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Mean dimension and a sharp embedding theorem: extensions of aperiodic subshifts

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Abstract. We show that if (X, T) is an extension of an aperiodic subshift (a subsystem of $(\{1, 2, \ldots, l\}^{\mathbb{Z}}, \text{shift})$ for some $l \in \mathbb{N}$) and has mean dimension $\text{mdim}(X, T) < (D/2)$, $D \in \mathbb{N}$, then it can be equivariantly embedded in $(([0, 1]^D)^{\mathbb{Z}}, \text{shift})$. The result is sharp. If (X, T) is an extension of an aperiodic zero-dimensional system then it can be equivariantly embedded in $(([0, 1]^{D+1})^{\mathbb{Z}}, \text{shift})$.

1. *Introduction*

Let (X, T) be a topological dynamical system, that is, X is a compact metric space and $T : X \to X$ is a homeomorphism. Let D be a positive integer, and let $[0, 1]^D$ be the D-dimensional unit cube. Let $(([0, 1]^D)^{\mathbb{Z}}, \sigma)$ be the shift on $[0, 1]^D$, that is, $\sigma(x)_n \triangleq x_{n+1}$ for $x = (x_n)_{n\in\mathbb{Z}} \in ([0, 1]^D)^{\mathbb{Z}}$. We study the following problem in this paper.

Problem 1.1. When does there exist an embedding $\phi : (X, T) \to (([0, 1]^D)^{\mathbb{Z}}, \sigma)$? Here ϕ is called an (equivariant) embedding if ϕ is a topological embedding and $\phi T = \sigma\phi$.

Jaworski [**Jaw74**] proved that if (X, T) is a finite-dimensional system and is aperiodic (i.e. it has no periodic points), then (X, T) can be embedded into the system $([0, 1]^{\mathbb{Z}}, \sigma)$. (See also Auslander [**Aus88**, Ch. 13, Theorem 9].) Clearly, if a system has periodic points, this may constitute an obstruction to embedding into $(([0, 1]^D)^{\mathbb{Z}}, \sigma)$. To quantify this we introduce an invariant called *periodic dimension*§ $\text{perdim}(X, T) = \sup_{n\in\mathbb{N}}(\dim(P_n)/n)$, where P_n denotes the set of points of period less than or equal to n. One readily checks that a necessary condition for embedding into $(([0, 1]^D)^{\mathbb{Z}}, \sigma)$ is $\text{perdim}(X, T) \le D$. However, one can construct systems such that $\text{perdim}(X, T) = D/2$ but the minimal d for

§ This is the definition of periodic dimension for finite-dimensional systems. For infinite-dimensional systems a slightly more complicated definition is used. See [**Gut12**, §2.4].

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which (X, T) is embedded in $(([0, 1]^D)^{\mathbb{Z}}, \sigma)$ is $d = D + 1$. In [**Gut12**] it is shown that if (X, T) is finite-dimensional and $\operatorname{perdim}(X, T) < D/2$, then (X, T) can be embedded into $(([0, 1]^D)^{\mathbb{Z}}, \sigma)$.

For infinite-dimensional systems the situation is very different. Lindenstrauss and Weiss [**LW00**, Proposition 3.3] constructed a minimal infinite-dimensional system which cannot be embedded into the system $([0, 1]^{\mathbb{Z}}, \sigma)$ by using the theory of mean dimension.

Mean dimension (denoted by $\operatorname{mdim}(X, T)$) is a topological invariant of dynamical systems introduced by Gromov [**Gro99**] and systematically investigated in [**LW00**]. The explicit formula appears at the end of §2. The mean dimension of the system $(([0, 1]^D)^{\mathbb{Z}}, \sigma)$ is equal to D. If a system (X, T) can be embedded into another system (X', T'), then $\operatorname{mdim}(X, T) \leq \operatorname{mdim}(X', T')$. So if (X, T) can be embedded into $(([0, 1]^D)^{\mathbb{Z}}, \sigma)$, then we must have $\operatorname{mdim}(X, T) \leq D$. Hence the mean dimension is also an obstruction to embedding a system into $(([0, 1]^D)^{\mathbb{Z}}, \sigma)$.

Surprisingly, Lindenstrauss [**Lin99**, Theorem 5.1] proved the following partial converse.

