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# Conditional expectation and stochastic integrals in non-commutative $L^p$ spaces

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## *Introduction*

The aim of the paper is to propose a general scheme for the consideration of non-commutative stochastic integrals. The role of a probability space is played by a couple $(\mathscr{A}, \phi_0)$, where $\mathscr{A}$ is a von Neumann algebra and $\phi_0$ is a faithful normal state on $\mathscr{A}$. Our processes live in the algebra $\tilde{\bar{\mathscr{A}}}$ of all measurable operators associated with the crossed product of $\mathscr{A}$ by the modular automorphism group $\sigma^{\phi_0}$. The algebra contains all the (Haagerup's) $L^p$ spaces over $\mathscr{A}$. The measure topology of the algebra has the nice feature of inducing the $L^p$ norm topology on the $L^p$ spaces, which makes it particularly suitable for defining stochastic integrals. The commutative theory fits smoothly into the scheme, although there exists no canonical way of embedding the algebra of (commutative) random variables into $\tilde{\bar{\mathscr{A}}}$. In fact, for any commutative stochastic process we have a family of different non-commutative stochastic processes corresponding to the process. This arbitrariness seems to be quite natural in the non-commutative context. An appropriate example can be found at the end of the paper (Section 6, C4).

The present theory applies to the Itô–Clifford integrals and to the quantum stochastic integrals based on non-Fock, quasi-free representations of the CCR and CAR algebras together with the associated filtrations. Theories of this kind were considered by many authors [**14**, **18**]; the basic definitions are to be found in a series of papers of Barnett, Streater and Wilde [**2–5**]. Our approach allows a unified treatment of the theories. It is presented in the last section of the paper. All the preceding sections can be treated as a technical introduction to the subject. They are mainly concerned with the different constructions of conditional expectation in the Haagerup's $L^p$ spaces. The main result of this part, Theorem 5·4, seems to be of an independent interest. The possibility of extending conditional expectation to the crossed product has already been shown (in a slightly more general situation) by Lance [**17**]. The proof given here is very straightforward and is combined with the verification of some categorical properties of the extension.

The paper is almost self-contained. Preliminary material on positive generalized operators, measurable operators, crossed products and $L^p$ spaces (see [**12**, **22**, **21**, **11**]) is gathered at the beginning of the corresponding sections. It should be noted that the 'product' and 'sum' of operators can have different meaning in different contexts (there are four different 'products', for example). That is why we give proofs of some 'evident' facts. As for the notation, bar stands for 'crossed product', hat for 'generalized positive', and tilde for 'measurable'.

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## 1. *Generalized positive operators*

Let $\mathscr{A}$ be a von Neumann algebra acting in a Hilbert space $H$. A generalized positive operator affiliated with $\mathscr{A}$ is, by definition, a map $\hat{x}: \mathscr{A}_*^+ \to [0, \infty]$ satisfying

(1) $\hat{x}(\lambda\phi) = \lambda\hat{x}(\phi)$ for $\phi \in \mathscr{A}_*^+, \lambda \geqslant 0$;

(2) $\hat{x}(\phi+\psi) = \hat{x}(\phi)+\hat{x}(\psi)$ for $\phi, \psi \in \mathscr{A}_*^+$;

(3) $\hat{x}$ is lower semicontinuous (with respect to the norm topology on $\mathscr{A}_*^+$).

All the generalized positive operators constitute the extended positive part of $\mathscr{A}$, denoted by $\mathscr{A}_+^\wedge$. $\mathscr{A}_+$ is identified with a subset of $\mathscr{A}_+^\wedge$ in a natural way – if $x \in \mathscr{A}_+$, then $x(\phi) = \phi(x)$ for $\phi \in \mathscr{A}_*^+$. Any $\hat{x} \in \mathscr{A}_+^\wedge$ may be uniquely represented as follows:

$$\hat{x} = \int_0^\infty \lambda \, de(\lambda) + \infty \cdot p, \quad p \in \mathscr{A}, \; e(\lambda) \in \operatorname{Proj} \mathscr{A},$$

which signifies that

$$\hat{x}(\omega_\xi) = \int_0^\infty \lambda \, d(e(\lambda)\xi, \xi) + \|p\xi\|^2 \cdot \infty \quad \text{for } \xi \in H.$$

It should be noted that the strong limit of $e(\lambda)$ as $\lambda \to \infty$ is equal to $1-p$, so that the integral represents a positive self-adjoint (densely defined) operator on $(1-p)H$. The square root of the operator is denoted by $\hat{x}^{\frac{1}{2}}$.

The generalized positive operators are added and multiplied by scalars in a natural way. For $z \in \mathscr{A}, \hat{x} \in \mathscr{A}_+^\wedge$, the element $z\hat{x}z^*$ is defined by

$$(z\hat{x}z^*)(\phi) = \hat{x}(z^*\phi z) \quad \text{for } \phi \in \mathscr{A}_*^+, \quad \text{where} \quad z^*\phi z(x) = \phi(zxz^*).$$

Any $\hat{x} \in \mathscr{A}_+^\wedge$ is the pointwise limit of an increasing sequence of operators from $\mathscr{A}_+$, and any increasing sequence of generalized positive operators converges pointwise to an element of $\mathscr{A}_+^\wedge$. Any normal weight $\phi$ on $\mathscr{A}$ has a unique extension $\hat{\phi}$ to $\mathscr{A}_+^\wedge$ which is (positively) homogeneous, additive and normal (i.e. $\hat{\phi}(\hat{x}_n) \nearrow \hat{\phi}(\hat{x})$ for $\hat{x}_n \nearrow \hat{x}$).

Suppose now that $\mathscr{A}$ admits a normal faithful semifinite trace $\tau$ and define $\tau(x \bullet y) = \tau(x^{\frac{1}{2}} y x^{\frac{1}{2}})$ for $x, y \in \mathscr{A}_+$. Then the map $(x, y) \mapsto \tau(x \bullet y)$ has a unique extention to $\mathscr{A}_+^\wedge \times \mathscr{A}_+^\wedge$ which is homogeneous, additive and normal in both variables. Moreover, the map $\hat{x} \mapsto \tau(\hat{x} \bullet \cdot)$ is a homogeneous, additive and normal bijection of $\mathscr{A}_+^\wedge$ onto the set of normal weights on $\mathscr{A}$.

Finally, if $\kappa: \mathscr{A}_1 \to \mathscr{A}_2$ is a von Neumann algebra isomorphism of some von Neumann algebras $\mathscr{A}_1$ and $\mathscr{A}_2$, then its restriction to $(\mathscr{A}_1)_+$ has a unique extension $\hat{\kappa}: (\mathscr{A}_1)_+^\wedge \to (\mathscr{A}_2)_+^\wedge$ which is homogeneous, additive and normal.

In the sequel, if $x$ is a positive self-adjoint operator on a closed linear subspace of $H$ and $\xi \in H \setminus \mathscr{D}(x)$, we define $\|x\xi\|$ to be $+\infty$.

We finish the section with two simple but useful lemmas.

LEMMA 1·1. *If* $h_n, h \in \mathscr{A}_+^\wedge$ $(n = 1, 2, \ldots)$, *then*

$$h_n \nearrow h \quad \textit{if and only if} \quad \|h_n^{\frac{1}{2}}\xi\| \nearrow \|h^{\frac{1}{2}}\xi\| \quad \textit{for each } \xi \in H.$$

*In particular,* $\mathscr{D}(h_n^{\frac{1}{2}}) \searrow \mathscr{D}(h^{\frac{1}{2}})$.

*Proof.* '$\Rightarrow$' Obvious, since $h_n(\omega_\xi) = \|h_n^{\frac{1}{2}}\xi\|^2$. '$\Leftarrow$' Let $\phi = \sum_i \omega_{\xi_i}$. Then

$$h_n(\phi) = \sum_i h_n(\omega_{\xi_i}) \nearrow \sum_i h(\omega_{\xi_i}) = h(\phi).$$

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Indeed, by assumption $h_n(\omega_\xi) \nearrow h(\omega_\xi)$ for each $\xi \in H$. Hence, by simple $\epsilon$-analysis, $\sum_i h_n(\omega_{\xi_i}) \nearrow \sum_i h(\omega_{\xi_i})$.

LEMMA 1·2. *If $x \in \mathscr{A}^{\wedge}_{+}$ and $x$ is a self-adjoint, densely defined operator, then $x\chi_{[0,n]}(x) \nearrow x$ in $\mathscr{A}^{\wedge}_{+}$.*

*Proof.* The result follows immediately from Lemma 1·1 by virtue of the Spectral Theorem.

## 2. *Measurable operators*

Let $\mathscr{A}$ be a semifinite von Neumann algebra acting in $H$ and let $\tau$ be a normal faithful semifinite trace on $\mathscr{A}$. A linear subspace $D$ of $H$ is called $\tau$-dense, if for any $\delta > 0$ there is a projection $p \in \mathscr{A}$ such that $pH \subset D$ and $\tau(1-p) \leqslant \delta$. A closed densely defined operator $x$ affiliated with $\mathscr{A}$ is said to be $\tau$-measurable, if its domain is $\tau$-dense. The set of all $\tau$-measurable operators is denoted by $\tilde{\mathscr{A}}$. Thus $\tilde{\mathscr{A}}_{+} \subset \mathscr{A}^{\wedge}_{+}$.

If $x, y \in \tilde{\mathscr{A}}$, then both $x+y$ and $xy$ are closable and their closures, denoted $x \dot{+} y$ and $x.y$ and called, respectively, the strong sum and the strong product of $x$ and $y$, are measurable. The sets

$$D(\epsilon, \delta) = \{x \in \tilde{\mathscr{A}};\ \text{there is a projection } p \in \mathscr{A}$$
$$\text{such that } \|xp\| \leqslant \epsilon \text{ and } \tau(1-p) \leqslant \delta\}, \epsilon > 0, \delta > 0,$$

form a basis of the neighbourhoods of 0 for the topological vector space topology on $\mathscr{A}$ called the measure topology. The set $\tilde{\mathscr{A}}$ endowed with the measure topology is a topological *-algebra.

We state an important consequence of the above definitions.

LEMMA 2·1. *If $h, k \in \tilde{\mathscr{A}}$ and $h|D = k|D$ for some dense subspace $D$ of $H$, then $h = k$.*

*Proof.* $h-k$ is a closable operator, equal to 0 on a dense subspace $D$ of its domain. Its closure has to be 0 on $H$, whence $h = k$ ($\tilde{\mathscr{A}}$ is a *-algebra with respect to . and $\dot{+}$).

We prove a number of technical results which will be useful in the sequel. For a slightly different version of the first one, see corollary 12·8 in [**19**].

