k^*$ for some $k \in \mathbb{Z}$. Then $y_k > y_k^*$ if $x_{k+1} > x_{k+1}^*$, or $y_k < y_k^*$ if $x_{k+1} < x_{k+1}^*$, in which $y_k = -D_1 H(x_k, x_{k+1})$ and $y_k^* = -D_1 H(x_k^*, x_{k+1}^*)$.*

Proof. From the assumption (1.3), we deduce that $-D_1 H(x, y)$ is strictly increasing with respect to y and hence the conclusions hold. □

LEMMA 5.4. *Assume that $\omega > 0$ is irrational. Let $0 \leq F_1 < F_2 \leq F_d(\omega)$ and $\ell_1 \subset \mathscr{X}^1_\omega$ be the strictly ordered circle invariant for both σ and $\phi^t_{F_1}$. Let $\Gamma_1 = \pi(\ell_1)$. Then there exists an orbit $\{(x_n^*, y_n^*)\}$ of the area-preserving twist map φ_{F_2} crossing the circle Γ_1, i.e., there exist $i, j \in \mathbb{Z}$ such that (x_i^*, y_i^*) and (x_j^*, y_j^*) lie on the opposite sides of Γ_1.*

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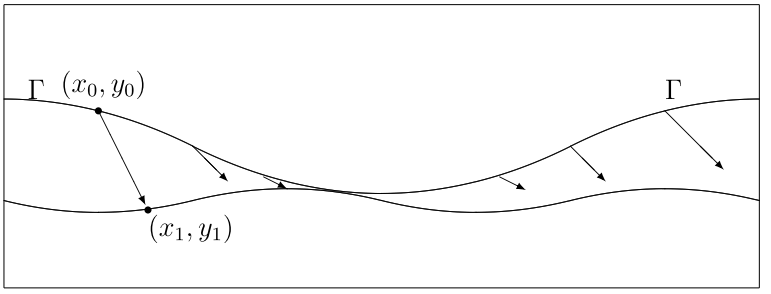


FIGURE 1. The image of each point in Γ under $\bar{\varphi}$ lies on or below Γ. So, if there is a point in Γ such that its image lies below Γ, then the flux does not vanish.

Proof. Let $\ell_2 \subset \mathscr{X}_\omega^1$ be the strictly ordered circle invariant for both σ and $\phi_{F_2}^t$ $(t \geq 0)$. Let h and h^* be the parametrizations of ℓ_1 and ℓ_2, respectively, satisfying (4.1) and (4.2). Then h_1 and h_1^* are two orientation-preserving circle homeomorphisms such that for each $\mathbf{x} = (x_n) \in \ell_1$ or $\mathbf{x}^* = (x_n^*) \in \ell_2$, one has that $x_n = h_1^n(x_0)$ or $x_n^* = h_1^{*n}(x_0^*)$ for $n \in \mathbb{Z}$.

Let ℓ_1^{rec} and ℓ_2^{rec} be defined as in (4.6). Then they are actually the sets of rotationally ordered equilibrium states of $\{\phi_{F_1}^t\}$ and $\{\phi_{F_2}^t\}$, respectively. Moreover, $\mathbf{x} = (x_n) \in \ell_1^{\text{rec}}$ $(\mathbf{x}^* = (x_n^*) \in \ell_2^{\text{rec}})$ if and only if $x_0 \in \text{Rec}(h_1)$ $(x_0^* \in \text{Rec}(h_1^*))$.

Note that h_1 and h_1^* have the same irrational rotation number ω. Then, from Lemma 2.1, it follows that either $\text{Rec}(h_1) = \text{Rec}(h_1^*)$ and $h_1|_{\text{Rec}(h_1)} = h_1^*|_{\text{Rec}(h_1^*)}$, or there exist $x_0 \in \text{Rec}(h_1)$ and $x_0^* \in \text{Rec}(h_1^*)$ such that the orbits $(x_n) = (h_1^n(x_0))$ and $(x_n^*) = (h_1^{*n}(x_0^*))$ cross infinitely often. Obviously, it is impossible for the former case to occur since $F_1 \neq F_2$. As a consequence, (x_n) and (x_n^*) cross infinitely often.