THEOREM 1.2. *If (X, T) is an extension of an aperiodic minimal system and satisfies $\operatorname{mdim}(X, T) < D/36$, then (X, T) can be embedded into the system $(([0, 1]^D)^{\mathbb{Z}}, \sigma)$.*

In [**Gut13**] it is shown one can replace the assumption of an aperiodic minimal system with the assumption of an aperiodic factor with a countable number of minimal subsystems or an aperiodic finite-dimensional factor. In [**Gut12**] the following theorem is proven.

THEOREM 1.3. *If (X, T) is an extension of an aperiodic finite-dimensional system and satisfies $\operatorname{mdim}(X, T) < D/16$, then (X, T) can be embedded into the system $(([0, 1]^{D+1})^{\mathbb{Z}}, \sigma)$.*

It is clear that both mean dimension and periodic dimension are obstructions for the embedding. Recently, Lindenstrauss and Tsukamoto conjectured that these are the only obstructions. Specifically, [**LT12**, Conjecture 1.2] states the following conjecture.

CONJECTURE 1.4. *If $\operatorname{mdim}(X, T) < D/2$ and $\operatorname{perdim}(X, T) < D/2$, then (X, T) can be embedded into the system $(([0, 1]^D)^{\mathbb{Z}}, \sigma)$.*

Our goal in this paper is to establish this conjecture in a special (infinite-dimensional aperiodic) case and show that this result is sharp. As mentioned before, [**Gut12**] contains a proof that the conjecture is true if X is finite-dimensional.

Throughout this paper we use the following notation: for a map $f : X \to [0, 1]^D$ we define $I_f : X \to ([0, 1]^D)^{\mathbb{Z}}$ by $I_f(x) \triangleq (f(T^n x))_{n \in \mathbb{Z}}$. Our main result is the following theorem.

THEOREM 1.5. *Let D be a positive integer. Let (Z, S) be an aperiodic zero-dimensional system, and let $\pi : (X, T) \to (Z, S)$ be an extension of (Z, S). If $\operatorname{mdim}(X, T) < D/2$, then there exists a continuous map $f : X \to [0, 1]^D$ such that*

$$(I_f, \pi) : X \to ([0, 1]^D)^{\mathbb{Z}} \times Z, \quad x \mapsto (I_f(x), \pi(x)),$$

is an embedding. Indeed, such continuous maps $f : X \to [0, 1]^D$ form a comeagre subset of $C(X, [0, 1]^D)$ (the space of continuous maps from X to $[0, 1]^D$).

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Example 1.6. Let (X, T) be an arbitrary dynamical system, and let (Z, S) be a zero-dimensional system. As $\mathrm{mdim}(X \times Z, T \times S) \leq \mathrm{mdim}(X, T) + \mathrm{mdim}(Z, S)$ [**LW00**, Proposition 2.8], the product system $(X, T) \times (Z, S)$ is an extension of (Z, S), whose mean dimension is equal to the mean dimension of (X, T).

Theorem 1.5 enables us to give a partial solution to Problem 1.1 by using the following corollary.

COROLLARY 1.7. *Let D be a positive integer, and let (X, T) be an extension of an aperiodic zero-dimensional system. If* $\mathrm{mdim}(X, T) < D/2$, *then (X, T) can be embedded into* $(([0, 1]^{D+1})^{\mathbb{Z}}, \sigma)$.

Proof. If (Z, S) is a zero-dimensional system, then (Z, S) can be embedded into $([0, 1]^{\mathbb{Z}}, \sigma)$. Indeed there exists a topological embedding $g : Z \to [0, 1]$. Hence $I_g : (Z, S) \to ([0, 1]^{\mathbb{Z}}, \sigma)$ gives an embedding of the system. Therefore, the product system $(([0, 1]^D)^{\mathbb{Z}}, \sigma) \times (Z, S)$ can be embedded into the system $(([0, 1]^{D+1})^{\mathbb{Z}}, \sigma)$. Corollary 1.7 now follows from Theorem 1.5. □

In the context of Corollary 1.7, it should be noted that for $\mathbb{Z}^k$-extensions of aperiodic zero-dimensional systems $(\mathbb{Z}^k, X)$, it was shown in [**Gut11**] that there exist constants $C(k) > 0$ so that $(\mathbb{Z}^k, X)$ can be embedded in $(([0, 1]^{\lfloor C(k)\mathrm{mdim}(\mathbb{Z}^k, X)\rfloor + 1})^{\mathbb{Z}^k}, \mathbb{Z}^k\text{-shift})$.