PROPOSITION 2·2. *For any $x \in \tilde{\mathscr{A}}$, $\hat{\tau}(x^*x) = \hat{\tau}(xx^*)$.*

*Proof.* Let $x = u|x|$ be the polar decomposition of $x$. Then $x^* = |x|u^*$, $x^*x = |x|^2$ and $xx^* = u|x|u^*$. Let $y_n = \chi_{[0,n]}(|x|^2)|x|^2$. Then, by 1·1 and 1·2,

$$\|y_n^{\frac{1}{2}} u^*\xi\| \nearrow \||x|u^*\xi\| \quad \text{for any } \xi \in H,$$

which yields

$$\||y_n^{\frac{1}{2}} u^*|\xi\| \nearrow \||x^*|\xi\| \quad \text{for any } \xi \in H.$$

We apply 1·1 once more to get $uy_nu^* \nearrow xx^*$. Hence, noting that $u^*uy_n = y_n$, we obtain

$$\hat{\tau}(xx^*) = \lim_n \tau(uy_nu^*) = \lim_n \tau(y_n) = \hat{\tau}(x^*x).$$

LEMMA 2·3. *If $h_n, h \in \tilde{\mathscr{A}}_{+}$, $h_n \nearrow$, $\mathscr{D}(h_n^{\frac{1}{2}}) \supset \mathscr{D}(h^{\frac{1}{2}})$ for each $n$ and $\|h_n^{\frac{1}{2}}\xi\| \nearrow \|h^{\frac{1}{2}}\xi\|$ for each $\xi \in \mathscr{D}(h^{\frac{1}{2}})$, then $h_n \nearrow h$.*

*Proof.* Let $h_n \nearrow k$. Then, by Lemma 1·1, $\|h^{\frac{1}{2}}\xi\| = \|k^{\frac{1}{2}}\xi\|$ for each $\xi \in \mathscr{D}(h^{\frac{1}{2}})$, which means that $\mathscr{D}(k^{\frac{1}{2}}) \supset \mathscr{D}(h^{\frac{1}{2}})$, so that $k^{\frac{1}{2}}$ is self-adjoint and densely defined. As $\mathscr{D}(h^{\frac{1}{2}})$ is

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$\tau$-dense, $k^{\frac{1}{2}} \in \tilde{\mathscr{A}}_+$. Let $u: h^{\frac{1}{2}}\xi \to k^{\frac{1}{2}}\xi$ for $\xi \in \mathscr{D}(h^{\frac{1}{2}})$; extend $u$ by continuity to the whole space $H$. Then $u$ is a partial isometry and $u^*u$ projects onto $(\ker h)^\perp$. If $u' \in \mathscr{A}'$, then for $\xi \in \mathscr{D}(h^{\frac{1}{2}}), u'\xi \in \mathscr{D}(h^{\frac{1}{2}})$ ($h^{\frac{1}{2}}$ being affiliated to $\mathscr{A}$) and

$$u'uh^{\frac{1}{2}}\xi = k^{\frac{1}{2}}u'\xi = uu'h^{\frac{1}{2}}\xi,$$

so that $u \in \mathscr{A}$. Hence, $u.h^{\frac{1}{2}} \in \tilde{\mathscr{A}}$ and $u.h^{\frac{1}{2}} | \mathscr{D}(h^{\frac{1}{2}}) = k^{\frac{1}{2}} | \mathscr{D}(h^{\frac{1}{2}})$. By Lemma 2·1, $u.h^{\frac{1}{2}} = k^{\frac{1}{2}}$, so that $k = h^{\frac{1}{2}}.u^*u.h^{\frac{1}{2}} = h$, which ends the proof.

LEMMA 2·4. *If $h, k \in \tilde{\mathscr{A}}_+$, $x_n \in \mathscr{A}$ and $x_n \nearrow h$, then $k.x_n.k \nearrow k.h.k$.*

*Proof.* By 2·3, it is enough to show that

$$k.x_n.k \nearrow, \mathscr{D}(x_n^{\frac{1}{2}}.k) = \mathscr{D}(|x_n^{\frac{1}{2}}.k|) = \mathscr{D}((k.x_n.k)^{\frac{1}{2}}) \supset \mathscr{D}((k.h.k)^{\frac{1}{2}}) = \mathscr{D}(h^{\frac{1}{2}}.k)$$

and that $\|(x_n^{\frac{1}{2}}.k)\xi\| \nearrow \|(h^{\frac{1}{2}}.k)\xi\|$ for $\xi \in \mathscr{D}(h^{\frac{1}{2}}.k)$.

Assume first that $\xi \in \mathscr{D}(h^{\frac{1}{2}}k)$. Then $\xi \in \mathscr{D}(x_n^{\frac{1}{2}}k)$ for all $n$ and $\|x_n^{\frac{1}{2}}k\xi\| \nearrow \|h^{\frac{1}{2}}k\xi\|$.

Take now an arbitrary $\xi \in \mathscr{D}(h^{\frac{1}{2}}.k)$. Then there is a sequence $(\xi_i)$ such that $\xi_i \in \mathscr{D}(h^{\frac{1}{2}}k)$, $\xi_i \to_i \xi$ and $h^{\frac{1}{2}}k\xi_i \to_i (h^{\frac{1}{2}}.k)\xi$. Hence

$$\|x_n^{\frac{1}{2}}k(\xi_i - \xi_j)\| \leqslant \|h^{\frac{1}{2}}k(\xi_i - \xi_j)\| \to_{i,j} 0,$$

so that $(x_n^{\frac{1}{2}}k\xi_i)_i$ is Cauchy and $\xi \in \mathscr{D}(x_n^{\frac{1}{2}}.k)$ for each fixed $n$. Furthermore,

$$\sup_n \|(x_n^{\frac{1}{2}}.k)(\xi_i - \xi)\| \leqslant \|(h^{\frac{1}{2}}.k)(\xi_i - \xi)\|$$

for each fixed $i$. Similarly, if $m > n$, then $\mathscr{D}(x_n^{\frac{1}{2}}.x) \supset \mathscr{D}(x_m^{\frac{1}{2}}.k)$ and $\|(x_n^{\frac{1}{2}}.k)\xi\| \leqslant \|(x_m^{\frac{1}{2}}.k)\xi\|$. Thus, $\mathscr{D}(x_n^{\frac{1}{2}}.k) \searrow \mathscr{D}(h^{\frac{1}{2}}.k)$ and $k.x_n.k \nearrow$.

It is now time to show that $\|(x_n^{\frac{1}{2}}.k)\xi\| \nearrow \|(h^{\frac{1}{2}}.k)\xi\|$ for $\xi \in \mathscr{D}(h^{\frac{1}{2}}.k)$. Take $\epsilon > 0$ and choose $j_0$ so that $\|(h^{\frac{1}{2}}.k)(\xi_{j_0} - \xi)\| \leqslant \epsilon$. For all sufficiently large $n$,

$$\begin{aligned}\|(h^{\frac{1}{2}}.k)\xi\| &\leqslant \|h^{\frac{1}{2}}k\xi_{j_0}\| + \epsilon \leqslant \|x_n^{\frac{1}{2}}k\xi_{j_0}\| + 2\epsilon \\ &\leqslant \|(x_n^{\frac{1}{2}}.k)\xi\| + \|(x_n^{\frac{1}{2}}.k)(\xi_{j_0} - \xi)\| + 2\epsilon \\ &\leqslant \|(x_n^{\frac{1}{2}}.k)\xi\| + 3\epsilon,\end{aligned}$$

which ends the proof.

LEMMA 2·5. *If $k \in \mathscr{A}_+$, $\hat{h} \in \mathscr{A}_+^\wedge$, $h \in \tilde{\mathscr{A}}_+$ and $\hat{h} = h$ as operators, then $k.h.k = k\hat{h}k$.*

*Proof.* Let $x_n \nearrow h = \hat{h}$. Then, by 2·4, $kx_nk \nearrow k.h.k$. It is also evident that $kx_nk \nearrow k\hat{h}k$.

PROPOSITION 2·6. *If $h, k \in \tilde{\mathscr{A}}_+$, then*

$$\tau(h \bullet k) = \hat{\tau}(k^{\frac{1}{2}}.h.k^{\frac{1}{2}}).$$

*Proof.* Let first $k \in \mathscr{A}_+$. Choose $(x_n)$, $x_n \in \mathscr{A}$ so that $x_n \nearrow h$. By 1·1, $\|x_n^{\frac{1}{2}}k^{\frac{1}{2}}\xi\| \nearrow \|h^{\frac{1}{2}}k^{\frac{1}{2}}\xi\|$ for $\xi \in H$, so that, again by 1·1, $kx_nk \nearrow k.h.k$.

Let now $k \in \tilde{\mathscr{A}}_+$ be arbitrary and $(x_n)$ as above. Then, by the first part of the proof, 2·2 and 2·4,

$$\begin{aligned}\tau(h \bullet k) &= \lim_n \hat{\tau}(x_n^{\frac{1}{2}}.k.x_n^{\frac{1}{2}}) = \lim_n \hat{\tau}(k^{\frac{1}{2}}.x_n.k^{\frac{1}{2}}) \\ &= \hat{\tau}(k^{\frac{1}{2}}.h.k^{\frac{1}{2}}).\end{aligned}$$

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### 3. *Norm one projections in $\mathscr{A}_+^\wedge$*

Let $\mathscr{A}$ be a von Neumann algebra and $\mathscr{B}$ its von Neumann subalgebra. We assume the existence of a $\sigma$-weakly continuous projection of norm one of $\mathscr{A}$ onto $\mathscr{B}$, which we denote by $E$.

We start with showing the possibility of extending $E$ to $\mathscr{A}_+^\wedge$.

PROPOSITION 3·1. *The mapping $E$ (restricted to $\mathscr{A}_+$) extends uniquely to a map $\hat{E}$ of $\mathscr{A}_+^\wedge$ onto $\mathscr{B}_+^\wedge$ which is positively homogeneous, additive, order-preserving and normal and satisfies $(\hat{E}\hat{x})(\phi) = \hat{x}(\phi \circ E)$.*

*Proof.* Let $\hat{x} \in \mathscr{A}_+^\wedge$, $x_n \in \mathscr{A}_+$ and $x_n \nearrow \hat{x}$. Since $E$ is positive, $Ex_n \nearrow \hat{y}$ for some $y \in \mathscr{B}_+^\wedge$. Put $\hat{E}\hat{x} = \hat{y}$. If $z_n \in \mathscr{A}_+$, $z_n \nearrow \hat{x}$, then, for each $\phi \in \mathscr{A}_*^+$, $\lim_n \phi(x_n) = \lim_n \phi(z_n)$, i.e. $x_n - z_n \to 0$ $\sigma$-weakly, whence $Ex_n - Ez_n \to 0$ $\sigma$-weakly and thus $\lim_n \phi(x_n) = \lim_n \phi(Ez_n)$ which in turn implies $Ez_n \nearrow \hat{E}\hat{x}$. We have

$$(\hat{E}\hat{x})(\phi) = \lim_n \phi(Ex_n) = \hat{x}(\phi \circ E).$$

It is obvious that $\hat{E}$ is positively homogeneous and additive. It is also order-preserving and normal due to the above formula.