Since $\mathbf{x} = (x_n) \in \ell_1$ and $\mathbf{x}^* = (x_n^*) \in \ell_2$ cross infinitely often, there is some $i \in \mathbb{Z}$ such that $x_i \leq x_i^*$ and $x_{i+1} > x_{i+1}^*$. Take $\bar{\mathbf{x}} = (\bar{x}_n) \in \ell_1$ such that $\bar{x}_i = x_i^*$. Then we have $\bar{x}_{i+1} \geq x_{i+1} > x_{i+1}^*$ since ℓ_1 is strictly ordered.

Since $\sigma^i \bar{\mathbf{x}} \in \ell_1$, $\pi(\sigma^i \bar{\mathbf{x}}) = (\bar{x}_i, \bar{y}_i) \in \Gamma_1$, where $\bar{y}_i = -D_1 H(\bar{x}_i, \bar{x}_{i+1})$. We construct by (2.3) an orbit $\{(x_n^*, y_n^*)\}$ of φ_{F_2} since (x_n^*) satisfies (1.4) with $F = F_2$. According to Lemma 5.3, we have $\bar{y}_i > y_i^*$. Note that $(\bar{x}_i, \bar{y}_i)$ lies on Γ_1. Then (x_i^*, y_i^*) lies below Γ_1.

Similarly, we have some $j \in \mathbb{Z}$ $(j > i)$ such that $x_j \geq x_j^*$ and $x_{j+1} < x_{j+1}^*$, yielding a point (x_j^*, y_j^*) lying above Γ_1. □

Proof of Theorem A. Assume that $F_d(\omega) = 0$. Then, from Lemma 5.1, we deduce the existence of a strictly ordered circle ℓ invariant for σ and ϕ_0^t for each $t \geq 0$. Moreover, for each $\mathbf{x} = (x_n) \in \ell$, it follows that $\mathcal{F}(\mathbf{x}, 0) \geq \mathbf{0}$.

Let $\Gamma = \pi(\ell)$. Then Γ is a circle on the cylinder. It follows from Lemma 5.2 that for each $(x_0, y_0) \in \Gamma$, $(x_1, y_1) = \varphi(x_0, y_0)$ lies below or on the circle Γ; see Figure 1. Note that the flux of $\bar{\varphi}$ vanishes, which means that the oriented area between a homotopically non-trivial circle and its image under $\bar{\varphi}$ is zero. The above discussion shows that none of the points in $\bar{\varphi}(\Gamma)$ lies above Γ. As a consequence, $\bar{\varphi}(x_0, y_0) \in \Gamma$ for all $(x_0, y_0) \in \Gamma$. Otherwise the flux does not vanish. Therefore, Γ is an invariant circle of rotation number ω for the area-preserving twist map $\bar{\varphi}$.

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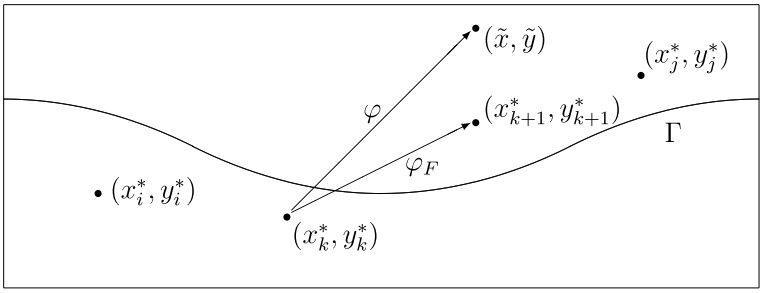


FIGURE 2. Construction of an orbit of φ crossing the invariant circle Γ of φ.

From the above discussion, we know that in fact each $\mathbf{x} \in \ell$ is an equilibrium point of (1.1) with $F = 0$. Otherwise there will be a point, the image of which under φ lies below Γ. In particular, for rational $\omega = p/q$, it follows that $\ell \subset \mathscr{X}_{p,q}^1$ and hence φ restricted on Γ is topologically conjugate to the rational rotation with rotation number p/q.