If we assume a stronger condition on the factor (Z, S) in Corollary 1.7, then we can get a sharp result as follows. Let l be a positive integer, and let $(\{1, 2, \ldots, l\}^{\mathbb{Z}}, \text{shift})$ be the shift on the alphabet $\{1, 2, \ldots, l\}$. A subsystem (closed shift-invariant subset) of such a topological dynamical system is called a *subshift* (see [**Wil04**]).

COROLLARY 1.8. *Let D be a positive integer, and let (X, T) be an extension of an aperiodic subsystem of* $(\{1, 2, \ldots, l\}^{\mathbb{Z}}, \text{shift})$. *If* $\mathrm{mdim}(X, T) < D/2$, *then (X, T) can be embedded into* $(([0, 1]^D)^{\mathbb{Z}}, \sigma)$.

Proof. Suppose that (Z, S) is an aperiodic subsystem of $(\{1, 2, \ldots, l\}^{\mathbb{Z}}, \text{shift})$ and that (X, T) is an extension of (Z, S). Since Z is zero-dimensional, Theorem 1.5 implies that (X, T) can be embedded into the system $(([0, 1]^D)^{\mathbb{Z}}, \sigma) \times (Z, S)$. From the assumption, the latter is a subsystem of

$$(([0, 1]^D \times \{1, 2, \ldots, l\})^{\mathbb{Z}}, \text{shift}). \tag{1.1}$$

The space $[0, 1]^D \times \{1, 2, \ldots, l\}$ can be topologically embedded into $[0, 1]^D$. Thus system (1.1) can be embedded into $(([0, 1]^D)^{\mathbb{Z}}, \sigma)$. □

Corollary 1.8 is analogous to the following classical result in dimension theory [**HW41**, p. 56, Theorem V 2]: if a compact metric space X satisfies $\dim X < D/2$, then X can be topologically embedded into $[0, 1]^D$. The following proposition shows that the condition $\mathrm{mdim}(X, T) < D/2$ in Corollary 1.8 is optimal.

PROPOSITION 1.9. *Let (Z, S) be a zero-dimensional system. For any positive integer D, there exists an extension (X, T) of (Z, S) such that* $\mathrm{mdim}(X, T) = D/2$ *and (X, T) cannot be embedded into* $(([0, 1]^D)^{\mathbb{Z}}, \sigma)$.

Indeed this proposition is a corollary of a stronger result (Proposition 4.2) given in §4.

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2. *Preliminaries*

In this section we present some basic facts about mean dimension theory. For more details, see Gromov [**Gro99**] and Lindenstrauss and Weiss [**LW00**]. Let (X, d) be a compact metric space, Y be a topological space and $f : X \to Y$ be a continuous map. Then f is called an ε-embedding for $\varepsilon > 0$ if $\operatorname{diam}(f^{-1}(y)) < \varepsilon$ for every $y \in Y$. We define $\operatorname{Widim}_\varepsilon(X, d)$ as the minimum of $n \geq 0$ such that there exist an n-dimensional polytope (finite simplicial complex) P and an ε-embedding $f : X \to P$. Equivalently, $\operatorname{Widim}_\varepsilon(X, d)$ is the minimum of $n \geq 0$ such that there exists an open covering $\{U_i\}_{i=1}^N$ of X satisfying $\operatorname{diam}(U_i) < \varepsilon$ $(1 \leq i \leq N)$ and $U_{i_1} \cap U_{i_2} \cap \cdots \cap U_{i_{n+2}} = \emptyset$ for all $1 \leq i_1 < i_2 < \cdots < i_{n+2} \leq N$ (i.e. the order of $\{U_i\}_{i=1}^N$ is at most n). $\operatorname{Widim}_\varepsilon(X, d)$ is monotone non-decreasing as $\varepsilon \to 0$.

The following is a key property of $\operatorname{Widim}_\varepsilon(X, d)$ in the study of our embedding problem.