In the sequel we assume additionally that $\mathscr{A}$ is semifinite, $\tau$ is a faithful normal semifinite trace on $\mathscr{A}$, and $E$ is $\tau$-invariant.

LEMMA 3·2. *$\hat{E}$ is $\hat{\tau}$-invariant, and*

$$\tau(h \bullet k) = \tau(\hat{E}h \bullet k) \quad \textit{for} \quad h \in \mathscr{A}_+^\wedge, k \in \mathscr{B}_+^\wedge.$$

*Proof.* Let $x_n \nearrow h$, $y_m \nearrow k$, $x_n \in \mathscr{A}_+$, $y_m \in \mathscr{B}_+$. Then, using 3·1, we get

$$\begin{aligned}\tau(h \bullet k) &= \lim_{m,n} \tau(E(y_m^{\frac{1}{2}} x_n y_m^{\frac{1}{2}})) \\ &= \lim_{m,n} \tau(y_m^{\frac{1}{2}}(Ex_n) y_m^{\frac{1}{2}}) \\ &= \tau(\hat{E}h \bullet k).\end{aligned}$$

PROPOSITION 3·3. *If $h \in \tilde{\mathscr{A}}_+$ is such that $\hat{E}h \in \tilde{\mathscr{B}}_+$, then*

$$\hat{E}(k^*.h.k) = k^*.\hat{E}h.k \quad \textit{for any} \quad k \in \tilde{\mathscr{B}}.$$

*Proof.* By 2·2, 2·6 and 3·2, for any $z \in \mathscr{B}_+$,

$$\begin{aligned}\tau(\hat{E}(k^*.h.k) \bullet z) &= \tau((k^*.h.k) \bullet z) \\ &= \tau(z^{\frac{1}{2}}.k^*.h.k.z^{\frac{1}{2}}) \\ &= \tau(h^{\frac{1}{2}}.k.z.k^*.h^{\frac{1}{2}}) \\ &= \tau(h \bullet (k.z.k^*)) \\ &= \tau(\hat{E}h \bullet (k.z.k^*)) \\ &= \tau((\hat{E}h)^{\frac{1}{2}}.k.z.k^*.(\hat{E}h^{\frac{1}{2}}) \\ &= \tau(z^{\frac{1}{2}}.k^*.\hat{E}h.k.z^{\frac{1}{2}}) \\ &= \tau((k^*.\hat{E}h.k) \bullet z).\end{aligned}$$

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Now it is enough to observe that $\tau(.\bullet z)$ for $z\in\mathscr{B}_+$ constitute a dense set of weights in $\mathscr{B}_*^+$.

PROPOSITION 3·4. *For any* $\hat{x}\in\mathscr{A}_+^\wedge, k\in\mathscr{B}_+$,

$$\hat{E}(k^*\hat{x}k) = k^*(\hat{E}\hat{x})\,k.$$

*Proof.* By 3·2, for $z\in\mathscr{B}_+$,

$$\begin{aligned}\tau(\hat{E}(k^*\hat{x}k)\bullet z) &= \tau(k^*\hat{x}k\bullet z)\\ &= \tau(\hat{x}\bullet kzk^*)\\ &= \tau(\hat{E}\hat{x}\bullet kzk^*)\\ &= \tau(k^*(\hat{E}\hat{x})\,k\bullet z).\end{aligned}$$

## 4. *Conditional expectation in crossed products*

Let $H$ be a Hilbert space. Denote by $\mathscr{C}_c(\mathbf{R})$ the space of complex continuous functions with compact support on $\mathbf{R}$. The tensor product $L^2(\mathbf{R},ds)\otimes H$ is spanned by elements of the form $f\otimes\xi$, where $f\in\mathscr{C}_c(\mathbf{R}), \xi\in H$. The Hilbert space completion of the linear combinations of such functions is denoted by $\bar{H}$.

Let now $\mathscr{A}$ be a von Neumann algebra acting in $H$ and $\phi_0$ a fixed faithful normal state on $\mathscr{A}$. Denote by $\sigma^{\mathscr{A}}$ the modular automorphism group associated with $\mathscr{A}$ and $\phi_0$. The crossed product of $\mathscr{A}$ by $\sigma^{\mathscr{A}}$ is the von Neumann algebra $\bar{\mathscr{A}}$ acting in $\bar{H}$ and generated by $\pi(x)$, $x\in\mathscr{A}$, and $\lambda(s), s\in\mathbf{R}$, where

$$\begin{aligned}(\pi(x)\,\xi)(t) &= \sigma_{-t}(x)\,\xi(t),\\ (\lambda(s)\,\xi)(t) &= \xi(t-s).\end{aligned}$$

Let $\bar{\mathscr{A}}_0$ be the *-algebra generated algebraically by operators $\pi(x), x\in\mathscr{A}$ and $\lambda(s)$, $s\in\mathbf{R}$. Then $\bar{\mathscr{A}}$ is the $\sigma$-weak closure of $\bar{\mathscr{A}}_0$. Every element $\bar{x}$ of $\bar{\mathscr{A}}_0$ may be represented as follows:

$$\bar{x} = \sum_k \lambda(s_k)\,\pi(x_k) \quad \text{for some } s_1,\dots,s_n\in\mathbf{R},\ x_1,\dots,x_n\in\mathscr{A}.$$

If $\kappa$ is a von Neumann algebra isomorphism from $\mathscr{A}_1$ onto $\mathscr{A}_2$, where $\mathscr{A}_i\,(i=1,2)$ is a von Neumann algebra with a faithful normal state $\phi_i$, and if

$$\kappa\circ\sigma_t^{\mathscr{A}_1} = \sigma_t^{\mathscr{A}_2}\circ\kappa \quad \text{for any } t\in\mathbf{R},$$

where $\sigma^{\mathscr{A}_i}$ is the modular automorphism group associated with $(\mathscr{A}_i,\phi_i)$, then there is a unique von Neumann algebra isomorphism $\bar{\kappa}$ of $\bar{\mathscr{A}}_1$ onto $\bar{\mathscr{A}}_2$ such that

$$\bar{\kappa}\circ\pi_{\mathscr{A}_1} = \pi_{\mathscr{A}_1}\circ\kappa \quad\text{and}\quad \bar{\kappa}\circ\lambda_{\mathscr{A}_1} = \lambda_{\mathscr{A}_2}.$$

The dual action $\theta$ on $\bar{\mathscr{A}}$ is the group $(\theta_s)_{s\in\mathbf{R}}$ of automorphisms of $\bar{\mathscr{A}}$ determined by

$$\begin{aligned}\theta_s(\pi(x)) &= \pi(x) \quad \text{for } x\in\mathscr{A}, s\in\mathbf{R};\\ \theta_s(\lambda(t)) &= e^{-ist}\lambda(t) \quad \text{for } s,t\in\mathbf{R}.\end{aligned}$$

Define $T:\bar{\mathscr{A}}_+\to\bar{\mathscr{A}}_+^\wedge$ by

$$(Tx)(\phi) = \int_{\mathbf{R}}\phi(\theta_s(x))\,ds \quad \text{for } x\in\bar{\mathscr{A}}_+, \phi\in\bar{\mathscr{A}}_*^+.$$

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Since $\hat{\theta}_s \circ T = T$ for each $s \in \mathbf{R}$, and $\theta_s(\bar{x}) = \bar{x}$ (with $\bar{x} \in \bar{\mathscr{A}}_+$) implies that $\bar{x} = \pi(x)$ for some $x \in \mathscr{A}$, the values of $T$ lie in $\pi(\mathscr{A})^\wedge_+$. For each normal weight $\phi$ on $\mathscr{A}$, we put

$$\bar{\phi} = \hat{\phi} \circ (\pi^{-1})^\wedge \circ T$$

and call $\bar{\phi}$ the dual weight of $\phi$. There exists a unique normal faithful semifinite trace $\tau$ on $\bar{\mathscr{A}}$ such that $\overline{\phi_0} = \tau(h_0 \bullet \cdot)$, where $h_0$ is the self-adjoint positive operator affiliated with $\bar{\mathscr{A}}$ for which $h_0^{it} = \lambda(t), t \in \mathbf{R}$. $\tau$ is called the relatively invariant trace since it satisfies $\tau \circ \theta_s = e^{-s}\tau$.

For any normal weight $\phi$ on $\mathscr{A}$ we denote by $h_\phi$ the element of $\bar{\mathscr{A}}^\wedge_+$ such that $\bar{\phi} = \tau(h_\phi \bullet \cdot)$. If $\phi$ is semifinite, then $h_\phi$ is a positive self-adjoint operator affiliated with $\mathscr{A}$ and

$$\tau(\chi_{]\lambda, \infty[}(h_\phi)) = \phi(1)/\lambda \quad \text{for } \lambda \in \mathbf{R}_+.$$

Hence, $h_\phi$ is $\tau$-measurable if and only if $\phi$ is finite. Moreover,

$$\{h_\phi : \phi \in \mathscr{A}_*^+\} = \{h \in \tilde{\mathscr{A}}_+ : \hat{\theta}_s(h) = e^{-s}h\}.$$

The automorphisms $\theta_s$ extend uniquely to continuous in measure *-automorphisms $\tilde{\theta}_s$ of $\tilde{\mathscr{A}}$. As $\tilde{\theta}_s(h) = \hat{\theta}_s(h)$ for $h \in \tilde{\mathscr{A}}_+$, we write simply $\theta_s$ instead of $\tilde{\theta}_s$ or $\hat{\theta}_s$.

It should be noted that for $h, k \in \tilde{\mathscr{A}}_+$, their strong sum $h \dot{+} k$ coincides with their form sum, that is, the sum of the generalized operators $h, k \in \mathscr{A}^\wedge_+$. Similarly, $x.h.x^* = x\hat{h}x^*$ for $x \in \bar{\mathscr{A}}$, $\hat{h} \in \bar{\mathscr{A}}^\wedge_+, h \in \tilde{\mathscr{A}}_+$, when $h = \hat{h}$ as operators. This enables us to extend the map $\phi \mapsto h_\phi$ of $\mathscr{A}_*^+$ into $\tilde{\mathscr{A}}_+$ to a linear bijection of $\mathscr{A}_*$ onto the set $\{h \in \tilde{\mathscr{A}} : \theta_s(h) = e^{-s}h \text{ for all } s\}$. We define a functional tr on the latter set by $\operatorname{tr}(h_\phi) = \phi(1)$.