Assume that ω is irrational and $\bar{\varphi}$ has an invariant circle Γ of rotation number ω. According to [**5**, Theorem 7.7], we know that $\Gamma = \pi(\mathscr{M}_\omega)$ and $p_0(\mathscr{M}_\omega) = \mathbb{R}$. We shall prove that $F_d(\omega) = 0$ by contradiction.

Assume that $F_d(\omega) > 0$. Let $F = F_d(\omega)$. Note that now $\mathscr{M}_\omega$ is actually a strictly ordered circle invariant for σ and (1.1) with $F = 0$. Indeed, according to [**5**, (4.2)], there is a lift f of an orientation-preserving circle homeomorphism such that

$$\mathscr{M}_\omega = \{\mathbf{x} = (x_n) \in \mathbb{R}^{\mathbb{Z}} \mid x_n = f^n(x_0),\ x_0 \in \mathbb{R},\ n \in \mathbb{Z}\}.$$

Setting $p_n(h(s)) = f^n(s)$, $s \in \mathbb{R}$, yields a parametrization h of $\mathscr{M}_\omega$.

From Lemma 5.4, we know that there exists an orbit $\{(x_n^*, y_n^*)\}$ of φ_F crossing Γ, i.e., there exist $i < j$ such that (x_i^*, y_i^*) lies below Γ while (x_j^*, y_j^*) lies on or above Γ.

Now we have some $k \in \mathbb{Z}$ such that (x_k^*, y_k^*) lies below Γ, while (x_{k+1}^*, y_{k+1}^*) is on or above Γ; see Figure 2. From Lemma 2.2, we deduce that $\varphi(x_k^*, y_k^*)$ lies above Γ, which is a contradiction to our assumption that Γ is an invariant circle of φ.

Assume that $\omega = p/q$ is rational and $\bar{\varphi}$ has an invariant circle Γ on which $\bar{\varphi}$ is topologically conjugate to the rational rotation with rotation number ω. According to [**5**, Theorem 7.7], it follows that $\Gamma \subset \pi(\mathscr{M}_\omega)$ and furthermore $\Gamma \subset \pi(\mathscr{M}_\omega^{\mathrm{per}})$. Consequently, we have that $p_0(\mathscr{M}_\omega^{\mathrm{per}}) = \mathbb{R}$.

We shall prove by contradiction that $F_d(\omega) = 0$. Assuming that $F_d(\omega) > 0$ and taking $0 < F \le F_d(\omega)$, we have an equilibrium state $\mathbf{x}^* = (x_n^*) \in \mathscr{X}_{p,q}^1$ of $\{\phi_F^t\}$ since $\bar{\nu}(\omega, F) = 0$. Let

$$\bar{\mathbf{x}} = \sup\{\mathbf{x} \in \mathscr{M}_\omega^{\mathrm{per}} \mid \mathbf{x} \le \mathbf{x}^*\}.$$

It follows from the fact that $p_0(\mathscr{M}_\omega^{\mathrm{per}}) = \mathbb{R}$ that $\bar{\mathbf{x}} = (\bar{x}_n) \in \mathscr{M}_\omega^{\mathrm{per}}$ and there exists $k \in \{1, 2, \ldots, q\}$ such that $\bar{x}_k = x_k^*$. Then we have $\bar{\mathbf{x}} \le \mathbf{x}^*$ but $\bar{\mathbf{x}} \ne \mathbf{x}^*$. As a consequence of Lemma 2.5, $\phi_F^t \bar{\mathbf{x}} \prec \phi_F^t \mathbf{x}^* = \mathbf{x}^*$ for $t > 0$. On the other hand, from Lemma 4.2, we know that for some $t > 0$, $\bar{\mathbf{x}} = \phi_0^t \bar{\mathbf{x}} \prec \phi_F^t \bar{\mathbf{x}} \prec \mathbf{x}^*$, which is a contradiction. Consequently, $F_d(\omega) = 0$. □