LEMMA 2.1. *Let (X, d) be a compact metric space, and let $f : X \to [0, 1]^M$ be a continuous map. Suppose that positive numbers δ and ε satisfy the following condition:*

$$d(x, y) < \varepsilon \Rightarrow \|f(x) - f(y)\|_\infty < \delta.$$

(Here $\|\cdot\|_\infty$ is the ℓ^∞-norm.) Under this condition, if $\operatorname{Widim}_\varepsilon(X, d) < M/2$, then there exists an ε-embedding $g : X \to [0, 1]^M$ satisfying $\sup_{x \in X} \|f(x) - g(x)\|_\infty < \delta$.

Proof. Set $a \triangleq \operatorname{Widim}_\varepsilon(X, d)$. Let $X = \bigcup_{i=1}^N U_i$ be an open covering of order a satisfying $\operatorname{diam}(U_i) < \varepsilon$. Fix $x_i \in U_i$, and let $\{\varphi_i\}_{i=1}^N$ be a partition of unity satisfying $\operatorname{supp} \varphi_i \subset U_i$. From the assumption on δ and ε, we have

$$s \triangleq \sup_{x \in \operatorname{supp} \varphi_i} \|f(x) - f(x_i)\|_\infty < \delta.$$

Since $2a < M$, we can choose $u_i \in [0, 1]^M$ $(1 \leq i \leq N)$ satisfying

$$\|u_i - f(x_i)\|_\infty < \delta - s$$

and the following condition: if there are $J, K \subset \{1, 2, \ldots, N\}$ with $|J|, |K| \leq a + 1$, $\{\lambda_i\}_{i \in J} \in (\mathbb{R} \backslash \{0\})^{|J|}$ and $\{\mu_i\}_{i \in K} \in (\mathbb{R} \backslash \{0\})^{|K|}$ such that

$$\sum_{i \in J} \lambda_i = \sum_{i \in K} \mu_i = 1, \quad \sum_{i \in J} \lambda_i u_i = \sum_{i \in K} \mu_i u_i,$$

then $J = K$ and $\lambda_i = \mu_i$ for all $i \in J = K$. (The existence of such $\{u_i\}_{i=1}^N$ follows from the fact that one can choose almost surely (with respect to Lebesgue measure) in $([0, 1]^M)^N$, $\{u_i\}_{i=1}^N$ to be affinely independent. See [**Gut12**, Lemma A.8].)

We define $g : X \to [0, 1]^M$ by setting $g(x) \triangleq \sum_{i=1}^N \varphi_i(x) u_i$. We have $g(x) - f(x) = \sum_i \varphi_i(x)(u_i - f(x))$. If $\varphi_i(x) \neq 0$ then, from the definition of u_i,

$$\|u_i - f(x)\|_\infty \leq \|u_i - f(x_i)\|_\infty + \|f(x_i) - f(x)\|_\infty < \delta.$$

Hence $\|g(x) - f(x)\|_\infty < \delta$ for all $x \in X$.

If $g(x) = g(y)$ for some $x, y \in X$, then the choice of $\{u_i\}$ implies that there exists $1 \leq i \leq N$ satisfying $\varphi_i(x) = \varphi_i(y) > 0$. Hence $x, y \in U_i$. Then $d(x, y) \leq \operatorname{diam}(U_i) < \varepsilon$. This shows that g is an ε-embedding. □

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Suppose that $T : X \to X$ is a homeomorphism. For two integers $a < b$ we define a new distance d_a^b by setting $d_a^b(x, y) \triangleq \max_{a \le i \le b} d(T^i x, T^i y)$. We define the mean dimension $\mathrm{mdim}(X, T)$ by

$$\mathrm{mdim}(X, T) \triangleq \sup_{\varepsilon > 0} \left(\lim_{n \to +\infty} \frac{\mathrm{Widim}_\varepsilon(X, d_0^{n-1})}{n} \right).$$

This limit always exists. (The limit value can be ∞.)