A von Neumann subalgebra $\mathscr{B}$ of $\mathscr{A}$ is said to be expected, if it is invariant with respect to the modular automorphism group associated with the pair $(\mathscr{A}, \phi_0)$. If $\mathscr{B}$ is expected, then there exists a unique conditional expectation $E$ of $\mathscr{A}$ onto $\mathscr{B}$, i.e. a $\sigma$-weakly continuous $\phi_0$-invariant projection of norm one from $\mathscr{A}$ onto $\mathscr{B}$: see [**20**].

In the rest of the section we assume (which does not cause any loss of generality) that $\phi_0(\cdot) = (\cdot \xi_0, \xi_0)$ for some cyclic and separating $\xi_0 \in H$. Let $\mathscr{B}$ be an expected subalgebra of $\mathscr{A}$ acting in $H$ and let $E$ be the corresponding conditional expectation. Denote by $K$ the closure of $\mathscr{B}\xi_0$ in $H$ and by $P$ the orthogonal projection of $H$ onto $K$. Note that $(Ex)\xi_0 = Px\xi_0$ for $x \in \mathscr{A}$. We treat $\bar{K}$ as a subspace of $\bar{H}$ and write $\bar{P}$ for the orthogonal projection of $\bar{H}$ onto $\bar{K}$. By the uniqueness of the modular automorphism group, $\sigma_t^{\mathscr{B}} = \sigma_t^{\mathscr{A}} | \mathscr{B}$ for $t \in \mathbf{R}$ ($\mathscr{B}$ considered with $\phi_0$). Hence, $\lambda_{\mathscr{B}} = \lambda_{\mathscr{A}}$ and $\pi_{\mathscr{B}} = \pi_{\mathscr{A}} | \mathscr{B}$. In particular, $\mathscr{B}$ is generated by the operators $\pi_{\mathscr{A}}(x), \lambda_{\mathscr{A}}(s)$ where $x \in \mathscr{B}, s \in \mathbf{R}$. For simplicity, we omit the indices $\mathscr{A}$ and $\mathscr{B}$ on $\sigma$, $\pi$ and $\lambda$. It is not difficult to show that $E \circ \sigma_t = \sigma_t \circ E$.

Let now $\kappa$ be a map from $\mathscr{B}$ into $\mathbf{B}(K)$ given by $\kappa(x) = P(x|K)$ for $x \in \mathscr{B}$. Then $\kappa$ is a von Neumann algebra isomorphism of $\mathscr{B}$ onto $\mathscr{C} = \kappa(\mathscr{B})$. The von Neumann algebra $\mathscr{C}$ acts in $K$; we consider it with the vector state $(\cdot \xi_0, \xi_0)$ given by the cyclic and separating vector $\xi_0$. Observe that $\sigma_t^{\mathscr{C}} \circ \kappa = \kappa \circ \sigma_t^{\mathscr{B}}$, so that there exists a von Neumann algebra isomorphism $\bar{\kappa}$ of $\bar{\mathscr{B}}$ onto $\bar{\mathscr{C}}$ satisfying

$$\bar{\kappa} \circ \pi = \pi_{\mathscr{C}} \circ \kappa \quad \text{and} \quad \bar{\kappa} \circ \lambda = \lambda_{\mathscr{C}}.$$

We easily check that $\bar{\kappa}(\bar{x}) = \bar{P}(\bar{x}|\bar{K})$ for $\bar{x} \in \bar{\mathscr{B}}$. In particular,

$$\lambda_{\mathscr{C}}(s)\xi = \lambda(s)\xi \quad \text{for } \xi \in \bar{K}, s \in \mathbf{R};$$

$$\pi_{\mathscr{C}}(\kappa(x))\xi = \pi(x)\xi \quad \text{for } \xi \in \bar{K}, x \in \mathscr{B}.$$

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LEMMA 4·1. *For every* $s\in\mathbf{R}, x\in\mathscr{A}, \xi\in\bar{K}$, *we have*

$$\lambda(s)\,\pi(x)\,\xi = \lambda(s)\,\pi(Ex)\,\xi.$$

*Proof.* Let $y\in\mathscr{B}, f\in\mathscr{C}_c(\mathbf{R}), \xi = f\otimes y\xi_0$. Then

$$\begin{aligned}(\lambda(s)\,\pi(Ex)\,\xi,\xi) &= \int((\lambda(s)\,\pi(Ex)\,\xi)(t),\xi(t))\,dt\\ &= \int((\pi(Ex)\,\xi)(t-s),\xi(t))\,dt\\ &= \int f(t-s)\overline{f(t)}\,(\sigma_{s-t}(Ex)\,y\xi_0, y\xi_0)\,dt;\end{aligned}$$

furthermore,
$$\begin{aligned}(\sigma_{s-t}(Ex)\,y\xi_0, y\xi_0) &= (E(y^*\sigma_{s-t}(x)\,y)\,\xi_0,\xi_0)\\ &= (Py^*\sigma_{s-t}(x)\,y\xi_0,\xi_0)\\ &= (\sigma_{s-t}(x)\,y\xi_0, y\xi_0),\end{aligned}$$

which gives
$$(\lambda(s)\,\pi(Ex)\,\xi,\xi) = (\lambda(s)\,\pi(x)\,\xi,\xi).$$

To end the proof, use the continuity of the scalar product and the polarization formula.

LEMMA 4·2. *Let* $s_1,\dots,s_n\in\mathbf{R}, x_1,\dots,x_n\in\mathscr{A}$. *Then*

$$\|\sum_k \lambda(s_k)\,\pi(Ex_k)\| \leqslant \|\sum_k \lambda(s_k)\,\pi(x_k)\|.$$

*Proof.* Since $\bar{\kappa}$ is a von Neuman algebra isomorphism, it is an isometry, so that, by 4·1,

$$\begin{aligned}\|\sum_k \lambda(s_k)\,\pi(Ex_k)\| &= \sup_{\xi\in\bar{K},\,\|\xi\|\leqslant 1}\|\sum_k \lambda_{\mathscr{C}}(s_k)\,\pi_{\mathscr{C}}(\kappa(Ex_k))\,\xi\|\\ &= \sup_{\xi\in\bar{K},\,\|\xi\|\leqslant 1}\|\sum_k \lambda(s_k)\,\pi(x_k)\,\xi\|\\ &\leqslant \|\sum_k \lambda(s_k)\,\pi(x_k)\|.\end{aligned}$$

By virtue of 4·2, the formula

$$\bar{E}(\sum_k \lambda(s_k)\,\pi(x_k)) = \sum_k \lambda(s_k)\,\pi(Ex_k)$$

defines a norm one projection of $\bar{\mathscr{A}}_0$ onto $\bar{\mathscr{B}}_0$.

PROPOSITION 4·3. *$\bar{E}$ extends to a $\sigma$-weakly continuous projection of norm one from $\bar{\mathscr{A}}$ to $\bar{\mathscr{B}}$ (still denoted by $\bar{E}$).*

*Proof.* Since $\bar{\kappa}^{-1}$ is $\sigma$-weakly continuous and $\bar{\kappa}(\bar{\mathscr{B}}_0) = \bar{\mathscr{C}}$, it is enough to check the $\sigma$-weak continuity for the map

$$\bar{\kappa}\circ\bar{E}:\sum_k \lambda(s_k)\,\pi(Ex_k)\mapsto\sum_k \lambda_{\mathscr{C}}(s_k)\,\pi_{\mathscr{C}}(\kappa(Ex_k)).$$

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Note that, by 4·1, for $\xi \in \bar{K}, \bar{x} = \sum_k \lambda(s_k)\pi(x_k) \in \bar{\mathscr{A}}_0$,

$$\begin{aligned}((\bar{\kappa} \circ \bar{E})(\bar{x}))\,\xi &= \sum_k \lambda(s_k)\,\pi_{\mathscr{G}}(\kappa(Ex_k))\,\xi \\ &= \sum_k \lambda(s_k)\,\pi(x_k)\,\xi \\ &= \bar{x}\xi.\end{aligned}$$

Hence, if $\bar{x}_n \in \bar{\mathscr{A}}_0$ and $\bar{x}_n \to 0$ $\sigma$-weakly, then $(\bar{\kappa} \circ \bar{E})(\bar{x}_n) \to 0$ $\sigma$-weakly.

LEMMA 4·4. *$\bar{E} \circ \theta_s = \theta_s \circ \bar{E}$ for each $s \in \mathbf{R}$.*

*Proof.* Let $\bar{x} = \sum_k \lambda(t_k)\pi(x_k)$. Then

$$\begin{aligned}(\bar{E} \circ \theta_s)(\bar{x}) &= E\Big(\sum_k e^{-ist_k}\lambda(t_k)\,\pi(x_k)\Big) \\ &= \sum_k e^{-ist_k}\lambda(t_k)\,\pi(Ex_k) \\ &= \theta_s\Big(\sum_k \lambda(s_k)\,\pi(Ex_k)\Big) \\ &= (\theta_s \circ \bar{E})(\bar{x}),\end{aligned}$$

so that the formula follows from the $\sigma$-weak continuity of $\bar{E}$ and $\theta_s$.

LEMMA 4·5. *$\hat{T} \circ \hat{\bar{E}} = \hat{\bar{E}} \circ \hat{T}$.*

*Proof.* By 4·4, 3·1, for $\phi \in \bar{\mathscr{A}}_*^+, \bar{x} \in \bar{\mathscr{A}}_+$,

$$\begin{aligned}T(\bar{E}\bar{x})(\phi) &= \int \phi(\theta_s(\bar{E}\bar{x}))\,ds \\ &= \int (\phi \circ \bar{E})(\theta_s \bar{x})\,ds \\ &= (T\bar{x})(\phi \circ \bar{E}) \\ &= \hat{\bar{E}}(T\bar{x})(\phi)\end{aligned}$$

and the result follows from the normality of $\hat{T}$ and $\hat{\bar{E}}$.