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6. *Proof of Theorem B*

First, we show that the depinning force $F_d(\omega)$ is upper semi-continuous with respect to the rotation number $\omega > 0$, that is,

$$\text{for each } \omega^* > 0, \text{ for all } \varepsilon > 0, \text{ there exists } \delta > 0 \text{ for all } \omega > 0$$
$$\text{with } |\omega - \omega^*| < \delta,\ F_d(\omega) < F_d(\omega^*) + \varepsilon.$$

Indeed, the average velocity of (1.1) with driving force $F = F_d(\omega^*) + \varepsilon$ in $\mathscr{X}^1_{\omega^*}$ is positive, i.e., $\bar{\nu}(\omega^*, F_d(\omega^*) + \varepsilon) > 0$. Since $\bar{\nu}$ is continuous with respect to ω by Theorem 4.5, we deduce that there exists $\delta > 0$ such that $\bar{\nu}(\omega, F_d(\omega^*) + \varepsilon) > 0$ for $|\omega - \omega^*| < \delta$, which implies that $F_d(\omega) < F_d(\omega^*) + \varepsilon$.

In what follows, we shall show that $F_d(\omega)$ is lower semi-continuous at irrational points, that is, for each irrational $\omega^* > 0$,

$$\text{for all } \varepsilon > 0, \text{ there exists } \delta > 0, \text{ for all } \omega > 0 \text{ with } |\omega - \omega^*| < \delta,\ F_d(\omega) > F_d(\omega^*) - \varepsilon.$$

The conclusion is true if $F_d(\omega^*) = 0$. Now we assume that $F_d(\omega^*) > 0$. In this case we prove that $F_d(\omega)$ is lower semi-continuous at ω^* by contradiction. Assume that $F_d(\omega)$ is not lower semi-continuous at some irrational ω^*, i.e., there exist $\varepsilon_0 > 0$, and a sequence $\{\omega_k\}$ with $\omega_k > 0$ and $\omega_k \to \omega^*$ as $k \to \infty$, such that

$$0 \le F_d(\omega_k) \le F_d(\omega^*) - \varepsilon_0.$$

We assume that $\omega_k \in [a, b]$ for $k = 1, 2, \ldots$ and $0 \le a < b$. From Lemma 5.1, we know that there exists a strictly ordered circle $\ell_k \subset \mathscr{X}^1_{[a,b]}$ invariant for σ and (1.1) with $F = F_d(\omega_k)$. There is a subsequence of $\{\omega_k\}$, not relabeled, such that

$$F_d(\omega_k) \to \bar{F} \le F_d(\omega^*) - \varepsilon_0 \quad \text{as } k \to \infty.$$

Meanwhile, $\ell_k \to \bar{\ell} \subset \mathscr{X}^1_{\omega^*}$ as $k \to \infty$ by Lemma 3.9, where $\bar{\ell}$ is a strictly ordered circle invariant for both σ and (1.1) with $F = \bar{F}$.

Note that by Lemma 5.1, $\mathcal{F}(\mathbf{x}, F_d(\omega_k)) \ge \mathbf{0}$ for each $\mathbf{x} \in \ell_k$. Then we have for each $\mathbf{x} \in \bar{\ell}$, $\mathcal{F}(\mathbf{x}, \bar{F}) \ge \mathbf{0}$. Let $\bar{\Gamma} = \pi(\bar{\ell})$. Then $\bar{\Gamma}$ is a circle on the cylinder having the properties stated in Lemma 5.2.

Since $0 \le \bar{F} < F^* = F_d(\omega^*)$, then by Lemma 5.4 there exists an orbit $\{(x_n^*, y_n^*)\}$ of the area-preserving twist map φ_{F^*} such that this orbit crosses the circle $\bar{\Gamma}$. Thus, we can find a point (x_k^*, y_k^*) on the orbit $\{(x_n^*, y_n^*)\}$ such that (x_k^*, y_k^*) is below $\bar{\Gamma}$ and (x_{k+1}^*, y_{k+1}^*) lies on or above $\bar{\Gamma}$. We deduce from Lemma 2.2 that $(\tilde{x}, \tilde{y}) = \varphi_{\bar{F}}(x_k^*, y_k^*)$ lies above $\bar{\Gamma}$.