3. *Proof of Theorem 1.5*

In this section we prove Theorem 1.5. The basic structure of the proof is the same as Lindenstrauss [**Lin99**, §5]. Lindenstrauss proved Theorem 1.2 by using a Rokhlin tower-like lemma [**Lin99**, Lemma 3.3]. The proof of this tower lemma uses the assumption that (X, T) is an extension of an aperiodic minimal system. Here we assume that (X, T) is an extension of an aperiodic zero-dimensional system. This assumption implies a much stronger tower lemma (Lemma 3.3), and thus we are able to prove a sharp result for the embeddability of (X, T). Throughout this section, (X, d) is a compact metric space and $T : X \to X$ is a homeomorphism such that there exist an aperiodic zero-dimensional system (Z, S) and a factor map $\pi : (X, T) \to (Z, S)$. Moreover, we assume that $\mathrm{mdim}(X, T) < D/2$ for a positive integer D. To keep our notation simple, we set $K \triangleq [0, 1]^D$.

For $\eta > 0$, we define a subset $A(\eta)$ of $C(X, K)$ by

$$A(\eta) \triangleq \{f \mid (I_f, \pi) : X \to K^{\mathbb{Z}} \times Z \text{ is an } \eta\text{-embedding with respect to } d\}.$$

It is easy to see that $A(\eta)$ is an open set. We will prove the following proposition.

PROPOSITION 3.1. *For any continuous $f : X \to K$, $\delta > 0$ and $\eta > 0$ there exists a continuous map $g : X \to K$ such that:*

(i) $\|f(x) - g(x)\|_\infty < \delta$ *for all* $x \in X$;

(ii) $(I_g, \pi) : X \to K^{\mathbb{Z}} \times Z$ *is an η-embedding with respect to the distance d.*

Assume this proposition. Then $A(\eta)$ is dense in $C(X, K)$. The Baire category theorem implies that

$$\bigcap_{n=1}^{\infty} A(1/n)$$

is comeagre and hence dense. Then every f in this set gives a desired solution. So the main problem is to prove Proposition 3.1.

3.1. *Representation lemma.*

Recall that (Z, S) is an aperiodic zero-dimensional system, and that the zero-dimensionality implies that clopen (closed and open) subsets form a basis of the topology of Z.

We start with a representation lemma for (Z, S). As it is similar to the well-known concept of a *Kakutani skyscraper decomposition* in measured dynamics (see [**Pet83**, Proof of Lemma 4.7 on p. 48]; the decomposition first appeared in [**Kak43**]) we call such a representation a *Kakutani skyscraper*.

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Definition 3.2. Let B be a zero-dimensional compact metric space. Let $T_B : B \to B$ be a homeomorphism. Let $h : B \to \mathbb{Z}_{>0}$ be continuous. Define $Q = \{(b, z) |\ b \in B,\ 0 \le z < h(b)\} \subset B \times \mathbb{Z}_{\ge 0}$. As h is bounded, this is a compact metric zero-dimensional space. Define $T : Q \to Q$ by

$$T(b, z) = \begin{cases} (b, z+1) & \text{for } z+1 < h(b), \\ (T_B(b), 0) & \text{for } z+1 = h(b). \end{cases}$$

Clearly T is a homeomorphism. B is referred to as the *base* of Q. $h : B \to \mathbb{Z}_{>0}$ is referred to as the *roof function* of Q. (Q, T) is referred to as a *Kakutani skyscraper* and it is said to be *induced* by (B, T_B, h).

LEMMA 3.3. *For any positive integer N there exist a clopen set $B \subset Z$, a continuous function $h : B \to R_N \triangleq \{N+1, \dots, 2N+1\}$ and a continuous homeomorphism $T_B : B \to B$ such that (Z, S) is isomorphic (as a dynamical system) to the Kakutani skyscraper (Q_N, T') induced by (B, T_B, h). We denote this isomorphism by $(b, n) : Z \to Q_N \subset B \times \mathbb{Z}_{\ge 0}$.*

Notation. We say that such B, T_B, h are *associated* with N.