PROPOSITION 4·6. *If $E$ is $\phi$-invariant for some normal weight $\phi$ on $\mathscr{A}$, then $\bar{E}$ is $\bar{\phi}$-invariant.*

*Proof.* By 4·5, for $\bar{x} \in \bar{\mathscr{A}}_+$,

$$\begin{aligned}\bar{\phi}(\bar{E}\bar{x}) &= (\hat{\phi} \circ (\pi^{-1})\hat{} \circ T)(\bar{E}\bar{x}) \\ &= (\hat{\phi} \circ (\pi^{-1})\hat{} \circ \hat{\bar{E}})(T\bar{x}).\end{aligned}$$

As $T\bar{x} \in \pi(\mathscr{A})_+^{\wedge}$, it follows that $\hat{\bar{E}}(T\bar{x}) = (\hat{\pi} \circ \hat{E} \circ (\pi^{-1})\hat{})(T\bar{x})$, so that

$$\bar{\phi}(\bar{E}\bar{x}) = \hat{\phi}(\hat{E}((\pi^{-1})\hat{}(T\bar{x}))) = \hat{\phi}((\pi^{-1})\hat{}T\bar{x}) = \bar{\phi}(\bar{x}).$$

THEOREM 4·7. *There is a unique faithful $\sigma$-weakly continuous norm one projection $\bar{E}$ of $\bar{\mathscr{A}}$ onto $\bar{\mathscr{B}}$ which is $\bar{\phi}_0$-invariant and $\tau$-invariant and satisfies $\theta_s \circ \bar{E} = \bar{E} \circ \theta_s$ for $s \in \mathbf{R}$.*

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*Proof.* The uniqueness and the faithfulness of $\bar{E}$ follows from the faithfulness of $\phi_0$ and the $\bar{\phi}_0$-invariance of $\bar{E}$. For the $\tau$-invariance of $\bar{E}$, see [17]. The rest follows by 4·3, 4·4 and 4·6.

LEMMA 4·8. *$\hat{\bar{E}}h_\phi = h_{\phi\circ E}$ for $\phi \in \mathscr{A}_*^+$.*

*Proof.* By 3·2 and 4·5, for $\bar{x} \in \bar{\mathscr{A}}_+$,

$$\begin{aligned}\tau(\hat{\bar{E}}h_\phi \bullet \bar{x}) &= \tau(h_\phi \bullet \bar{E}\bar{x}) \\ &= \bar{\phi}(\bar{E}\bar{x}) \\ &= (\hat{\phi} \circ (\pi^{-1})\hat{} \circ T \circ \bar{E})(\bar{x}) \\ &= ((\phi \circ E)\hat{} \circ (\pi^{-1})\hat{} \circ T)(\bar{x}) \\ &= \overline{\phi \circ E}(\bar{x}) \\ &= \tau(h_{\phi\circ E} \bullet \bar{x}).\end{aligned}$$

## 5. $L^p$ spaces

Let

$$L^p(\mathscr{A}) = \{h \in \tilde{\mathscr{A}} : \theta_s(h) = e^{-s/p}h\}, \quad p \in [1, \infty].$$

If $h \in L^p(\mathscr{A})$, $1 \leqslant p < \infty$, then $|h|^p \in L^1(\mathscr{A})$; we put

$$\|h\|_p = \operatorname{tr}(|h|^p)^{1/p}.$$

From the definition, $L^\infty(\mathscr{A}) = \pi(\mathscr{A})$. For $h = \pi(x)$,

$$\|h\|_\infty = \|\pi(x)\| = \|x\|.$$

With the above norms, the $L^p$ spaces are Banach spaces, with $L^\infty(\mathscr{A})$ isometrically isomorphic to $\mathscr{A}$ and $L^1(\mathscr{A})$ isometrically isomorphic to $\mathscr{A}_*$, the isometry being the map $\phi \mapsto h_\phi$.

If $h \in L^p(\mathscr{A})$ and $1/p + 1/q = 1/r$, then $h.k \in L^r(\mathscr{A})$ and $\|h.k\|_r \leqslant \|h\|_p \|k\|_q$. Moreover, if $r = 1$, then $\operatorname{tr}(h.k) = \operatorname{tr}(k.h)$. The space $L^2(\mathscr{A})$ is a Hilbert space with the scalar product $(h, k) = \operatorname{tr}(h.k^*)$. For any $\alpha, 0 \leqslant \alpha \leqslant 1$, and $p, q \geqslant 1, 1/p + 1/q = 1/r$, the set $h_0^{1/p}.\pi(\mathscr{A}).h_0^{1/q}$ is dense in $L^r(\mathscr{A})$.

Define now $L_\alpha^p(\mathscr{A}, \phi_0) = h_0^{\alpha/q}.L^p(\mathscr{A}).h_0^{(1-\alpha)/q}$ for $0 \leqslant \alpha \leqslant 1$. The norms on $L_\alpha^p(\mathscr{A}, \phi_0)$ are given by

$$\|h_0^{\alpha/q}.k.h_0^{(1-\alpha)/q}\|_{p,\alpha} = \|k\|_p \quad \text{for } k \in L^p(\mathscr{A}).$$

Let us note that

$$L_\alpha^\infty(\mathscr{A}, \phi_0) \subset L_\alpha^p(\mathscr{A}, \phi_0) \subset L_\alpha^q(\mathscr{A}, \phi_0) \subset L_\alpha^1(\mathscr{A}, \phi_0) = L^1(\mathscr{A})$$

and $\|h\|_{q,\alpha} \leqslant \|h\|_{p,\alpha}$ for $p \geqslant q \geqslant 1$, with $h \in L_\alpha^p(\mathscr{A}, \phi_0)$. In particular, all the spaces $L_\alpha^p(\mathscr{A}, \phi_0)$ are dense in $L^1(\mathscr{A})$ in the norm $\|\cdot\|_1$.

For each $\alpha \in [0, 1]$, $(L_\alpha^\infty(\mathscr{A}, \phi_0), L^1(\mathscr{A}))$ is a pair of compatible Banach spaces, and so is the pair $(L_0^p(\mathscr{A}, \phi_0), L_1^p(\mathscr{A}, \phi_0))$ for each $p \in [1, \infty]$. Let $C_\alpha, 0 \leqslant \alpha \leqslant 1$, be the interpolation functor of type $\alpha$ defined by Calderon's complex interpolation method [7]. The non-commutative Stein–Weiss interpolation theorem of Kosaki [16] states that for each $p \in [1, \infty]$ and $\alpha \in [0, 1]$,

$$L_\alpha^p(\mathscr{A}, \phi_0) = C_{1/p}(L_\alpha^\infty(\mathscr{A}, \phi_0), L^1(\mathscr{A})) = C_\alpha(L_0^p(\mathscr{A}, \phi_0), L_1^p(\mathscr{A}, \phi_0)).$$

Let $E_1 : L^1(\mathscr{A}) \to L^1(\mathscr{A}), h_\phi \mapsto h_{\phi\circ E}$.

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LEMMA 5·1. *$E_1|_{L^p_{\frac{1}{2}}(\mathscr{A},\phi_0)}: L^p_{\frac{1}{2}}(\mathscr{A},\phi_0) \to L^p_{\frac{1}{2}}(\mathscr{A},\phi_0)$ and $\|E_1|_{L^p_{\frac{1}{2}}(\mathscr{A},\phi_0)}\|_{p,\frac{1}{2}} = 1$.*

*Proof.* By 4·8, $\hat{E}h_0 = h_0$, hence $h_0^{\frac{1}{2}} \in L^2(\mathscr{B})$ and $\hat{E}h_0^{\frac{1}{2}} = h_0^{\frac{1}{2}}$. Further, by 4·8 and 3·3,

$$\begin{aligned} E_1(h_0^{\frac{1}{2}}.\pi(x).h_0^{\frac{1}{2}}) &= h_0^{\frac{1}{2}}.\hat{E}(\pi(x)).h_0^{\frac{1}{2}} \\ &= h_0^{\frac{1}{2}}.\pi(Ex).h_0^{\frac{1}{2}} \in L^\infty_{\frac{1}{2}}(\mathscr{A},\phi_0), \\ \|E_1(h_0^{\frac{1}{2}}.\pi(x).h_0^{\frac{1}{2}})\|_{\infty,\frac{1}{2}} &= \|\pi(Ex)\|_\infty = \|Ex\| \leqslant \|x\|, \end{aligned}$$

with equality for $x \in \mathscr{B}$. Thus the conclusions of the lemma follow immediately from the basic theorems of interpolation theory.

The above lemma justifies the following definition. For each $h \in L^p(\mathscr{A})$, the operator $E_p h$ is given by

$$h_0^{1/2q}.E_p h.h_0^{1/2q} = E_1(h_0^{1/2q}.h.h_0^{1/2q}).$$

It is obvious that $E_p: L^p(\mathscr{A}) \to L^p(\mathscr{A})$, $\|E_p\| = 1$ and $E_p h = h$ for $h \in L^p(\mathscr{B})$. Moreover, $E_1 = \hat{E}$ on $L^1_+(\mathscr{A})$, $E_\infty = \hat{E}$ on $L^\infty_+(\mathscr{A})$ and

$$\hat{E}(h_0^{1/2q}.k.h_0^{1/2q}) = h_0^{1/2q}.(E_p k).h_0^{1/2q}$$

for $k \in L^p(\mathscr{A})$, $1 \leqslant p \leqslant \infty$.

It would be perfectly possible to use the above mentioned Stein–Weiss theorem to define $E_p^{(\alpha)}: L^p(\mathscr{A}) \to L^p(\mathscr{A})$ for any $\alpha \in [0,1]$ by the formula

$$h_0^{\alpha/q}.E_p^{(\alpha)}h.h_0^{(1-\alpha)/q} = E_1(h_0^{\alpha/q}.h.h_0^{(1-\alpha)/q}).$$

It is not difficult to check that $E_p^{(\alpha)} = E_p$ for any $\alpha$. The choice $\alpha = \frac{1}{2}$ is a matter of convenience only. It is also evident that the results on norm convergence of martingales (see [**8–10**, **13**]) hold in Haagerup's spaces $L^p(\mathscr{A})$.

LEMMA 5·2. *If $k \in L^p_+(\mathscr{A})$, then $\hat{E}k \in \tilde{\mathscr{A}}$.*

*Proof.* Let $k = \int_0^\infty \lambda\, de(\lambda)$. Then

$$\begin{aligned} \hat{E}k &\leqslant \hat{E}\left(\int_0^1 \lambda\, de(\lambda) + \int_1^\infty \lambda^p\, de(\lambda)\right) \\ &\leqslant \hat{E}\left(\int_0^1 \lambda\, de(\lambda)\right) + \hat{E}k^p \\ &\leqslant \bar{E}\left(\int_0^1 \lambda\, de(\lambda)\right) + E_1 k^p \in \bar{\mathscr{A}} + L^1(\mathscr{A}) \subset \tilde{\mathscr{A}}, \end{aligned}$$

with the inequalities in the sense of generalized positive operators. It is easy to conclude that $\hat{E}k$ is a positive self-adjoint densely defined operator, and that it is measurable.