From Lemma 5.2, we know that $\varphi_{\bar{F}}(x_0, y_0)$ is on or below $\bar{\Gamma}$ for each $(x_0, y_0) \in \bar{\Gamma}$. If $\bar{\Gamma}$ is invariant for $\varphi_{\bar{F}}$ ($\bar{F} = 0$), then we have a contradiction since we have constructed an orbit of $\varphi_{\bar{F}}$ crossing $\bar{\Gamma}$. If $\bar{\Gamma}$ is not invariant for $\varphi_{\bar{F}}$, there exists a point $(x_0, y_0) \in \bar{\Gamma}$ such that $(\hat{x}, \hat{y}) = \varphi_{\bar{F}}(x_0, y_0)$ lies below $\bar{\Gamma}$. Draw a curve $\mathcal{C}$ from (x_0, y_0) to (x_k^*, y_k^*) such that $\mathcal{C}$ lies below $\bar{\Gamma}$. Then $\varphi_{\bar{F}}(\mathcal{C})$ is a continuous curve connecting $(\hat{x}, \hat{y})$ and $(\tilde{x}, \tilde{y})$, and

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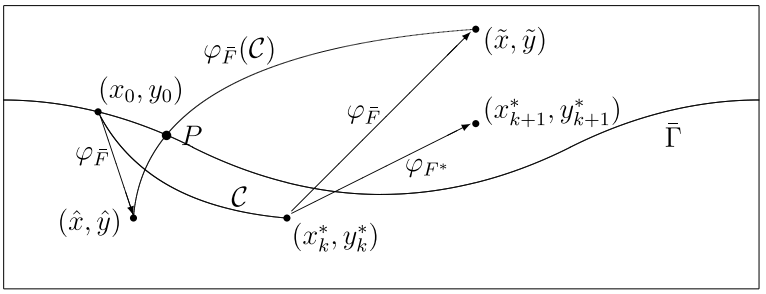


FIGURE 3. Construction of a point P in $\bar{\Gamma}$ such that $\varphi_{\bar{F}}^{-1}(P)$ lies below $\bar{\Gamma}$.

intersecting $\bar{\Gamma}$ at some point, see Figure 3, which is a contradiction since the preimage of this intersection point lies on $\bar{\Gamma}$ or above $\bar{\Gamma}$ by Lemma 5.2. □

7. *Discussion*

For each $F \geq 0$, we have constructed a strictly ordered circle $\ell \subset \mathscr{X}^1_\omega$ invariant for $\{\sigma_{m,k}\}$ and (1.1). If $F = 0$, then ℓ is a ghost circle defined by Golé [**20**]. We remark that for periodic boundary conditions (3.1), the ghost circles were also obtained by Le Calvez; see [**24**, Ch. 10].

Recently, Mather's criterion [**30**] has been generalized to models in statistical mechanics by de la Llave and Valdinoci [**14**, **15**], and the existence of ghost circles has been extended to monotone lattice recurrence relations by Mramor and Rink [**35**]. We remark that the construction of strictly ordered circles invariant for the shift map and the gradient flow in this paper is easy to generalize to monotone lattice recurrence relations if the models satisfy the conditions posed in [**35**] or [**14**, **15**]. What intrigues us is whether the main conclusions can be generalized, that is, roughly speaking, that the minimal energy configurations form a foliation is equivalent to that the depinning force vanishes. We should mention that the proof in this paper depends heavily on a property that if an exact area-preserving twist map on the cylinder has an invariant circle, then there are no orbits crossing this circle.

An obvious question is whether we can obtain stronger estimates for the continuity of the depinning force as Mather did in [**31**] and as Mramor and Rink did for monotone recurrence relations in [**36**]. We believe that they are obtainable as long as we have some basic estimates for Birkhoff periodic configurations as in [**36**].

It is also interesting to study in the partial differential equations setting the relationship between gaps of the Birkhoff minimizers and the depinning force of the heat flow associated with an elliptic equation. Related results in the elliptic integrand setting have been obtained by Bessi [**6**, **7**], de la Llave and Valdinoci [**13**], and Matano and Rabinowitz [**28**].

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