Proof. There exist clopen sets $U_1, \dots, U_m$ such that $Z = U_1 \cup \cdots \cup U_m$ and $U_i \cap S^k U_i = \emptyset$ for all $1 \le i \le m$ and $1 \le k \le N$. We define clopen sets V_l $(1 \le l \le m)$ by $V_1 \triangleq U_1$ and $V_{l+1} \triangleq V_l \cup (U_{l+1} \setminus \bigcup_{k=-N}^{N} S^k V_l)$. We can directly check the following two conditions by induction on $l = 1, \dots, m$:

$$V_l \cap S^k V_l = \emptyset \quad (1 \le l \le m,\ 1 \le k \le N),$$
$$U_1 \cup \cdots \cup U_l \subset \bigcup_{k=-N}^{N} S^k V_l \quad (1 \le l \le m).$$

Then the clopen set $B \triangleq V_m$ satisfies $B \cap S^k B = \emptyset$ $(1 \le k \le N)$ and $Z = \bigcup_{k=1}^{2N+1} S^k B$. Define the continuous function $h : B \to R_N \triangleq \{N+1, \dots, 2N+1\}$ by $h(b) = \min\{n \ge 1 \mid S^n b \in B\}$. Clearly $N < h(b) \le 2N+1$. Define $T_B : B \to B$ by $T_B b = S^{h(b)} b$. T_B is clearly continuous as h is locally constant. T_B is invertible as $T_B^{-1} b = S^{-h'(b)} b$, where $h'(b) = \min\{l \ge 1 \mid S^{-l} b \in B\}$, and clearly $h(T_B^{-1} b) = h'(b)$. We define $n : Z \to \{0, \dots, 2N\}$ by $n(z) \triangleq \min\{l \ge 0 \mid S^{-l} z \in B\}$. Let (Q_N, T') be the Kakutani skyscraper induced by (B, T_B, h). Define $b : Z \to B$ by $b(z) = S^{-n(z)} z$. Then $(b, n) : (Z, S) \to (Q_N, T')$ is an equivariant homeomorphism. □

3.2. *Proof of Proposition 3.1.* Throughout this subsection we fix a continuous function $f : X \to K$ (recall that $K = [0, 1]^D$) and positive numbers δ and η. Fix $0 < \varepsilon < \eta$ such that

$$d(x, y) < \varepsilon \Rightarrow \|f(x) - f(y)\|_\infty < \delta. \tag{3.1}$$

Since $\mathrm{mdim}(X, T) < D/2$, we can take a positive integer N such that, for every $k \ge N$,

$$\frac{\mathrm{Widim}_\varepsilon(X, d_0^{k-1})}{k} < \frac{D}{2}.$$

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Applying Lemma 3.3 to (Z, S), let $Q = Q_N(B, T_B, h)$ be the Kakutani skyscraper induced by (B, T_B, h) associated with N. Let $B_k = \pi^{-1}(h^{-1}(\{k\}))$, $k \in R_N \triangleq \{N + 1, \ldots, 2N + 1\}$. These are clopen sets. We consider the function $(I_f|_0^{k-1})|_{B_k}(x) \triangleq (f(x), f(Tx), \ldots, f(T^{k-1}x)) \in K^k$ on the metric space (B_k, d_0^{k-1}). Notice that, by (3.1), $d_0^{k-1}(x, y) < \varepsilon$ implies that $\|I_f|_0^{k-1}(x) - I_f|_0^{k-1}(y)\|_\infty < \delta$. By Lemma 2.1 there is an ε-embedding $F_k : (B_k, d_0^{k-1}) \to K^k$ satisfying

$$\sup_{x \in B_k} \|F_k(x) - I_f(x)|_0^{k-1}\|_\infty < \delta. \tag{3.2}$$

Define $g : X \to K$ by $g(T^l x) \triangleq F_k(x)|_l$ for $x \in B_k$ and $0 \le l < k$. Clearly g is continuous and by (3.2) $\sup_{x \in X} \|g(x) - f(x)\|_\infty < \delta$.

Now assume that $(I_g(x), \pi(x)) = (I_g(y), \pi(y))$. As $\pi(x) = \pi(y)$ we have $b(\pi(x)) = b(\pi(y))$ and $n \triangleq n(\pi(x)) = n(\pi(y))$. (Here $b(\cdot)$ and $n(\cdot)$ are the maps introduced in Lemma 3.3.) Conclude that there exists $k \in R_N$ so that $0 \le n < k$ and $T^{-n}x, T^{-n}y \in B_k$. From $I_g(T^{-n}x)|_0^{k-1} = I_g(T^{-n}y)|_0^{k-1}$ we conclude that $F_k(T^{-n}x) = F_k(T^{-n}y)$. As $F_k : (B_k, d_0^{k-1}) \to K^k$ is an ε-embedding we conclude that $d_0^{k-1}(T^{-n}x, T^{-n}y) < \varepsilon$ which implies that $d(x, y) < \varepsilon$. □