LEMMA 5·3. *Let $k \in L^p_+(\mathscr{A})$ and let*

$$D = h_0^{1/2q}(\mathscr{D}((E_p k)^{\frac{1}{2}} h_0^{1/2q})).$$

*Then $D$ is dense in $\bar{H}$, $\mathscr{D}((\hat{E}k)^{\frac{1}{2}}) \supset D$ and $\|(E_p k)^{\frac{1}{2}}\eta\| = \|(\hat{E}k)^{\frac{1}{2}}\eta\|$ for $\eta \in D$.*

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*Proof.* Note that the operator $(E_p k)^{\frac{1}{2}} h_0^{1/2q}$ is premeasurable (i.e. measurable after taking closure), hence its domain is dense in $H$ (even $\tau$-dense). If $\xi \perp D$, then

$$0 = (h_0^{1/2q}\eta, \xi) = (\eta, h_0^{1/2q}\xi)$$

for $\eta$ from a dense set $\mathscr{D}((E_p k)^{\frac{1}{2}} h_0^{1/2q})$, which implies $h_0^{1/2q}\xi = 0$, so that $\xi = 0$.

Let $x_n \in \mathscr{A}, x_n \nearrow k$. Then

$$h_0^{1/2q}.x_n.h_0^{1/2q} \nearrow h_0^{1/2q}.k.h_0^{1/2q}$$

and, by 3·3,

$$\begin{aligned} h_0^{1/2q}.\bar{E}x_n.h_0^{1/2q} &= \hat{\bar{E}}(h_0^{1/2q}.x_n.h_0^{1/2q}) \\ &\nearrow \hat{\bar{E}}(h_0^{1/2q}.k.h_0^{1/2q}) \\ &= h_0^{1/2q}.E_p k.h_0^{1/2q}. \end{aligned}$$

Hence

$$\|(\bar{E}x_n)^{\frac{1}{2}}.h_0^{1/2q}\xi\| \nearrow \|(E_p k)^{\frac{1}{2}}.h_0^{1/2q}\xi\| \quad \text{for } \xi \in H.$$

If $\xi \in \mathscr{D}((E_p k)^{\frac{1}{2}} h_0^{1/2q})$ and $\eta = h_0^{1/2q}\xi$, then

$$\|(\bar{E}x_n)^{\frac{1}{2}}\eta\| \nearrow \|(E_p k)^{\frac{1}{2}}\eta\| < \infty.$$

On the other hand,

$$\|(Ex_n)^{\frac{1}{2}}\eta\| \nearrow \|(\hat{\bar{E}}k)^{\frac{1}{2}}\eta\| \quad \text{for } \eta \in H.$$

The results stated follow.

THEOREM 5·4. *If $k \in L_+^p(\mathscr{A})$, then $E_p k = \hat{\bar{E}}k$.*

*Proof.* Put $u((\hat{\bar{E}}k)^{\frac{1}{2}}\eta) = (E_p k)^{\frac{1}{2}}\eta$ for $\eta \in D$ (with $D$ defined in 5·3). Extend $u$ by continuity on $\bar{H}_1 = (\hat{\bar{E}})^{\frac{1}{2}}(D)$ and put $u\eta = 0$ for $\eta \in \bar{H} \ominus \bar{H}_1$, so that $u$ is a partial isometry. Using the premeasurability of $(E_p k)^{\frac{1}{2}} h_0^{1/2q}$, 5·3 and the premeasurability of $(\hat{\bar{E}}k)^{\frac{1}{2}} h_0^{1/2q}$ (which, in turn, follows from 5·2), we get, for $\xi \in \mathscr{D}((E_p k)^{\frac{1}{2}} h_0^{1/2q})$ and $a' \in \mathscr{A}'$,

$$\begin{aligned} a'u(\hat{\bar{E}}k)^{\frac{1}{2}} h_0^{1/2q}\xi &= a'(E_p k)^{\frac{1}{2}} h_0^{1/2q}\xi = (E_p k)^{\frac{1}{2}} h_0^{1/2q} a'\xi \\ &= u(\hat{\bar{E}}k)^{\frac{1}{2}} h_0^{1/2q} a'\xi = ua'(\hat{\bar{E}}k)^{\frac{1}{2}} h_0^{1/2q}\xi. \end{aligned}$$

If $\eta \in \bar{H} \ominus \bar{H}_1$, then, for each $a' \in \mathscr{A}', \xi \in \mathscr{D}((E_p k)^{\frac{1}{2}} h_0^{1/2q})$,

$$(a'\eta, (\hat{\bar{E}}k)^{\frac{1}{2}} h_0^{1/2q}\xi) = (\eta, (\hat{\bar{E}}k)^{\frac{1}{2}} h_0^{1/2q}(a')^*\xi) = 0,$$

since $(a')^*\xi \in \mathscr{D}((\hat{\bar{E}}k)^{\frac{1}{2}} h_0^{1/2q})$. Hence $a'(\bar{H} \ominus \bar{H}_1) \subset H \ominus H_1$, that is, $a'u\eta = 0 = ua'\eta$ for $\eta \in \bar{H} \ominus \bar{H}_1$. Thus, $u \in \mathscr{A}'$ and

$$(E_p k)^{\frac{1}{2}}|D = u(\hat{\bar{E}}k)^{\frac{1}{2}}|D.$$

By 2·1,

$$(E_p k)^{\frac{1}{2}} = u.(\hat{\bar{E}}k)^{\frac{1}{2}} = (\hat{\bar{E}}k)^{\frac{1}{2}}u^*.$$

If $\xi \in D$, then $(\hat{\bar{E}}k)^{\frac{1}{2}}\xi \in u^*u(\bar{H})$. It follows that

$$E_p k|D = ((\hat{\bar{E}}k)^{\frac{1}{2}}u^*u.(\hat{\bar{E}}k)^{\frac{1}{2}})|D = (\hat{\bar{E}}k)|D.$$

Apply 2·1 once more to end the proof of the theorem.

## 6. *Stochastic integrals*

Let $\mathscr{A}$ be a von Neumann algebra with a faithful normal state $\phi_0$, $\mathscr{B}$ its expected von Neumann subalgebra, and $E$ the conditional expectation of $\mathscr{A}$ onto $\mathscr{B}$. We shall be concerned with the canonical extensions of $E$ to some subsets of $\tilde{\mathscr{A}}$, for example

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to $L^p(\mathscr{A})$. To simplify the notation, we identify a linear subspace of $\tilde{\mathscr{A}}$ which is big enough to contain all the $L^p$ spaces and to which $E$ can easily be extended. We denote the extended conditional expectation by $\mathscr{E}$. More precisely,

$$\mathscr{D}(\mathscr{E}) = \{z = x - y; x, y \in \tilde{\mathscr{A}}_+, \hat{E}x, \hat{E}y \in \tilde{\mathscr{A}}_+\}$$

and

$$\mathscr{E}z = \hat{E}x - \hat{E}y \quad \text{for} \quad z = x - y \in \mathscr{D}(\mathscr{E}).$$

It is clear that $\mathscr{E}z$ does not depend on the way in which $z$ is written as a difference of two positive operators. By 5·4, we have $\mathscr{E}|\tilde{\mathscr{A}}_+ = \hat{E}$ $\mathscr{E}|L^p(\mathscr{A}) = E_p$. It should be noted that $\mathscr{B}^\wedge_+$, $\tilde{\mathscr{B}}$ and $L^p(\mathscr{B})$ are canonically embedded into $\mathscr{A}^\wedge_+$, $\tilde{\mathscr{A}}$ and $L^p(\mathscr{A})$, respectively. Moreover, $\mathscr{E}$ takes $L^p(\mathscr{A})$ onto $L^p(\mathscr{B})$.

Let $T$ be an index set. A stochastic process on $T$ is a family $(X_t)_{t\in T}$ with $X_t \in \tilde{\mathscr{A}}$ for each $t \in T$. Assuming the index set $T$ directed, we define a filtration as an increasing family $(\mathscr{A}_t)$ of von Neumann subalgebras of $\mathscr{A}$ (with a common unit element 1 being also a unit of $\mathscr{A}$). The triple $(\mathscr{A}, \phi_0, (\mathscr{A}_t))$ is called a stochastic base. A filtration $(\mathscr{A}_t)$ is expected, if each of the algebras $\mathscr{A}_t$ is expected. We write $\mathscr{E}_t$ for the corresponding conditional expectation. A stochastic process $(X_t)$ is adapted (to the filtration $(\mathscr{A}_t)$), if $X_t \in \tilde{\mathscr{A}}_t$ for each $t \in T$. A process $(X_t)$ adapted to an expected filtration $(\mathscr{A}_t)$ is a martingale, if $X_t \in \mathscr{D}(\mathscr{E}_s)$ and $\mathscr{E}_s X_t = X_s$ for $s \prec t, s, t \in T$.

In the rest of the section, if not stated otherwise, $T$ is an interval $[t_0, t] \subset \mathbf{R}_+$ and $f, X$ are adapted processes on $T$.

We say that $f$ is simple if it can be written in the form

$$f = \sum_{i=0}^{n-1} f(t_i)\, \chi_{[t_i, t_{i+1}[} + f(t)\, \chi_{\{t\}},$$

where $t_0 < \ldots < t_n = t$. If $t_0 < t' < t$, then $f|[t_0, t]$ and $f|[t', t]$ are simple processes on $[t_0, t]$ and $[t', t]$, respectively. The class of simple adapted processes on $T$ is denoted by $\mathscr{S}$.

The right and the left stochastic integral of a simple process $f$ with respect to $(X_s)$ are defined by

$$\int_{t_0}^{t} f(s)\, dX_s = \sum_{i=0}^{n-1} f(t_i)\,(X_{t_{i+1}} - X_{t_i})$$

and

$$\int_{t_0}^{t} dX_s\, f(s) = \sum_{i=0}^{n-1} (X_{t_{i+1}} - X_{t_i}) f(t_i),$$

respectively.

The following properties of the integrals follow from the definition.

LEMMA 6·1. *Let $f, g \in \mathscr{S}$ and $a \in \mathbf{C}$. Then*

(1) $$\int_{t_0}^{t} (af + g)(s)\, dX_s = a \int_{t_0}^{t} f(s)\, dX_s + \int_{t_0}^{t} g(s)\, dX_s,$$

(2) $$\left( \int_{t_0}^{t} f(s)\, dX_s \right)^* = \int_{t_0}^{t} dX_s^*\, f(s)^*,$$

(3) $$\int_{t_0}^{t} f(s)\, dX_s = \int_{t_0}^{t'} f(s)\, dX_s + \int_{t'}^{t} f(s)\, dX_s \quad \text{for } t_0 < t' < t,$$

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*and, if* $(X_s)$ *is a martingale,*

$$\mathscr{E}_r\left(\int_{t_0}^{t} f(s)\,dX_s\right) = 0 \quad \text{for } r \leqslant t_0, \tag{4}$$

*so that the process*

$$t \mapsto \int_{t_0}^{t} f(s)\,dX_s$$

*on* $[t_0, \infty[$ *is a martingale. Of course, the properties* (2) (3) *and* (4) *are also valid, mutatis mutandis, for left stochastic integrals.*

We next present a general scheme of stochastic integration, which can be applied in a variety of situations.