4. *Proof of Proposition 1.9*

The argument in this section is a slight modification of the argument of Lindenstrauss and Tsukamoto [**LT12**, Lemma 3.3]. Let Y be the triod graph (the graph of the shape 'Y'). Let d_Y be the graph distance on Y where all three edges have length 1. Consider Y^n with the distance $d_{\ell^\infty}(x, y) \triangleq \max_{1 \le i \le n} d_Y(x_i, y_i)$. The following fact is proven in [**LT12**, Proposition 2.5] by using the method of Matoušek [**Mat03**].

LEMMA 4.1. *For any $0 < \varepsilon < 1$ there does not exist an ε-embedding from the space (Y^n, d_{ℓ^∞}) to $\mathbb{R}^{2n-1}$.*

Let D be a positive integer. In [**LT12**, §3] a compact metric space (X, d) with a homeomorphism $T : X \to X$ satisfying the following two conditions was constructed.

(i) $\mathrm{mdim}(X, T) = D/2$.

(ii) There exist sequences of positive integers $\{c_n\}_{n \ge 1}$ and $\{b_n\}_{n \ge 1}$ such that†

$$\lim_{n \to \infty} b_n = \infty, \quad \lim_{n \to \infty} \left(c_n - \frac{Db_n}{2} \right) = \infty,$$

and so that there exists a distance-increasing‡ continuous map from $(Y^{c_n}, d_{\ell^\infty})$ to $(X, d_0^{b_n - 1})$ for every $n \ge 1$ [**LT12**, Lemma 3.1]. Moreover, one can assume that (X, T) is minimal. But this is not used below.

Proposition 1.9 follows from the discussion in Example 1.6 and the following proposition. (We set $K = [0, 1]^D$ as in §3.)

PROPOSITION 4.2. *For any dynamical system (Z, S), the product system $(X, T) \times (Z, S)$ cannot be embedded into $(K^{\mathbb{Z}}, \sigma)$.*

† In the notation of [**LT12**, §3], we have $c_n = Db_n \prod_{k=1}^{n}(1 - p_k)$.

‡ A map $f : (Y^{c_n}, d_{\ell^\infty}) \to (X, d_0^{b_n - 1})$ is said to be distance-increasing if $d_0^{b_n - 1}(f(x), f(y)) \ge d_{\ell^\infty}(x, y)$.

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Proof. Let d_Z be the distance on Z. We define a distance on $X \times Z$ by $\mathrm{dist}((x, z), (x', z')) \triangleq \max(d(x, x'), d_Z(z, z'))$. Fix $p \in Z$. The map

$$X \to X \times Z, \quad x \mapsto (x, p),$$

gives an isometric embedding from (X, d_0^{n-1}) to $(X \times Z, \mathrm{dist}_0^{n-1})$ for every $n \geq 1$. From property (ii) of (X, T), there exists a distance-increasing continuous map from $(Y^{c_n}, d_{\ell\infty})$ to $(X \times Z, \mathrm{dist}_0^{b_n-1})$ for every $n \geq 1$. From this point on one can use the proof of [**LT12**, Lemma 3.3] with X replaced by $X \times Z$ verbatim.

Suppose that there exists an embedding $f : (X \times Z, T \times S) \to (K^{\mathbb{Z}}, \sigma)$. Let d' be a distance on $K^{\mathbb{Z}}$. There exists $\varepsilon > 0$ such that if $x, y \in X \times Z$ satisfy $d'(f(x), f(y)) < \varepsilon$ then $\mathrm{dist}(x, y) < 1/2$. Then, for any $N \geq 1$, $d'^{N-1}_0(f(x), f(y)) < \varepsilon$ implies that $\mathrm{dist}_0^{N-1}(x, y) < 1/2$.

We can take $L = L(\varepsilon) > 0$ such that if $x, y \in K^{\mathbb{Z}}$ satisfy $x_n = y_n$ $(-L \leq n \leq L)$ then $d'(x, y) < \varepsilon$. This implies that, for any $N \geq 1$, if $x, y \in K^{\mathbb{Z}}$ satisfy $x_n = y_n$ $(-L \leq n \leq N + L - 1)$, then we have $d'^{N-1}_0(x, y) < \varepsilon$. Let $\pi_{[-L,N+L-1]} : K^{\mathbb{Z}} \to K^{\{-L,-L+1,\ldots,N+L-1\}}$, $x \mapsto (x_n)_{-L \leq n \leq N+L-1}$, be the projection. From the above discussion, $\pi_{[-L,N+L-1]} \circ f : (X \times Z, \mathrm{dist}_0^{N-1}) \to K^{\{-L,-L+1,\ldots,N+L-1\}}$ is a $(1/2)$-embedding. Since there exists a distance-increasing continuous map from $(Y^{c_n}, d_{\ell\infty})$ to $(X \times Z, \mathrm{dist}_0^{b_n-1})$, we conclude that there exists a $(1/2)$-embedding from $(Y^{c_n}, d_{\ell\infty})$ to $K^{\{-L,-L+1,\ldots,b_n+L-1\}} = [0, 1]^{D(b_n+2L)}$ for every $n \geq 1$. Applying Lemma 4.1 to this situation, we get $2c_n \leq D(b_n + 2L)$. Hence

$$c_n - \frac{Db_n}{2} \leq LD.$$

This contradicts condition (ii): $\lim_{n\to\infty}(c_n - Db_n/2) = \infty$. □

References

[Aus88] J. Auslander. *Minimal Flows and their Extensions (North-Holland Mathematics Studies: Notas de Matemática [Mathematical Notes], 153)*. North-Holland, Amsterdam, 1988.

[Gro99] M. Gromov. Topological invariants of dynamical systems and spaces of holomorphic maps. I. *Math. Phys. Anal. Geom.* **2**(4) (1999), 323–415.

[Gut11] Y. Gutman. Embedding $\mathbb{Z}^k$-actions in cubical shifts and $\mathbb{Z}^k$-symbolic extensions. *Ergod. Th. & Dynam. Sys.* **31**(2) (2011), 383–403.

[Gut12] Y. Gutman. Mean dimension and Jaworski-type theorems. *Preprint*, 2012, http://arxiv.org/abs/1208.5248.

[Gut13] Y. Gutman. Dynamical embedding in cubical shifts and the topological Rokhlin and small boundary properties. *Preprint*, 2013, http://arxiv.org/abs/1301.6072.

[HW41] W. Hurewicz and H. Wallman. *Dimension Theory (Princeton Mathematical Series, 4)*. Princeton University Press, Princeton, NJ, 1941.

[Jaw74] A. Jaworski. The Kakutani–Beboutov theorem for groups. *PhD Dissertation*, University of Maryland, 1974.

[Kak43] S. Kakutani. Induced measure preserving transformations. *Proc. Imp. Acad. Tokyo* **19** (1943), 635–641.

[Lin99] E. Lindenstrauss. Mean dimension, small entropy factors and an embedding theorem. *Publ. Math. Inst. Hautes Études Sci.* **89**(1) (1999), 227–262.

https://doi.org/10.1017/etds.2013.30 Published online by Cambridge University Press

[LT12] E. Lindenstrauss and M. Tsukamoto. Mean dimension and an embedding problem: an example. *Israel J. Math.*, to appear. *Preprint*, 2012.

[LW00] E. Lindenstrauss and B. Weiss. Mean topological dimension. *Israel J. Math.* **115** (2000), 1–24.

[Mat03] J. Matoušek. Using the Borsuk–Ulam theorem. *Universitext (Lectures on Topological Methods in Combinatorics and Geometry)*. Springer, Berlin, 2003, written in cooperation with Anders Björner and Günter M. Ziegler.

[Pet83] K. Petersen. *Ergodic Theory (Cambridge Studies in Advanced Mathematics, 2)*. Cambridge University Press, Cambridge, 1983.

[Wil04] S. G. Williams. Introduction to symbolic dynamics. *Symbolic Dynamics and its Applications (Proceedings of Symposia in Applied Mathematics, 60)*. American Mathematical Society, Providence, RI, 2004, pp. 1–11.

https://doi.org/10.1017/etds.2013.30 Published online by Cambridge University Press