Let $M$ be an index set, let $\mathscr{F}_\mu$, for each $\mu \in M$, be a linear space of adapted processes endowed with a linear topology $\mathscr{T}_\mu$, and let $\mathscr{F} \subset \bigcup_{\mu \in M} \mathscr{F}_\mu$. Suppose that $(X_s)$ is an adapted process on $T$. We say that $\mathscr{F}$ is an admissible class of (left) integrands for the (right) integrator $(X_s)$, if the following conditions are satisfied:

(i) for any $f \in \mathscr{F}$, there is $\mu \in M$ and a sequence $(f_n)$ with $f \in \mathscr{F}_\mu, f_n \in \mathscr{S} \cap \mathscr{F}_\mu, f_n \to f$ in $\mathscr{T}_\mu$, such that the sequence $(\int_{t_0}^{t} f_n(s)\,dX_s)$ is Cauchy in $\tilde{\mathscr{A}}$ (with measure topology);

(ii) if, for some $\mu, \nu \in M$, $f \in \mathscr{F}_\mu \cap \mathscr{F}_\nu \cap \mathscr{F}$, $f_n \in \mathscr{S} \cap \mathscr{F}_\mu$, $g_n \in \mathscr{S} \cap \mathscr{F}_\nu$, $f_n \to f$ in $\mathscr{T}_\mu$ and $g_n \to f$ in $\mathscr{T}_\nu$, and if the limits

$$\lim \int_{t_0}^{t} f_n(s)\,dX_s, \lim \int_{t_0}^{t} g_n(s)\,dX_s$$

exist, then they are equal.

Now, we are able to define the right stochastic integral $\int_{t_0}^{t} f(s)\,dX_s$ as a limit (in measure) of a sequence from part (i) of the above definition. Part (ii) of the definition guarantees that the value of the integral does not depend on the approximating sequence of simple processes. The left stochastic integral can be defined analogously. It should be noted that the integral just constructed satisfies condition (1) of 6·1 and does not, in general, satisfy (2) and (4).

Let now $\mathscr{F}$ and $\mathscr{F}_\mu$ be some sets of adapted processes on $\mathbf{R}_+$. Denote by $\mathscr{F}^{(T)}$ and $\mathscr{F}_\mu^{(T)}$ the sets of their restrictions to a closed interval $T$. Suppose we are given a family of topologies $\mathscr{T}_\mu^{(T)}$ on $\mathscr{F}_\mu^{(T)}$ satisfying the following condition:

$$\text{if} \quad T' \subset T \quad \text{and} \quad f_n \to f \quad \text{in } \mathscr{T}_\mu^{(T)}, \quad \text{then} \quad f_n | T' \to f | T' \quad \text{in } \mathscr{T}_\mu^{(T')}.$$

We say that $\mathscr{F}$ is an admissible class of integrands for the integrator $(X_s), s \in \mathbf{R}_+$, with respect to the inductive families $(\mathscr{F}_\mu^{(T)}, \mathscr{T}_\mu^{(T)})$, if, for any $T$, $\mathscr{F}^{(T)}$ is an admissible class of integrands for the integrator $(X_s), s \in T$, with respect to $(\mathscr{F}_\mu^{(T)}, \mathscr{T}_\mu^{(T)})$. It is clear that the condition (3) of 6·1 holds.

If $\mathscr{F}, \mathscr{G}$ are classes of admissible integrants for some integrators $(X_s), (Y_s)$, and if $f \in \mathscr{F} \cap \mathscr{G}$, we can define the stochastic integral of $f$ with respect to any linear combination of the integrators in an obvious way.

We shall give several specific examples of such a theory.

A. *Commutative theory – integration with respect to a continuous semimartingale*

Let $(\Omega, \Sigma, P, (\Sigma_t)_{t \in R_+})$ be a (commutative) stochastic base. We assume the filtration to be continuous and the set $\Sigma_0$ to contain all $P$-negligible elements in $\Sigma$. Let $(\mathscr{A}, \tau_0, (\mathscr{A}_t))$ be the corresponding non-commutative stochastic base. We identify $\mathscr{A}$ with

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$\mathscr{A} \otimes L^\infty(\mathbf{R})$ and the canonical trace $\tau$ on $\tilde{\mathscr{A}}$ with $\tau_0 \otimes e^{-s}\,ds$ (see [22]). We use the mapping $x \mapsto x \otimes \chi_{]0,\infty[}$ to obtain a homeomorphic image of $\mathscr{A}$ in $\tilde{\mathscr{A}}$. Hence, we have an injective mapping of the set of equivalence classes of random variables on $(\Omega, \Sigma, P)$ onto a subset of $\tilde{\mathscr{A}}$ which preserves the measure topology. By a slight abuse of notation, we shall identify commutative stochastic processes with their non-commutative counterparts.

Suppose $(X_t)_{t\in R_+}$ is a continuous semimartingale with the decomposition, P-a.s., $X_t = X_0 + M_t + A_t^+ - A_t^-$, where $M_0 = A_0^+ = A_0^- = 0$, $(M_t)$ is a continuous local martingale and $(A_t^\pm)$ are continuous, increasing, adapted processes. Put $A_t = A_t^+ + A_t^-$. Let $\mathscr{F}$ (resp. $\mathscr{G}$) denote the class of non-commutative processes corresponding to adapted processes $f$ satisfying

$$P\left(\left\{\int_0^t f(s)^2\, d\langle M\rangle_t < \infty\right\}\right) = 1 \quad \text{for each } t \in \mathbf{R}_+,$$

where $\langle M\rangle_t$ is the quadratic variation of $(M_t)$, (resp.

$$P\left(\left\{\int_0^t f(s)^2\, dA_t < \infty\right\}\right) = 1 \quad \text{for each } t \in \mathbf{R}_+).$$

We endow $\mathscr{F}^{(T)}$ with the following topology $\mathscr{T}^{(T)}$:

$$f_n \to f \quad \text{in } \mathscr{T}^{(T)} \quad \text{if and only if} \quad \int_{t_0}^t |f_n(s) - f(s)|^2\, d\langle M\rangle_t \to 0 \text{ in probability.}$$

Using 3·2·26 and 3·2·27 of [15], we easily check that $\mathscr{F}$ is an admissible class of integrands for $(M_t)$. Similarly, we can make $\mathscr{G}$ into an admissible class of integrands for $(A_t)$. That allows to integrate any element of $\mathscr{F} \cap \mathscr{G}$ with respect to $(X_t)$.

B. *Integration with respect to an increasing process*

Let $(\mathscr{A}, \phi_0, (\mathscr{A}_t)_{t\in R_+})$ be a stochastic base. We assume the filtration to be right continuous. Suppose $(A_t)$ is an increasing, right continuous, adapted $L_1$ process. Then $(A_t)$ defines a countably additive $L^1(\mathscr{A})$-valued measure $\mu$ on Borel subsets of $\mathbf{R}_+$, such that $\mu(]s,t]) = A_t - A_s$, $\mu(\{0\}) = 0$. Moreover, we have

$$\|\mu\|(]s,t]) = |\mu|(]s,t]) = \operatorname{tr}(A_t - A_s),$$

where $\|\mu\|$ and $|\mu|$ denote the semivariation and variation of $\mu$ on $\mathbf{R}_+$, respectively (see [2, 6]). Let $\mathscr{F}$ denote the class of these adapted $L^\infty$ processes $f$ which satisfy:

(i) for any $T$, there is a sequence $f_n$ of simple adapted $L^\infty$ processes on $T$, such that $f_n \to f\,|\,T$ in $|\mu|$-measure (with respect to the $L^\infty$ topology);

(ii) $\sup_{0\leqslant s\leqslant t} \|f(s)\|_\infty < \infty$ for each $t \in \mathbf{R}_+$.

Put $M = \mathbf{R}_+$ and let $\mathscr{F}_\nu^{(T)}$ denote the set of adapted $L^\infty$ processes $f$ on $T$ satisfying $\|f(s)\| \leqslant \nu$ for $s \in T$. We endow it with the $|\mu|$-measure topology (with respect to the $L^\infty$ topology). It is easy to check, using theorem 7 of [6], that $\mathscr{F}$ is an admissible class of integrands for $(A_t)$ and the integral $\int_{t_0}^t f(s)\, dA_s$ is always equal to the Bartle integral $\int_{t_0}^t f(s)\, d\mu(s)$.

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C. *Integration with respect to a dominated martingale*

Let $(\mathscr{A}, \phi_0, (\mathscr{A}_t)_{t\in\mathbf{R}_+})$ be a stochastic base. Assume the filtration to be expected. We call a Radon measure $\mu$ on $\mathbf{R}_+$ (right) dominating for an $L^p$ process $(X_t)$, $(2 \leqslant p \leqslant \infty)$ (which is then dominated by the measure) if, for all $s, t \in \mathbf{R}_+$ with $s < t$,

$$\|\mathscr{E}_s(X_t X_t^*) - X_s X_s^*\|_{p/2} \leqslant \mu([s, t[).$$

(The definition of a left dominating measure is obtained by an obvious modification.)

Let $(X_t)$ be a dominated $L^p$ martingale and let $q$ be such that $1/p + 1/q = \frac{1}{2}$. Denote by $M$ the set of all dominating measures of the process, and by $\mathscr{F}_\mu$ the set of adapted processes in the space $\mathscr{L}^2_{\mathrm{loc}}(\mathbf{R}_+, d\mu; L^q(\mathscr{A}))$ of all $\mu$-measurable maps $f$ from $\mathbf{R}_+$ into $L^q(\mathscr{A})$ such that, for each compact interval $T = [t_0, t]$,

$$\|f\|_{\mu, T} = \left(\int_{t_0}^{t} \|f(s)\|_q^2 \, d\mu(s)\right)^{\frac{1}{2}} < \infty.$$

Put $\mathscr{F} = \bigcup_{\mu\in M} \mathscr{F}_\mu$ and endow $\mathscr{F}_\mu^{(T)}$ with the topology given by the seminorm $\|\cdot\|_{\mu, T}$. We shall show that $\mathscr{F}$ is an admissible class of integrands for $(X_t)$. We start with two lemmas.

LEMMA 6.2. *For any $f \in \mathscr{F}_\mu^{(T)}$ there is a sequence $(f_n)$ of simple adapted $L^q$ processes such that $\|f_n - f\|_{\mu, T} \to 0$.*

*Proof.* For given $\epsilon > 0$, choose a compact set $C \subset T$ such that

$$\int_{T\setminus C} \|f(s)\|_q^2 \, d\mu(s) \leqslant \epsilon/2$$

and that $f|C$ is continuous. Let $K = \sup_{s\in C} \|f(s)\|_q$. Put

$$h(s) = \begin{cases} f(s) & \text{for } s \in C \\ 0 & \text{for } s \in T\setminus C. \end{cases}$$

Note that $h$ is $\mu$-measurable and adapted. Choose now a compact set $D \subset T$ such that $h|D$ is continuous and $\mu(T\setminus D)^{\frac{1}{2}} \leqslant \epsilon/8K$. Since $h|D$ is uniformly continuous on $D$, there is $t_0 < t_1 < \ldots < t_n = t$ such that: $t_1, \ldots, t_{n-1} \in D$; if $t_0 \notin D$, then $D \cap [t_0, t_1[ = \varnothing$; and, finally, $\|h(s) - h(t_k)\|_q \leqslant \epsilon/4\mu(D)$ whenever $t_k \leqslant s < t_{k+1}$ and $s \in D$. In fact, there is $\delta > 0$ such that $\|h(s_1) - h(s_2)\|_q \leqslant \epsilon/4\mu(D)$ for $|s_1 - s_2| < \delta$, $s_1, s_2 \in D$. If $\min D \neq t_0$, put $t_1 = \min D$. Suppose $t_0, \ldots, t_k$ have already been defined. Put $t_{k+1} = \min(D \cap [t_k + \delta, t])$ if $D \cap [t_k + \delta, t] \neq \varnothing$ and $t_{k+1} = t$ otherwise. Define now $g(t) = \sum_{k=0}^{n-1} h(t_k) \chi_{[t_k, t_{k+1}[} + h(t) \chi_{\{t\}}$. It is clear that $g$ is $\mu$-measurable, adapted and simple. We have

$$\begin{aligned} \|f - g\|_{\mu, T} &\leqslant \|f - h\|_{\mu, T} + \|h - g\|_{\mu, T} \\ &\leqslant \|f\|_{\mu, T\setminus C} + \|h - g\|_{\mu, D} + \|h\|_{\mu, T\setminus D} + \|g\|_{\mu, T\setminus D} \\ &\leqslant \epsilon/2 + \mu(D) \cdot \epsilon/4\mu(D) + 2K \cdot \epsilon/8K = \epsilon, \end{aligned}$$

which ends the proof of the lemma.

LEMMA 6.3. *For any simple adapted $L^q$ process $f$ on $T$,*

$$\left\| \int_{t_0}^{t} f(s) \, dX_s \right\|_2^2 \leqslant \int_{t_0}^{t} \|f(s)\|_q^2 \, d\mu(s).$$

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*Proof.* The following standard calculation (see, for example, [2]) yields the result.

$$\begin{aligned}
\left\| \int_{t_0}^{t} f(s)\, dX_s \right\|_2^2 &= \sum_{i,j=1}^{n} \operatorname{tr}\left((X_{t_j}^* - X_{t_{j-1}}^*) f(t_{j-1})^* f(t_{i-1}) (X_{t_i} - X_{t_{i-1}})\right) \\
&= \sum_{i=1}^{n} \operatorname{tr}\left((X_{t_i} - X_{t_{i-1}})(X_{t_i}^* - X_{t_{i-1}}^*) f(t_{i-1})^* f(t_{i-1})\right) \\
&= \sum_{i=1}^{n} \operatorname{tr}\left((\mathscr{E}_{t_{i-1}}(X_{t_i} X_{t_i}^*) - X_{t_{i-1}} X_{t_{i-1}}^*) f(t_{i-1})^* f(t_{i-1})\right) \\
&\leqslant \sum_{i=1}^{n} \operatorname{tr}\left((\mathscr{E}_{t_{i-1}}(X_{t_i} X_{t_i}^*) - X_{t_{i-1}} X_{t_{i-1}}^*)^{p/2}\right)^{2/p} \| f(t_{i-1}) \|_q^2 \\
&\leqslant \sum_{i=1}^{n} \mu([t_{i-1}, t_i[)\, \| f(t_{i-1}) \|_q^2 \\
&= \int_{t_0}^{t} \| f(s) \|_q^2 \, d\mu(s).
\end{aligned}$$

It is an immediate consequence of 6·2 and 6·3 that the set $\mathscr{F}$ satisfies condition (i) of the definition of an admissible class of integrands. We shall show now that condition (ii) is also satisfied. For clarity, we omit the indices $T$. Suppose that $f \in \mathscr{F}_\mu \cap \mathscr{F}_\nu, f_n, g_n \in \mathscr{S}, f_n \to f$ in $\| \cdot \|_\mu$ and $g_n \to f$ in $\| \cdot \|_\nu$. Evidently, $f \in \mathscr{F}_{\mu+\nu}$. By 6·1, there is a sequence $(h_n)$ with $h_n \in \mathscr{S}, h_n \to f$ in $\| \cdot \|_{\mu+\nu}$. Let $c = \lim \int_{t_0}^{t} h_s(s)\, dX_s$. It is clear that the sequence $f_1, h_1, f_2, h_2, \ldots$ converges in $\| \cdot \|_\mu$ to $f$ and the sequence $g_1, h_1, g_2, h_2, \ldots$ converges in $\| \cdot \|_\nu$ to $f$. Hence,

$$\lim \int_{t_0}^{t} f_n(s)\, dX_s = c = \lim \int_{t_0}^{t} g_n(s)\, dX_s,$$

so that the limit does not depend on the choice of the approximating sequence. Thus, $\mathscr{F}$ is an admissible class of integrands for $(X_t)$.

A process $(X_t)$ is right dominated by a measure $\mu$ if and only if the process $(X_t^*)$ is left dominated by $\mu$. It is evident that the admissible class of left integrands for $(X_t)$ given above is also an admissible class of right integrands for $(X_t^*)$, and the formula (2) is satisfied.

There are two important classes of dominated $L^p$ martingales, for which there exists a 'best' dominating measure $\mu_0$, namely, $L^2$ martingales and quasi-Wiener processes. We shall now have a closer look at these two classes.

Assume first that $(X_t)$ is a left continuous $L^2$ martingale. (Any $L^p$ martingale is automatically left continuous if the filtration is assumed to be left continuous.) Put

$$\mu_0([s, t[) = \operatorname{tr}(X_t X_t^* - X_s X_s^*).$$

It is readily checked that $\mu_0$ can be extended to a Radon measure on $\mathbf{R}_+$ which dominates the martingale. The class of admissible integrands $\mathscr{F} = \mathscr{F}_{\mu_0}$ consists then of all the adapted processes from $\mathscr{L}^2_{\mathrm{loc}}(\mathbf{R}_+, d\mu_0; L^\infty(\mathscr{A}))$. The theory obtained generalizes that from [4] in that it does not require the algebra to be finite and also that the class of integrands is larger than that in [4]. If we relax our definition so that the a.e. defined adapted processes are admitted, then the class of integrands can be made into a Hilbert space, as in [2].

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We say that an adapted $L^p$ process is (right) quasi-Wiener, if, for some left continuous increasing function $F: \mathbf{R}_+ \to \mathbf{R}$, $X_t X_t^* - F(t)\, h_0^{2/p}$ is a martingale (cf. [1]). Our 'best' measure is given now by $\mu_0([s,t[) = F(t) - F(s)$. The class of admissible integrands $\mathscr{F} = \mathscr{F}_{\mu_0}$ consists of all the adapted processes in $\mathscr{L}^2_{\mathrm{loc}}(\mathbf{R}_+, d\mu_0; L^q(\mathscr{A}))$. Important examples of such processes are listed below (the description of appropriate filtrations can be found in the papers cited with each entry).

1. The classical Wiener process treated as a non-commutative $L^2$ process [5] (the embedding into $\tilde{\mathscr{A}}$ is different from that considered in point $A$): $F(t) = t$.

2. The fermion field process $\Psi$ in a Clifford algebra treated as an $L^\infty$ process [2]: $F(t) = t$.

3. Wick martingales in a Clifford algebra treated as $L^\infty$ processes [2]: for the $L^\infty$ martingale

$$W(u_1, \ldots, u_n; s) =: \Psi(u_1 \chi_{[0,s]}) \ldots \Psi(u_n \chi_{[0,s]}),$$

where $u_1, \ldots, u_n \in L^2_{\mathrm{loc}}(\mathbf{R}_+)$ and $\Psi$ is the fermion field process, we have

$$F(t) = \int_0^t \ldots \int_0^t |u(t_1, \ldots, t_n)|^2 \, dt_1 \ldots dt_n.$$

4. Linear combinations of the creation and annihilation processes in a quasi-free representation of the CAR and CCR algebras [5]: Let $\omega$ be the gauge-invariant quasi-free state on the CAR algebra, determined by the multiplication operator by some function $\rho$ on $\mathbf{R}_+$ with values in the interval $[0, 1]$ and bounded away (a.e.) from 0 and 1. Define an $L^p$ process

$$X_t = (\alpha_1 A^*(u\chi_{[0,t]}) + \alpha_2 A(u\chi_{[0,t]}))\, h_0^{1/p},$$

where $\alpha_1, \alpha_2 \in \mathbf{C}$, $u \in L^2_{\mathrm{loc}}(\mathbf{R}_+)$ and $A^*, A$ are the creation and annihilation processes, respectively. We have

$$F(t) = |\alpha_1|^2 \int_0^t |u(s)|^2 \rho(s)\, ds + |\alpha_2|^2 \int_0^t |u(s)|^2 (1 - \rho(s))\, ds.$$

In [5], for the definition of the left stochastic integral $(X_t)$ is treated as an $L^\infty$ process, and that of the right stochastic integral as an $L^2$ process. To relate the two integrals to each other, some additional information is needed, in this case supplied by the canonical anticommutation relations (see [5], lemma 3·3).

The case of the CCR algebra can be treated similarly. The integrator is now an $L^2$ process defined by

$$X_t = (\alpha_1 A^*(u\chi_{[0,t]}) + \alpha_2 A(u\chi_{[0,t]}))\, h_0^{\frac{1}{2}},$$

where $A^*(u\chi_{[0,t]})$ and $A(u\chi_{[0,t]})$ are treated as unbounded operators acting in $L^2(\mathscr{A})$.

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